ORDINAL INVARIANTS FOR TOPOLOGICAL SPACES

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0. INTRODUCTION

Cardinal invariants such as weight, density, and dimension have been widely used in the classification of topological spaces. More rarely (see for example Maurice [10] and Stone [13]), ordinal invariants have been employed. In this paper, we introduce two related ordinal invariants, σ and κ, first in the categories of sequential and k-spaces (Section 1) and later in arbitrary spaces (Section 6). (For an informed opinion of the importance of the category of k-spaces, see Steenrod [12]). Our main result is the existence, for each $\alpha \leq \omega_1$, of a countable, zero-dimensional Hausdorff space X with $\sigma(X) = \kappa(X) = \alpha$ (Theorems 4.1 and 5.1).

1. PRELIMINARIES

A topological space X is a *k-space* (see Arhangel'skiĭ [2], Cohen [4], and Steenrod [12]) if a subset F of X is closed whenever its intersection with each bicompact subset K of X is closed in K. For each subset A of X, we shall write $x \in A^{\sim}$ if and only if $x \in \mathrm{cl}_K(A \cap K)$ for some bicompact subset K of X. Now let $A^0 = A$, and for each nonlimit ordinal $\alpha = \beta + 1$, let $A^\alpha = (A^\beta)^{\sim}$. If α is a limit ordinal, let $A^\alpha = \bigcup\{A^\beta \mid \beta < \alpha\}$. For an arbitrary space X, let $\kappa(X)$ denote the infimum of the ordinals α such that $A^\alpha = \mathrm{cl}_X A$ for each subset A of X. A straightforward argument, involving only cardinality in one direction and the fact that a single point may be added to a bicompact set without destroying bicompactness, establishes the following result.

1.1. PROPOSITION. *X is a k-space if and only if $\kappa(X)$ exists.*

Since the definition of k-spaces was given in terms of closure only, the following proposition is obvious.

1.2. PROPOSITION. *κ is a topological invariant in the category of k-spaces.*

$\kappa(X) = 0$ if and only if X is discrete, and $\kappa(X) \leq 1$ is precisely the criterion that determines the k'-spaces (see Arhangel'skiĭ [2]).

We now restrict our attention to a special case. A subset U of a topological space X is *sequentially open* if each sequence converging to a point in U is eventually in U. The space X is *sequential* if each sequentially open subset of X is open (see Franklin [8], [9]). For each subset A of X, we shall denote by $A^{\wedge}$ the set of all limits of sequences in A. Now let $A^0 = A$, and for each ordinal $\alpha = \beta + 1$, let $A^\alpha = (A^\beta)^{\wedge}$. If α is a limit ordinal, let $A^\alpha = \bigcup\{A^\beta \mid \beta < \alpha\}$. (Whether A^α refers to the sequential closure $\wedge$ or the k-closure $\sim$ will always be clear from the context.) Denote by $\sigma(X)$ the infimum of the ordinals α with the property that $A^\alpha = \mathrm{cl}_X A$ for all $A \subseteq X$. The following is a folk theorem.

1.3. PROPOSITION. *X is sequential if and only if $\sigma(X)$ exists. In this case,* $\sigma(X) \leq \omega_1$ (where ω_1 is the first uncountable ordinal).

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For a proof of the second assertion, see, for example, Dolcher [5, equation (22)] or Vaidyanathaswamy [14, p. 278].

1.4. PROPOSITION. *σ is a topological invariant in the category of sequential spaces.*

Again, $\sigma(X) = 0$ if and only if X is discrete, and $\sigma(X) \leq 1$ is the criterion that defines the Fréchet spaces (see [3], [8], [9]).

1.5. PROPOSITION. *Every sequential space is a* k-*space. Conversely, every countable Hausdorff* k-*space* X *is sequential and satisfies the condition* $\sigma(X) = \kappa(X)$.

