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Orientability and energy minimization in liquid crystal models

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Abstract

Uniaxial nematic liquid crystals are modelled in the Oseen-Frank theory through a unit vector field n. This theory has the apparent drawback that it does not respect the head-to-tail symmetry in which n should be equivalent to -n. This symmetry is preserved in the constrained Landau - de Gennes theory that works with the tensor $Q = s\big(n \otimes n - \frac{1}{3} Id\big)$.

We study the differences and the overlaps between the two theories. These depend on the regularity class used as well as on the topology of the underlying domain. We show that for simply-connected domains and in the natural energy class $W^{1,2}$ the two theories coincide, but otherwise there can be differences between the two theories, which we identify.

In the case of planar domains with holes and various boundary conditions, for the simplest form of the energy functional, we completely characterise the instances in which the predictions of the constrained Landau - de Gennes theory differ from those of the Oseen-Frank theory.

1. Introduction

The challenge of describing nematic liquid crystals by a model that is both comprehensive and simple enough to manipulate efficiently has led to the existence of several major competing theories. One of the most simple and successful is the Oseen-Frank theory that describes nematics using relatively simple tools, namely vector fields, but has the deficiency of ignoring a physical symmetry of the material. A more complex theory was proposed by de Gennes and uses matrix-valued functions (Q-tensors). In the simplest constrained case of uniaxial Q-tensors with a constant scalar order parameter these Q-tensors can be interpreted as line fields. We study in this paper the differences between these two theories in this constrained case and establish when the more complicated, but physically more realistic, theory of de Gennes can be replaced by the simpler Oseen-Frank theory, and when this cannot be done. Of particular interest to our study are liquid crystal 'defects' and we examine the instances when one would detect fake 'defects' in the material resulting from using the more simplistic Oseen-Frank model instead of the constrained Landau - de Gennes model (we refer to such a situation as having a *non-orientable line field*).

In nematic liquid crystals the rod-like molecules tend to align, locally, along a preferred direction. This is modeled by assigning at each material point x in the region Ω occupied by the liquid crystal a probability measure $\mu(x,\cdot):\mathcal{L}(\mathbb{S}^2)\to[0,1]$ for describing the orientation of the molecules, where $\mathcal{L}(\mathbb{S}^2)$ denotes the family of Lebesgue measurable sets on the unit sphere. Thus $\mu(x,A)$ gives the probability that the molecules with centre of mass in a very small neighbourhood of the point $x\in\Omega$ are pointing in a direction contained in $A\subset\mathbb{S}^2$.

Nematic liquid crystals are locally invariant with respect to reflection in the plane perpendicular to the preferred direction. This is commonly referred to as the 'head-to-tail' symmetry, see [16]. This means that $\mu(x,A)=\mu(x,-A)$, for all $x\in\Omega, A\subset\mathcal{L}(\mathbb{S}^2)$. Note that because of this symmetry the first moment of the probability measure vanishes:

$$\langle p\rangle \stackrel{\text{def}}{=} \int_{\mathbb{S}^2} p\,d\mu(p)=\frac{1}{2}\left[\int_{\mathbb{S}^2} p\,d\mu(p)+\int_{\mathbb{S}^2} -p\,d\mu(-p)\right]=0.$$

The first nontrivial information on μ comes from the tensor of second moments:

$$M_{ij}\stackrel{\text{def}}{=}\int_{\mathbb{S}^2} p_ip_j\,d\mu(p),\ i,j=1,2,3.$$

We have $M=M^T$ and $\operatorname{tr} M=\int_{\mathbb{S}^2} d\mu(p)=1$. Let e be a unit vector. Then

$$e\cdot Me=\int_{\mathbb{S}^2}(e\cdot p)^2\,d\mu(p)=\langle\cos^2(\theta)\rangle$$

where θ is the angle between p and e.

If the orientation of the molecules is equally distributed in all directions we say that the distribution is *isotropic* and then $\mu=\mu_0$ where $d\mu_0(p)=\frac{1}{4\pi}dA$. The corresponding second moment tensor is

$$M_0\stackrel{\text{def}}{=}\frac{1}{4\pi}\int_{\mathbb{S}^2} p\otimes p\,dA=\frac{1}{3}Id$$

(since $\int_{\mathbb{S}^2} p_1p_2\,d\mu(p)=0$, $\int_{\mathbb{S}^2} p_1^2\,d\mu(p)=\int_{\mathbb{S}^2} p_2^2\,d\mu(p)=\int_{\mathbb{S}^2} p_3^2\,d\mu(p)$ and $\operatorname{tr} M_0=1$).

The de Gennes order-parameter tensor Q is defined as

$$Q\stackrel{\text{def}}{=}M-M_0=\int_{\mathbb{S}^2}\left(p\otimes p-\frac{1}{3}Id\right)\,d\mu(p) \tag{1}$$

and measures the deviation of the second moment tensor from its isotropic value.

Since Q is symmetric and $\operatorname{tr} Q = 0$, Q has, by the spectral theorem, the representation:

$$Q = \lambda_1 \hat{e}_1 \otimes \hat{e}_1 + \lambda_2 \hat{e}_2 \otimes \hat{e}_2 - (\lambda_1 + \lambda_2)\hat{e}_3 \otimes \hat{e}_3$$

where $\hat{e}_1$, $\hat{e}_2$, $\hat{e}_3$ is an orthonormal basis of eigenvectors of Q with the corresponding eigenvalues λ_1, λ_2 and $\lambda_3 = -(\lambda_1 + \lambda_2)$.

When two of the eigenvalues are equal (and non-zero) the order parameter Q is called *uniaxial*, otherwise being *biaxial*. The case $Q = 0$ is the *isotropic* state. Equilibrium configurations of liquid crystals are obtained, for instance, as energy minimizers, subject to suitable boundary conditions. The simplest commonly used energy functional is

$$\mathcal{F}_{LG}[Q] = \int_\Omega \left[\frac{L}{2} \sum_{i,j,k=1}^{3} Q_{ij,k} Q_{ij,k} - \frac{a}{2} \operatorname{tr} Q^2 - \frac{b}{3} \operatorname{tr} Q^3 + \frac{c}{4} \left(\operatorname{tr} Q^2\right)^2 \right] dx \tag{2}$$

where a, b, c are temperature and material dependent constants and $L > 0$ is the elastic constant. In the physically significant limit $L \to 0$ (and for appropriate boundary conditions) we have that the energy minimizers are suitably approximated by minimizers of the corresponding *'Oseen-Frank energy functional'*

$$\mathcal{F}_{OF}[Q] = \int_\Omega \sum_{i,j,k=1}^{3} Q_{ij,k} Q_{ij,k} \, dx$$

in the restricted class of $Q \in W^{1,2}$, with Q uniaxial a.e., so that

$$Q = s\left(n \otimes n - \frac{1}{3} Id\right) \tag{3}$$

with $s \in \mathbb{R}$ (an explicit constant depending on a, b and c) and $n(x) \in \mathbb{S}^2$ a.e. $x \in \Omega$.

This convergence, as $L \to 0$, was studied initially in [29] and further refined in [25]. In the following we will restrict ourselves to studying tensors Q that admit a representation as in (3) and we will further refer to this as *the constrained Landau - de Gennes theory*. A related model was briefly studied in [22] (see in particular pp. 590 − 592).

Taking into account the definition (1) of the tensor Q we have that

$$Qn \cdot n = \frac{2}{3} s = \langle (p \cdot n)^2 - \frac{1}{3} \rangle = \langle \cos^2 \theta - \frac{1}{3} \rangle$$

where θ is the angle between p and n. Hence $s = \frac{3}{2}\langle \cos^2 \theta - \frac{1}{3} \rangle$ and so we must necessarily have $-\frac{1}{2} \leq s \leq 1$ with $s = 1$ when we have perfect ordering parallel to n, $s = -\frac{1}{2}$ when all molecules are perpendicular to n and $s = 0$ iff $Q = 0$ (which occurs if μ is isotropic). Thus s is a measure of order and is called the *scalar order parameter associated to the tensor* Q. In the physical literature it is often assumed that s is constant almost everywhere. For experimental determinations of s see for instance [12].

We continue working under this assumption, that s is a non-zero constant, independent of $x \in \Omega$. Thus, taking into account the representation (3) of the Q-tensor we have that, for constant s, there exists a bijective correspondence between Q that have the representation (3) and pairs $\{n, -n\}$ with $n \in \mathbb{S}^2$. Hence we can think of Q as the line joining n and $-n$. We can thus identify the space of Q as in (3) with the real projective space $\mathbb{R}P^2$; see the beginning of Section 3 for more details.

Traditionally in the mathematical modeling of liquid crystals the Q-tensor (3) has been replaced by an *oriented* line, a line with a direction, i.e. $n \in \mathbb{S}^2$. This is done in the Oseen-Frank theory, [15].

In this paper we analyze the consequences of this assumption, from the point of view of energy minimization. We show that taking into account the *possible* unorientedness of the locally preferred direction produces a theory that includes the traditional approach (an oriented vector field has an unoriented counterpart but not necessarily the other way around) and exhibits features not seen by the traditional approaches.

We analyze first the relation between line and vector fields by determining when a line field can be 'oriented' into a vector field, *without changing its regularity class*. We show that this depends both on the regularity of the line field and the topology of the domain. We also show that, under suitable assumptions, the orientability of a line field on a $2D$ bounded domain can be determined just by knowing the orientability of the line field on the boundary of the domain.

When the class of line fields is strictly larger then the class of vector fields it is possible that a global energy minimizer is a line field that is 'non-orientable' i.e. cannot be reduced to a vector field. These are precisely the instances in which the traditional, Oseen-Frank, theory would fail to recognize a global minimizer, and would indicate as a global minimizer one that is physically just a local minimizer, a minimizer in the class of orientable line fields. We analyze this situation in the case of planar line fields for a simple energy functional, and provide necessary and sufficient conditions for the global energy minimizer to be non-orientable, in terms of an integer programming problem.

The paper is organized as follows: in Section 2 we define rigorously the notions used, in particular orientability in the Sobolev space setting, and study the relation between the regularity of a line field and that of the corresponding vector field (when it exists). In Section 3 we determine the conditions under which a line field can be oriented into a vector field and show that on $2D$ domains orientability can be checked at the boundary. In Section 4 we provide analytic orientability criteria on $2D$ domains; that is we show how to reduce the topological problem of checking the orientability of the line fields to an analytic computation. In Section 5 we study the orientability of the global energy minimizers for planar line fields.

The majority of the results of this paper were announced in [3], [4]. Readers not familiar with Sobolev spaces and related analysis may find [4] a helpful introduction.

2. Preliminaries and notation

For the rest of the paper we fix $s \in [-\frac{1}{2}, 1]$ to be a given non-zero constant and Q will denote a 3×3 traceless and symmetric matrix that admits the representation:

$$Q = s\left(n \otimes n - \frac{1}{3}Id\right) \tag{4}$$

where $n \in \mathbb{R}^3, |n| = 1$ and s is the given non-zero constant.

We denote by $\mathcal{M}^{3\times3}(\mathbb{R})$ the set of 3×3 matrices with real values. In general for an arbitrary symmetric and traceless $Q \in \mathcal{M}^{3\times3}(\mathbb{R})$ there might be no $n \in \mathbb{S}^2$ so that Q has a representation as in (4). The necessary and sufficient conditions for a 3×3 matrix to have such a representation are:

Proposition 1. *For fixed $s \in \mathbb{R}$ and a matrix $Q \in \mathcal{M}^{3\times3}(\mathbb{R})$ that is symmetric with trace zero the following conditions are equivalent:*
(i) $Q = s(n \otimes n - \frac{1}{3}Id)$ for some $n \in \mathbb{S}^2$,
(ii) Q has two equal eigenvalues equal to $-\frac{s}{3}$,
(iii) $\det Q = -\frac{2s^3}{27}$, $\mathrm{tr}\, \mathrm{Q}^2 = \frac{2s^2}{3}$.

Remark 1. From *(iii)* it follows that a necessary and sufficient condition for Q to have the represenation *(i)* for some $s \in \mathbb{R}$ is that $\left(\mathrm{tr}\, Q^2\right)^3 = 54\,(\det Q)^2$.

Proof. By the spectral theorem *(ii)* holds if and only if

$$Q = -\frac{s}{3}\left(Id - n \otimes n\right) + \frac{2s}{3} n \otimes n$$

for some $n \in \mathbb{S}^2$, which is *(i)*.

If the eigenvalues of Q are $\lambda_1, \lambda_2, \lambda_3$, so that $\mathrm{tr}\, Q = \lambda_1 + \lambda_2 + \lambda_3 = 0$, the characteristic equation for Q is

$$\det(\lambda Id - Q) = (\lambda - \lambda_1)(\lambda - \lambda_2)(\lambda - \lambda_3) = \lambda^3 - \frac{1}{2}\mathrm{tr}(Q^2)\lambda - \det Q = 0$$

since $\lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_3\lambda_1 = \frac{1}{2}[(\lambda_1 + \lambda_2 + \lambda_3)^2 - (\lambda_1^2 + \lambda_2^2 + \lambda_3^2)]$. So the eigenvalues are $-\frac{s}{3}, -\frac{s}{3}, \frac{2s}{3}$ if and only if *(iii)* holds. □

For the fixed non-zero constant $s \in [-\frac{1}{2}, 1]$ we denote

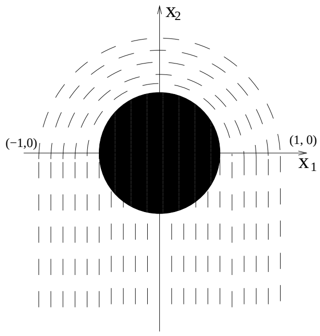

Fig. 1. A non-orientable director field

$$\boldsymbol{\mathcal{Q}} = \left\{ Q \in \mathcal{M}^{3\times 3}(\mathbb{R}); Q = s\left(n\otimes n - \frac{1}{3}Id\right) \text{ for some } n \in \mathbb{S}^2 \right\}. \tag{5}$$

Also, let us denote

$$\boldsymbol{\mathcal{Q}}_2 = \left\{ Q \in \boldsymbol{\mathcal{Q}};\ Q = s\left((n_1,n_2,0)\otimes(n_1,n_2,0) - \frac{1}{3}Id \right) \right\} \tag{6}$$

corresponding to the 'planar' unit vectors $n = (n_1, n_2, 0)$.

Note that the bijective identification of $\mathbb{R}P^2$ with $\boldsymbol{\mathcal{Q}}$ allows one to endow $\boldsymbol{\mathcal{Q}}$ with the structure of a two-dimensional manifold. Similarly $\boldsymbol{\mathcal{Q}}_2$ can be bijectively identified with $\mathbb{R}P^1$, and thus can be given the structure of a one-dimensional manifold; see the beginning of Section 3 for details.

We define the projection operator $P : \mathbb{S}^2 \to \boldsymbol{\mathcal{Q}}$ by:

$$P(n) \stackrel{\text{def}}{=} s\left(n\otimes n - \frac{1}{3}Id\right). \tag{7}$$

Note that one has $P(n) = P(-n)$. Thus the operator P provides us with a way of 'unorienting' an $\mathbb{S}^2$-valued vector field.

In order to know when the study of unoriented director fields can be reduced to that of oriented ones one needs to know when the opposite is possible, that is to 'orient' a $\boldsymbol{\mathcal{Q}}$-valued vector field. This should be done without creating 'artificial defects', that is discontinuities of the vector field that were not present in the line field. Orienting a line field is sometimes impossible as can be seen by the example in the Figure 1 where a line field is defined outside of the circle centered at $(0,0)$ of radius $\frac{1}{2}$. Heuristically one sees that by trying to orient the line field for $x_2 > 0$ and then for $x_2 < 0$ one would get a discontinuity. A rigorous proof of the non-orientability of the line field in Figure 1 will be provided in Section 4, Lemma 11.

We define an open set in $\mathbb{R}^m$ to be of class $C^k, k \geq 0$, respectively Lipschitz, if for any point $x \in \partial\Omega$ there exist a $\delta > 0$ and an orthonormal coordinate system $Y = (y', y_m) = (y_1, y_2, \ldots, y_m)$ with origin at x together with a function of class C^k (respectively Lipschitz) $f : \mathbb{R}^{m-1} \to \mathbb{R}$ such that

$$\mathcal{U}_\delta := \{y \in \Omega : |y| < \delta\} = \{y \in \mathbb{R}^m : y_m > f(y'), |y| < \delta\}. \tag{8}$$

This definition allows one to consider the open sets of class C^k (respectively Lipschitz) as manifolds with boundary and ensures that the topological boundary of the set coincides with the boundary of the manifold with boundary (see the discussion in the Appendix for more details).

