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ORDINARY, ABSOLUTE AND STRONG SUMMABILITY AND MATRIX TRANSFORMATIONS

ABDULLAH M. JARRAH AND EBERHARD MALKOWSKY

Abstract. **Many important sequence spaces arise in a natural way from various concepts of summability, namely ordinary, absolute and strong summability. In the first two cases they may be considered as the domains of the matrices that define the respective methods of summability; the situation, however, is different and more complicated in the case of strong summability.**

Given sequence spaces X and Y, we find necessary and sufficient conditions for the entries of a matrix to map X into Y, and characterize the subclass of those matrices that are compact operators.

This paper gives a survey of recent research in the field of matrix transformations at the University of Niš, Serbia and Montenegro, in the past four years.

1. Introduction and the Basic Theory

Classical summability theory deals with a generalization of the concept of convergence of sequences or series of real or complex numbers. The idea is to assign a *limit* to divergent sequences or series by considering a *transform* rather than the original sequence or series. Most popular are *matrix transformations* and we shall confine to them.

1.1. **Concepts of summability and summability methods.** Here we deal with three different concepts of summability, ordinary, absolute and strong summability and mention the most important methods of summability that are given by matrices.

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Let $A = (a_{nk})_{n,k=0}^{\infty}$ be an infinite matrix of complex numbers. Then a sequence $x = (x_k)_{k=0}^{\infty}$ of complex numbers is said to be *summable* A to a complex number ℓ, if the series $A_n(x) = \sum_{k=0}^{\infty} a_{nk}x_k$ converge for all n and $\lim_{n\to\infty} A_n(x) = l$, denoted by $x \to \ell(A)$; this is the concept of *ordinary summability.*

Let $0 < p < \infty$. A sequence x is said to be *absolutely summable* A *with index* p *to a complex number* ℓ if the series $A_n(x)$ converge for all n and $\sum_{n=0}^{\infty} |A_n(x)|^p = \ell$; this is denoted by $x \to \ell|A|^p$. A sequence x is said to be *strongly summable* A *with index* p *to a complex number* ℓ if the series $\sum_{k=0}^{\infty} a_{nk}|x_k - l|^p$ converge for all n and $\lim_{n\to\infty} \sum_{k=0}^{\infty} a_{nk}|x_k - l|^p = 0$; this is denoted by $x \to \ell[A]^p$.

The most important matrix transformations are given by *Hausdorff matrices* and their special cases, the *Cesàro*, *Hölder* and *Euler matrices*, and by *Nörlund matrices.* The matrices are *triangles* that is $a_{nk} = 0$ for $k > n$ and $a_{nn} = 1$ $(n = 0, 1, \dots)$.

Let $\mu = (\mu_n)_{n=0}^{\infty}$ be a given complex sequence, M be the diagonal matrix with $m_{nn} = \mu_n$ for all n and D be the matrix with $d_{nk} = (-1)^k \binom{n}{k}$ where $\binom{n}{k}$ are the binomial coefficients. Then the matrix $H = H(\mu) = DMD$ is called the *Hausdorff matrix associated with the sequence* μ; its entries are given by $h_{nk} = \sum_{j=k}^{n} (-1)^{j+k} \binom{n}{j}\binom{j}{k}$.
The *Cesàro matrix* C_α *of order* $\alpha > -1$ is the Hausdorff matrix associated with the sequence μ where $\mu_n = A_n^\alpha = \binom{n+\alpha}{n}$ for $n = 0, 1, \dots$; its entries are given by $(C_\alpha)_{n,k} = A_{n-k}^{\alpha-1}/A_n^\alpha$.
The *Hölder matrix* H^α *of order* $\alpha > -1$ is the Hausdorff matrix associated with the sequence μ where $\mu_n = (n+1)^{-\alpha}$ for $n = 0, 1, \dots$; no explicit formula is known for the entries of the matrices H^α.
The *Euler matrix* E_q *of order* $q > 0$ is the Hausdorff matrix associated with the matrix μ where $\mu_n = (q+1)^{-n}$; its entries are given by $(E_q)_{n,k} = \binom{n}{k} q^{n-k}/(q+1)^n$.

Finally, let $q = (q_k)_{k=0}^{\infty}$ be a sequence such that $Q_n = \sum_{k=0}^{n} q_k \neq 0$ for all n. Then the *Nörlund matrix* (N, q) is given by $(N, q)_{n,k} = q_{n-k}/Q_n$.

We refer the reader to [5, 11, 19, 23] for the concepts of summability and summability methods.

1.2. **Matrix domains and strong matrix domains.** In this subsection, we introduce the concepts of *matrix domains* and *strong matrix domains.*

Let ω denote the set of all complex sequences $x = (x_k)_{k=0}^{\infty}$. We write l_∞, c, c_0 and ϕ, and cs and bs for the sets of all bounded, convergent, null and finite sequences, and for the sets of all convergent and bounded series, respectively. Furthermore, we write $\ell_p = \{x \in \omega : \sum_{k=0}^{\infty} |x_k|^p < \infty\}$. As usual, e and $e^{(n)}$ $(n = 0, 1, \dots)$ are the sequences with $e_k = 1$ for all k,

$e_n^{(n)} = 1$ and $e_k^{(n)} = 0$ $(k \neq n)$.

Given any subset $X \subset \omega$ and any infinite matrix $A = (a_{nk})_{n,k=0}^{\infty}$, we write

$$X_A = \{x \in \omega : A(x) = (A_n(x))_{n=0}^{\infty} \in X\}$$

for the *matrix domain of A in X* and

$$X_{[A]^p} = \left\{ x \in \omega : A(|x|^p) = \left(\sum_{k=0}^{\infty} a_{nk}|x_k|^p \right)_{n=0}^{\infty} \in X \right\} \text{ for } 0 < p < \infty$$

for the *strong matrix domain of A with index p in X*. (We assume that all the series converge.) In the special case of $X = c$, the set c_A is called the *convergence domain of A*; a sequence x is summable A to ℓ if and only if $x \in c_A$. Also a sequence x is absolutely summable with index p if and only if $x \in (\ell_p)_A$.

In the special case of $X = c_0$, the set $(c_0)_{[A]^p}$ is the set of all sequences that are strongly summable A with index p to zero. When $A = C_1$, the Cesàro matrix of order 1, Maddox [10] introduced and studied the sets $w_0^p = (c_0)_{[C_1]^p}$, $w^p = \{x \in \omega : x - le \in w_0^p\}$ and $w_\infty^p = (\ell_\infty)_{[C_1]^p}$, the sets of sequences that are strongly summable to zero, strongly summable and strongly bounded with index p by the C_1 method.

1.3. **Mapping theorems in BK spaces.** It is quite natural to try and find those infinite matrices which transform every convergent sequence into a convergent sequence, that is to characterize the class (c, c) of all matrices A that map c into c by giving necessary and sufficient conditions for the entries of such a matrix A. Matrices that belong to the class (c, c) are said to be *conservative* or *convergence preserving*, conservative matrices that leave the limits unchanged are said to be regular.

The famous *Toeplitz theorem* (1911) [21] gives the following necessary and sufficient conditions for regular matrices

$$\sup_n \sum_{k=0}^{\infty} |a_{nk}| < \infty, \ \lim_{n\to\infty} A_n(e) = 1 \text{ and } \lim_{n\to\infty} A_n(e^{(k)}) = 0 \text{ for every } k.$$

More generally, given sequence spaces X and Y, it is interesting to establish *mapping theorems* that characterize the class (X, Y) of all infinite matrices A that map X into Y. So $A \in (X, Y)$ if and only if $X \subset Y_A$.

The theory of BK spaces is the most powerful tool in the characterization of matrix transformations between sequence spaces. A *BK space* X is a Banach sequence space with continuous coordinates $P_k : X \to \mathbb{C}$ where $P_k(x) = x_k$ $(k = 0, 1, \dots)$ for all $x \in X$.

Example 1.1. *The sets ℓ_∞, c and c_0 are BK spaces with* $\|x\|_\infty = \sup_k |x_k|$; *the sets ℓ_p $(1 \le p < \infty)$ are BK spaces with* $\|x\|_p = (\sum_{k=0}^{\infty} |x_k|^p)^{1/p}$.

The following result shows why BK spaces are so important.