Proof. The first assertion follows from the bicompactness of the union of a convergent sequence and one of its limit points. For the first part of the converse, note that if a subset A of X is not closed, there exist by [2, Section 10] a bicompact set $K \subseteq X$ and a point $p \in \mathrm{cl}(K \cap A) \setminus (K \cap A)$. Since K satisfies the first countability axiom, some sequence in $K \cap A$ converges to $p \notin A$. Hence A is not sequentially closed (that is, its complement is not sequentially open). Thus X is a sequential space. To complete the proof, we note that the inequality $\kappa(X) \leq \sigma(X)$ always holds, and we again use the fact that countable bicompact Hausdorff spaces satisfy the first countability axiom.

The one-point compactification of $M \setminus \mathbf{N}$, where M is the space of Example 5.1 of [9], is a countable T_1-k-space that is not sequential. (This is in fact the one-point compactification of S_2 without the level-one points, as described in Section 3.) There exists a sequential bicompact Hausdorff space Ψ^* that is not a Fréchet space [9, Example 7.1]. Hence $\kappa(\Psi^*) = 1 < \sigma(\Psi^*)$ (= 2, as it happens). Also, there exist countable, bicompact, sequential T_1-spaces that are not Fréchet spaces [9, Example 5.3]. Hence the cardinality and separation hypotheses of Proposition 1.5 are actually needed. We state the following for future reference.

1.6. PROPOSITION. *If* X *is the disjoint topological sum of a family* $\{X_\alpha\}$ *of* k-*spaces (or sequential spaces), then* $\kappa(X) = \sup \kappa(X_\alpha)$ $(\sigma(X) = \sup \sigma(X_\alpha))$.

2. THE SEQUENTIAL SUM

Let $S = \{0\} \cup \{1/n \mid n \in \mathbf{N}\} \subseteq \mathbf{R}$ have the relative topology; that is, let S be a convergent sequence with its limit point. For each i $(0 < i < \omega_0)$, let $\langle X_i, 0_i \rangle$ be a T_1-space with a base point. We define the sequential sum $\sum \langle X_i, 0_i \rangle$ as follows. Let X be the disjoint topological sum of the X_i, and let $A = \{0_i \mid i < \omega_0\}$. Then A is a closed subspace of X, and the function $f\colon A \to S$ defined by $f(0_i) = 1/i$ is continuous. Let $\sum \langle X_i, 0_i \rangle$ be the adjunction space $X \cup_f S$. The pertinent facts about the sequential sum are as follows.

2.1. PROPOSITION. *If each* X_i *is a* k-*space (or a sequential space), then so is* $\sum \langle X_i, 0_i \rangle$, *for each choice of* 0_i. *If* $\kappa_i = \kappa(X_i)$ $(\sigma_i = \sigma(X_i))$ *is a nonlimit ordinal for each* X_i, *then there exist* 0_i *such that* $\kappa(X) = (\sup \kappa_i) + 1$ $(\sigma(X) = (\sup \sigma_i) + 1)$.

Proof. In the sequential case, the first assertion follows from [8, Propositions 1.2 and 1.6], and in the case of k-spaces it can be proved similarly. Since $X \setminus \{0\}$ is the disjoint topological sum of the X_i, $\kappa(X \setminus \{0\}) = \sup \kappa_i$ by Proposition 1.6. From the fact that $\{0\} \cup \{0_i\}$ is bicompact, it follows immediately that $\kappa(X) \leq (\sup \kappa_i) + 1$. We shall construct a subset M of $X \setminus \{0\}$ such that $0 \in M^{(\sup \kappa_i)+1} \setminus M^{\sup \kappa_i}$. Choose a subsequence $\{\kappa_j\}$ of the $\{\kappa_i\}$ that converges upwards to $\sup \kappa_i$ in the order topology. For each j, let $\theta_j + 1 = \kappa_j$. Then there exist $0_j \in X$ and $M_j \subseteq X_j$ such that $0_j \in (M_j)^{\kappa_j} \setminus (M_j)^{\theta_j}$ (the remaining 0_i may be chosen arbitrarily). Let $M = \bigcup M_j$. Let β be the least ordinal such that $0 \in M^{\beta+1} \setminus M^{\beta}$. Then $0 \in cl_X(B \cap M^{\beta})$ for some bicompact B. Letting $K = \{0_j\}$, we see that $K \cap B \cap M^{\beta} = \bigcup (K \cap B \cap (M_j)^{\beta})$ is infinite. But $K \cap (M_j)^{\beta} \neq \emptyset$ only if $\beta \geq \kappa_j$. Hence $\beta \geq \sup \kappa_i$, and thus $0 \notin M^{\sup \kappa_i}$. Hence $\kappa(X) = (\sup \kappa_i) + 1$.