In the rest of the paper an open and connected set is called a *domain*.

Let us define, in a standard manner (see also [19] and the references there) the Sobolev spaces $W^{1,p}(M, N)$ where M is a C^0 manifold and N is a smooth manifold isometrically embedded in $\mathbb{R}^l$:

$$W^{1,p}(M, N) = \{f : f \in W^{1,p}(M, \mathbb{R}^l), f(x) \in N, \, a.e. \, x \in M\}.$$

In the case when M is a Lipschitz manifold with non-empty boundary, we define (see also [20]):

$$W^{1,p}_\varphi(M, N) = \{f : f \in W^{1,p}(M, \mathbb{R}^l), f(x) \in N, \, a.e. \, x \in M, \, \mathrm{Tr}\, f = \varphi\}$$

where Tr denotes the trace operator ([1]).

We will also sometimes denote the Sobolev space $W^{1,p}(M, N)$, respectively the L^p space $L^p(M, N)$, just by $W^{1,p}(M)$, respectively $L^p(M)$, when it is clear from the context what the target space is.

We define now orientability for line fields in a Sobolev space:

Definition 1. Let $\Omega \subset \mathbb{R}^d$ be a domain. We say that $Q \in W^{1,p}(\Omega, \boldsymbol{\mathcal{Q}})$, $1 \leq p \leq \infty$ is *orientable* if there exists an $n \in W^{1,p}(\Omega, \mathbb{S}^2)$ such that $P(n) = Q$ $\mathcal{L}^d$ a.e. in Ω. Otherwise we call Q *non-orientable*.

Furthermore, if Ω is Lipschitz, then $Q \in W^{1-\frac{1}{p},p}(\partial\Omega, \boldsymbol{\mathcal{Q}})$, $1 < p \leq \infty$ is *orientable* if there exists an $n \in W^{1-\frac{1}{p},p}(\Omega, \mathbb{S}^2)$ such that $P(n) = Q$ $\mathcal{H}^{d-1}$ a.e. in $\partial\Omega$. Otherwise we call Q *non-orientable*.

We show first that if a line field is orientable there can be just two orientations, that differ by change of sign.

Proposition 2. *Let $\Omega \subset \mathbb{R}^3$ be a domain. An orientable line field $Q(x) \in W^{1,p}(\Omega, \boldsymbol{\mathcal{Q}})$, $1 \leq p \leq \infty$, can have only two orientations. More precisely, if $n, m \in W^{1,p}(\Omega, \mathbb{S}^2)$ with $P(n) = P(m) = Q$ and $n \neq m$ a.e. we have that $m = -n$ a.e. in Ω.*

Proof. We have that $m = \tau n \in W^{1,p}(\Omega, \mathbb{S}^2)$ with $\tau^2(x) = 1$ a.e. In order to obtain the conclusion it suffices to show that $\tau(x)$ has constant sign almost everywhere.

For a.e. x_2, x_3 we have that $\tau(x)n(x)$ has a representative that is absolutely continuous in x_1 and also $n(x)$ has a representative that is absolutely continuous in x_1. Hence $\tau(x)n(x) \cdot n(x) = \tau(x)$ is absolutely continuous in x_1. A similar statement holds for x_2 and x_3.

For any $\bar{x} = (\bar{x}_1, \bar{x}_2, \bar{x}_3) \in \Omega$ let $\bar{\varepsilon} > 0$ be some number so that $B_{\bar{\varepsilon}}(\bar{x}) \subset \Omega$. On the intersection of almost any line $x_1 = const$ with the ball $B_{\bar{\varepsilon}}(\bar{x})$ we have that $\tau(x)$ is absolutely continuous and, since $\tau^2(x) = 1$, we get that τ is constant. We take an arbitrary $\varphi \in C_0^\infty(B_{\bar{\varepsilon}}(\bar{x}))$ and using Fubini's theorem we obtain:

$$\int_{B_{\bar{\varepsilon}}(\bar{x})} \tau(x)\varphi_{,1}(x)\,dx = \int_{B_{\bar{\varepsilon}}(\bar{x})} (\tau\varphi)_{,1}(x)\,dx = 0. \tag{9}$$

We obtain thus that the weak derivative $\tau_{,1}$ is zero in $B_{\bar{\varepsilon}}(\bar{x})$. Similarly $\tau_{,2}, \tau_{,3}$ are zero in $B_{\bar{\varepsilon}}(\bar{x})$. Thus $\nabla\tau = 0$ and τ is constant in $B_{\bar{\varepsilon}}(\bar{x})$. The connectedness of Ω implies that τ is constant in Ω. Hence $\tau \equiv 1$ a.e. in Ω or $\tau \equiv -1$ a.e. in Ω. □

Orientability in the open set Ω implies orientability at the boundary:

Proposition 3. *Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with Lipschitz boundary. Let $Q \in W^{1,p}(\Omega, \boldsymbol{\mathcal{Q}})$, $1 < p \leq \infty$ be orientable, so that $Q = s\left(n \otimes n - \frac{1}{3}Id\right)$ a.e. in Ω for some $n \in W^{1,p}(\Omega, \mathbb{S}^2)$. Then $\mathrm{Tr}\,\mathrm{n} \in \mathrm{W}^{1-\frac{1}{p},p}(\partial\Omega, \mathbb{S}^2)$ and Q is orientable on the boundary, i.e.*

$$\mathrm{Tr}\,\mathrm{Q} = \mathrm{s}\left(\mathrm{Tr}\,\mathrm{n} \otimes \mathrm{Tr}\,\mathrm{n} - \frac{1}{3}\mathrm{Id}\right) \tag{10}$$

with $\mathrm{Tr}\,\mathrm{Q} \in \mathrm{W}^{1-\frac{1}{p},p}(\partial\Omega, \boldsymbol{\mathcal{Q}})$.

Proof. Choose some open ball B containing $\bar{\Omega}$. We can extend n to an $\tilde{n} \in W^{1,p}(B, \mathbb{R}^3)$. Truncate each component $\tilde{n}_i$ by 1, i.e. define

$$\bar{n}_i(x) = \begin{cases} 1 & \text{if } \tilde{n}_i(x) \geq 1 \\ \tilde{n}_i(x) & \text{if } |\tilde{n}_i(x)| < 1 \\ -1 & \text{if } \tilde{n}_i(x) \leq -1 \end{cases}. \tag{11}$$

Then $\bar{n} \in L^\infty(B, \mathbb{R}^3) \cap W^{1,p}(B, \mathbb{R}^3)$ and $\bar{n} = n$ in Ω. Mollify $\bar{n}$ to get $\bar{n}^{(j)} \in C^1(\bar{\Omega}, \mathbb{R}^3)$ with $\bar{n}^{(j)} \to n$ in $W^{1,p}(\Omega, \mathbb{R}^3)$.

Let us define now an extension of P to the whole $\mathbb{R}^3$, namely the function $\tilde{P} : \mathbb{R}^3 \to \mathcal{M}^{3\times 3}(\mathbb{R})$ with $\tilde{P}(m) \stackrel{\text{def}}{=} s\left(m \otimes m - \frac{1}{3}Id\right)$. Then

$$\tilde{P}(\bar{n}^{(j)}) = s\left(\bar{n}^{(j)} \otimes \bar{n}^{(j)} - \frac{1}{3}Id\right) \to \tilde{P}(n) \text{ in } W^{1,p} \tag{12}$$

since

$$|\bar{n}_i^{(j)}\bar{n}_k^{(j)} - n_i n_k| \leq |\bar{n}_i^{(j)}(\bar{n}_k^{(j)} - n_k)| + |n_k(\bar{n}_i^{(j)} - n_i)| \leq C|\bar{n}^{(i)} - n|$$

and, for example,

$$\bar{n}_i^{(j)}\bar{n}_{k,\alpha}^{(j)} - n_i n_{k,\alpha} = \underbrace{\bar{n}_i^{(j)}(\bar{n}_{k,\alpha}^{(j)} - n_{k,\alpha})}_{\to 0 \text{ in } L^p} + \underbrace{(\bar{n}_i^{(j)} - n_i)n_{k,\alpha}}_{\to 0 \text{ in } L^p}.$$

Recalling that $\bar{n}^{(j)}$ are C^1 functions on $\bar{\Omega}$ we have:

$$\mathrm{Tr}\,\tilde{\mathrm{P}}(\bar{\mathrm{n}}^{(\mathrm{j})}) = \tilde{\mathrm{P}}(\mathrm{Tr}\,\bar{\mathrm{n}}^{(\mathrm{j})}) \tag{13}$$

As $\bar{n}^{(j)} \to n$ in $W^{1,p}(\bar{\Omega};\mathbb{R}^3)$ we have that $\mathrm{Tr}\,\bar{\mathrm{n}}^{(\mathrm{j})} \to \mathrm{Tr}\,\mathrm{n}$ and $\tilde{P}(\mathrm{Tr}\,\bar{\mathrm{n}}^{(\mathrm{j})}) \to \tilde{\mathrm{P}}(\mathrm{Tr}\,\mathrm{n})$ in $L^p(\partial\Omega)$. Taking into account this convergence together with (12), the continuity of the trace operator Tr and the fact that $\tilde{P}(n) = P(n) = Q$ we can pass to the limit (in $L^p(\partial\Omega)$) in both sides of (13) and obtain relation (10).

In order to finish the proof we just need to check that $\mathrm{Tr}\,\mathrm{n} \in \mathbb{S}^2$, $\mathcal{H}^2$ a.e. $x \in \partial\Omega$. We know that $\mathrm{Tr}\,\mathrm{n} \in \mathrm{W}^{1-\frac{1}{\mathrm{p}},\mathrm{p}}(\partial\Omega;\mathbb{R}^3)$. Hence recalling (see for instance [14], p. 133) that we have

$$\lim_{r\to 0} \frac{1}{|B_r(x)\cap\Omega|} \int_{B_r(x)\cap\Omega} |\mathrm{n}(\mathrm{y}) - \mathrm{Tr}\,\mathrm{n}(\mathrm{x})|\,\mathrm{dy} = 0, \text{ for } \mathcal{H}^2 \text{ a.e. } \mathrm{x} \in \partial\Omega \tag{14}$$

and since

$$\begin{aligned}\int_{B_r(x)\cap\Omega} \Big|1 - |\mathrm{Tr}\,\mathrm{n}(\mathrm{x})|\Big|\,\mathrm{dy} &= \int_{\mathrm{B_r(x)\cap\Omega}} \Big||\mathrm{n}(\mathrm{y})| - |\mathrm{Tr}\,\mathrm{n}(\mathrm{x})|\Big|\,\mathrm{dy} \\ &\le \int_{B_r(x)\cap\Omega} |\mathrm{n}(\mathrm{y}) - \mathrm{Tr}\,\mathrm{n}(\mathrm{x})|\,\mathrm{dy},\end{aligned}$$

we get that $|\mathrm{Tr}\,\mathrm{n}| = 1$ $\mathcal{H}^2$ a.e. $x \in \partial\Omega$. □

Remark 2. One can easily check, by straightforward modifications of the proofs, that the results of Proposition 2 and Proposition 3 hold for domains $\Omega \subset \mathbb{R}^2$ with $\boldsymbol{\mathcal{Q}}$ replaced by $\boldsymbol{\mathcal{Q}}_2$.

Orientability is preserved by weak convergence:

Proposition 4. *Let $\Omega \subset \mathbb{R}^d$ be a bounded domain with boundary of class C^0. For $1 \le p \le \infty$ let $Q^{(k)} \in W^{1,p}(\Omega, \boldsymbol{\mathcal{Q}}), k \in \mathbb{N}$ be a sequence of orientable maps with the corresponding $n^{(k)} \in W^{1,p}(\Omega, \mathbb{S}^2)$ such that $P(n^{(k)}) = Q^{(k)}$. If $Q^{(k)}$ converges weakly to Q in $W^{1,p}$, where $1 \le p < \infty$ (or weak* when $p = \infty$), then Q is orientable and $Q = P(n)$ for some $n \in W^{1,p}$.*

Proof. We start by proving an auxiliary result:

Lemma 1. *Let $\Omega \subset \mathbb{R}^d$ be an open and bounded set. If $n \in W^{1,p}(\Omega, \mathbb{S}^2)$, $1 \le p \le \infty$ then $Q = P(n) \in W^{1,p}(\Omega, \boldsymbol{\mathcal{Q}})$. Conversely, assume that $Q \in W^{1,p}(\Omega, \boldsymbol{\mathcal{Q}})$, $1 \le p \le \infty$ and let n be a measurable function on Ω with values in $\mathbb{S}^2$ such that $P(n) = Q$. Moreover assume that n is continuous along almost every line parallel to the coordinate axes and intersecting Ω. Then $n \in W^{1,p}(\Omega, \boldsymbol{\mathcal{Q}})$.*

Moreover:

$$Q_{ij,k} n_j = s n_{i,k}. \tag{15}$$

Proof of the lemma. For $g, h \in W^{1,1}(\Omega) \cap L^\infty(\Omega)$ we have $gh \in W^{1,1}(\Omega) \cap L^\infty(\Omega)$ and $(gh)_{,i} = gh_{,i} + g_{,i}h$ (see [1]). Hence, if $n \in W^{1,p}(\Omega)$, we have $Q \in W^{1,1}(\Omega)$ and $Q_{ij,k} = s(n_i n_{j,k} + n_{i,k} n_j)$ from which we obtain $\nabla Q \in L^p(\Omega)$ and then $Q \in W^{1,p}(\Omega)$. Also

$$Q_{ij,k} n_j = s\left[n_i(n_{j,k} n_j) + n_{i,k}\right] = s[\frac{n_i}{2}\underbrace{(n_j n_j)}_{=1}{}_{,k} + n_{i,k}] = s n_{i,k}.$$

Conversely, suppose that $Q \in W^{1,p}$. Let $x \in \Omega$ with n continuous along the line $(x + \mathbb{R}e_k) \cap \Omega$. Let $x + te_k \in \Omega$. As $Q \in W^{1,1}$ we can suppose that Q is differentiable at x in the direction e_k. Then

$$\begin{aligned} &\tfrac{Q_{ij}(x+te_k) - Q_{ij}(x)}{t} = s\left[\tfrac{n_i(x+te_k)n_j(x+te_k) - n_i(x)n_j(x)}{t}\right] \\ &\quad = s \cdot n_i(x + te_k)\left[\tfrac{n_j(x+te_k) - n_j(x)}{t}\right] + s \cdot \left[\tfrac{n_i(x+te_k) - n_i(x)}{t}\right] n_j(x). \end{aligned}$$

Multiply both sides by $\frac{1}{2}\left[n_j(x + te_k) + n_j(x)\right]$. Then, since

$$\begin{aligned} &\left[n_j(x + te_k) - n_j(x)\right]\left[n_j(x + te_k) + n_j(x)\right] \\ &\quad = n_j(x + te_k)n_j(x + te_k) - n_j(x)n_j(x) = 1 - 1 = 0 \end{aligned} \tag{16}$$

we have that

$$\begin{aligned} &\frac{Q_{ij}(x + te_k) - Q_{ij}(x)}{t} \cdot \frac{1}{2}\left[n_j(x + te_k) + n_j(x)\right] \\ &\qquad = s \cdot \left[\tfrac{n_i(x+te_k) - n_i(x)}{t}\right] n_j(x)\tfrac{1}{2}\left[n_j(x + te_k) + n_j(x)\right]. \end{aligned} \tag{17}$$

Letting $t \to 0$ and using the assumed continuity of n we deduce that

$$s \cdot \lim_{t \to 0} \frac{n_i(x + te_k) - n_i(x)}{t} = Q_{ij,k}(x)n_j(x).$$

Hence the partial derivatives of n exist almost everywhere in Ω and satisfy

$$s n_{i,k} = Q_{ij,k} n_j$$

and since $\nabla Q \in L^p$ it follows that $n \in W^{1,p}(\Omega, \mathbb{S}^2)$ as required. □

We continue with the proof of the proposition. As the sequence $Q^{(k)}$ is weakly convergent in $W^{1,p}$ for $1 \le p < \infty$ (weak* for $p = \infty$), using (15) we have that $n^{(k)}$ is bounded in the $W^{1,p}$ norm (equi-integrable if $p = 1$). Thus there exists $n \in W^{1,p}(\Omega, \mathbb{S}^2)$ such that on a subsequence $(n^{(k_l)})_{k_l \in \mathbb{N}}$ we have $n^{(k_l)} \rightharpoonup n$ in $W^{1,p}(\Omega)$ and $n^{(k_l)} \to n$ a.e. which implies that $P(n) = Q$. □

Remark 3. Lemma 1 shows that in order to have $n \in W^{1,p}(\Omega, \mathbb{S}^2)$ such that $P(n) = Q$ for $Q \in W^{1,p}(\Omega, \boldsymbol{\mathcal{Q}})$ it suffices to have only $n \in W^{1,1}(\Omega, \mathbb{S}^2)$ since then n will have the same regularity as Q.