Theorem 1.2. ([14, Theorem 1.17. p. 153], [22, Theorem 4.2.8, p. 57]) *Any matrix map between BK spaces is continuous.*

A *Schauder–basis* or *basis*, for short, of a normed space X is a sequence (b_n) of elements of X such that, for every $x \in X$, there is a unique sequence (λ_n) of scalars with $x = \sum_{n=0}^{\infty} \lambda_n b_n = \lim_{m\to\infty} \sum_{n=0}^{m} \lambda_n b_n$; a BK space $X \supset \phi$ is said to have AK, if, for every sequence $x = (x_k)_{k=0}^{\infty} \in X$, we have $x = \sum_{k=0}^{\infty} x_k e^{(k)} = \lim_{m\to\infty} x^{[m]}$ where $x^{[m]} = \sum_{k=0}^{m} x_k e^{(k)}$ is the *m-section of the sequence x*; it is said to have AD, if ϕ is dense in X.

Example 1.3. *The spaces c_0 and ℓ_p* $(1 \le p < \infty)$ *have AK. The sequence $(e, e^{(0)}, e^{(1)}, \dots)$ is a basis of c, more precisely, if $x = (x_k)_{k=0}^{\infty} \in c$ is a sequence with* $\lim_{k\to\infty} x_k = \ell$*, then x has a unique representation $x = \ell \cdot e + \sum_{k=0}^{\infty}(x_k - \ell)e^{(k)}$. The space ℓ_∞ has no Schauder–basis. Every BK space with AK obviously has AD. An example of a BK space with AD which does not have AK can be found in* [22, Example 5.2.5, p. 78].

We refer the reader to [11, 22] for the theory of mapping theorems, and to the survey article in [20]. An extension to infinite matrices of operators can be found in [12]. Most of the theory of BK spaces extends to the more general *FK spaces*, that is complete linear metric sequence spaces with continuous coordinates.

1.4. **Multipliers and β–duals.** The concept of the β–dual of a set of sequences naturally arises in connection with the characterization of matrix transformations; β–duals are special cases of *multipliers*.

Let x and z be complex sequences, and X and Y be subsets of ω. We write $xz = (x_k z_k)_{k=0}^{\infty}$, $z^{-1} * X = \{x \in \omega : xz \in X\}$, $z^{\beta} = z^{-1} * cs$, $M(X,Y) = \cap_{x\in X} x^{-1} * Y = \{a \in \omega : ax \in Y \text{ for all } x \in X\}$ for the *multiplier of X and Y*, and $X^{\beta} = M(X, cs)$ for the *β–dual of X*.

Example 1.4. *We have $\omega^{\beta} = \phi$, $\phi^{\beta} = \omega$, $\ell_\infty^{\beta} = c^{\beta} = c_0^{\beta} = \ell_1$, $\ell_p^{\beta} = \ell_q$ for $1 < p < \infty$ where $q = p/(p-1)$ and $\ell_1^{\beta} = \ell_\infty$.*

The multiplier of a BK space also is a BK space; this result does not extend to FK spaces.

Theorem 1.5. ([22, Theorem 4.3.15, p. 64]) *Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be BK spaces, $X \supset \phi$ and $Z = M(X,Y)$. Then Z is a BK space with $\|z\| = \sup\{\|xz\|_Y : \|x\|_X = 1\}$ for all $z \in Z$. In particular, if X is a BK space, so is X^{β} with $\|a\|^{\beta} = \sup\{\|ax\|_{bs} : \|x\|_X =\}$ where $\|ax\|_{bs} = \sup_n |\sum_{k=0}^{n} a_k x_k|$.*

There also is a close relation between β–duals and continuous *duals*. Let X and Y be Banach spaces. Then $B(X,Y)$ denotes the set of all bounded linear operators $L: X \to Y$ and $B(X,Y)$ is a Banach space with the *operator norm* $\|L\| = \sup\{\|L(x)\| : \|x\| \leq 1\}$ for all $L \in B(X,Y)$. We write $X^* = B(X,\mathbb{C})$ for the *continuous dual of* X, that is the set of all continuous linear functionals f on X, with the norm $\|f\| = \sup\{|f(x)| : \|x\| \leq 1\}$ for all $f \in X^*$.

Theorem 1.6. ([22, Theorem 7.2.9, p. 107], [14, Theorem 1.34, p. 159]) *Let $X \supset \phi$ be a BK space. Then there is a linear one-to–one map $T: X^\beta \to X^*$; we denote this by $X^\beta \subset X^*$. If X has AK, then T is also onto.*

If X is a BK space then $\|a\|_X^* = \sup\{|\sum_{k=0}^\infty a_k x_k| : \|x\| \leq 1\}$ is defined and finite for all $a \in X^\beta$ by Theorem 1.6.

Example 1.7. *Let X be any of the spaces ℓ_∞, c c_0 and ℓ_p for $1 \leq p < \infty$. Then the norms $\|\cdot\|_{X^\beta}$, $\|\cdot\|^\beta$ and $\|\cdot\|_X^*$ are equivalent on X^β.*

By Theorem 1.2, there also is a close relation between the classes (X,Y) and $B(X,Y)$, namely if X and Y are BK spaces, then every matrix $A \in (X,Y)$ defines an operator $L_A \in B(X,Y)$ by $L_A(x) = A(x)$ for all $x \in X$; we denote this by $(X,Y) \subset B(X,Y)$. We apply Theorem 1.2 and Examples 1.1, 1.4 and 1.7 to obtain

Example 1.8. *Let X be a BK space. Then $A \in (X,\ell_\infty)$ if and only if*

(1.1)
$$\|A\|_{(X,\ell_\infty)}^* = \sup_n \|A_n\|_X^* < \infty \text{ where } A_n = (a_{nk})_{k=0}^\infty \text{ for all } n$$

denotes the sequence in the n–th row of the matrix A. If $A \in (X,Y)$ then $\|L_A\| = \|A\|_{(X,\ell_\infty)}^*$. In particular, we obtain $A \in (\ell_p,\ell_\infty)$ if and only if

$$\sup_{n,k} |a_{nk}| < \infty \quad \text{for } p = 1,$$
$$\sup_n \sum_{k=0}^\infty |a_{nk}|^q < \infty \quad \text{for } 1 < p < \infty \text{ and } q = \frac{p}{p-1},$$
$$\sup_n \sum_{k=0}^\infty |a_{nk}| < \infty \quad \text{for } p = \infty.$$

Proof. We write $\|A\|^* = A_{(X,\ell_\infty)}^*$ for short, and $B_X = \{x \in X : \|x\| \leq 1\}$ for the *unit ball in* X.
First we assume that condition (1.1) holds. Then, for all $x \in B_X$, the series $A_n(x)$ converge for all n, and $A(x) \in \ell_\infty$, which obviously implies $A_n \in X^\beta$ for all n and $A(x) \in Y$ for all $x \in X$. Thus we have $X \subset Y_A$, and so $A \in (X,Y)$.

Conversely, if $A \in (X, Y)$ then $L_A \in B(X, Y)$ by Theorem 1.2, since ℓ_∞ is a BK space by Example 1.1, and consequently there is a constant M such that $\|A(x)\|_\infty = \|L_A(x)\|_\infty = \sup_n |A_n(x)| \leq M\|x\|$ for all $x \in B_X$. This implies condition (1.1) and also that $\|L_A\| = \|A\|^*$ by the definitions of the operator norm $\|\cdot\|$ and of $\|\cdot\|^*$.
Finally, the conditions for $A \in (\ell_p, \ell_\infty)$ are an immediate consequence of condition (1.1) and Examples 1.4 and 1.7. □

If X and Y be BK spaces then $(X, Y) \subset B(X, Y)$, as we already know. Now we study when an operator $L \in B(X, Y)$ can be given by an infinite matrix A in which case we write $B(X, Y) \subset (X, Y)$.