An even simpler proof may be given in the sequential case. In addition, one may easily verify the following assertion.

2.2. PROPOSITION. *If each* X_i *is zero-dimensional, then the sequential sum of the* X_i *is zero-dimensional.*

3. CONSTRUCTION OF THE S_n

In this section we shall construct (in two distinct ways) a countable space S_n that satisfies the condition $\kappa(S_n) = \sigma(S_n) = n$ for each $n < \omega_0$, and has all 'nice' topological properties except local bicompactness. Moreover, the space S_n will be minimal in a sense to be made explicit in Proposition 3.1.

Let $S_0 = \{0\}$, and, having already defined S_{n-1} with base point 0, let S_n be the sequential sum of countably many copies of $\langle S_{n-1}, 0 \rangle$, choosing 0 again as base point. S_n is then defined recursively for each $n < \omega_0$. Clearly, $S_1 = S$, and S_2 is the space of Arens (see [1] and [9, Example 5.1]).

We now define the level $\ell_n(x)$ for points $x \in S_n$. For $n = 0$, let $\ell_0(0) = 0$. Having defined the level of each point in S_{n-1}, choose $x \in S_n$. If $x = 0$, let $\ell_n(x) = 0$. If not, then $x \in S_{n-1}$, and we let $\ell_n(x) = \ell_{n-1}(x) + 1$.

Now, for each point x of level n in S_n, take a copy S_x of S, and let X be the disjoint topological sum of the sets S_x. Let $A = \{0_x \in S_x \mid \ell_n(x) = n\}$, and define $f: A \to S_n$ by $f(0_x) = x$. Then the adjunction space $X \cup_f S_n$ is homeomorphic to S_{n+1}, and we have the second construction.

Suppose that for each $k < n$ we have defined a partial order $\leq_k$ on S_k, with 0 as maximal element. Then let $\leq_n$ be the partial order on S_n generated by $\leq_{n-1} \cup \{(y, x) \mid y \in S_x\}$. We shall use these orders in Section 5.

Also for later use, note that the second construction yields a natural embedding $\phi_n: S_n \to S_{n+1}$.

The properties claimed for the S_n in the opening paragraph of this section follow immediately from Propositions 2.1 and 2.2 and the following assertion.

3.1. PROPOSITION. *If a Hausdorff sequential space* X *contains a copy of* S_n, *then* $\sigma(X) \geq n$. *Conversely, if* $\sigma(X) \geq n$, *then* X *contains a subspace whose sequential closure is homeomorphic to* S_n.

Proof. Let L_n be the points of level n in $S_n \subseteq X$. If $\sigma(X) = k$ for $k < n$, there are countably many points $x_j \in L_n^{\wedge} \setminus (L_n \cup L_{n-1})$ with 0 (the zero-level point of S_n) in $\{x_j \mid j \in \mathbf{N}\}^{k-1}$. Let A_j be the range of a sequence in L_n converging to x_j, and for each $y_i \in L_{n-1}$, let B_i be the points of L_n under y_i. Since X is a Hausdorff space, $A_j \cap B_i$ is finite for all $i, j \in \mathbf{N}$. Hence there exist disjoint sets A and B such that $A_j \setminus A$ and $B_i \setminus B$ are finite for all $i, j \in \mathbf{N}$. The set $B \cup (S_n \setminus L_n)$ is open in S_n and contains 0. Hence there is an open set U in X such that

$U \cap S_n = B \cup (S_n \setminus L_n)$. Hence $A \cap U = \emptyset$. Thus $x_j \notin U$ for each j. This contradicts $0 \in cl_X \{x_j | j \in \mathbf{N}\}$.