The previous result allows us to show that non-orientability is a stable property with respect to the metric induced by the distance $d(Q, \tilde{Q}) = \|Q - \tilde{Q}\|_{W^{1,p}(\Omega,\mathbb{R}^9)}$. More precisely we have:

Lemma 2. *Let $Q \in W^{1,p}(\Omega, \boldsymbol{\mathcal{Q}}), 1 \leq p \leq \infty$ be non-orientable. Then there exists $\varepsilon > 0$, depending on Q, so that for all $\tilde{Q} \in W^{1,p}(\Omega, \boldsymbol{\mathcal{Q}})$ with $\|\tilde{Q} - Q\|_{W^{1,p}(M,\mathbb{R}^9)} < \varepsilon$ the line field $\tilde{Q}$ is also non-orientable.*

Taking into account the previous proposition and reasoning by contradiction the proof is straightforward.

The line-field theory we have presented represents a generalization of the Oseen-Frank theory, which uses vector fields for describing uniaxial nematic liquid crystals. The Oseen-Frank theory has been successful in predicting the equilibrium states as local or global minimizers of an energy functional:

$$E_{OF} = \int_{\Omega} W(n, \nabla n)\, dx, \tag{18}$$

where

$$\begin{aligned} W(n, \nabla n) = K_1(\operatorname{div} n)^2 + K_2(n \cdot \operatorname{curl} n)^2 + K_3|n \wedge \operatorname{curl} n|^2 \\ +(K_2 + K_4)(\operatorname{tr}(\nabla n)^2 - (\operatorname{div} n)^2) \end{aligned} \tag{19}$$

and the K_i are elastic constants.

As described in [4] we consider a special case of the Landau - de Gennes theory, in which the elastic energy density is defined by

$$\psi(Q, \nabla Q) = L_1 I_1 + L_2 I_2 + L_3 I_3 + L_4 I_4,$$

where the L_i are constants and the four elastic invariants $I_1, \dots, I_4$ are given by

$$I_1 = Q_{ij,j}Q_{ik,k},\ I_2 = Q_{ik,j}Q_{ij,k},\ I_3 = Q_{ij,k}Q_{ij,k},\ I_4 = Q_{lk}Q_{ij,l}Q_{ij,k},$$

where we have used the summation convention with $i, j, k \in \{1, 2, 3\}$.

It can be checked that the Oseen-Frank energy is expressible in terms of the constant s Landau - de Gennes Q-tensors (see [32]). We have that

$$I_1 = s^2 \left(|\operatorname{div} n|^2 + |n \wedge \operatorname{curl} n|^2\right),\ I_2 = s^2 \left(|n \wedge \operatorname{curl} n|^2 + \operatorname{tr}(\nabla n)^2\right),$$

$$I_3 = 2s^2 \left(\operatorname{tr}(\nabla n)^2 + |n \cdot \operatorname{curl} n|^2 + |n \wedge \operatorname{curl} n|^2\right),$$

$$I_4 = 2s^3 \left(\frac{2}{3}|n \wedge \operatorname{curl} n|^2 - \frac{1}{3}\operatorname{tr}(\nabla n)^2 - \frac{1}{3}|n \cdot \operatorname{curl} n|^2\right).$$

We let

$$K_1 = L_1 s^2 + L_2 s^2 + 2L_3 s^2 - \frac{2}{3}L_4 s^3,\quad K_2 = 2L_3 s^2 - \frac{2}{3}L_4 s^3,$$

$$K_3 = L_1 s^2 + L_2 s^2 + 2L_3 s^2 + \frac{4}{3}L_4 s^3,\quad K_4 = L_2 s^2,$$

and observe that the L_i can also be expressed in terms of the K_i. Then we have that

$$W(n, \nabla n) = \psi(Q, \nabla Q),$$

and thus the Oseen-Frank elastic energy is the same as the Landau - de Gennes elastic energy.

For more information concerning the form of the Landau-de Gennes energy ψ and its relationship to the Oseen-Frank energy see [28], [37].

3. Orientability issues

The existence of an oriented, $\mathbb{S}^2$-valued, version of a $\mathcal{Q}$-valued field is essentially a topological question. It amounts to checking if there exists a lifting of a map that takes values into $\mathcal{Q}$ to one that takes values into its covering space $\mathbb{S}^2$, so that the lifting has the same regularity as the map. If a continuous map in with values in $\mathcal{Q}$ has a lifting to a continuous map with values in $\mathbb{S}^2$ we call the map *orientable*. Otherwise we call the map *non-orientable*.

Let us recall (see [23],[26]) that a continuous map $Q : \Omega \to \mathcal{Q}$ is said to have a lifting $\varphi^Q : \Omega \to \mathbb{S}^2$ if $P \circ \varphi^Q = Q$ where $P : \mathbb{S}^2 \to \mathcal{Q}$ is a covering map, defined in our case as

$$P(n) = s(n \otimes n - \frac{1}{3}Id). \tag{20}$$

In order to show that P is a covering map, in a topological sense, we need to endow $\mathcal{Q}$ with an appropriate topological structure. To this end let us first recall that $\mathbb{R}P^2 \stackrel{\text{def}}{=} \mathbb{S}^2/\sim$ is the quotient of $\mathbb{S}^2$ with respect to the equivalence relation $n \sim m \Leftrightarrow n = \pm m$ (see [11], [35]). We define the map $b : \mathcal{Q} \to \mathbb{R}P^2$ by

$$b(s(n \otimes n - \frac{1}{3}Id)) = \{n, -n\} \in \mathbb{R}P^2, \quad \text{for all } n \in \mathbb{S}^2.$$

One can then define in a standard manner a topological structure on $\mathcal{Q}$ so that b is a continuous map. Moreover one can endow $\mathcal{Q}$ with a Riemannian structure so that b is an isometry. Therefore in the remainder of the paper we are able to use for Q-valued functions theorems that were proved for functions with values into a Riemannian manifold.

The map $P : \mathbb{S}^2 \to \mathcal{Q}$ is then easily seen to be continuous. Also P is surjective and one can easily check that every point $M \in \mathcal{Q}$ has an evenly covered neighbourhood (that is there exists an open $U \subset \mathcal{Q}$, with $M \in U$ and each component of $P^{-1}(U)$ is mapped homeomorphically onto U by P). Thus P is a covering map (see also [26] for more details about covering maps in general).

In the case of planar line fields, that is $\mathcal{Q}_2$-valued fields, one has a similar lifting problem by identifying, analogously, $\mathcal{Q}_2$ with $\mathbb{R}P^1$, and denoting (without loss of generality, regarding $\mathbb{R}P^1$ as embedded in $\mathbb{R}P^2$ and $\mathbb{S}^1$ in $\mathbb{S}^2$) also with P, the covering map from $\mathbb{S}^1$ to $\mathbb{R}P^1$.

There exists a well-developed theory, in algebraic topology, that shows when it is possible to have a lifting. Both to avoid the reader having to enter into the technical details of this theory and related topological ideas, and for its intrinsic interest, we give in the next subsection a self-contained treatment of the orientability of continuous line fields that uses only elementary point-set topology (but of course with ingredients than have counterparts in the algebraic topology approach). We use this to show that for a large class of two-dimensional domains G one can check orientability of a continuous line field Q (on $\bar{G}$) just by determining the orientability of $Q|_{\partial G}$.

However, we face the significant additional difficulty that these topological results are restricted to continuous functions, while functions in the Sobolev space $W^{1,p}(\Omega, \boldsymbol{\mathcal{Q}})$ (with $\Omega \subset \mathbb{R}^d$, $d = 2, 3$) are not necessarily continuous for $p \leq d$. Also we are interested only in liftings that preserve the Sobolev regularity class. We begin to address these questions in Section 3.2.

3.1. Continuous line fields on arbitrary domains

We first show how to lift continuous line fields along continuous paths:

Lemma 3. *If $-\infty < t_1 < t_2 < \infty$ and $Q : [t_1, t_2] \to \boldsymbol{\mathcal{Q}}$ is continuous then there exist exactly two continuous maps (liftings) $n^+, n^- : [t_1, t_2] \to \mathbb{S}^2$, so that*

$$Q(t) = s(n^\pm(t) \otimes n^\pm(t) - \frac{1}{3} Id), \tag{21}$$

and $n^+ = -n^-$. (Equivalently, given either of the two possible initial orientations $\bar{n} \in \mathbb{S}^2$, so that $Q(t_1) = s(\bar{n} \otimes \bar{n} - \frac{1}{3} Id)$, there exists a unique continuous lifting $n : [t_1, t_2] \to \mathbb{S}^2$ with $n(t_1) = \bar{n}$.)

Suppose in addition that $\bar{Q} = s(\bar{m} \otimes \bar{m} - \frac{1}{3} Id), \bar{m} \in \mathbb{S}^2$, and that $|Q(t) - \bar{Q}| \leq \varepsilon |s|$ for all $t \in [t_1, t_2]$ where $0 < \varepsilon < \sqrt{2}$. Then one of the liftings, n^+ say, satisfies

$$|n^+(t) - \bar{m}| \leq \varepsilon, \ \ \textit{for all } t \in [t_1, t_2] \tag{22}$$

and the other $n^- = -n^+$ satisfies

$$|n^-(t) + \bar{m}| \leq \varepsilon, \ \ \textit{for all } t \in [t_1, t_2] \tag{23}$$

Proof. Let $0 < \varepsilon < \sqrt{2}$. Given $n, \bar{m} \in \mathbb{S}^2$ with $|n \otimes n - \bar{m} \otimes \bar{m}| \leq \varepsilon$ we have that $|n \otimes n - \bar{m} \otimes \bar{m}|^2 = 2(1 - (n \cdot \bar{m})^2) \leq \varepsilon^2$ and so

$$n \cdot \bar{m} \geq \sqrt{1 - \frac{\varepsilon^2}{2}} > 0 \text{ or } n \cdot \bar{m} \leq -\sqrt{1 - \frac{\varepsilon^2}{2}} < 0. \tag{24}$$

Thus $n \otimes n = n^+ \otimes n^+ = n^- \otimes n^-$, where $n^+ \cdot \bar{m} > 0$ and $n^- = -n^+$ satisfies $n^- \cdot \bar{m} < 0$.

Now let $Q(\tau) = s(n(\tau)\otimes n(\tau) - \frac{1}{3}Id)$ be continuous on $[t_1, t_2]$. Then there exists $\delta > 0$ such that $|n(\tau)\otimes n(\tau) - n(\sigma)\otimes n(\sigma)| < \sqrt{2}$ for all $\sigma, \tau \in [t_1, t_2]$ with $|\sigma - \tau| \le \delta$, and we may suppose that $t_2 - t_1 = M\delta$ for some integer M. First take $\bar{m} \stackrel{\text{def}}{=} n(t_1)$ and for each $\tau \in [t_1, t_1 + \delta]$ choose $n^+(\tau)$ as above so that $n^+(\tau)\otimes n^+(\tau) = n(\tau)\otimes n(\tau)$ and $n^+(\tau)\cdot \bar{m} > 0$. We claim that $n^+ : [t_1, t_1+\delta] \to \mathbb{S}^2$ is continuous. Indeed let $\sigma_j \to \sigma$ in $[t_1, t_1+\delta]$ and suppose for contradiction that $n^+(\sigma_j) \not\to n^+(\sigma)$. Then since $n^+(\sigma_j)\otimes n^+(\sigma_j) \to n^+(\sigma)\otimes n^+(\sigma)$ there is a subsequence σ_{j_k} such that $n^+(\sigma_{j_k}) \to -n^+(\sigma)$. But then $-n^+(\sigma)\cdot m \ge 0$ a contradiction which proves the claim. Repeating this procedure with $\bar{n} = n^+(t_1+\delta)$ we obtain a continuous lifting $n^+ : [t_1, t_1 + 2\delta] \to \mathbb{S}^2$, and thus inductively a continuous lifting $n^+ : [t_1, t_2] \to \mathbb{S}^2$. Setting $n^- = -n^+$ gives a second continuous lifting.

If $n^* : [t_1, t_2] \to \mathbb{S}^2$ is a continuous lifting then we may suppose that $n^*(t_1) = n^+(t_1)$ say. We claim that then $n^*(\tau) = n^+(\tau)$ for all $\tau \in [t_1, t_2]$. If not, by continuity there would be a first $T > t_1$ with $n^*(T) = n^-(T)$. But then $n^*(T) = \lim_{\tau\to T_-} n^*(\tau) = \lim_{\tau\to T_-} n^+(\tau) = n^+(T)$, a contradiction. Thus there are exactly two continuous liftings.

Finally suppose that $|Q(t) - \bar{Q}| \le \varepsilon|s|$ for all $t \in [t_1, t_2]$ with $0 < \varepsilon < \sqrt{2}$. Then by (24) and the continuity of the lifting one of the liftings, n^+ say, satisfies $n^+(t)\cdot \bar{m} \ge \sqrt{1 - \frac{\varepsilon^2}{2}}$ and so $|n^+(t) - \bar{m}|^2 = 2(1 - n^+(t)\cdot\bar{m}) \le 2(1 - \sqrt{1-\frac{\varepsilon^2}{2}}) \le \varepsilon^2$ and the result follows. □

Proposition 5. *Let $\Omega \subset \mathbb{R}^d$ be a simply-connected domain, and let $Q : \Omega \to \mathcal{Q}$ be continuous. Then there exists a continuous lifting $n_Q : \Omega \to \mathbb{S}^2$.*

Proof. This is standard, and follows, for example, from [23] p. 61, Prop. 1.33. To give a direct argument, fix $x_0 \in \Omega$ and let $n^0 \in \mathbb{S}^2$ be one of the two possible orientations of $Q(x_0)$. Given any $x \in \Omega$, let $\gamma : [0,1] \to \Omega$ be a continuous path with $\gamma(0) = x_0, \gamma(1) = x$. By Lemma 3 there is a unique continuous lifting $n : [0,1] \to \mathbb{S}^2$ of $Q(\gamma(\cdot))$ with $n(0) = n^0$. Define $n^Q(x) = n(1)$. Then the method of proof of Theorem 1 below, which for simply-connected domains does not depend on the dimension d, shows that n^Q is well defined and continuous. (Note that local path connectedness holds because Ω is open, and that none of the complications concerning the ω_i are needed, so that $h^*(\lambda,\cdot) = h(\lambda,\cdot)$.) □.

In the rest of this subsection we restrict ourselves to a class of topologically non-trivial domains in $2D$ and show that one can check orientability at the boundary.

Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with $\partial\Omega$ a Jordan curve. For $1 \le i \le N$ let $\omega_i \subset \mathbb{R}^2$ be a bounded domain with $\partial\omega_i$ a Jordan curve and $\overline{\omega_i} \subset \Omega$, $\overline{\omega_i} \cap \overline{\omega_j} = \emptyset$ if $i \ne j$. Note that Ω and ω_i are simply connected (for example by the proof of Lemma 5 (iii)). Let

$$G = \Omega \setminus \cup_{i=1}^N \overline{\omega_i}. \tag{25}$$

We call such a domain G a *domain with holes.*

Theorem 1. *Let $Q : \overline{G} \to \mathcal{Q}_2$ be continuous with $Q|_{\partial\omega_i}$ orientable as a continuous function for $1 \leq i \leq N$. Then Q is orientable as a continuous function.*

The proof of Theorem 1 requires some preparation. Let $f : [0,1] \to \overline{\Omega}$ be a continuous path with $f(0)$ and $f(1)$ belonging to $\overline{G}$. We define a new continuous path $f^* : [0,1] \to \overline{G}$ with $f^*(0) = f(0), f^*(1) = f(1)$ by replacing the parts of f where it lies in some ω_i by corresponding paths on $\partial\omega_i$. If $f(t) \in \overline{G}$ we set $f^*(t) = f(t)$. Otherwise consider an interval $[t_1, t_2] \subset [0,1]$ such that $f(t) \in \omega_i$ for $t_1 < t < t_2$, $f(t_1) \in \partial\omega_i$, $f(t_2) \in \partial\omega_i$, for some $i \in \{1, \dots, N\}$.