Theorem 1.9. *Let X and Y be BK spaces and X have AK. Then $B(X, Y) \subset (X, Y)$.*

Proof. Let $L \in B(X, Y)$ be given. We write $L_n = P_n \circ L$ for $n = 0, 1, \dots$. Then $L_n \in X^*$ for all n since Y is a BK space. We put $a_{nk} = L_n(e^{(k)})$ for $n, k = 0, 1, \dots$. Let $x = (x_k)_{k=0}^\infty \in X$ be given. Since X has AK and $L_n \in X^*$ for all n, it follows that $x = \sum_{k=0}^\infty x_k e^{(k)}$ and $L_n(x) = \sum_{k=0}^\infty x_k L_n(e^{(k)}) = \sum_{k=0}^\infty a_{nk} x_k$ for all n, hence $L(x) = A(x)$. □

Example 1.10. *We have $(\ell_1, \ell_1) = B(\ell_1, \ell_1)$ and $A \in (\ell_1, \ell_1)$ if and only if*

$$\|A\|^*_{(1,1)} = \sup_k \sum_{n=0}^\infty |a_{nk}| < \infty; \tag{1.2}$$

furthermore, if $A \in (\ell_1, \ell_1)$ then $\|L_A\| = \|A\|_{(1,1)}$.

Proof. We write $\|A\|^* = \|A\|^*_{(1,1)}$ for short.
First we observe that $(\ell_1, \ell_1) = B(\ell_1, \ell_1)$ by Theorems 1.2 and 1.9, since ℓ_1 is a BK space with AK by Example 1.3.
Now we assume that condition (1.2) holds. Then, in particular, $\sup_k |a_{nk}| < \infty$ for all n, that is $A_n \in \ell_\infty = \ell_1^\beta$ for all n by Example 1.4. Let $x \in \ell_1$ be given. Then it also follows that

$$\begin{aligned} \|A(x)\|_1 &= \sum_{n=0}^\infty |A_n(x)| = \sum_{n=0}^\infty \left| \sum_{k=0}^\infty a_{nk} x_k \right| \\ &\leq \sum_{k=0}^\infty |x_k| \sum_{n=0}^\infty |a_{nk}| \leq \|A\|^* \, \|x\|_1 < \infty, \end{aligned} \tag{1.3}$$

hence $A(x) \in \ell_1$. Thus we have shown $\ell_1 \subset (\ell_1)_A$, that is $A \in (\ell_1, \ell_1)$; (1.3) also shows $\|L_A\| \leq \|A\|^*$.
Conversely, if $A \in (\ell_1, \ell_1)$, then $L_A \in B(\ell_1, \ell_1)$, and so there is a constant M such that $\|L_A(x)\|_1 = \|A(x)\|_1 \leq M\|x\|_1$ for all $x \in B_{\ell_1}$. In particular, it follows for $e^{(k)} \in B_{\ell_1}$ $(k = 0, 1, \dots)$ that $\|L_A(e^{(k)})\|_1 = \sum_{n=0}^\infty |a_{nk}| \leq M$ for all k, and this implies condition (1.3) and also that $\|L_A\| \geq \|A\|^*$. □

1.5. **Compact operators and measures of noncompactness.** If X and Y are BK spaces, then, since $(X,Y) \subset B(X,Y)$, it is interesting to characterize the subclass $(X,Y)_K$ of (X,Y) for which L_A is a compact operator. One way of achieving this is by applying the *Hausdorff measure of noncompactness.*

We recall the following definitions and results. If M and S are subsets of a metric space (X,d) and $\varepsilon > 0$ then S is called an *ε-net of M* if, for every $x \in M$, there exists an $s \in S$ such that $d(x,s) < \varepsilon$; if S is finite then the ε-net S of M is called a *finite ε-net of M*.

Let X and Y be Banach spaces. A linear operator $L : X \to Y$ is called *compact* if its domain is all of X and, for every bounded sequence (x_n) in X, the sequence $(L(x_n))$ has a convergent subsequence in Y.

If Q is a bounded subset of the metric space X, then the *Hausdorff measure of noncompactness of Q* is defined as

$$\chi(Q) = \inf\{\varepsilon > 0 : Q \text{ has a finite } \varepsilon\text{–net in } X\};$$

χ is called the *Hausdorff measure of noncompactness.*

Theorem 1.11. ([14, Theorem 2.15, p. 170])
Let Q be a bounded subset of the normed space X where $X = \ell_p$ for $1 \le p < \infty$ or $X = c_0$. If $P_n : X \to X$ is the operator defined by $P_n(x) = x^{[n]} = (x_1, x_2, \dots, x_n, 0, \dots)$ for all $x = (x_k)_{k=0}^\infty \in X$ then

$$\chi(Q) = \lim_{n\to\infty} (\sup_{x\in Q} \|(I - P_n)(x)\|).$$

Theorem 1.12 (Goldenštein, Gohberg, Markus). ([14, Theorem 2.23, p. 173])
Let X be a Banach space with Schauder basis (b_n), Q be a bounded subset of X and $P_n : X \to X$ the projector onto the linear span of $b_1, \dots, b_n$. Then

$$\frac{1}{a} \limsup_{n\to\infty} \left(\sup_{x\in Q} \|(I - P_n)(x)\| \right) \le \chi(Q)$$
$$\le \inf_n \left(\sup_{x\in Q} \|(I - P_n)(x)\| \right) \le \limsup_{n\to\infty} \left(\sup_{x\in Q} \|(I - P_n)(x)\| \right)$$

where $a = \limsup_{n\to\infty} \|I - P_n\|$.

If X and Y are Banach spaces and χ_1 and χ_2 are Hausdorff measures of noncompactness on X and Y then an operator $L : X \to Y$ is called *(χ_1, χ_2)-bounded* if $L(Q)$ is a bounded subset of Y for every bounded subset Q of X and there exists a positive constant K such that $\chi_2(L(Q)) \le K\chi_1(Q)$ for every bounded subset Q of X. If an operator$L$ is (χ_1, χ_2)–bounded then the number $\|L\|_{(\chi_1,\chi_2)} = \inf\{K > 0 : \chi_2(L(Q)) \le K\chi_1(Q)$ for all bounded $Q \subset$

$X\}$ is called the (χ_1, χ_2)-*measure of noncompactness of* L. If $\chi_1 = \chi_2 = \chi$ then we write $\|L\|_\chi = \|L\|_{(\chi,\chi)}$.

The following result gives a formula to find the Hausdorff measure of an operator $L \in B(X, Y)$; this is the reason why the Hausdorff measure is suitable to characterize classes $(X, Y)_K$.

Theorem 1.13. ([14, Theorem 2.25, p. 175])
Let X and Y be Banach spaces and $L \in B(X, Y)$ and $S_X = \{x \in X : \|x\| = 1\}$ denote the unit sphere in X. *Then $\|L\|_\chi = \chi(L(B_X)) = \chi(L(S_X))$.*

Example 1.14. *Let $1 < p < \infty$ and $q = p/(p-1)$. If $A \in (\ell_p, c_0)$ then*

$$\|L_A\|_\chi = \lim_{r\to\infty} \left(\sup_{n>r} \left(\sum_{k=0}^{\infty} |a_{nk}|^q \right)^{1/q} \right). \tag{1.4}$$

Proof. Let us remark that the limit in (1.4) exists. We write $B = B_{\ell_p}$ for short. Let $P_r : c_0 \to c_0$ $(r = 0, 1, \dots)$ be the projector to the first $r + 1$ coordinates, that is $P_r(y) = y^{[r]}$. Then we have by Theorems 1.12 and 1.13

$$\|L_A\|_\chi = \chi(L_A(B)) = \lim_{r\to\infty} \left(\sup_{x\in B} \|(I - P_n)(A(x))\|_\infty \right), \tag{1.5}$$

since $\|I - P_r\| = 1$ for all r. Let $A^{(r)}$ be the matrix with rows $A_n^{(r)} = 0$ for $0 \le n \le r$ and $A_n^{(r)} = A_n$ for $n > r$. Then, since $(\ell_p, c_0) \subset (\ell_p, \ell_\infty)$, we have by Example 1.8

$$\begin{aligned}\sup_{x\in S} \|(I - P_r)(A(x))\|_\infty &= \sup_{x\in S} \|A^{(r)}(x)\| = \|A^{(r)}\|^*_{(\ell_p,c_0)} = \|L_{A^{(r)}}\| \\ &= \sup_{n>r} \left(\sum_{k=0}^{\infty} |a_{nk}|^q \right)^{1/q},\end{aligned}$$

and this and (1.5) together imply (1.4). □

We also have

Corollary 1.15. ([14, Corollary 2.26, p. 175])
Let X and Y be Banach spaces and $L \in B(X, Y)$. Then $\|L\|_\chi = 0$ if and only if L is compact.