The second assertion is obvious for $n = 0$ and $n = 1$, and for $n = 2$ it follows from Proposition 7.3 of [9]. In order to complete the induction, we prove something a little stronger: *if* A *is a subset of a sequential Hausdorff space* X *and if* $x \in A^n \setminus A^{n-1}$, *there exists a subset* S'_n *of* A^n *whose sequential closure is homeomorphic to* S_n, *and whose points of level* k *lie in* $A^{n-k} \setminus A^{n-k-1}$. With the assertion stated in this form, the inductive proof is trivial if we note that a sequentially bicontinuous bijection is a homeomorphism from the sequential closure of its domain to that of its range. The second assertion of the proposition now follows immediately.

4. CONSTRUCTION OF THE K_α

In this section we shall construct, for each $\alpha < \omega_1$, a countable space K_α (again with 'nice' properties) such that $\kappa(K_\alpha) = \sigma(K_\alpha) = \alpha$.

Let $K_0 = S_0 = \{0\}$, and suppose K_β is defined for each $\beta < \alpha$. If α is a limit ordinal, let K_α be the disjoint topological sum of the K_β with $\beta < \alpha$. By Proposition 1.6, $\kappa(K_\alpha) = \sigma(K_\alpha) = \alpha$. If $\alpha = \beta + 1$, choose a sequence of nonlimit ordinals $\{\beta_i\}$ with supremum β. By Proposition 2.1, we may choose $0_i \in K_{\beta_i}$ so that $\kappa(K_\alpha) = (\sup \beta_i) + 1 = \alpha$, where K_α is the sequential sum of the K_{β_i}. By Proposition 1.5, $\sigma(K_\alpha) = \alpha$ also. We recapitulate:

4.1. THEOREM. *For each ordinal* $\alpha < \omega_1$, *there exists a countable, zero-dimensional Hausdorff space* K_α *such that* $\kappa(K_\alpha) = \sigma(K_\alpha) = \alpha$.

Note that we may also define the space K_{ω_1} as the disjoint topological sum of the K_α for $\alpha < \omega_1$. Then K_{ω_1} is a zero-dimensional Hausdorff space of cardinality and local weight $\aleph_1$, with $\kappa(K_{\omega_1}) = \sigma(K_{\omega_1}) = \omega_1$. In the next section we shall construct another such space that is not only countable but also homogeneous.

5. CONSTRUCTION OF S_ω

Using the maps $\phi_n: S_n \to S_{n+1}$ defined in Section 3, we define for each pair $m < n < \omega_0$ a map $\phi_m^n: S_m \to S_n$ by $\phi_m^n = \phi_{n-1}^n \circ \cdots \circ \phi_m^{m+1}$, creating an inductive system $\langle S_n, \phi_m^n \rangle$ of spaces and maps. Denote by S_ω the inductive limit of this system.

5.1. THEOREM. S_ω *is a countable, sequential, zero-dimensional, homogeneous Hausdorff space with* $\kappa(S_\omega) = \sigma(S_\omega) = \omega_1$, *and it contains a copy of* K_α *for each* $\alpha < \omega_1$.

Proof. S_ω is clearly countable, and it is sequential by [8, Corollary 1.7]. Hence, by Propositions 1.3 and 1.5, $\kappa(S_\omega) = \sigma(S_\omega) \leq \omega_1$ (S_ω is clearly a T_1-space and is therefore a Hausdorff space, since it will be shown to be zero-dimensional). The opposite inequality will follow from the relation $K_\alpha \subseteq S_\omega$ for each $\alpha < \omega_1$.

Denoting by $\psi_n: S_n \to S$ the canonical map, we define a partial order on S_ω by the rule that $x \leq y$ if and only if there exists a triple n, a, b such that $a \in \psi_n^{-1}(x)$, $b \in \psi_n^{-1}(y)$, and $a \leq_n b$ (see Section 3).