Let $\gamma_i : \mathbb{S}^1 \to \mathbb{R}^2$ parametrize $\partial\omega_i$, so that $\gamma_i = \gamma_i(\theta)$ can be identified with a 2π-periodic function $\gamma_i : \mathbb{R} \to \mathbb{R}^2$, $\gamma_i(\theta+2\pi) = \gamma_i(\theta)$, where θ denotes the polar angle of $\mathbb{S}^1$. Then $\gamma_i(\theta_1) = f(t_1), \gamma_i(\theta_2) = f(t_2)$, where $\theta_i \in [0, 2\pi)$. For $t \in (t_1, t_2)$ we define

$$f^*(t) = \gamma_i(\theta(t)) \tag{26}$$

where

$$\theta(t) = \theta_1 + (\tilde{\theta}_2 - \theta_1)\left(\frac{t - t_1}{t_2 - t_1}\right) \tag{27}$$

and

$$\tilde{\theta}_2 = \begin{cases} \theta_2 & \text{if } |\theta_2 - \theta_1| \leq \pi \\ \theta_2 - \operatorname{sgn}(\theta_2 - \theta_1)2\pi & \text{if } |\theta_2 - \theta_1| > \pi \end{cases}. \tag{28}$$

Thus f^* is continuous from $[t_1, t_2] \to \partial\omega_i$ and traces the image under γ_i of the minor (shorter) arc joining θ_1 and θ_2 (with an unimportant specific choice of the arc if θ_1, θ_2 represent opposite points of $\mathbb{S}^1$).

Lemma 4. *$f^* : [0,1] \to \overline{G}$ is continuous.*

Proof. Let $\sigma_j \to \sigma$ in $[0,1]$. We show that $f^*(\sigma_j) \to f^*(\sigma)$. If $f(\sigma) \in G \cup \partial\Omega$ then this is obvious, since then $f(\sigma_j) \in \overline{G}$ for sufficiently large j, and so

$$\lim_{j\to\infty} f^*(\sigma_j) = \lim_{j\to\infty} f(\sigma_j) = f(\sigma) = f^*(\sigma). \tag{29}$$

If $f(\sigma) \in \omega_i$ then, since $f(0) = f(1) \in \bar{G}$, there exists an interval $[t_1, t_2] \subset [0,1]$ as above with $t_1 < \sigma < t_2$, so that $\lim_{j\to\infty} f^*(\sigma_j) = f^*(\sigma)$ follows from (26)-(28). Thus, arguing by contradiction, we may assume that $f(\sigma) \in \partial\omega_i$ for some i. If $f(\sigma_j) \in \partial\omega_i$ for all j then again we have (29). Thus it suffices to consider the case when $f(\sigma_j) \in \omega_i$ for all j and either $(i)\sigma_j > \sigma$ for all j, or $(ii)\sigma_j < \sigma$ for all j.

We assume (i) with (ii) being treated similarly. If $f(t) \in \omega_i$ for all $t > \sigma$ with $t - \sigma$ sufficiently small, then we can take $t_1 = \sigma$ and again $\lim_{j\to\infty} f^*(\sigma_j) = f^*(\sigma)$ follows from (26)-(28).

Otherwise there exist $t_{2j} > \sigma_j > t_{1j} > \sigma$ with $f(t) \in \omega_i$ for all $t \in (t_{1j}, t_{2j}), f(t_{1j}) \in \partial\omega_i$, $f(t_{2j}) \in \partial\omega_i$ and $t_{2j} \to \sigma$. We have that $f(t_{1j}) =$

$\gamma_i(\theta_{1j}), f(t_{2j}) = \gamma_i(\theta_{2j})$ with $\theta_{1j}, \theta_{2j} \in [0, 2\pi)$ and $\theta_{1j} \stackrel{\mathbb{S}^1}{\to} \theta, \theta_{2j} \stackrel{\mathbb{S}^1}{\to} \theta$, where $\gamma_i(\theta) = f(\sigma)$, since otherwise, for example, we would have a subsequence θ_{1j_k} with $\theta_{1j_k} \stackrel{\mathbb{S}^1}{\to} \bar\theta \neq \theta \bmod (2\pi)$ and $\gamma_i(\theta) = \gamma_i(\bar\theta)$, contradicting that $\partial\omega_i$ is a Jordan curve. Thus from (26), (27) we have

$$f^*(\sigma_j) = \gamma_i \left(\theta_{1j} + (\tilde\theta_{2j} - \theta_{1j}) \left(\frac{\sigma_j - t_{1j}}{t_{2j} - t_{1j}} \right) \right)$$

and so $\lim_{j\to\infty} f^*(\sigma_j) = \gamma_i(\theta) = f(\sigma) = f^*(\sigma)$ as required. □

Lemma 5.

(*i*) *$\overline{\Omega}$ is path-connected and simply connected.*

(*ii*) *$\overline{G}$ is path-connected.*

(*iii*) *$\bar G$ is locally path-connected: given any $x \in \overline{G}$ and $\bar\varepsilon > 0$, there exists $\delta > 0$ such that for any $z \in \overline{G}$ with $|x - z| < \delta$ there is a continuous path $\tilde\gamma : [0, 1] \to \overline{G}$ with $\tilde\gamma(0) = x, \tilde\gamma(1) = z$ and $|\tilde\gamma(t) - x| < \bar\varepsilon$ for all $t \in [0, 1]$.*

Proof. By the Schoenflies theorem [7] there is a homeomorphism $u : \mathbb{R}^2 \to \mathbb{R}^2$ with $u(D) = \Omega, u(\partial D) = \partial\Omega$ and $u(\overline{D}^c) = \overline{\Omega}^c$ where $D = B(0, 1)$ is the unit disk. Since $\overline{D}$ is path connected and simply connected this proves (*i*). Given $x, z \in \overline{G}$ there thus exists a continuous path $f : [0, 1] \to \overline{\Omega}$ with $f(0) = x, f(1) = z$. By Lemma 4, $f^* : [0, 1] \to \overline{G}$ is continuous with $f^*(0) = x, f^*(1) = z$, which proves (*ii*). To prove (*iii*), note that this is obvious if $x \in G$ so we may suppose that $x \in \partial\Omega$. The argument in the case $x \in \partial\omega_i$ for some i is similar. Note that $u^{-1}(x) \in \partial D$. Let $\sigma > 0$ be sufficiently small so that $B(u^{-1}(x), \sigma) \cap \cup_{i=1}^N u^{-1}(\partial\omega_i)$ is empty and $|y - u^{-1}(x)| < \sigma$ implies $|u(y) - x| < \bar\varepsilon$. Then let $\delta > 0$ be such that $|z - x| < \delta$ implies $|u^{-1}(z) - u^{-1}(x)| < \sigma$. Then $\tilde\gamma(t) = u(tu^{-1}(z) + (1-t)u^{-1}(x))$ defines a suitable path. □

Before starting the proof of Theorem 1 we need one more technical lemma:

Lemma 6. *Let G be a domain with holes as above and let $Q \in C(\bar G, \boldsymbol{Q})$. There exists $\nu > 0$ such that if $\bar x \in \bar G$, $-\infty < t_1 < t_2 < \infty$, $f^{(j)}, f \in C([t_1, t_2]; \bar G)$, $f^{(j)}([t_1, t_2]) \subset B(\bar x, \nu)$, $f^{(j)}(t_2) \to f(t_2)$, and if $n^{(j)}, n : [t_1, t_2] \to \mathbb{S}^2$ are continuous liftings of $Q(f^{(j)}(\cdot)), Q(f(\cdot))$ respectively with $|n^{(j)}(t_1) - n(t_1)| < 1$, then $n^{(j)}(t_2) \to n(t_2)$.*

Proof. Choose ν sufficiently small such that $|Q(x) - Q(y)| \leq |s|$ if $x, y \in \bar G$ with $|x - y| \leq 2\nu$. In Lemma 3 set $\bar m = n(t_1)$. Then by (22), (23) we have that

$$|n(t) + n(t_1)| > 1 \text{ for all } t \in [t_1, t_2]. \tag{30}$$

Also

$$|n^{(j)}(t_1) + n(t_1)| \geq 2|n(t_1)| - |n^{(j)}(t_1) - n(t_1)| > 1 \tag{31}$$

and so by Lemma 3

$$|n^{(j)}(t) - n(t_1)| \leq 1 \text{ for all } t \in [t_1, t_2]. \tag{32}$$

Suppose that $n^{(j)}(t_2) \not\to n(t_2)$. Then since $Q(f^{(j)}(t_2)) \to Q(f(t_2))$ there exists a subsequence j_k such that $n^{(j_k)}(t_2) \to -n(t_2)$. But then, from (32), $|n(t_2)+n(t_1)| \leq 1$, contradicting (30). □

Proof of Theorem 1. Let $x_0 \in G$ and choose one of the two possible orientations $(m^0, 0)$ for $Q(x_0)$, where $m^0 \in \mathbb{S}^1$. Let $x \in \overline{G}$ be arbitrary. By Lemma 5(ii) there exists a continuous path $\gamma : [0,1] \to \overline{G}$ with $\gamma(0) = x_0$, $\gamma(1) = x$. By Lemma 3 there exists a unique continuous lifting $n : [0,1] \to \mathbb{S}^1$ such that $Q(\gamma(t)) = s\left((n(t),0)\otimes(n(t),0) - \frac{1}{3}Id\right)$, $t \in [0,1]$ and $n(0) = m^0$. We define $N(x) \stackrel{\text{def}}{=} n(1)$. To show that $N(x)$ is well defined, suppose that $\gamma' : [0,1] \to \overline{G}$ is another continuous path with $\gamma'(0) = x_0, \gamma'(1) = x$, let $n' : [0,1] \to \mathbb{S}^1$ be the corresponding continuous lifting, and suppose for contradiction that $n'(1) \neq N(x)$, so that $n'(1) = -N(x)$. Define the continuous loop $\Gamma : [0,1] \to \overline{G}$ by

$$\Gamma(t) = \begin{cases} \gamma(2t) & \text{if } 0 \leq t \leq \frac{1}{2} \\ \gamma'(2(1-t)) & \text{if } \frac{1}{2} \leq t \leq 1 \end{cases}$$

so that $\Gamma(0) = \Gamma(1) = x_0$. Then $\tilde{N}(t)$ defined by

$$\tilde{N}(t) = \begin{cases} n(2t) & \text{if } 0 \leq t \leq \frac{1}{2} \\ -n'(2(1-t)) & \text{if } \frac{1}{2} \leq t \leq 1 \end{cases}$$

is a continuous lifting such that

$$Q(\Gamma(t)) = s\left((\tilde{N}(t),0)\otimes(\tilde{N}(t),0) - \frac{1}{3}Id\right) \tag{33}$$

and $\tilde{N}(0) = m^0, \tilde{N}(1) = -m^0$. The bulk of the proof will be to show that such a lifting cannot exist, so that $N(x)$ is well defined. Assuming this, we claim that $N : \overline{G} \to \mathbb{S}^1$ is a continuous lifting of Q, that is

$$Q(x) = s\left((N(x),0)\otimes(N(x),0) - \frac{1}{3}Id\right)$$

for all $x \in \overline{G}$ and N is continuous in x. We only need to prove the continuity.

Let $z_j \in \overline{G}$ with $z_j \to x$. By Lemma 5 (iii) there exist continuous paths $\tilde{\gamma}_j : [0,1] \to \bar{G}$ with $\tilde{\gamma}_j(0) = x, \tilde{\gamma}_j(1) = z_j$ and $\tilde{\gamma}_j(\cdot) \to \gamma_x$ in $C([0,1];\bar{G})$ where $\gamma_x : [0,1] \to \bar{G}, \gamma_x(t) = x, \forall t \in [0,1]$. Then we may consider the path

$$\hat{\gamma}_j(t) = \begin{cases} \gamma(2t) & \text{if } 0 \leq t \leq \frac{1}{2} \\ \tilde{\gamma}_j(2t-1) & \text{if } \frac{1}{2} \leq t \leq 1 \end{cases}$$

so that $N(x) = \tilde{n}^j(0), N(z_j) = \tilde{n}^j(1)$, where $\tilde{n}^j : [0,1] \to \mathbb{S}^1$ is such that $(\tilde{n}^j, 0)$ is the unique continuous lifting of $Q(\tilde{\gamma}_j(\cdot))$ with $\tilde{n}^j(0) = N(x)$. By Lemma 6 we deduce that $N(z_j) \to N(x)$ as required.

It remains to prove that there is no continuous loop $\Gamma : [0,1] \to \overline{G}$ with $\Gamma(0) = \Gamma(1) = x_0$ and a corresponding continuous lifting $\tilde{N} : [0,1] \to \mathbb{S}^1$ so that (33) holds and $\tilde{N}(0) = m^0, \tilde{N}(1) = -m^0$.

Since $\overline{\Omega}$ is simply connected, there exists a continuous homotopy $h : [0,1]^2 \to \overline{\Omega}, h = h(\lambda, t)$ with $h(0,t) = \Gamma(t), h(1,t) = x_0$ for all $t \in [0,1]$ and $h(\lambda, 0) = h(\lambda, 1) = x_0$ for all $\lambda \in [0,1]$. For each λ we consider the path $h^*(\lambda, \cdot) : [0,1] \to \overline{G}$, which is continuous by Lemma 4. By Lemma 3, for each $\lambda \in [0,1]$ there is a unique continuous lifting $n^\lambda : [0,1] \to \mathbb{S}^1$ such that

$$Q(h^*(\lambda,t)) = s\left((n^\lambda(t),0) \otimes (n^\lambda(t),0) - \frac{1}{3}Id\right) \tag{34}$$

and $n^\lambda(0) = m^0$. We know that $n^0(1) = \tilde{N}(1) = -m^0$. We will prove that $n^\lambda(1)$ is a continuous function of $\lambda \in [0,1]$ so that $n^\lambda(1) = -m^0$ for all $\lambda \in [0,1]$. In particular $n^1(1) = -m^0$ contradicting $h(1,t) = x_0$ for all t and $n^1(0) = m^0$ (since $n^1(t) \in \{m^0, -m^0\}$ is continuous). This contradiction proves the theorem.

To prove the continuity of $n^\lambda(1)$ in λ we make use of the assumption that $Q|_{\partial\omega_i}$ is orientable for each i. Let $\lambda_k \to \lambda$ in $[0,1]$. Define

$$T = \sup\{t \in [0,1] : h(\lambda,t) \in G \cup \partial\Omega, n^{\lambda_k}(t) \to n^\lambda(t)\}.$$

Since $h(\lambda,t) \in \overline{G}$ for all $\lambda \in [0,1]$ and all sufficiently small t it follows from Lemma 6 that $T > 0$. Suppose for contradiction that $T < 1$. If $h(\lambda,T) \in G \cup \partial\Omega$ then by the continuity of h there exists a $\sigma > 0$ such that $h(\lambda_k,t) \in B(h(\lambda,T),\delta)$ for all sufficiently large k and for all t with $|t-T| < \sigma$, where $\delta > 0$ is small enough so that $B(h(\lambda,T),\delta) \cap \overline{\Omega} \subset \overline{G}$ and $\delta < \nu$, where ν is as given in Lemma 6. Hence if $|t-T| < \sigma$, $h(\lambda,t) = \lim_{k\to\infty} h(\lambda_k,t) \in \overline{G}$, and so $h^*(\lambda,t) = h(\lambda,t)$. Also by the definition of T, there exists $\tau \in (T-\sigma,T)$ with $n^{\lambda_k}(\tau) \to n^\lambda(\tau)$, and so by Lemma 6 $n^{\lambda_k}(t) \to n^\lambda(t)$ for $t \in (T, T+\sigma)$, contradicting the definition of T.

Thus we may suppose that $h(\lambda,T) \in \partial\omega_i$ for some i. Let $[T,\bar{T}]$ be the maximal closed interval, containing T, such that $h(\lambda,t) \in \overline{\omega_i}$ for all $t \in [T,\bar{T}]$.