Example 1.16. *We have $A \in (\ell_1, \ell_1)_K$ if and only if condition (1.2) holds and*

$$\lim_{m\to\infty} \sum_{n=m}^{\infty} |a_{nk}| = 0. \tag{1.6}$$

Proof. We write $S = S_{\ell_1}$ for short.
By Example 1.10, we have $(\ell_1, \ell_1) = B(\ell_1, \ell_1)$, and by Corollary 1.15, the operator L_A is compact if and only if $\|L\|_\chi = 0$. But we have by Theorems 1.11 and 1.13, and by (1.4)

$$\|L_A\|_\chi = \chi(L_A(S)) = \lim_{m\to\infty}\left(\sup_{x\in S}\|(I-P_n)(A(x))\|_1\right) = \lim_{m\to\infty}\sum_{n=m}^{\infty}|a_{nk}|.$$

□

We refer the reader to [1, 2, 14] for the general theory of measures of noncompactness; applications of the Hausdorff measure of noncompactness in the characterizations of various classes $(X, Y)_K$ can be found in [14].

2. Matrix Domains of Triangles

In this section, we study matrix domains of *triangles*. We give some general results that reduce the study of matrix domains X_T of triangles and matrix transformations between them to the study of the spaces X and matrix transformations between them.

We need the following well–known result on the inverse of a triangle.

Proposition 2.1. ([3, Remark 2 (a), p. 22] and [22, 1.4.8, p. 9])
Every triangle T has a unique right inverse S, that is $TS = I$ where I is the identity matrix, and S is also a left inverse of T, that is $ST = I$. Moreover S is also a triangle with $s_{nn} = 1/t_{nn}$ for all n and S is the only inverse of T. Therefore we have $S = T^{-1}$.

Throughout, let T be a triangle and S be its inverse.

2.1. The topological properties of matrix domains of triangles.

First we note that the matrix domain of a triangle in a BKspace is a BK space.

Theorem 2.2. ([14, Theorems 3.3, 3.5, pp. 178, 179], [22, Theorems 4.3.12, 4.3.14, pp. 63, 64])
Let X be a BK space. Then X_T is a BK space with $\|x\|_T = \|T(x)\|$.
If X is a closed subspace of Y then X_T is a closed subspace of Y_T .

It turns out that the matrix domain of a triangle in a space with a (Schauder) basis also has a basis.

Theorem 2.3. *If $(b^{(n)})_{n=0}^\infty$ is a basis of the normed sequence space X, then $(S(b^{(n)}))_{n=0}^\infty$ is a basis of X_T.*

Proof. We write $Y = X_T$ and put $c^{(n)} = S(b^{(n)})$ for all $n = 0, 1, \dots$. First we note that $c^{(n)} \in Y$ for all n, since $T(c^{(n)}) = T(S(b^{(n)})) = b^{(n)}$ by Proposition 2.1. Let $y \in Y$ be given. Then $x = T(y) \in X$ and

$x^{<m>} = \sum_{n=0}^{m} \lambda_n b^{(n)} \to x$ $(m \to \infty)$ for a unique sequence $(\lambda_n)_{n=0}^{\infty}$ of scalars. We put $y^{<m>} = \sum_{n=0}^{m} \lambda_n c^{(n)}$ for $m = 0, 1, \ldots$. Then $T(y^{<m>}) = \sum_{n=0}^{m} \lambda_n T(c^{(n)}) = \sum_{n=0}^{m} \lambda_n b^{(n)} = x^{<m>}$, and consequently $\|y^{<m>} - y\|_T = \|T(y^{<m>} - y)\| = \|T(y^{<m>}) - T(y)\| = \|x^{<m>} - x\| \to 0$ as $(m \to \infty)$. □

Since $X = (X_T)_S$ by Proposition 2.1, an application of Theorem 2.3 to X_T yields

Remark 2.4. *The matrix domain X_T of a normed sequence space has a basis if and only if X has a basis.*

An application of Theorem 2.3 yields bases for the matrix domains of triangles in BK spaces with AK and in the convergence domains of triangles.

Corollary 2.5. *Let X be a BK space with AK and the sequences $c^{(n)}$ $(n = 0, 1, \ldots)$ and $c^{(-1)}$ be defined by $c_k^{(n)} = 0$ for $0 \leq k \leq n-1$ and $c_k^{(n)} = s_{kn}$ for $k \geq n$, and $c_k^{(-1)} = \sum_{j=0}^{k} s_{kj}$ $(k = 0, 1, \ldots)$.*
(a) Then every sequence $y = (y_n)_{n=0}^{\infty} \in Y = X_T$ has a unique representation

$$y = \sum_{n=0}^{\infty} T_n(y) c^{(n)}. \tag{2.1}$$

(b) Then every sequence $z = (z_n)_{n=0}^{\infty} \in Z = X_T \oplus e$ has a unique representation

$$z = \ell e + \sum_{n=0}^{\infty} T_n(z - \ell e) \tag{2.2}$$

where ℓ is the uniquely determined complex number such that $z = y + \ell e$ for $y \in Y = X_T$.
(c) Then every sequence $w = (w_n)_{n=0}^{\infty} \in W = (X \oplus e)_T$ has a unique representation

$$w = lc^{(-1)} + \sum_{n=0}^{\infty} (T_n(w) - l) c^{(n)} \tag{2.3}$$

where ℓ is the uniquely determined complex number such that $T(w) - \ell e \in X$.

Proof. First we note that $c^{(n)} = S(e^{(n)})$ $(n = 0, 1, \ldots)$ and $c^{-1} = S(e)$, hence the sequences $(c^{(n)})_{n=0}^{\infty}$ and $(c^{(n)})_{n=-1}^{\infty}$ are bases of Y and W, respectively, by Theorem 2.3.
(a) Let $y = (y_n)_{n=0}^{\infty} \in Y$ be given. Then $x = T(y) \in X$ and (2.1) follows if we take $\lambda_n = T_n(y)$ $(n = 0, 1, \ldots)$ in the proof of Theorem 2.3.

(b) Let $z = (z_n)_{n=0}^{\infty} \in Z = Y \oplus e$ be given. Then there are uniquely determined $y \in Y$ and $\ell \in \mathbb{C}$ such that $z = y + \ell e$, and we have $y = \sum_{n=0}^{\infty} T_n(y) c^{(n)}$ by Part (a), and so $z = \ell e + y = \ell e + \sum_{n=0}^{\infty} T_n(z - \ell e) c^{(n)}$.

(c) Let $w = (w_n)_{n=0}^\infty \in W$. Then $v = T(z) \in V = X \oplus e$ and there are unqiuely determined $x \in X$ and $\ell \in \mathbb{C}$ such that $v = x + \ell e$. We put $y = w - \ell c^{(-1)}$. Then $y \in X_T$, since $T(y) = T(w) - \ell e = v - \ell e = x \in X$, and so we have by $y = \sum_{n=0}^\infty T_n(y)c^{(n)} = \sum_{n=0}^\infty (T_n(w) - l)c^{(n)}$ by Part (a). Now (2.3) is an immediate consequence, since $w = y + \ell c^{(-1)}$. □

2.2. **The β–duals of matrix domains of triangles.** The determination of the β–dual $(X_T)^\beta$ of a BK space with AK can be reduced to that of X^β and the characterization of the class (X, c_0).

Theorem 2.6. ([17, Theorem 2.4])
Let X be a B space with AK and $R = S^t$, the transpose of S. Then $a \in (X_T)^\beta$ if and only if $a \in (X^\beta)_R$ and $W \in (X, c_0)$ where where the triangle W is defined by $w_{mk} = \sum_{j=m}^\infty a_j s_{jk}$; moreover, if $a \in (X_T)^\beta$ then $\sum_{k=0}^\infty a_k z_k = \sum_{k=0}^\infty R_k(a)T_k(z)$ for all $z \in Z = X_T$.

Remark 2.7. ([17, Remark 2.5])
The conclusion of Theorem 2.6 also holds for $X = c$ or $X = \ell_\infty$.