Noting that $\ell_n(x) = k$ implies that $\ell_{n+1}(\phi_n(x)) = k$, we may unambiguously define the level $\ell(x)$ of a point x in S_ω by choosing some n and a with $a \in \psi_n^{-1}(x)$ and setting $\ell(x) = \ell_n(a)$. It is easy to verify that $x \leq y$ implies $\ell(x) \geq \ell(y)$.

For each $x \in S_\omega$, let $I(x) = \{y \in S_\omega \mid y \leq x\}$; that is, let I(x) be the principal ideal generated by x. We shall show by an induction on the level of x that *each* I(x) *is homeomorphic to* S_ω. For $\ell(x) = 0$, the assertion is trivial. Suppose $\ell(x) = 1$, and let $T_n = \psi_n^{-1}(I(x))$ for each $n < \omega_0$. Then $T_0 = \emptyset$, and for $n > 0$, T_n is homeomorphic to S_{n-1}. But clearly I(x) is the inductive limit of the system $\langle T_n, \phi_n^m \mid T_n \rangle$, and hence it is homeomorphic to S_ω. Now suppose our assertion is true for points at level n - 1 and that $\ell(x) = n$. Then there exists exactly one $y \in S_\omega$ with $\ell(y) = n - 1$ and $x < y$. The point x is at level 1 with respect to I(y), which is homeomorphic to S_ω by the inductive assumption, and hence $I(x) \cong S_\omega$ by the level-one argument.

Denote the level-one points of S_ω by 0_i. Then S_ω is the sequential sum of the family $\langle I(0_i), 0_i \rangle$, and so S_ω *is the sequential sum of countably many copies of itself* with the level-zero point of each as base point.

It is easily verified that a sequence $\{x_n\} \subseteq S_\omega$ of distinct points converges to $x_0 \in S_\omega$ if and only if eventually $\ell(x_n) = \ell(x_0) + 1$ and eventually $x_n \leq x_0$. We shall write $x \sim y$ if $\ell(x) = \ell(y)$ and there exists a z such that $\ell(z) = \ell(x) - 1$, $x \leq z$, and $y \leq z$. Hence $\{x_n\} \to x_0$ implies that eventually $x_n \sim x_m$ or $x_n = x_0$. In fact, in order that a sequence of distinct points in S_ω converge, it is necessary and sufficient that it be eventually composed of points pairwise related by $\sim$. Using this characterization of sequential convergence and the fact that S_ω is sequential, one sees that not only is each I(x) open and closed, but for each family $\{x_i\}$ no infinite subfamily of which is related by $\sim$, $\bigcup I(x_i)$ is open and closed. It then follows immediately that S_ω *is zero-dimensional.*

Let $x, y \in S_\omega$ be distinct points. If x and y are not comparable, then I(x) and I(y) are homeomorphic, disjoint, open and closed neighborhoods of x and y, respectively. If $x \leq y$, then I(x) and $I(y) \setminus I(x)$ are such neighborhoods, and so S_ω *is homogeneous.*

We shall now recursively imbed each K_α in S_ω. Suppose this has been accomplished for each $\beta < \alpha$, so that the base point 0_β of K_β is the level-zero point of S_ω whenever β is not a limit ordinal. For each such β, let L_β be a copy of S_ω with K_β so embedded. If α is a limit ordinal, then K_α is the disjoint topological sum of the K_β and is homeomorphic to a subset of any sequential sum of the L_β. If $\alpha = \beta + 1$, K_α is the sequential sum of some sequence K_{β_i}. Then K_α is embedded in the sequential sum of the corresponding $\langle L_{\beta_i}, 0_{\beta_i} \rangle$, which is again S_ω.

Since *for each nonlimit ordinal* α, K_α *is homeomorphic to a closed subspace of* S_ω, $\sigma(S_\omega) = \omega_1$ and the proof is complete.