Lemma 7. $n^{\lambda_k}(T) \to n^\lambda(T)$, $n^{\lambda_k}(\bar{T}) \to n^\lambda(\bar{T})$

Proof. *Step 1.* We show that given $\delta > 0$ there exist $\sigma > 0$ and k_0 such that

$$|h^*(\lambda_k,t) - h(\lambda,T)| < \delta \tag{35}$$

whenever $|t-T| < \sigma$ and $k \geq k_0$.

If this were not true there would be a sequence $t_j \to T$ and a subsequence $k_j \to \infty$ such that

$$|h^*(\lambda_{k_j},t_j) - h(\lambda,T)| \geq \delta \text{ for all } j.$$

Since $\lim_{j\to\infty} h(\lambda_{k_j},t_j) = h(\lambda,T)$ we may suppose that $h(\lambda_{k_j},t_j) \in \omega_i$ for all j, and thus we have $t_{1j} < t_j < t_{2j}$ with $h(\lambda_{k_j},t) \in \omega_i$ for $t \in (t_{1j},t_{2j})$, $h(\lambda_{k_j},t_{1j}), h(\lambda_{k_j},t_{2j}) \in \partial\omega_i$. By the definition of T there exists a sequence $T_l \to T-$ such that $h(\lambda,T_l) \in G$ (and $n^{\lambda_k}(T_l) \to n^\lambda(T_l)$). In particular $h(\lambda_{k_j},T_l) \in G$ for j sufficiently large, and so $t_{1j} \geq T_l$ for large enough j. Hence $t_{1j} \to T$ and so $h^*(\lambda_{k_j}, t_{1j}) = h(\lambda_{k_j}, t_{1j}) \to h(\lambda, T)$. Thus $h^*(\lambda_{k_j}, t_{1j}) = \gamma_i(\theta_{1j}), h^*(\lambda_{k_j}, t_{2j}) = \gamma_i(\theta_{2j})$, where $\theta_{1j} \to \theta$ and $h(\lambda, T) = \gamma_i(\theta)$. From (26)-(28)

$$h^*(\lambda_{k_j}, t_j) = \gamma_i\left(\theta_{1j} + (\tilde{\theta}_{2j} - \theta_{1j})\right)\left(\frac{t_j - t_{1j}}{t_{2j} - t_{1j}}\right)\bigg).$$

Considering separately the cases when $t_{2j} - t_{1j} \to 0$ and when $t_{2j} - t_{1j} \geq \mu > 0$ we get $\lim_{j\to\infty} h^*(\lambda_{k_j}, t_j) = h(\lambda, T)$.

This contradiction proves the claim, which by a similar argument also holds if T is replaced by $\bar{T}$.

Step 2. We prove that $n^{\lambda_k}(T) \to n^\lambda(T)$. In Step 1, we choose $\delta \in (0, \nu)$, where ν is given in Lemma 6, and note that $h^*(\lambda_k, T) \to h^*(\lambda, T) = h(\lambda, T)$. Since $n^{\lambda_k}(T_l) \to n^\lambda(T_l)$ the result follows from Lemma 6 applied on the interval $[T_l, T]$.

Step 3. We prove that $n^{\lambda_k}(\bar{T}) \to n^\lambda(\bar{T})$.

If $T = \bar{T}$ there is nothing to prove and so we assume $\bar{T} > T$. First we note that

$$\lim_{k\to\infty} \sup_{t\in[T,\bar{T}]} \operatorname{dist}(h^*(\lambda_k, t), \partial\omega_i) = 0, \tag{36}$$

since otherwise there would exist a subsequence $k_j \to \infty$ and $t_j \to t$ in $[T, \bar{T}]$ with $h^*(\lambda_{k_j}, t_j) = h(\lambda_{k_j}, t_j) \to h(\lambda, t) \notin \overline{\omega_i}$, a contradiction.

Since $Q|_{\partial\omega_i}$ is orientable, there is a unique continuous lifting $\hat{N} : \partial\omega_i \to \mathbb{S}^1$ such that $\hat{N}(h(\lambda, T)) = n^\lambda(T)$. Given $\varepsilon > 0$ sufficiently small, we choose $\delta \in (0, \varepsilon), k_0$ such that (from (36)) if $k \geq k_0$

$$\operatorname{dist}(h^*(\lambda_k, t), \partial\omega_i) < \delta, \text{ for all } t \in [T, \bar{T}], \tag{37}$$

$$x, y \in \bar{G} \text{ with } |x - y| < 4\delta \text{ implies } |Q(x) - Q(y)| < \varepsilon|s|, \tag{38}$$

$$|\hat{N}(z) - \hat{N}(\bar{z})| < \varepsilon, \text{ if } z, \bar{z} \in \partial\omega_i \text{ with } |z - \bar{z}| < 3\delta, \tag{39}$$

and

$$|n^{\lambda_k}(T) - n^\lambda(T)| < \varepsilon,\ |h^*(\lambda_k, T) - h(\lambda, T)| \leq \delta. \tag{40}$$

For $k \geq k_0$ define

$$\begin{aligned} S_k = \{t \in [T, \bar{T}] :\ & \text{there exists } z = z(t) \in \partial\omega_i \\ & \text{with } |z - h^*(\lambda_k, t)| \leq \delta, |n^{\lambda_k}(t) - \hat{N}(z)| \leq 2\varepsilon\}. \end{aligned}$$

It is easily seen that S_k is closed. Also $T \in S_k$ because we can take $z = h(\lambda, T)$ and use (40). We show that if $t \in S_k$ with $t < \bar{T}$ then $t + \tilde{t} \in S_k$ for $\tilde{t} > 0$ sufficiently small, so that $S_k = [T, \bar{T}]$.

Given $\tilde{t} < T - t$, by (37) there exists $z(t + \tilde{t}) \in \partial\omega_i$ with

$$|z(t + \tilde{t}) - h^*(\lambda_k, t + \tilde{t})| < \delta.$$

If $\tilde{t}$ is chosen small enough so that $|h^*(\lambda_k, t+\tilde{t}) - h^*(\lambda_k, t)| < \delta$ we have that

$$|h^*(\lambda_k, t+\tilde{t}) - z(t)| \leq 2\delta, \tag{41}$$

$$|z(t+\tilde{t}) - z(t)| < 3\delta. \tag{42}$$

Thus by (38)

$$|Q(h^*(\lambda_k, t+\tilde{t})) - Q(z(t))| < \varepsilon|s|.$$

Also $|n^{\lambda_k}(t) - \hat{N}(z(t))| \leq 2\varepsilon$ and so by Lemma 3 we have $|n^{\lambda_k}(t+\tilde{t}) - \hat{N}(z(t))| \leq \varepsilon$ and hence, by (39),(42) $|n^{\lambda_k}(t+\tilde{t}) - \hat{N}(z(t+\tilde{t}))| \leq 2\varepsilon$. Hence $t+\tilde{t} \in S_k$ as required.

Since $\bar{T} \in S_k$ and since $h^*(\lambda_k, \bar{T}) \to h(\lambda, \bar{T})$ letting $\varepsilon \to 0$ we deduce that $n^{\lambda_k}(\bar{T}) \to \hat{N}(h(\lambda, \bar{T}))$. But $h^*(\lambda, t) \in \partial\omega_i$ and is continuous in t for $t \in [T, \bar{T}]$. Hence $n^\lambda(t) = \hat{N}(h^*(\lambda, t))$ for all $t \in [T, \bar{T}]$ and in particular $\hat{N}(h(\lambda, \bar{T})) = n^\lambda(\bar{T})$. Thus $n^{\lambda_k}(\bar{T}) \to n^\lambda(\bar{T})$ as required. □

To complete the proof of the theorem we note that by the definition of $\bar{T}$ there exists a sequence $\bar{T}_r \to \bar{T}+$ with $h(\lambda, \bar{T}_r) \in G$. By (35) for $\bar{T}$, given $\delta \in (0, \nu)$, ν as in Lemma 6, we have $|h^*(\lambda_k, t) - h(\lambda, \bar{T})| < \delta$ for $k \geq k_0$ and $|t - \bar{T}| < \sigma$. Let r be large enough so that $|\bar{T}_r - \bar{T}| < \sigma$. Then for k large enough $h^*(\lambda_k, \bar{T}_r) = h(\lambda_k, \bar{T}_r) \in G$, and so $h^*(\lambda_k, \bar{T}_r) \to h^*(\lambda, \bar{T}_r)$. Since also $n^{\lambda_k}(\bar{T}) \to n^\lambda(\bar{T})$ we deduce from Lemma 6 that $n^{\lambda_k}(\bar{T}_r) \to n^\lambda(\bar{T}_r)$ for r sufficiently large, contradicting the definition of T. Thus $T = 1$, and using the same argument as just after the definition of T we deduce that $n^{\lambda_k}(1) \to n^\lambda(1)$. □

Remark 4. Theorem 1 could alternatively have been proved using algebraic topology notions. The orientability of a continuous line field on $\bar{G}$ needs to be checked only on a set of generators of the fundamental group $\pi_1(\bar{G})$ of $\bar{G}$. It seems to be well known (though tricky to find in the literature) that for a domain with holes as defined before, the boundary loops $\partial\omega_i, i = 1, \ldots, n$ constitute a family of generators of $\pi_1(\bar{G})$ (this could be checked for instance by reducing the domain to a standard domain with circular regions, through a conformal mapping; see for instance [13]). These observations suffice to give the result of Theorem 1.

3.2. Arbitrary line fields on simply-connected domains

In this case we have that there is a lifting in $W^{1,p}$ for $p \geq 2$ but not for $p < 2$:

Theorem 2. *Let Ω be a bounded simply-connected domain of class C^0 in $\mathbb{R}^d, d = 2, 3$. Let $Q \in W^{1,p}(\Omega, \boldsymbol{\mathcal{Q}}), 1 \leq p \leq \infty$. If $p \geq 2$ there exists a lifting $n \in W^{1,p}(\Omega, \mathbb{S}^2)$ so that $P \circ n = Q$.*

Moreover we have the estimate

$$c_1\|\nabla Q\|_{L^p} \leq \|\nabla n\|_{L^p} \leq c_2\|\nabla Q\|_{L^p} \tag{43}$$

with c_1, c_2 constants that depend only on p.

For $p < 2$ there exist line fields for which there is no lifting.

Proof. *The case* $p \geq 2$.

Let us recall the following result of M. Pakhzad and T. Rivière:

Proposition ([33], p.225) *Let* M, N *be compact smooth manifolds, with* M *simply connected. For* $u \in W^{1,2}(M,N)$ *there exists a sequence* $\left\{u^{(k)}\right\}_{k\in\mathbb{N}}$ *with* $u^{(k)} \in C^\infty(M,N)$ *so that* $u^{(k)}$ *converges weakly to* u.

Let us assume first that Ω is a domain with smooth boundary. Using the above theorem with $M = \bar{\Omega}$ and $N = \boldsymbol{\mathcal{Q}}$ we can find a sequence of smooth functions $Q^{(k)}$ converging weakly to Q. Each $Q^{(k)}$ is orientable, by Proposition 5 as the domain Ω is simply connected. Using Proposition 4 we obtain that the limit function Q is also orientable.

We obtain thus that for $Q \in W^{1,p}(\Omega, \boldsymbol{\mathcal{Q}}) \subset W^{1,2}(\Omega, \boldsymbol{\mathcal{Q}}), p \geq 2$ there exists $n \in W^{1,2}(\Omega, \mathbb{S}^2)$ with $P(n) = Q$. But then n is continuous along almost any line parallel with the axis of coordinates, hence by (15) we get $n \in W^{1,p}(\Omega, \mathbb{S}^2)$.

In order to extend the theorem to less smooth domains we need the following

Lemma 8. ([2]) *Let* $\Omega \subset \mathbb{R}^n$ *be a bounded simply-connected domain of class* C^0. *There exists* $\varepsilon_0 > 0$ *so that for any* $\varepsilon > 0$ *with* $\varepsilon < \varepsilon_0$ *there exists a domain* $\Omega_\varepsilon \subset \Omega$ *with smooth boundary and such that* $d_H(\Omega_\varepsilon, \Omega) < \varepsilon$ *where* d_H *denotes the Hausdorff distance. Moreover* Ω_ε *can be chosen so that it is simply-connected and* $\Omega_{\varepsilon'} \subset \Omega_\varepsilon$ *if* $\varepsilon < \varepsilon'$.

Using the lemma one finds a sequence of simply-connected smooth domains $\Omega_{\varepsilon_k} \subset \Omega, k \in \mathbb{N}$, with $\Omega_{\varepsilon_k} \subset \Omega_{\varepsilon_{k+1}}$ and $\cup_{k\in\mathbb{N}}\Omega_{\varepsilon_k} = \Omega$. Then for Ω_{ε_1} one has, by the previous arguments, that there exists $n_{\varepsilon_1} \in W^{1,p}(\Omega_{\varepsilon_1}, \mathbb{S}^2)$ so that $P(n_{\varepsilon_1}) = Q$ on Ω_{ε_1}. On Ω_{ε_2} one has two possibilities of orienting Q, and one chooses $n_{\varepsilon_2} \in W^{1,p}(\Omega_{\varepsilon_2}, \mathbb{S}^2)$ so that $n_{\varepsilon_2}(x) = n_{\varepsilon_1}(x)$, a.e. $x \in \Omega_{\varepsilon_1}$. One continues similarly defining inductively $n_{\varepsilon_k}, k \in \mathbb{N}$.

We can define now $n \in W^{1,2}(\Omega, \mathbb{S}^2)$ by $n(x) = n_{\varepsilon_k}(x)$, for all $x \in \Omega_{\varepsilon_k}$.

The formula (43) is straightforward by taking into account the relation between n and Q as well as (15).

The case $1 \leq p < 2$.

An example is provided in Figure 2. The line field

$$Q(x) = s\left(n(x) \otimes n(x) - \frac{1}{3}Id\right)$$

on $\Omega = (-1,1)^3 \subset \mathbb{R}^3$ corresponds to what is called in the physical literature 'an index one-half singularity', where

$$n(x_1,x_2,x_3) = \begin{cases} \left(\frac{x_2}{\sqrt{x_1^2+x_2^2}}, -\frac{x_1}{\sqrt{x_1^2+x_2^2}}, 0\right) & \text{if } (x_1,x_3) \in (-1,1)^2, x_2 \in [0,1) \\ (0,1,0) & \text{if } (x_1,x_3) \in (-1,1)^2, x_2 \in (-1,0) \end{cases}.$$

That the line field in Fig. 2 is not orientable follows from the argument in Lemma 11. It is an easy exercise to adapt the previous example to provide an example of a non-orientable line field, in $W^{1,p}(\Omega)$ for an arbitrary domain Ω as in the hypothesis. □

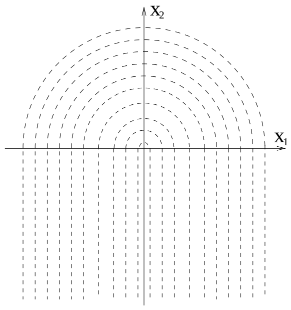

Fig. 2. A non-orientable director field on a simply-connected domain, for $p < 2$

Remark 5. As explained in [4] an interesting consequence of the theorem is that the line field in Fig.2 cannot be modified in a cylindrical core $x_1^2 + x_2^2 \leq \varepsilon^2$ so that it has finite Landau-de Gennes energy, whereas line disclinations of infinite energy can be so modified by 'escape into the third dimension' (see [5]).

4. Analytic orientability criteria in $2D$

In this section we restrict ourselves to planar line fields, i.e. the domain Ω is a subset of $\mathbb{R}^2$ and the line field takes values only in $\mathcal{Q}_2$, not in the whole of $\mathcal{Q}$.

It is important to know, from a PDE perspective, if it is possible to detect the orientability (or non-orientability) of a line field just by knowing its boundary values.

Let us first recall Remark 2 (after Proposition 3) which shows that orientability in a domain implies orientability at the boundary. Thus, in particular, if a line field in $W^{1,2}(\Omega)$ is orientable then its trace on the boundary, a line field in $H^{1/2}(\partial\Omega)$, must be orientable as well. We will see, in the next section, in Proposition 7, that the converse is true as well, namely that orientability at the boundary implies orientability in the interior. We recall that it was already shown in Section 3.1, Theorem 1, that for continuous line fields on domains with holes G (as defined in Section 3.1) orientability can be checked at the boundary.