Now we consider two important special cases. Let Σ, Δ and Δ^+ be the matrices with $\Sigma_{nk} = 1$ $(0 \le k \le n)$, $\Sigma_{nk} = 0$ $(k > n)$, $\Delta_{n,n-1} = \Delta^+_{n,n+1} = -1$, $\Delta_{n,n}\Delta^+_{n,n} = 1$ and $\Delta_{nk} = \Delta^+_{n,k} = 0$ otherwise. A subset X of ω is said to be normal if $x \in X$ and $|y_k| \le |x_k|$ $(k = 0, 1, \dots)$ for some sequence y together imply $y \in X$.

Corollary 2.8. ([13, Corollary 2.2])
Let X be a normal subset of ω. Then $(X_\Sigma)^\beta = (X^\beta)_{\Delta^+} \cap M(X, c_0)$.

Corollary 2.9. ([13, Theorem 2.5])
Let X be a normal BK space with AK and the matrix E be defined by $e_{nk} = 0$ for $0 \le k \le n$ and $e_{nk} = 1$ for $k \ge n$ $(n = 0, 1, \dots)$. Then $(X_\Delta)^\beta = (X^\beta \cap M(X_\Delta, c_0))_E$.

Remark 2.10. *Given any sequence $a = (a_n)_{n=0}^\infty$, we write C for the matrix with rows $C_n = a_n e^{[n]}$ for $n = 0, 1, \dots$. Then $a \in M(X_\Delta, c_0)$ if and only if $C \in (X, c_0)$. In particular, $a \in M(\ell_p, c_0)$ for $1 < p < \infty$ if and only if*

$$\sup_n (n+1)^{1/q}|a_n| < \infty \text{ where } q = p/(p-1). \tag{2.4}$$

Proof. The first part is obvious, since $y \in Y = X_\Delta$ if and only if $x = \Delta(y) \in X$ and $a\Sigma(x) = C(x) = ay$.
If $C \in (\ell_p, c_0) \subset (\ell_p, \ell_\infty)$ then $\sup_n(\sum_{k=0}^\infty |c_{nk}|^q)^{1/q} = \sup_n (n+1)^{1/q}|a_n| < \infty$ by Example 1.8, which is condition (2.4).
Conversely let condition (2.4) hold. Then there is a constant M such that $(n+1)^{1/p}|a_n| \le K$ for all n, and so $\lim_{n\to\infty} c_{nk} = \lim_{n\to\infty} a_n = 0$ for each k.

This and condition (2.4) together imply $C \in (\ell_p, c_0)$ by [22, Example 8.4.5D, p. 129]. $\square$

Example 2.11. *Let* $1 < p < \infty$*,* $bv^p = (\ell_p)_\Delta$ *and* $q = p/(p-1)$*. Then it follows from Corollary 2.9 and Remark 2.10 that* $a \in (bv^p)^\beta$ *if and only if* $\sum_{k=0}^\infty |\sum_{j=k}^\infty a_j|^q < \infty$ *and* $\sup_k (k+1)^{1/q} |\sum_{j=k}^\infty a_j| < \infty$*.*

2.3. **Matrix transformations between matrix domains of triangles.** In this subsection, we give some general results that reduce the characterization of the classes (X, Y_T) and (X_T, Y) to that of the class (X, Y).

Theorem 2.12. ([14, Theorem 3.8, p. 180])
Let X *and* Y *be arbitrary subsets of* ω*. Then* $A \in (X, Y_T)$ *if and only if* $C = TA \in (X, Y)$*. Furthermore, if* X *and* Y *are* BK *spaces and* $A \in (X, Y_T)$ *then* $\|L_A\| = \|L_C\|$*.*

Theorem 2.13. [17, Thereom 3.1])
Let X *be a* BK *space with* AK*,* Y *be an arbitrary subset of* ω *and* $R = S^t$*. Then* $A \in (X_T, Y)$ *if and only if* $B^A \in (X, Y)$ *and* $W^{A_n} \in (X, c_0)$ *for all* $n = 0, 1, \dots$ *where* B^A *is the matrix with rows* $B_n(A) = R(A_n)$ *for* $n = 0, 1, \dots$ *and the triangles* W^{A_n} $(n = 0, 1, \dots)$ *are defined by* $w_{mk}^{A_n} = \sum_{j=m}^\infty a_{nj} s_{jk}$*.*

Remark 2.14. *Theorem 2.13 also holds when* $X = c$ *or* $X = \ell_\infty$*. If* Y *is a linear space, then we can also show that* $A \in (c_T, Y)$ *if and only if* $A \in ((c_0)_T, Y)$ *and* $A(S(e)) \in Y$*.*

As at the end of the previous subsection, we give the special results when $T = \Sigma$ or $T = \Delta$.

Corollary 2.15. ([13, Theorem 2.6])
Let X *and* Y *be subsets of* ω *and* X *be normal. Then* $A \in (X_\Sigma, Y)$ *if and only if* $A_n \in M(X, c_0)$ *for all* $n = 0, 1, \dots$ *and* $B \in (X, Y)$ *where* $B_n = \Delta^+(A_n)$ *for all* $n = 0, 1, \dots$*.*

Corollary 2.16. ([13, Theorem 2.7])
Let $X \supset \phi$ *be a normal* FK *space with* AK*,* Y *be a linear space. Then* $A \in (X_\Delta, Y)$ *if and only if* $R^A \in (X, Y)$ *where* $r_{nk}^A = \sum_{j=k}^\infty a_{nj}$ *for all* n, k *and* $R_n^A \in (M(X_\Delta, c_0)$ *for all* n*.*

Example 2.17. *Let* $1 < p < \infty$ *and* $q = p/(p-1)$*. Then it follows from Corollary 2.16, Example 2.11 and* [22, Example 8.4.5D, p. 129] *that* $A \in (bv^p, c_0)$ *if and only if*

$$\|A\|_{(bv^p, \ell_\infty)} = \sup_n \left(\sum_{k=0}^\infty \left| \sum_{j=k}^\infty a_{nj} \right|^q \right)^{1/q} < \infty, \tag{2.5}$$

$\lim_{n\to\infty} \sum_{j=k}^\infty a_{nj} = 0$ *for each* k *and* $\sup_k (k^{1/q} |\sum_{j=k}^\infty a_{nj}|) < \infty$ *for all* n*.*

2.4. **The Hausdorff measure of noncompactness in matrix domains of triangles.** Here we give a result to find the Hausdorff measure of noncompactness of bounded subsets in matrix domains of triangles.

Theorem 2.18. *Let X be a normed sequence space and χ_T and χ denote the Hausdorff measures of noncompactness on $\mathcal{M}_{X_T}$ and $\mathcal{M}_X$, the collection of all bounded ets in X_T and X, respectively. Then $\chi_T(Q) = \chi(T(Q))$ for all $Q \in \mathcal{M}_{X_T}$.*

Proof. We write $Y = X_T$, $B(x,r)$ and $B_T(y,r)$ for the open balls of radius r in X and Y, centred at x and y, respectively, and observe that $Q \in \mathcal{M}_Y$ if and only if $P = T(Q) \in \mathcal{M}_X$ by the definition of the norm $\|\cdot\|_T$. Thus $\chi_T(Q)$ is defined if and only if $\chi(T(Q))$ is defined.
First we show that $\chi_T(Q) \leq \chi(T(Q))$ for all $Q \in \mathcal{M}_Y$. We put $t = \chi_T(Q)$ and $s = \chi(T(Q))$ and assume $t > s$ for some $Q \in \mathcal{M}_Y$. Then there are a real ε with $s < \varepsilon < t$, $x_1, \dots, x_n \in X$ and $r_1, \dots, r_n < \varepsilon$ such that $T(Q) \subset \bigcup_{k=1}^n B(x_k, r_k)$. Let $q \in Q$ be given. We put $p = T(q) \in X$, and so there are $y_j \in Y$ with $x_j = T(y_j)$ and $r_j < \varepsilon$ such that $p \in B(x_j, r_j)$, that is $\|p - x_j\| = \|T(q) - T(y_j)\| = \|T(q - y_j)\| = \|q - y_j\|_T < r_j$, hence $q \in B_T(y_j, r_j) \subset \bigcup_{k=1}^n B_T(y_k, r_k)$. Since $q \in Q$ was arbitrary, we have $Q \subset \bigcup_{k=1}^n B_T(y_k, r_k)$. Therefore we have $\chi_T(Q) \leq \varepsilon < t$ which is a contradiction to $\chi_T(Q) = t$, and consequently we must have $\chi_T(Q) \leq \chi(T(Q))$ for all $Q \in \mathcal{M}_Y$.
Applying what we have just shown with X and Y replaced by Y and $Y_S = X$ where $S = T^{-1}$, we obtain $\chi(T(Q)) = (\chi_T)_S(T(Q)) \leq \chi_T(S(T(Q))) = \chi_T(Q)$ for all $Q \in \mathcal{M}_Y$. □