Dudley has shown [6, Theorem 7.8] that the sequential closure (that is, the smallest sequential topology containing the given one) of the weak topology of a separable, infinite-dimensional Banach space is the 'bounded topology' (see [7, pp. 425-430]). We shall apply Theorem 5.1 to show that $\sigma(\ell_2) = \omega_1$ if ℓ_2 is provided with its bounded topology. The authors are indebted to C. V. Coffman for a key idea in the proof.

5.2. THEOREM. S_ω *can be embedded as a sequentially closed subset of* ℓ_2 *taken with its bounded topology. Hence,* $\sigma(\ell_2) = \omega_1$.

Proof. Using the second description of S_n in Section 3, we shall recursively define an embedding $\theta_n: S_n \to \ell_2$ of each S_n into ℓ_2 in such a way that $\theta_m = \theta_n \circ \phi_m^n$ for each $m < n$. Since S_ω is the inductive limit of the S_n, this will map S_ω into ℓ_2.

We first represent each S_n as a collection of finite sequences of natural numbers, as follows. Represent the single point of S_0 by the empty sequence. Let $S_1 = S_0 \cup \{(i) \mid i \in \mathbf{N}\}$. Supposing that S_n has been defined, and for $x = (i_1, i_2, \cdots, i_n) \in S_n$ and of level n, let $S_x = \{x\} \cup \{(i_1, \cdots, i_n, j) \mid j > i_n\}$. Now construct S_{n+1} as in Section 3. S_ω can be thought of as the union of the S_n in this representation, in other words, as the collection of all finite, strictly increasing sequences of natural numbers.

Convergence of sequences in S_ω (or in any S_n) can easily be described in terms of this representation: essentially, a sequence converges if and only if it is eventually of the same level (that is length), say n, and eventually constant in each of the first $n - 1$ coordinates, and if it is further either unbounded in its eventual last coordinate or eventually constant there. In the first case, the limit point is represented by the sequence of the first $n - 1$ eventual values, and in the second case the sequence is eventually constant. We shall now embed each S_n as a sequentially closed subset of ℓ_2, by a θ_n such that sequential convergence in $\theta_n(S_n)$ has this same description. Hence each θ_n will be a homeomorphism (see for example Moore [11, Theorem 6.13]); therefore the limit θ of the θ_n will also be a homeomorphism.

Let $\{b^i\}$ be the standard orthonormal basis for ℓ_2, defined by $b_k^i = 0$ if $i \neq k$ and $b_i^i = 1$. Define θ_0 by $\theta_0(\emptyset) = 0$, where 0 denotes the origin in ℓ_2. Define θ_1 by $\theta_1(i) = b^i$ and $\theta_1(\emptyset) = \theta_0(\emptyset) = 0$. Having defined θ_n as an extension of θ_{n-1}, let $\theta_{n+1} = \theta_n$ on S_n, and for $(i_1, \cdots, i_n, i_{n+1}) \in S_{n+1} \setminus S_n$, let

$$\theta_{n+1}(i_1, \cdots, i_n, i_{n+1}) = \theta_n(i_1, \cdots, i_n) + i_n b^{n+1}.$$

We must show that each θ_n is a sequential homeomorphism onto $\theta_n(S_n)$, and that $\theta_n(S_n)$ is sequentially closed in ℓ_2.

From the fact that $\theta_n(i_1, \cdots, i_n)$ is nonzero only in the $i_1, i_2, \cdots, i_n$th places, it is clear that each θ_n is one-to-one.

That θ_1 is a sequential homeomorphism follows from the well-known fact that the sequence $\{b^i\}$ converges weakly to zero. Clearly, $\theta_1(S_1)$ is sequentially closed in ℓ_2. If we suppose that θ_n has been shown to be a sequential homeomorphism and that $\{x_k\}$ is a convergent sequence in S_{n+1}, then we may assume all x_k to have the same level. If $\ell_{n+1}(x_k) \leq n$, then $\theta_{n+1}(x_k) = \theta_n(x_k)$, and the convergence is preserved. If $\ell_{n+1}(x_k) = n + 1$, we may assume that $x_k = (i_1, \cdots, i_n, j_k)$, with the j_k unbounded. (Otherwise, $\{x_k\}$ is an eventually constant sequence and convergence is preserved.) Then $\lim x_k = (i_1, \cdots, i_n)$, and

$$\theta_{n+1}(x_k) = \theta_n(i_1, \cdots, i_n) + i_n b^{j_k},$$

which converges weakly to $\theta_n(i_1, \cdots, i_n) = \theta_{n+1}(i_1, \cdots, i_n)$. Thus θ_{n+1} is sequentially continuous.