In order to obtain an analogue of the previous theorem for less regular functions, in $W^{1,2}(G)$, we need first to understand the relation between the orientability in the class of continuous line fields and that in $W^{1,2}(G)$. We study first this question at the boundary and consider a line field on ∂G that is in $C(\partial G) \cap H^{1/2}(\partial G)$. We claim that if the line field is non-orientable, as a continuous line field, then it is also non-orientable in $H^{1/2}(\partial G)$.

More precisely, for $G \subset \mathbb{R}^2$, a domain with holes as defined in Section 3.1, let $Q \in C(\partial G, \mathcal{Q}_2) \cap H^{1/2}(\partial G, \mathcal{Q}_2)$ be a line field non-orientable in the class of continuous fields. We assume for contradiction that Q is orientable as a function in $H^{1/2}(\partial G)$ i.e. that $Q_{ij} = s(n_i n_j - \frac{\delta_{ij}}{3}), i, j = 1, 2, 3$ with $n_i \in H^{1/2}(\partial G, \mathbb{R}), i = 1, 2, 3$ and $(n_1(x), n_2(x), n_3(x)) \in \mathbb{S}^1$ a.e. $x \in \partial G$. If we can show that $n_i, i = 1, 2, 3$ has a continuous representative, we obtain a contradiction which proves our claim. As G has a smooth boundary we can, without loss of generality, assume that there exists locally a smooth transformation that takes functions in $H^{1/2}(B_\delta(P) \cap \partial G)$ (with $P \in \partial G$) into functions in $H^{1/2}(I)$ where I is an open interval and thus we need to show that if $Q_{ii} = s n_i n_i \in C(I)$ and $n_i \in H^{1/2}(I; \mathbb{R})$ for $i = 1, 2, 3$ then there exists a continuous $\bar{n}_i, i = 1, 2, 3$ such that $\bar{n}_i(x) = n_i(x)$ a.e. $x \in I$.

Lemma 9. *Let $I \subset \mathbb{R}$ be an open set. Take $f : I \to \mathbb{R}$ be such that $f \in H^{1/2}(I; \mathbb{R})$ and $f^2 \in C(I; \mathbb{R})$. Then there exists $f^* \in C(I, \mathbb{R})$ so that $f^* = f$ a.e. on I.*

Proof. We claim first that if $f(a) \neq 0$ there exists a $\bar{\delta} = \bar{\delta}(a) > 0$ so that on $(a - \bar{\delta}, a + \bar{\delta})$ the function f has constant sign almost everywhere.

Assuming the claim the proof is straightforward. Indeed, let $Z(f)$ denote the zero set of f in I. We define $s : I \setminus Z(f) \to \{1, -1\}$ such that $s(y) = 1$ if there exists a $\delta_0(y) > 0$ so that f is positive almost everywhere on the interval $(y - \delta, y + \delta)$ for any $\delta < \delta_0$, and $s(y) = -1$ otherwise. One can easily check that s is constant on the connected component of y for any $y \in I \setminus Z(f)$.

Recalling that $f^2 \in C(I; \mathbb{R})$ and so is defined everywhere we let

$$f^*(y) \stackrel{\text{def}}{=} \begin{cases} s(y)\sqrt{f^2} & \text{if } y \in I \setminus Z(f) \\ 0 & \text{if } y \in Z(f) \end{cases}$$

and one can easily check that f^* is continuous. Indeed, if $y \in I \setminus Z(f)$ there exists an open interval around y, say $(y - \delta, y + \delta)$ on which $s(y)$ is constant hence on $(y - \delta, y + \delta)$ we have that f^* is either plus or minus $\sqrt{f^2}$, and $\sqrt{f^2}$ is a continuous function. If $y \in Z(f)$ let us take $(y_n)_{n \in \mathbb{N}}$ an arbitrary sequence of points so that $y_n \to y$. The continuity of f^2 implies that for any $\varepsilon > 0$ there exists a $n(\varepsilon)$ such that $|f^2(y_n)| < \varepsilon$ if $n > n(\varepsilon)$ so that $|f(y_n)| = | \pm f^*(y_n)| = |f^*(y_n)| \leq \sqrt{\varepsilon}$, which proves the continuity of f^* at y.

We continue by proving the claim and start by assuming without loss of generality that $f(a) = l > 0$. As $f^2 \in C(I, \mathbb{R})$ there exists a $\delta_0 > 0$ such that

$$|f^2(x) - l^2| < \frac{l^2}{4}, \text{ for all } x \in (a - \delta_0, a + \delta_0). \tag{44}$$

Note that $H^{1/2}(I, \mathbb{R}) \subset VMO(I, \mathbb{R})$ (see for instance [9],[10],[36]). Recall that if $f \in VMO(I)$ then for any $\varepsilon > 0$ there exists a $\tilde{\delta} > 0$ such that:

$$\frac{1}{|B(x,\delta)|} \int_{B(x,\delta)} \left| f(s) - \frac{1}{|B(x,\delta)|} \int_{B(x,\delta)} f(t)dt \right| ds < \varepsilon$$

for all $\delta < \min\{\tilde{\delta}, \frac{1}{2}(x, \partial I)\}$.

We show that there exists a $\delta_1 < \tilde{\delta}$ so that for any $I(x)_\delta = (x-\delta, x+\delta) \subset (a-\delta_0, a+\delta_0)$ with $\delta < \delta_1$ we have

$$\left|\frac{1}{|I(x)_\delta|}\int_{I(x)_\delta} f(y)dy\right| > \frac{l}{8}. \tag{45}$$

Indeed, if (45) were false there would exist two sequences $(\delta_k)_{k\in\mathbb{N}}, (x_k)_{k\in\mathbb{N}}$, with $\delta_k \to 0$ so that $I(x_k)_{\delta_k} = (x_k - \delta_k, x_k + \delta_k) \subset (a-\delta_0, a+\delta_0)$ and

$$-\frac{l}{8} \le \frac{1}{|I(x_k)_{\delta_k}|}\int_{I(x_k)_{\delta_k}} f(s)ds \le \frac{l}{8}. \tag{46}$$

From the VMO characterization of f we have

$$\frac{1}{|I(x_k)_{\delta_k}|}\int_{I(x_k)_{\delta_k}} \left|f(s) - \frac{1}{|I(x_k)_{\delta_k}|}\int_{I(x_k)_{\delta_k}} f(t)dt\right|ds < \frac{l}{4}$$

for δ_k small enough. However, the last inequality cannot hold simultaneously with (44) and (46). This contradiction proves (45).

Let us denote

$$g(x,\delta) \stackrel{\text{def}}{=} \frac{1}{|I(x)_\delta|}\int_{I(x)_\delta} f(y)dy.$$

As f^2 (and thus f) is bounded on $[a-\delta_0, a+\delta_0]$ one can easily check that $g(x,\delta)$ is continuous as a function of two variables on the set $\{(x,\delta); (x,\delta) \in (a-\delta_0+\delta_1, a+\delta_0-\delta_1)\times[0,\delta_1]\}$ and has no zeros on this set (because of (45)). Thus g has constant sign on $\{(x,\delta); (x,\delta) \in (a-\delta_0+\delta_1, a+\delta_0-\delta_1)\times[0,\delta_1]\}$ and then by using the Lebesgue differentiation theorem we obtain that f also has constant sign almost everywhere on $(a-\delta_0, a+\delta_0)$. $\square$

In order to study the orientability of planar line fields for Q a $\boldsymbol{\mathcal{Q}}_2$-valued function we define the *auxiliary* complex-valued map $A(Q)$:

$$A(Q) \stackrel{\text{def}}{=} \frac{2}{s}Q_{11} - \frac{1}{3} + i\frac{2}{s}Q_{12},\ A(Q) \in \mathbb{S}^1 \subset \mathbb{C}. \tag{47}$$

The motivation for this definition is that if Q has the form in (6) and $\mathcal{Z}(n) \stackrel{\text{def}}{=} n_1 + in_2$ then $A(Q) = \mathcal{Z}^2(n)$. The auxiliary map allows one to associate to a planar line field an *auxiliary* unit-length vector field. We shall determine the orientability of the line field in terms of topological properties of this auxiliary vector field. We provide first a necessary and sufficient condition for orientability along the boundary of bounded smooth sets. This does not suffice for a line field to be orientable on the whole domain but provides a necessary condition for it.

Before stating the orientability criterion, we need to fix some notations about the degree. Let us recall [24], pp. $120-130$, that one can define an integer degree for a smooth function $f: M \to N$ at a regular value $y = f(x)$ where M and N are boundaryless, compact and oriented manifolds of the same dimension. We work only with a connected target manifold ($N = \mathbb{S}^1$), so the degree is independent of the regular value chosen [24], Lemma 1.4, p. 124. In the case when M has several connected components $M_1, \dots, M_k$ we denote $\deg(f, M) = \sum_{i=1}^k \deg(f, M_i)$ where each M_i is given the orientation induced by the inclusion $M_i \hookrightarrow M$. In the case when M is connected and its orientation is the standard one induced from the ambient space we omit the M and simply write $\deg f$.

However, sometimes the degree can be defined for functions that are not necessarily smooth. Let us recall Theorem $A.3$ in [31] that gives a formula for the degree of a complex-valued function $f \in H^{1/2}(\mathbb{S}^1, \mathbb{S}^1)$, namely:

$$\deg f = \frac{1}{2\pi i} \int_{\mathbb{S}^1} f^{-1} \frac{\partial f}{\partial \theta} \, d\theta. \tag{48}$$

(note that the integral is defined in the sense of distributions since $f^{-1} = \bar{f} \in H^{\frac{1}{2}}(\mathbb{S}^1, \mathbb{S}^1)$ and $\frac{\partial f}{\partial \theta} \in H^{-1/2}(\mathbb{S}^1, \mathbb{S}^1)$).

Proposition 6. *Let Ω be a smooth, bounded domain in $\mathbb{R}^2$ and let $Q \in W^{1,2}(\Omega, \boldsymbol{\mathcal{Q}}_2)$. We denote by $(\partial\Omega)_i$, $i = 1, \dots, k$ the connected components of the boundary.*

For any $i \in \{1, 2, \dots, k\}$ the function $\operatorname{Tr} Q|_{(\partial\Omega)_i} \in H^{1/2}((\partial\Omega)_i, \boldsymbol{\mathcal{Q}}_2)$, is orientable (in the space $H^{1/2}$) if and only if $\deg(A(\operatorname{Tr} Q), (\partial\Omega)_i) \in 2\mathbb{Z}$. Moreover if there exists a unit-length vector field n such that $\mathcal{Z}(n) = n_1 + in_2 \in H^{1/2}((\partial\Omega)_i, \mathbb{S}^1)$ and $P(n) = \operatorname{Tr} Q$ a.e. on $(\partial\Omega)_i$ then $\deg(n, (\partial\Omega)_i) = \frac{1}{2} \deg(A(\operatorname{Tr} Q), (\partial\Omega)_i)$.

Proof. We can regard $\bar{\Omega}$ as a manifold with boundary and then the topological boundary of the set coincides with the boundary as a manifold. The boundary is then again a manifold. More precisely $\partial\Omega$ is a one-dimensional closed manifold without boundary. Taking into account the classification theorem for one-dimensional manifolds (see [30]) we have that each connected component of $\partial\Omega$ is diffeomorphic to $\mathbb{S}^1$. We continue thus by assuming, without loss of generality, that for $i \in \{1, 2, \dots, k\}$ we have $(\partial\Omega)_i = \mathbb{S}^1$.

It is easily seen that $\operatorname{Tr} Q \in H^{1/2}(\mathbb{S}^1, \boldsymbol{\mathcal{Q}}_2)$ is orientable if and only if for the function $A(\mathrm{Tr\, Q}) \in \mathrm{H}^{1/2}(\mathbb{S}^1, \mathbb{S}^1)$ there exists a unit-length vector field n such that $\mathcal{Z}(n) \in H^{1/2}(\mathbb{S}^1, \mathbb{S}^1)$ and $A(\operatorname{Tr} Q) = \mathcal{Z}^2(n)$.

We claim now that a necessary and sufficient condition for the existence of a unit-length vector field n so that $\mathcal{Z}(n) \in H^{1/2}(\mathbb{S}^1, \mathbb{S}^1)$ and $A(\operatorname{Tr} Q|_{\mathbb{S}^1}) = \mathcal{Z}^2(n)$ is $\deg(A(\operatorname{Tr} Q), \mathbb{S}^1) \in 2\mathbb{Z}$.

We prove first the necessity. It is known ([31], p.21) that for any function $v \in H^{1/2}(\mathbb{S}^1, \mathbb{S}^1)$ there exists a number $k = \deg\, v \in \mathbb{Z}$ and a unique (up to an integral multiple of 2π) $V \in H^{1/2}(\mathbb{S}^1, \mathbb{R})$ so that $v(z) = z^k \cdot e^{iV(z)}$ a.e. $z \in \mathbb{S}^1$. If we assume that $A(\operatorname{Tr} Q|_{\mathbb{S}^1}) = \mathcal{Z}^2(n)$ for some unit-length vector field n with $\mathcal{Z}(n) \in H^{1/2}(\mathbb{S}^1, \mathbb{S}^1)$ using the quoted result we have that there exist $\alpha = \deg\, A(\operatorname{Tr} Q|_{\mathbb{S}^1}) \in \mathbb{Z}$, $\beta = \deg\, n \in \mathbb{Z}$ and $g, h \in H^{1/2}(\mathbb{S}^1, \mathbb{R})$ so that $A(\operatorname{Tr} Q|_{\mathbb{S}^1})(z) = z^\alpha \cdot e^{ig(z)}$ and $\mathcal{Z}\big(n(z)\big) = z^\beta \cdot e^{ih(z)}$. The equality

$A(\mathrm{Tr}\,Q)|_{\mathbb{S}^1}) = \mathcal{Z}^2(n)$ implies that, a.e. on $\mathbb{S}^1$, one has:

$$z^{\alpha-2\beta} = e^{i(2h-g)}. \tag{49}$$

We claim that the last equality implies $\alpha = 2\beta$. Indeed, we have $2h-g \in H^{1/2}(\mathbb{S}^1, \mathbb{R})$ and thus (see for instance [8], Thm. 2) $e^{i(2h-g)} \in H^{1/2}(\mathbb{S}^1, \mathbb{S}^1)$.

Using formula (48) we find that the expression on the right hand side of (49) has degree 0, while the one on the left hand side has degree $\alpha - 2\beta \in \mathbb{Z}$, hence our claim. In order to prove the sufficiency let us assume that $\deg A(\mathrm{Tr}\,Q) = 2k$, $k \in \mathbb{Z}$. Then, by the previously quoted representation formula in ([31], p.21) there exists a $W \in H^{1/2}(\mathbb{S}^1, \mathbb{R})$ so that $A(\mathrm{Tr}Q)(z) = z^{2k}e^{iW(z)}$ and thus there exists a vector field n such that $\mathcal{Z}(n)(z) = z^k e^{iW(z)/2} \in H^{1/2}(\mathbb{S}^1, \mathbb{S}^1)$.

The same representation formula immediately gives the last part of the Proposition. □

We can now provide a necessary and sufficient condition for orientability *on the whole domain*, in the case of a planar domain with holes.

Proposition 7. *Let G be a planar domain with holes as defined in* (25)*, Section* 3.1*. Assume moreover that ∂G is smooth. Let $Q \in W^{1,2}(G, \boldsymbol{\mathcal{Q}}_2)$. Then Q is orientable if and only if*

$$\deg(A(\mathrm{Tr}\ Q|_{\partial\Omega}), \partial\Omega) \in 2\mathbb{Z},\ \deg(A(\mathrm{Tr}\ Q|_{\partial\omega_i}), \partial\omega_i) \in 2\mathbb{Z}, i = 1, \dots, n.$$

Proof. The necessity of the condition is a consequence of Proposition 6 together with Proposition 3. We show the sufficiency. As $G \subset \mathbb{R}^2$ we have, from [34], that there exists a sequence of functions $Q_k \in C^1(\bar{G}; \boldsymbol{\mathcal{Q}}_2)$ so that $Q_k \to Q$ in $W^{1,2}(G, \boldsymbol{\mathcal{Q}}_2)$. We show that for k large enough Q_k is orientable. First let us observe that we have

Lemma 10. *Let G be an open set in $\mathbb{R}^2$. The function $Q \in W^{1,2}(G; \boldsymbol{\mathcal{Q}}_2) \cap C(\bar{G}; \boldsymbol{\mathcal{Q}}_2)$ is orientable as a function in $W^{1,2}$ if and only if it is orientable as a continuous function.*

Proof of the lemma. We assume first that Q is orientable in $W^{1,2}$ and show that it is orientable in C. Let $n \in W^{1,2}(G; \mathbb{S}^1)$ be such that $P(n) = Q$. Note that this implies $n_i^2 \in W^{1,2}(G; \mathbb{R}) \cap C(\bar{G}; \mathbb{R}), i = 1, 2, 3$.