Example 2.19. *Let $T = \Delta$ and $X = \ell_p$ for $1 \leq p < \infty$ or $X = c_0$. Then it follows from Theorems 2.18 and 1.11 that $\chi_T(Q) = \chi(T(Q)) = \lim_{n\to\infty} \sup_{x\in T(Q)} \|(I - P_n)(x)\| = \lim_{n\to\infty} \sup_{y\in Q} \|(I - P_n)(T(y))\|$ for all $Q \in \mathcal{M}_{X_T}$ where $\|(I - P_n)(T(y))\| = (\sum_{k=n}^\infty |y_k - y_{k-1}|^p)^{1/p}$ for $X = \ell_p$ and $\|(I - P_n)(T(y))\| = \sup_{k\geq n} |y_k - y_{k-1}|$ for $X = c_0$.*

Example 2.20. *Let $1 < p < \infty$ and $q = p/(p-1)$. Then it follows from Corollary 2.16 and Example 2.17 that if $A \in (bv^p, c_0)$ then*

$$\|L_A\|_\chi = \lim_{r\to\infty} (\sup_{n>r} (\sum_{k=0}^\infty |\sum_{j=k}^\infty a_{nj}|^q)^{1/q}.$$

Further results concerning the characterizations of matrix transformations and compact operators between the matrix domains of triangles can be found in [14, 15, 16, 17, 18].

3. Strong Matrix Domains and Mixed Norm Spaces

In this section, we study mixed norm spaces. Strong matrix domains can be considered as special cases of mixed norm spaces.

Let $1 \leq p < \infty$ throughout.

In 1968, Maddox [10] introduced and studied the sets $w_0^p = (c_0)_{[C_1]^p} = \{x \in \omega : \lim_{n\to\infty} 1/n \sum_{k=1}^n |x_k|^p = 0\}$ and $w_\infty^p = (\ell_\infty)_{[C_1]^p}$ of sequences that are strongly summable C_1 with index p to zero and strongly bounded C_1 with index p. He also observed that the *sections* $1/n \sum_{k=1}^n$ can be replaced by the blocks $1/2^{\nu+1} \sum_{k=2^\nu}^{2^{\nu+1}-1}$, and that the *section* and *block norms* $\|x\| = \sup_n (1/n \sum_{k=1}^n |x_k|^p)^{1/p}$ and $\|x\|' = \sup_{\nu\geq 0}(1/2^{\nu+1} \sum_{k=2^\nu}^{2^{\nu+1}-1} |x_k|^p)^{1/p}$ are equivalent.

In 1974, Jagers [7] studied the *Cesáro sequence spaces* $ces(p) = (\ell_p)_{[C]^1}$ which are Banach spaces with the norm

$$\|x\|_{ces(p)} = (\sum_{n=1}^\infty (1/n \sum_{k=1}^n |x_k|)^p)^{1/p}.$$

It can be found in [4] that an equivalent norm on $ces(p)$ is

$$\|x\| = (\sum_{\nu=0}^\infty 2^{\nu(1-p)} (\sum_{k=2^\nu}^{2^{\nu+1}-1} |x_k|)^p)^{1/p}.$$

The *mixed norm spaces* $\ell(p,q) = \{x \in \omega : \sum_{\nu=0}^\infty (\sum_{k=2^\nu}^{2^{\nu+1}-1} |x_k|^p)^{q/p} < \infty\}$ were introduced by Hedlund [6] in 1969; see also *Kellog* [9].

Now we generalize the concept of mixed norm spaces.

Throughout, let $(k(\nu))_{\nu=0}^\infty$ be a strictly increasing sequence of integers with $k(0) = 0$, and I_ν be the set of all integers k with $k(\nu) \leq k \leq k(\nu+1)-1$ ($\nu = 0, 1, \dots$). Given any sequence $x = (x_k)_{k=1}^\infty$, then, for every $\nu = 0, 1, \dots$, we write $x^{<\nu>} = \sum_{k\in I_\nu} x_k e^{(k)}$ for the *ν–block of the sequence x*. Let $X, Y \supset \phi$ be sequence spaces, normed with $\|\cdot\|_X$ and $\|\cdot\|_Y$. We define the *generalized mixed norm spaces* $Z = [Y, X]^{<k(\nu)>} = \{z \in \omega : (\|z^{<\nu>}\|_X)_{\nu=0}^\infty \in Y\}$, and put

$$g(z) = \|(\|z^{<\nu>}\|_X)_{\nu=0}^\infty\|_Y \text{ for all } z \in Z. \tag{3.1}$$

Since $\phi \subset X$, $\|z^{<\nu>}\|_X$ is defined for every $z \in \omega$ and for all $\nu = 0, 1, \dots$. Hence the sequence $y = (y_\nu)_{\nu=0}^\infty$ with $y_\nu = \|z^{<\nu>}\|_X$ ($\nu = 0, 1, \dots$) is defined. Furthemore, since $\phi \subset X, Y$, we obviously have $\phi \subset Z$.

Example 3.1. *Let $1 \le p, r < \infty$. Then we obtain*

$$[\ell_r, \ell_p]^{<k(\nu)>} = \left\{ z \in \omega : \sum_{\nu=0}^{\infty} (\sum_{k \in I_\nu} |z_k|^p)^{r/p} < \infty \right\},$$

$$[\ell_r, \ell_\infty]^{<k(\nu)>} = \left\{ z \in \omega : \sum_{\nu=0}^{\infty} (\max_{k \in I_\nu} |z_k|)^r < \infty \right\},$$

$$[c_0, \ell_p]^{<k(\nu)>} = \left\{ z \in \omega : \lim_{\nu \to \infty} \sum_{k \in I_\nu} |z_k|^p = 0 \right\} \text{ and}$$

$$[\ell_\infty, \ell_p]^{<k(\nu)>} = \left\{ z \in \omega : \sup_{\nu \ge 0} \sum_{k \in I_\nu} |z_k|^p < \infty \right\}.$$

In the special case of $r = p$ and $1 \le p \le \infty$, we have $[\ell_r, \ell_p]^{<k(\nu)>} = \ell_p$. If $k(\nu) = 2^\nu$ for $\nu = 0, 1, \ldots$, then $[\ell_r, \ell_p]^{<k(\nu)>} = \ell(r, p)$, the mixed norm spaces in [6, 9].
Let $1 \le p < \infty$ and $k(\nu) = 2^\nu$ for all ν. If $d_\nu = (1/k(\nu+1))^{1/p}$ for $\nu = 0, 1, \ldots$ then $[d^{-1} * c_0, \ell_p]^{<k(\nu)} = w_0^p$ and $[d^{-1} * \ell_\infty, \ell_p]^{<k(\nu)>} = w_\infty^p$ [10]. *If $d_\nu = 2^{\nu(1/p-1)}$ for $\nu = 0, 1, \ldots$ we obtain the Cesáro sequence spaces or weighted mixed norm spaces $[d^{-1} * \ell_\infty, \ell_1]^{<k(\nu)>}$* [7].

3.1. **The topological properties of generalized mixed norm spaces.** First, we study the topological properties of the generalized mixed norm spaces $Z = [Y, X]^{<k(\nu)>}$.

We say that a norm $\|\cdot\|$ on a sequence space is *monotonous* if $|x_k| \le |\tilde{x}_k|$ $(k = 1, 2, \ldots)$ for $x, \tilde{x} \in X$ implies $\|x\| \le |\tilde{x}\|$. Given a sequence $z \in \omega$, we write $y = (y_\nu)_{\nu=0}^{\infty}$ for the sequence $y_\nu = \|z^{<\nu>}\|_X$ $(\nu = 0, 1, \ldots)$.

Proposition 3.2. ([8, Proposition 3.1])
Let $X, Y \supset \phi$ be normed sequence spaces and $Z = [Y, X]^{<k(\nu)>}$.
(a) If Y is normal and $\|\cdot\|_X$ is monotonous then Z is normal.
(b) If $\|\cdot\|_Y$ is monotonous then Z is normed with respect to g defined in (3.1). If, however, $\|\cdot\|$ is not monotonous, then g does not satisfy the triangle inequality in general.