Conversely, suppose $\{x^k\}$ is any sequence in $\theta(S_\omega)$ that converges weakly to x^0 in ℓ_2. Since $\{x^k\}$ must be bounded in norm, there exists a uniform bound, say q, on the number of nonzero coordinates of the x^k. Thus $\{x^k\} \subseteq \theta_q(S_q)$. Hence, *if each $\theta_n(S_n)$ is sequentially closed, so is $\theta(S_\omega)$.* Since weak convergence implies pointwise convergence, and since each coordinate of each x^k is an integer, the sequence $\{x^k\}$ must be eventually constant in each coordinate. Let r be the number of eventually nonzero coordinates, and suppose the theorem is proved for $r < n$. If $r = n$, we may assume that $x^k = \theta_n(i_1^k, \cdots, i_n^k)$, where eventually $i_1^k = i_k, \cdots, i_{n-1}^k = i_{n-1}$. Then $x^0 = \theta_n(i_1, \cdots, i_{n-1}) \in \theta_n(S_n)$, which is therefore sequentially closed. Clearly, $\{i_n^k\}$ is unbounded, and so $(i_1^k, \cdots, i_n^k)$ converges in S_n to $(i_1, \cdots, i_{n-1})$. This completes the proof.

Note that 0 is in the weak closure of $\theta_2(S_2) \setminus \theta_1(S_1)$ but is not the weak limit of any sequence therein. This is the well-known example of von Neumann.

6. SOME REMARKS AND QUESTIONS

As Theorem 5.2 suggests, the functions σ and κ can be extended to the category of all topological spaces and continuous maps by means of the co-reflective functors s and k that assign to each space X the spaces sX and kX, where the underlying set is the same and the new topologies are the smallest sequential and k-space topologies containing the original. We may then define $\sigma(X) = \sigma(sX)$ and $\kappa(X) = \kappa(kX)$. Propositions 1.5 and 1.6 extend immediately. What else can be said?

Proposition 3.1 establishes the S_n as test spaces for spaces X with $\sigma(X) = n$. It seems that, permitting all possible choices of the β_i (see the second paragraph of Section 4), we could use the K_α as test spaces for $\sigma(X) = \alpha$. Are there test spaces for κ?

The disjoint topological sum K of the K_α fpr $\alpha < \omega_1$ satisfies the condition $\sigma(K) = \omega_1$ but contains no copy of S_ω. Can this happen with a countable space, or with a homogeneous space?

Is there for each $\alpha > \omega_1$ a space K_α with $\kappa(K_\alpha) = \alpha$? More particularly, if α is an ordinal corresponding to a cardinal $\tau(\alpha) > \aleph_1$, is there a k-space K_α with $\kappa(K_\alpha) = \alpha$ and $\overline{\overline{K}}_\alpha \leq \tau(\alpha)$ $(K_\alpha \leq 2^{\tau(\alpha)})$? Is there for each $\alpha < \beta \leq \omega_1$ a space $X_{\alpha,\beta}$ with $\kappa(X_{\alpha,\beta}) = \alpha$ and $\sigma(X_{\alpha,\beta}) = \beta$?

What are the permanence properties of a space with $\sigma(X) = \alpha$ and $\kappa(X) = \alpha$?

S_ω is something of a topological curiosity in itself. Are there other countable Hausdorff k-spaces with no point of first countability? If so, are there others that are homogeneous and sequential?

It is easily seen that the proof of Theorem 5.2 depends only on the existence of a sequence bounded away from 0 and converging weakly to 0. For what linear topological spaces do such sequences exist?

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