We prove first that $n_i \in W^{1,2}(G; \mathbb{R}), i = 1, 2, 3$ and $n_i^2 \in C(\bar{G}; \mathbb{R})$ imply $n_i = n_i^*$ a.e. for some $n_i^* \in C(\bar{G}; \mathbb{R}), i = 1, 2, 3$. To prove this we claim first that:

(*C*) *If $x_0 \in \bar{G}$ is such that $n_i^2(x_0) \neq 0$ then there is a neighbourhood of x_0 on which n_i has constant sign almost everywhere*

Assuming (C) it is straightforward to construct n_i^*, in a manner nearly identical to the proof of a similar claim in the proof of Lemma 9. We continue by proving the claim (C). Let $l^2 \stackrel{\text{def}}{=} n_i^2(x_0), l > 0$. There exists $\varepsilon > 0$ such that if $|x - x_0| < \varepsilon$ then $n_i(x) \in (-\frac{5}{4}l, -\frac{3}{4}l) \cup (\frac{3}{4}l, \frac{5}{4}l)$. From $n_i \in$

$W^{1,2}(G,\mathbb{R})$ we have that n_i is continuous along almost all lines parallel with the coordinate axes, in a suitably chosen reference frame. This suffices for concluding that n_i has constant sign almost everywhere in $\{x \in G; |x-x_0| < \varepsilon\}$.

Assume on the other hand that Q is orientable as a continuous function, i.e. there exists a $n \in C(\bar{G}, \mathbb{S}^1)$ so that $P(n) = Q$. Using Lemma 1 we have that $n \in W^{1,2}$. □

Continuing the proof of the theorem let us recall [9] that for a unit-length vector field $n \in H^{1/2}(\mathbb{S}^1, \mathbb{S}^1)$ there exists a $\delta > 0$ (depending on n) such that for any other unit-length vector field $m \in H^{1/2}(\mathbb{S}^1, \mathbb{S}^1)$ with $\|n - m\|_{BMO} < \delta$ we have that m has the same degree as n. Taking into account the relation between the $BMO(\mathbb{S}^1, \mathbb{S}^1)$ semin-norm and the $H^{1/2}(\mathbb{S}^1, \mathbb{S}^1)$ norm we have that there exists $\delta_0 > 0$ so that if $\|n - m\|_{H^{1/2}(\mathbb{S}^1,\mathbb{R})} < \delta_0$ then n and m have the same degree. Thus for k large enough we have that $\deg(A(\operatorname{Tr} Q_k|_{\partial\Omega}), \partial\Omega) \in 2\mathbb{Z}$, $\deg(A(\operatorname{Tr} Q_k|_{\partial\omega_i}), \partial\omega_i) \in 2\mathbb{Z}, i = 1, \ldots, n$.

Proposition 6 shows that $\operatorname{Tr} Q_k|_{\partial\Omega}, \operatorname{Tr} Q_k|_{\partial\omega_i}, i = 1, \ldots, n$ are orientable in $H^{1/2}$. Using Lemma 9 we have that $\operatorname{Tr} Q_k|_{\partial\Omega}, \operatorname{Tr} Q_k|_{\partial\omega_i}, i = 1, \ldots, n$ are also orientable in the class of continuous functions. Using Theorem 1 we have that Q_k is orientable in the class of continuous functions. Using Lemma 10 we obtain that for large enough k the function Q_k is orientable in $W^{1,2}$. Since strong convergence preserves orientability (see Proposition 4) we conclude that Q is orientable. □

Remark 6. It is known (see for instance [1]) that for functions with values in $\mathbb{R}^d$ we have that $W^{1,2}(\Omega) \setminus C(\Omega) \neq \emptyset$ (for $\Omega \subset \mathbb{R}^2$). However one may ask if for functions with values in $\mathbb{S}^1$ the situation is different. This is not the case, as shown by the vector field: $n(x) = (n_1(x), n_2(x), n_3(x))$ with $n_1(x) = \frac{1}{2}\sin\big(\ln\ln(\frac{k}{|x|})\big)$, $n_2(x) = \sqrt{1 - n_1(x)^2}$, $n_3(x) = 0$, on $D = \{x \in \mathbb{R}^2, |x| \leq 1\}$ (we take $k > 1$). Then one can easily check that $n \in W^{1,2}(D; \mathbb{S}^1) \setminus C(D; \mathbb{S}^1)$.

The previous proposition shows that we can determine the orientability to computing certain numbers. However, in specific cases, it may be simpler to just use Lemma 10 and check the orientability at the continuous level of regularity, where topological tools can be more efficient.

As an example, consider an analytic description of the line field in Figure 1. Let

$$\tilde{Q} = s(\tilde{n} \otimes \tilde{n} - \frac{1}{3} Id) \in W^{1,2}(\tilde{\Omega}, \boldsymbol{\mathcal{Q}}_2) \tag{50}$$

where

$$\tilde{\Omega} \stackrel{\text{def}}{=} \{(x,y) \in [-1,1] \times [-1,0], \sqrt{x^2+y^2} \geq \frac{1}{2}\} \\ \cup \{(x,y); y \geq 0, \frac{1}{2} \leq \sqrt{x^2+y^2} \leq 1\} \tag{51}$$

and

$$\tilde{n}(x,y) = \begin{cases} (0,1,0) & \text{if } (x,y) \in ([-1,1]\times[-1,0]) \cap \tilde{\Omega} \\ (-\frac{y}{\sqrt{x^2+y^2}}, \frac{x}{\sqrt{x^2+y^2}}, 0) & \text{if } y \geq 0, \frac{1}{2} \leq \sqrt{x^2+y^2} \leq 1 \end{cases} \tag{52}$$

Lemma 11. *The line field $\tilde{Q}$ as in* (50), (52) *on the domain $\tilde{\Omega}$ as in* (51) *is not orientable in $W^{1,2}(\tilde{\Omega};\boldsymbol{Q}_2)$ or in $C(\tilde{\Omega};\boldsymbol{Q}_2)$.*

Proof. Lemma 10 shows that it suffices to prove the non-orientability in the class of continuous line fields. Let us consider the following subsets of $\tilde{\Omega}$: $\Omega_1 \stackrel{\text{def}}{=} \{(x,y) \in \tilde{\Omega},\, y \leq 0\}$, $\Omega_2 \stackrel{\text{def}}{=} \{(x,y) \in \tilde{\Omega},\, x \leq 0\}$, $\Omega_3 \stackrel{\text{def}}{=} \{(x,y) \in \tilde{\Omega},\, x \geq 0\}$. We assume for contradiction that the continuous line field is orientable and try to find an orientation. In Ω_1 there are only two possible orientations (see also Proposition 2), that is all the unit vectors are $(0,1,0)$ or all are $(0,-1,0)$. Let us assume that we pick the orientation $(0,1,0)$. There are two possible orientations in Ω_2 but since $\Omega_1 \cap \Omega_2 \neq \emptyset$ and we have already chosen an orientation in Ω_1 we can only pick the orientation $(\frac{y}{\sqrt{x^2+y^2}}, -\frac{x}{\sqrt{x^2+y^2}})$ in Ω_2. Also there are two possible orientations in Ω_3 but since $\Omega_1 \cap \Omega_3 \neq \emptyset$ and we have already chosen an orientation in Ω_1 we can only pick the orientation $(-\frac{y}{\sqrt{x^2+y^2}}, \frac{x}{\sqrt{x^2+y^2}})$ in Ω_3. Thus on the line $\{(0,y); y \in [\frac{1}{2},1]\}$ we have both the orientation $(-1,0,0)$ and $(1,0,0)$. Similarily, if we start with the other possible orientation in Ω_1 we also reach a contradiction. □

5. The minimizing Q-harmonic maps versus minimizing harmonic maps in the plane

We saw in the previous sections that in order to have a geometry in which both orientable and non-orientable energy minimizers exist we need to allow for a domain that is not simply connected. Propositions 6 and 7 show that if the boundary data on all components of the boundary is orientable then any line field with that boundary data will be orientable. Moreover, if the boundary data on at least one component of the boundary is non-orientable then any line field with that boundary data will necessarily be non-orientable. Thus full knowledge of the boundary data completely determines the orientability of the line fields with that boundary data. In order to allow for a geometry with both orientable and non-orientable energy minimizers we need to fix orientable boundary data on only one part of the boundary.

The simplest situation one could conceive is to consider a domain with one hole. However in such a domain putting orientable boundary data on one component of the boundary would imply that any line field with that boundary data is orientable (indeed, let $G = \Omega \setminus \overline{\omega}_1$ and $g : \partial\Omega \to \boldsymbol{\mathcal{Q}}_2$ be orientable, so that degree of $A(g)$ is even; for any $h : \partial\omega_1 \to \boldsymbol{\mathcal{Q}}_2$ we need to have $\deg(A(g)) + \deg(A(h)) = 0$, see for instance [24], p. 126, and hence h is orientable). Thus we need to take at least two holes. If one puts orientable boundary data on two of the components of the boundary, leaving the third component free, a degree argument as before shows that the boundary data on the third component of the boundary must be orientable as well, hence we can only have orientable line fields.

Thus we are led to considering the case of a domain with two holes and orientable boundary data on only one connected component of the boundary. Such a situation is presented in Fig. 3. More precisely let us consider the domains (for $\delta > 1$):

$$\begin{aligned}
M_1 &\stackrel{\text{def}}{=} \{x = (x_1, x_2) \in \mathbb{R}^2 : x_1^2 + (x_2 - \delta)^2 < 1\} \\
M_2 &\stackrel{\text{def}}{=} \{x = (x_1, x_2) \in \mathbb{R}^2 : x_1^2 + (x_2 + \delta)^2 < 1\} \\
M_3 &\stackrel{\text{def}}{=} \{x = (x_1, x_2) \in \mathbb{R}^2 : |x_1| < 1; |x_2| \le \delta\} \\
M_4 &\stackrel{\text{def}}{=} \{x = (x_1, x_2) \in \mathbb{R}^2 : x_1^2 + (x_2 - \delta)^2 \le \tfrac{1}{2}\} \\
M_5 &\stackrel{\text{def}}{=} \{x = (x_1, x_2) \in \mathbb{R}^2 : x_1^2 + (x_2 + \delta)^2 \le \tfrac{1}{2}\}
\end{aligned} \tag{53}$$

and we define the stadium domain:

$$M_\delta = M_1 \cup M_2 \cup M_3 \setminus (M_4 \cup M_5). \tag{54}$$

On the outer boundary we impose as boundary conditions lines tangent to the boundary, which can be oriented clockwise (as shown in Fig. 3b) or anticlockwise. Thus we have a simple geometry with boundary conditions that allow both orientable and non-orientable line fields. We compare the minimizers of

$$\mathcal{I}_\delta(Q) = \int_{M_\delta} |\nabla Q(x)|^2 \, dx$$

(in $W^{1,2}(M_\delta, \boldsymbol{\mathcal{Q}}_2)$, subject to the indicated line field boundary conditions on the outer boundary) with the minimizers of

$$\mathcal{J}_\delta(n) = 2s^2 \int_{M_\delta} |\nabla n(x)|^2 \, dx$$

(in $W^{1,2}(M_\delta; \mathbb{S}^1)$, subject to tangent vector-field boundary conditions on the outer boundary). Note that $\mathcal{J}_\delta(n) = \mathcal{I}_\delta(Q)$ when Q is orientable.

We have:

Lemma 12. *Let $\bar{n}_\delta \in W^{1,2}(M_\delta, \mathbb{S}^1)$ be any global energy minimizer of $\mathcal{J}_\delta(n)$ in $W^{1,2}(M_\delta; \mathbb{S}^1)$ (subject to tangent vector-field boundary conditions on the outer boundary, as in Fig. 3b). Let $\bar{Q}_\delta \in W^{1,2}(M_\delta; \boldsymbol{\mathcal{Q}}_2)$ be any global energy minimizer of $\mathcal{I}_\delta(Q)$ in $W^{1,2}(M_\delta; \boldsymbol{\mathcal{Q}}_2)$ (subject to tangent line-field boundary conditions on the outer boundary, as in Fig. 3a).*

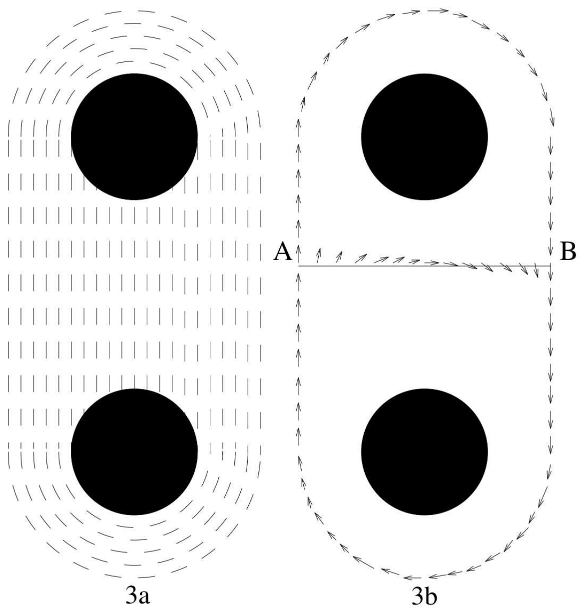

Fig. 3. A situation in which the energy minimizer is non-orientable

There exists a $\delta_0 > 1$ *so that for any* $\delta > \delta_0$ *we have*

$$\mathcal{I}_\delta(\bar{Q}_\delta) < \mathcal{J}_\delta(\bar{n}_\delta).$$

Proof. Let us observe first that the sets in which we do the minimization, in either the oriented or non-oriented context, are non-empty. Indeed, let us take

$$\tilde{Q}(x) = \begin{cases} s\left((0,1,0) \otimes (0,1,0) - \frac{1}{3}Id\right), \; x \in M_3 \\ s\left(n_\delta(x) \otimes n_\delta(x) - \frac{1}{3}Id\right), \; x \in M_1 \setminus M_4, \; x_2 \geq \delta \\ s\left(m_\delta(x) \otimes m_\delta(x) - \frac{1}{3}Id\right), \; x \in M_2 \setminus M_5, \; x_2 \leq -\delta \end{cases}$$

where

$$n_\delta(x) \stackrel{\text{def}}{=} \left(\frac{x_2 - \delta}{|(x_1, x_2 - \delta)|}, -\frac{x_1}{|(x_1, x_2 - \delta)|}, 0\right)$$

$$m_\delta(x) \stackrel{\text{def}}{=} \left(\frac{x_2 + \delta}{|(x_1, x_2 + \delta)|}, -\frac{x_1}{|(x_1, x_2 + \delta)|}, 0\right).$$

Then $\tilde{Q} \in W^{1,2}$ and satisfies the boundary conditions. Let us observe that $\tilde{Q}$ is exactly the line field shown in Fig. 3a. It is also straightforward to see that in the case of vector-field boundary conditions (see Fig. 3b) there exist vector fields $n_\delta \in W^{1,2}(M_\delta; \mathbb{S}^1)$ on the whole M_δ that match the boundary conditions.

For any such vector field the orientation along a line AB as in Fig. 3b changes from up to down or down to up, and we can find a lower bound for the energy along such a line. Indeed, for almost all $x_2 \in [-\delta, \delta]$ we have $n_\delta(\cdot, x_2) \in W^{1,2}([-1,1]; \mathbb{S}^1)$ and $n_\delta(-1, x_2) = (0,1,0)$, $n_\delta(1, x_2) = (0,-1,0)$, and it is easy to check that $\int_{[-1,1]\times\{x_2\}} |\partial_{x_1} n_\delta(z, x_2)|^2\, dz \geq \frac{\pi^2}{2}$.