Theorem 3.3. ([8, Theorem 3.2])
Let $X \supset \phi$ be a normed sequence space, $Y \supset \phi$ be a normal BK space and $\|\cdot\|_Y$ be monotonous. Then Z is a BK space with $\|\cdot\|_Z = g$ where g is defined in (3.1). Furthermore, if Y has AK and $\|\cdot\|_X$ is monotonous then Z also has AK.

Example 3.4. *(a) Let $1 \le r, p < \infty$. Then $[\ell_r, \ell_p]^{<k(\nu)>}$ and $[c_0, \ell_p]^{<k(\nu)>}$ are BK spaces with AK with $\|z\|_{(r,p)} = (\sum_{\nu=0}^{\infty} (\sum_{k \in I_\nu} |z_k|^p)^{r/p})^{1/r}$ and $\|z\|_{(\infty,p)}$*

$= \sup_{\nu \geq 0}(\sum_{k \in I_\nu} |z_k|^p)^{1/p}$, *and* $[\ell_\infty, \ell_p]^{<k(\nu)>}$ *is a* BK *space with* $\|\cdot\|_{(\infty,p)}$; *moreover,* $[c_0, \ell_p]^{<k(\nu)>}$ *is a closed subspace of* $[\ell_\infty, \ell_p]^{<k(\nu)>}$; $[\ell_r, \ell_\infty]^{<k(\nu)>}$ *are* BK *spaces with* AK *with* $(\sum_{\nu=0}^\infty (\max_{k \in I_\nu} |z_k|)^r)^{1/r}$.
Let $1 \leq p < \infty$, $k(\nu) = 2^\nu$ *and* $d_\nu = (1/k(\nu+1))^{1/p}$ *for* $\nu = 0, 1, \ldots$. *Then* w_0^p *and* w_∞^p *are* BK *spaces with* $\|z\|' = \sup_{\nu=0}(1/2^{\nu+1} \sum_{k=2^\nu}^{2^{\nu+1}-1} |x_k|^p)^{1/p}$, *and* w_0^p *has* AK; *moreover* w_0^p *is a closed subspace of* w_∞^p.

3.2. **The β–duals of generalized mixed norm spaces.** Now we determine the β–duals of the spaces $[Y, X]^{<k(\nu)>}$.

If X is a normed sequence space and $a \in \omega$, we write $\|a\|_{X,\alpha} = \sup_{x \in B_X} \sum_{k=0}^\infty |a_k x_k|$ and $\|a\|_{X,\beta} = \sup_{x \in B_X} |\sum_{k=0}^\infty a_k x_k|$ provided the expressions exist and are finite which is the case whenever X is a BK space and $a \in X^\alpha$ or $a \in X^\beta$ (cf. [22, Theorems 4.3.15 and 7.2.9, pp. 64 and 107]).

A norm on a sequence space X is said to be KB if the set $\mathcal{P} = \{P_k : X \to \mathbb{C} : P_k(x) = x_k \ (x \in X) \ (k = 1, 2, \ldots)\}$ of coordinates is equicontinuous, that is if there is a constant K such that $|x_k| \leq K\|x\|$ for all $x \in X$ and all k. If X is a Banach sequence space with a norm which is KB then it is obviously a BK space. Conversely the norm of a BK space need not be KB in general. To see this, we choose $X = (\ell_\infty)_\Delta$ with $\|x\| = \sup_k |x_k - x_{k-1}|$, a BK space, and the sequence x with $x_k = k$ for $k = 1, 2, \ldots$.

If X is a normed sequence space then we write $X^\delta = \{a \in \omega : \|a\|_{X,\alpha} < \infty\}$.

Theorem 3.5. ([8, Theorem 4.1])
Let X *and* Y *be normed sequence spaces with* $X, Y \supset \phi$ *and* $\|\cdot\|_Y$ *be monotonous.*
(a) Then $[Y^\delta, X^\delta]^{<k(\nu)>} \subset ([Y, X]^{<k(\nu)>})^\delta$.
(b) If, in addition, the norms $\|\cdot\|_X$ *and* $\|\cdot\|_X$ *are both* KB, $\|\cdot\|_X$ *is monotonous and* Y *is normal then* $([Y, X]^{<k(\nu)>})^\delta \subset [Y^\delta, X^\delta]^{<k(\nu)>}$.

If X is a BK space then $X^\alpha = X^\delta$ by [22, Theorem 4.3.15, p. 64], and if X is normal then $X^\alpha = X^\beta$. Therefore we obtain from Proposition 3.2 and Theorems 3.3 and 3.5

Corollary 3.6. ([8, Corollary 4.2])
Let X *be a normed sequence space,* Y *be a normal* BK *space and the norms* $\|\cdot\|_X$ *and* $\|\cdot\|_Y$ *be monotonous and* KB. *Then* $([Y, X]^{<k(\nu)>})^\alpha = [Y^\alpha, X^\alpha]^{<k(\nu)>}$.
If, in addition, X *is normal then* $([Y, X]^{<k(\nu)>})^\beta = [Y^\beta, Y^\beta]^{<k(\nu)>}$.

Example 3.7. *Let* $1 \leq r, p < \infty$, s *and* q *be the* conjugate numbers of r and p, *that is* $s = \infty$ *for* $r = 1$ *and* $s = r/(r-1)$ *for* $1 < r < \infty$ *and* q *defined similarly. Since the norms* $\|\cdot\|_{\ell_p,\beta}$ *and* $\|\cdot\|_q$ *and* $\|\cdot\|_{\ell_\infty,\beta}$ *and* $\|\cdot\|_1$ *are*

equivalent on ℓ_p^β *and on* $\ell_\infty^\beta = c_0^\beta$, *we have* $([\ell_r, \ell_p]^{<k(\nu)>})^\beta = [\ell_s, \ell_q]^{<k(\nu)>}$ *and* $([c_0, \ell_p]^{<k(\nu)>})^\beta = ([\ell_\infty, \ell_p]^{<k(\nu)>})^\beta = [\ell_1, \ell_q]^{<k(\nu)>}$.
Let $\mathcal{U}$ *denote the set of all sequences* u *with* $u_k \neq 0$ *for all* k. *If* $u \in \mathcal{U}$ *then we write* $1/u = (1/u_k)_{k=1}^\infty$, *and it is obvious that* $(u^{-1} * X)^\beta = (1/u)^{-1} * X^\beta$ *for arbitrary subsets* X *of* ω. *Let* $k(\nu) = 2^\nu$ *and* $d_\nu = (1/k(\nu+1))^{1/p}$ *for* $\nu = 0, 1, \dots$. *Then* $(w_0^p)^\beta = (w_\infty^p)^\beta = \mathcal{M}_p$ *where*

$$\mathcal{M}_p = \begin{cases} \left\{ a \in \omega : \sum_{\nu=0}^\infty 2^{\nu+1} \max_{k \in I_\nu} |a_k| < \infty \right\} & (p = 1) \\ \left\{ a \in \omega : \sum_{\nu=0}^\infty 2^{\nu+1} \left(\sum_{k \in I_\nu} |a_k|^q \right)^{1/q} < \infty \right\} & (1 < p < \infty) \end{cases}.$$

3.3. **Matrix transformations in generalized mixed norm spaces.** Now we characterise some classes of matrix transformations between mixed norm spaces.

Let $(m(\mu))_{\mu=0}^\infty$ be a strictly increasing sequence of integers with $m(0) = 1$ and $M_\mu = \{m \in \mathbb{N} : m(\mu) \leq m \leq m(\mu+1)-1\}$ $(\mu = 0, 1, \dots)$. Furthermore, let T denote the set of all sequences $(t_\mu)_{\mu=0}^\infty$ of integers such that for each μ there is one and only one $t_\mu \in M_\mu$.

First we give a result that characterises the classes (X, Y) where X is any BK space and Y is any of the spaces ℓ_∞, c_0, ℓ_1, $[\ell_\infty, \ell_1]^{<m(\mu)>}$, $[\ell_1, \ell_\infty]^{<m(\mu)>}$ or $[c_0, \ell_1]^{<m(\mu)>}$.

Theorem 3.8. ([8, Theorem 4.4])
Let X *be a* BK *space, or a* BK *space with* AK *in the cases marked* $*$. *We write* $\sup_N$ *for the supremum taken over all finite subsets* N *of* $\mathbb{N}_0$. *Then the conditions for* $A \in (X, Y)$ *when* Y *is any of the spaces* ℓ_∞, c_0, ℓ_1, $[\ell_\infty, \ell_1]^{<m(\mu)>}$, $[\ell_1, \ell_\infty]^{<m(\mu)>}$ *or* $[c_0, \ell_1]^{<m(\mu)>}$ *can be read from the table*

To / From	ℓ_∞	c_0	ℓ_1	$[\ell_\infty, \ell_1]^{<m(\mu)>}$	$[\ell_1, \ell_\infty]^{<m(\mu)>}$	$[c_0, \ell_1]^{<m(\mu)>}$
X	(1.)	* (2.)	(3.)	(4.)	(5.)	* (6.)

where

(1.) (1.1) where (1.1) $\sup_n \|A_n\|_{X,\beta} < \infty$

(2.) (1.1) and (2.1) where (2.1) $\lim_{n\to\infty} a_{nk} = 0$ for each k

(3.) (3.1) where (3.1) $\sup_N \left\|\sum_{n\in N} A_n\right\|_{X,\beta} < \infty$

(4.) (4.1) where (4.1)

$$\sup_\mu \left(\max_{M(\mu)\subset M_\mu} \left\| \sum_{m\in M(\mu)} A_m \right\|_{X,\beta} \right) < \infty$$

(5.) (5.1) where (5.1) $\sup_N \left(\sup_{t\in T} \left\| \sum_{\mu\in N} A_{t_\mu} \right\|_{X,\beta} \right) < \infty$

(6.) (4.1) and (6.1) where (6.1) $\lim_{\mu\to\infty} \sum_{n\in M_\mu} |a_{nk}| = 0$ for each k.

We obtain as an immediate consequence of Example 3.7 and Theorem 3.8

Corollary 3.9. ([8, Corollary 4.5])
Let $1 < r < \infty$ and $1 < p \le \infty$ and s and q be the conjugate numbers of r and p. Then the conditions for $A \in ([\ell_r, \ell_p]^{<k(\nu)>}, Y)$ where Y is any of the spaces in Theorem 3.8 can be read from the table

From \ To	ℓ_∞	c_0	ℓ_1	$[\ell_\infty, \ell_1]^{<m(\mu)>}$	$[\ell_1, \ell_\infty]^{<m(\mu)>}$	$[c_0, \ell_1]^{<m(\mu)>}$
$[\ell_r, \ell_p]^{<k(\nu)>}$	(1.)	(2.)	(3.)	(4.)	(5.)	(6.)

where

(1.) (1.1) where (1.1) $\sup_n \sum_{\nu=0}^\infty \left(\sum_{k \in I_\nu} |a_{nk}|^q \right)^{s/q} < \infty$

(2.) (1.1) and (2.1) where (2.1) is (2.1) in Theorem 3.8

(3.) (3.1) where (3.1) $\sup_N \sum_{\nu=0}^\infty \left(\sum_{k \in I_\nu} \left| \sum_{n \in N} a_{nk} \right|^q \right)^{s/q} < \infty$

(4.) (4.1) where (4.1)

$$\sup_\mu \left(\max_{M(\mu) \subset M_\mu} \sum_{\nu=0}^\infty \left(\sum_{k \in I_\nu} \left| \sum_{m \in M(\mu)} a_{mk} \right|^q \right)^{s/q} \right) < \infty$$

(5.) (5.1) where (5.1)

$$\sup_N \left(\sup_{t \in T} \sum_{\nu=0}^\infty \left(\sum_{k \in I_\nu} \left| \sum_{\mu \in N} a_{t_\mu, k} \right|^q \right)^{s/q} \right) < \infty$$

(6.) (4.1) and (6.1) where (6.1) is (6.1) in Theorem 3.8

If $r = 1$ or $p = 1$ replace $\sum_{\nu=0}^\infty$ or $\sum_{k \in I_\nu}$ by $\sup_{\nu \ge 0}$ or $\max_{k \in I_\nu}$ in conditions (1.1), (3.1), (4.1) and (5.1) in **(1.)–(6.)**. *The conditions for $A \in ([c_0, \ell_p]^{<k(\nu)>}), Y)$ are those in* **(1.)–(6.)** *with $s = 1$ in (1.1), (3.1), (4.1) and (5.1). Finally, we have $([\ell_\infty, \ell_p]^{<k(\nu)>}, Y) = ([c_0, \ell_p]^{<k(\nu)>}, Y)$ for $Y \ne c_0, [c_0, \ell_1]^{<m(\mu)>}$.*

Now we give the *dual result* of Theorem 3.8. We write T' for the set of all strictly increasing sequences $t = (t_\nu)_{\nu=0}^\infty$ of integers such that for each ν there is one and only one $t_\nu \in I_\nu$.

Theorem 3.10. ([8, Theorem 4.6])
Let W be a BK space with AK and $Y = W^\beta$. Then the conditions for $A \in (X, Y)$ where X is any of the spaces ℓ_∞, c_0, ℓ_1, $[\ell_1, \ell_\infty]^{<k(\nu)>}$, $[\ell_\infty, \ell_1]^{<k(\nu)>}$ or $[c_0, \ell_1]^{<k(\nu)>}$ can be read from the table

To \ From	ℓ_∞	c_0	ℓ_1	$[\ell_\infty, \ell_1]^{<k(\nu)>}$	$[\ell_1, \ell_\infty]^{<k(\nu)>}$	$[c_0, \ell_1]^{<k(\nu)>}$
Y	(1.)	(2.)	(3.)	(4.)	(5.)	(6.)

where

(1.) (1.1) where (1.1) $\sup_N \| \sum_{n\in N} A^n\|_Y < \infty$

(2.) (1.1)

(3.) (3.1) where (3.1) $\sup_n \|A^n\|_Y < \infty$

(4.) (4.1) where (4.1) $\sup_N \left(\sup_{t\in T'} \| \sum_{\nu\in N} A^{t_\nu}\|_Y\right) < \infty$

(5.) (5.1) where (5.1) $\sup_N \left(\max_{K(\nu)\subset K_\nu} \| \sum_{m\in K(\nu)} A^m\|_Y\right) < \infty$

(6.) (4.1)

We obtain as an immediate consequence of Theorem 3.10

Corollary 3.11. ([8, Corollary 4.7])
Let $1 < r < \infty$, $1 < p < \infty$ and X be any of the spaces in Theorem 3.8. Then the conditions for $A \in (X, [\ell_r, \ell_p]^{<m(\mu)>})$ can be read from the table

From / To	ℓ_∞	c_0	ℓ_1	$[\ell_\infty, \ell_1]^{<k(\nu)>}$	$[\ell_1, \ell_\infty]^{<k(\nu)>}$	$[c_0, \ell_1]^{<k(\nu)>}$
$[\ell_r, \ell_p]^{<m(\mu)>}$	(1.)	(2.)	(3.)	(4.)	(5.)	(6.)

where

(1.) (1.1) where (1.1) $\sup_N \sum_{\mu=0}^\infty \left(\sum_{k\in M_\mu} \left|\sum_{n\in N} a_{kn}\right|^p\right)^{r/p} < \infty$

(2.) (1.1)

(3.) (3.1) where (3.1) $\sup_n \sum_{\mu=0}^\infty \left(\sum_{k\in M_\mu} |a_{kn}|^p\right)^{r/p} < \infty$

(4.) (4.1) where (4.1)

$$\sup_N \left(\sup_{t\in T'} \sum_{\mu=0}^\infty \left(\sum_{k\in M_\mu} \left|\sum_{\nu\in N} a_{k,t_\nu}\right|^p\right)^{r/p}\right) < \infty$$

(5.) (5.1) where (5.1)

$$\sup_N \left(\max_{k(\nu)\in K_\nu} \sum_{\nu=0}^\infty \left(\sum_{k\in M_\mu} \left|\sum_{m\in K(\nu)} a_{km}\right|^p\right)^{r/p}\right) < \infty$$

(6.) (4.1)

Further results on matrix transformations and compact operators between mixed norm spaces can be found in [14, 8, 15, 16].

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