Then

$$\int_{M_\delta} |\nabla n_\delta(x)|^2\, dx \geq \int_{M_\delta} |\partial_{x_1} n_\delta(x)|^2\, dx$$
$$\geq \int_{M_3 \setminus (M_4 \cup M_5)} |\partial_{x_1} n_\delta(x)|^2\, dx \geq (\delta - \frac{1}{2})\pi^2. \tag{55}$$

Thus we have that $\int_{M_\delta} |\nabla \bar{n}_\delta(x)|^2\, dx \geq (\delta - \frac{1}{2})\pi^2$ and, noting the way $\tilde{Q}$ is defined we have that $\int_{M_\delta} |\nabla \tilde{Q}(x)|^2\, dx$ is independent of δ. Hence there exists $\delta_0 > 0$ so that for any $\delta > \delta_0$ we have

$$2s^2 \int_{M_\delta} |\nabla \bar{n}_\delta(x)|^2\, dx \geq s^2(2\delta - 1)\pi^2 > \int_{M_\delta} |\nabla \tilde{Q}(x)|^2\, dx \geq \int_{M_\delta} |\nabla \bar{Q}_\delta(x)|^2\, dx$$

which proves the claim. □

The previous theorem shows that for δ large enough the Oseen-Frank theory fails to capture the global energy minimizer and detects just a local energy minimizer, the energy minimizer in the class of oriented line fields. In the following we completely characterize the instances in which the Oseen-Frank theory fails in this way.

We consider a smooth planar domain $G = \Omega \setminus \cup_{i=1}^n \overline{\omega_i}$ with $n \geq 1$ holes, $\omega_i, i = 1, \ldots, n$, as defined in (25), Section 3.1. We consider the problem of minimizing the energy

$$\mathcal{I}_G(Q) = \int_G |\nabla Q(x)|^2\, dx, \tag{56}$$

on this domain in the class of $\mathcal{Q}_2$-valued functions whose gradients are square integrable and that satisfy $Q|_{\partial\Omega} = g$ with g smooth. We shall provide necessary and sufficient conditions for the global minimizers to be non-orientable. This is the most interesting situation as it is precisely that in which the Oseen-Frank theory would fail to see the right energy minimizer and would only provide a local energy minimizer, a minimizer in the class of orientable line fields.

In order to encode the complexity of the domain and its relationship with the prescribed boundary data g, we need $n+1$ functions $h_1, \ldots, h_n$ and $h(g)$.The functions $h_i, i = 1, 2, \ldots, n$ encode the characteristics of the holes and their relations with the set Ω. Each function $h_i, i = 1, \ldots, n$ is the solution of the equation

$$\begin{cases} \Delta h_i = 0 & \text{on G} \\ h_i = 1 & \text{on } \partial\omega_i \\ h_i = 0 & \text{on } \partial\omega_j,\ j \neq i \\ \frac{\partial h_i}{\partial \nu} = 0 & \text{on } \partial\Omega \end{cases} \tag{57}$$

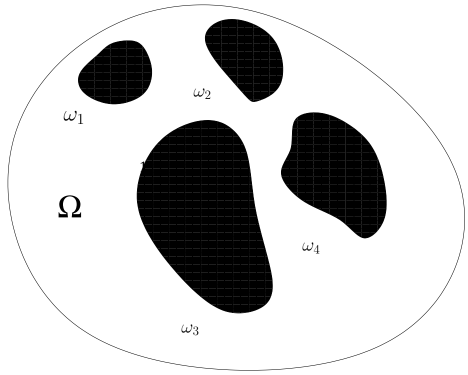

Fig. 4. A domain with holes

We define the matrix $D = (D_{ij}), i, j = 1, \dots, n$ *depending only on the domain*, by $D_{ij} \stackrel{\text{def}}{=} \frac{1}{2\pi} \int_{\partial\omega_i} \frac{\partial h_j}{\partial \nu}(\sigma)\, d\sigma$. Note that $D_{ij} = \frac{1}{2\pi} \int_G \nabla h_i(x) \cdot \nabla h_j(x)\, dx$ so that D is symmetric.

It will be important, in later calculations, to know explicitly the nullspace of the matrix D:

Lemma 13. *Let* $e \stackrel{\text{def}}{=} (\underbrace{1, 1, \dots, 1}_{n \text{ times}})$. *Then the nullspace of* D *is* $N(D) = \mathbb{R}e$.

Proof. Let $z \in \mathbb{R}^n$, $h(x) \stackrel{\text{def}}{=} (h_1(x), \dots, h_n(x))$ and denote $v(x) \stackrel{\text{def}}{=} h(x) \cdot z$. Then

$$Dz = 0 \Leftrightarrow \int_{\partial\omega_i} \frac{\partial v}{\partial \nu}\, d\sigma = 0, \quad \text{for all } i \in \{1, 2, \dots, n\}.$$

The last relation implies that $\int_G |\nabla v(x)|^2\, dx = \int_{\partial G} v \cdot \frac{\partial v}{\partial \nu}\, d\sigma = 0$ and hence v is a constant function. But $v = z_i$ on $\partial\omega_i$ and so $z_1 = z_2 = \dots = z_n$.

Conversely if $z_1 = \dots = z_n = a$ then

$$\begin{cases} \Delta v = 0 & \text{on } G \\ v = a & \text{on each } \partial\omega_i \\ \frac{\partial v}{\partial n} = 0 & \text{on } \partial\Omega \end{cases}$$

and so by uniqueness $v \equiv a$ and $Dz = 0$. □

In order to define the function $h(g)$ we need to use the auxiliary vector-field $A(g) = \frac{2}{s} g_{11} - \frac{1}{3} + i\frac{2}{s} g_{12}$ associated to the line field g, as defined in (47). This is a complex-valued function but from now on, until the end of the paper, we identify in a standard way the complex-valued function $A(g)$ with a vector-valued real function. The function $h(g)$ describes the relation between the domain and the boundary data and is defined as the solution of the equation

$$\begin{cases} \Delta h(g) = 0 & \text{on G} \\ \frac{\partial h(g)}{\partial \nu} = A(g) \times \frac{\partial A(g)}{\partial \tau} & \text{on } \partial\Omega \\ h(g) = 0 & \text{on } \partial G \setminus \partial\Omega \end{cases} \tag{58}$$

(note that in the above the vector-valued real function $A(g) \times \frac{\partial A(g)}{\partial \tau}$ is identified, in a standard way, with a scalar, real-valued function). The derivative $\frac{\partial}{\partial \tau}$ is the tangential derivative on the boundary.

We define the vector $J(g) \stackrel{\text{def}}{=} (J(g)^1, \ldots, J(g)^n)$ (depending on both the domain and the boundary data), where $J(g)^i = \frac{1}{2\pi} \int_{\partial \omega_i} \frac{\partial h(g)}{\partial \nu} \, d\sigma$.

Let

$$\mathcal{D}(g) \stackrel{\text{def}}{=} \{(d_1, \ldots, d_n) \in \mathbb{Z}^n, \sum_{i=1}^n d_i = -\deg(A(g), \partial \Omega)\}$$

$$\mathcal{D}_{even}(g) \stackrel{\text{def}}{=} \{(d_1, \ldots, d_n) \in (2\mathbb{Z})^n, \sum_{i=1}^n d_i = -\deg(A(g), \partial \Omega)\}.$$

We can now state a necessary and sufficient criterion for determining the orientability of the global minimizer of the Q-harmonic maps problem:

Theorem 3. *Let $g \in W^{1,2}(\partial \Omega, \mathcal{Q}_2)$ be orientable, and assume that*

$$W_g^{1,2}(G, \mathcal{Q}_2) = \{Q : G \to \mathcal{Q}_2; \mathcal{I}_G(Q) < \infty, \, Q|_{\partial \Omega} = g\}$$

is nonempty. Then the infimum of $\mathcal{I}_G(Q)$ in $W_g^{1,2}(G, \mathcal{Q}_2)$ is attained.

For $d \in \mathcal{D}(g)$ let $c(d) \stackrel{\text{def}}{=} (c_1(d), \ldots, c_n(d))$ be a solution of the equation

$$D \cdot c = d - J(g). \tag{59}$$

Then a necessary and sufficient condition for all global minimizers to be non-orientable is

$$\inf_{d \in \mathcal{D}(g)} c(d) \cdot Dc(d) < \inf_{d \in \mathcal{D}_{even}(g)} c(d) \cdot Dc(d). \tag{60}$$

Proof. For any $Q \in W_g^{1,2}(G, \mathcal{Q}_2)$ we have $A(Q) \in \mathbb{S}^1$ and moreover

$$\|\nabla Q\|_{L^2(G)} = \frac{\sqrt{2}}{s} \|\nabla A(Q)\|_{L^2(G)}.$$

Observing that A is a bijective operator we have that our minimization problem reduces to

$$\inf_{m \in W^{1,2}_{A(g)}(G, \mathbb{S}^1)} \frac{2}{s^2} \int_G |\nabla m(x)|^2 \, dx. \tag{61}$$

It is well known that the minimum of the energy for the last problem is attained by a function $m_{min} \in W^{1,2}_{A(g)}(G, \mathbb{S}^1)$ satisfying a harmonic map equation (see [6]).

In order to determine this function we first claim that for any $m \in W^{1,2}_{A(g)}(G,\mathbb{S}^1)$ we have

$$\deg(A(g),\partial\Omega) = -\Sigma_{i=1}^n \deg(m,\partial\omega_i). \tag{62}$$

Indeed, by [34] there exists a sequence $m_k \in W^{1,2}_{A(g)}(G,\mathbb{S}^1)\cap C(G,\mathbb{S}^1)$ so that $m_k \to m$ in $W^{1,2}$. Taking into account that the function m_k is continuous, and the properties of the degree for continuous functions, [24], p. 126 we have

$$\deg(A(g),\partial\Omega) = -\sum_{i=1}^n \deg(m_k,\partial\omega_i).$$

Using the continuity of the trace operator we let $k\to\infty$ in the last relation and we obtain the claimed relation (62).

Thus we can divide the function space $W^{1,2}_{A(g)}$ into countably many disjoint subsets corresponding to maps with given degrees d_i on each $\partial\omega_i, i = 1,\dots,n$. A way of solving the minimization problem (61) is to obtain first the minimizer on each such subset as before and thus obtain countably many functions $m_1, m_2,\dots,m_l,\dots, l\in\mathbb{N}$. The solution of (61) is then that m_k with $\|\nabla m_k\|_{L^2(G)} = \inf_{i\in\mathbb{N}}\|\nabla m_i\|_{L^2(G)}$ (such an m_k exists because there exists a global minimizer for the problem (61)).

Thus we need to study first the minimization problem

$$\inf_{\substack{m\in W^{1,2}_{A(g)}(G,\mathbb{S}^1)\\ \deg(m,\partial\omega_i)=d_i, i=1,\dots,n}} \frac{2}{s^2}\int_G |\nabla m(x)|^2\,dx \tag{63}$$

for each set of $d_i, i=1,\dots,n$ so that $d=(d_1,\dots,d_n)\in\mathcal{D}(g)$.

The advantage in studying (63) rather than (61) is that the determination of the minimum for (63) can be reduced to a simpler, scalar, problem. Indeed, it is shown in [6] that for a fixed $d\in\mathcal{D}(g)$ if we denote by m^* a minimizer of (63) then $\|\nabla \mathrm{m}^*\|_{L^2(G)} = \|\nabla\Phi\|_{L^2(G)}$ where Φ is the unique solution (up to an additive constant) of the scalar problem:

$$\begin{cases} \Delta\Phi = 0 & \text{on G}\\ \Phi = c_i & \text{on } \partial\omega_i,\ i=1,2,\dots,n\\ \int_{\partial\omega_i}\frac{\partial\Phi}{\partial\nu}\,d\sigma = 2\pi d_i & i=1,2,\dots,n\\ \frac{\partial\Phi}{\partial\nu} = A(g)\times\frac{\partial A(g)}{\partial\tau} & \text{on } \partial\Omega \end{cases} \tag{64}$$

where d_i are prescribed (with $d=(d_1,\dots,d_n)\in\mathcal{D}(g)$), but not the c_i.

Taking Φ to be a solution of the above problem let us denote $h = \Phi - \sum_{i=1}^n c_i h_i$. We have that h is a solution of the problem (58). Since (58) has a unique solution we get that $h\equiv h(g)$, thus

$$\Phi = h(g) + \sum_{i=1}^n c_i h_i. \tag{65}$$

Taking into account the representation (65) of Φ as well as the equation it satisfies, (64), we get

$$\sum_{j=1}^{n} D_{ij}c_j + J^i(g) = d_i, i = 1, \dots n. \tag{66}$$

Lemma 13 shows that the last system has a one-dimensional affine space of solutions, so that the function $c(d)$ introduced in the statement is multivalued. We claim that, however, the value of $c(d) \cdot Dc(d)$ is independent of the particular representative of $c(d)$ used. Indeed observe that multiplying (59) by e we obtain $Dc \cdot e = d \cdot e - J(g) \cdot e$ and since D is symmetric and $De = 0$ we obtain $d \cdot e = J(g) \cdot e$, thus proving our claim.

In order to finish the proof it suffices to recall the orientability criterion given by Proposition 7 and observe that from (65) we have

$$\begin{aligned}
\|\nabla\Phi\|^2_{L^2(G)} &= \sum_{i,j=1}^{n} c_i c_j \int_G \nabla h_i \cdot \nabla h_j \, dx + 2\sum_{j=1}^{n} c_j \int_G \nabla h_j \cdot \nabla h(g) \, dx \\
&\qquad + \int_G |\nabla h(g)|^2 \, dx \\
&= \sum_{i,j=1}^{n} c_i c_j \int_{\partial G} \frac{\partial h_i}{\partial \nu} h_j \, d\sigma + 2\sum_{j=1}^{n} c_j \int_{\partial G} \frac{\partial h_j}{\partial \nu} h(g) \, d\sigma + \int_G |\nabla h(g)|^2 \, dx \\
&= c(d) \cdot Dc(d) + \int_G |\nabla h(g)|^2 \, dx,
\end{aligned}$$

where we used the definitions of $h(g), h_i, i = 1, \dots, n$ and $D_{ij}, i, j = 1, \dots, n$. □

Remark 7. One can see, carefully following the proof, that one also has that if

$$\inf_{d \in \mathcal{D}(g) \setminus \mathcal{D}_{even}(g)} c(d) \cdot Dc(d) > \inf_{d \in \mathcal{D}_{even}(g)} c(d) \cdot Dc(d) \tag{67}$$

then all global energy minimizers must necessarily be orientable.

Moreover, if

$$\inf_{d \in \mathcal{D}(g) \setminus \mathcal{D}_{even}(g)} c(d) \cdot Dc(d) = \inf_{d \in \mathcal{D}_{even}(g)} c(d) \cdot Dc(d) \tag{68}$$

then there exist both an orientable and a non-orientable global energy minimizer.

Remark 8. One can easily see that for g smooth the set $W^{1,2}_g$ is non-empty by recalling [18], [24] that a degree zero smooth map $\tilde{g} : \partial G \to \mathbb{S}^1$ can be extended to a smooth map on G. On the other hand one can alway choose some suitable smooth vector field $h : \partial G \setminus \partial\Omega \to \mathbb{S}^1$ so that

$$\tilde{g}(x) = \begin{cases} g(x) & \text{if } x \in \partial\Omega \\ h(x) & \text{if } x \in \partial G \setminus \partial\Omega \end{cases}$$

has degree zero.

In general, for g not smooth, the space $W^{1,2}_g$ may be empty (see [21]).

We continue with a more detailed analysis of the case when the domain G has only two holes, by using the tools developed in the previous Proposition.

Proposition 8. (i) *Let $G = \Omega \setminus \cup_{i=1}^2 \overline{\omega_i}$ be a domain with two holes, ω_1 and ω_2, as defined in* (25), *Section* 3.1. *We take a boundary data $g \in W^{1,2}(\partial\Omega, \boldsymbol{\mathcal{Q}}_2)$ that is an orientable line field and assume that the space $W^{1,2}_g(\Omega, \boldsymbol{\mathcal{Q}}_2)$ is non-empty. Then* $\mathrm{dist}(\mathrm{J(g)}^1, \mathbb{Z}) = \mathrm{dist}(\mathrm{J(g)}^2, \mathbb{Z})$, $\mathrm{dist}(\mathrm{J(g)}^1, 2\mathbb{Z}) = \mathrm{dist}(\mathrm{J(g)}^2, 2\mathbb{Z})$ *and all the global energy minimizers are non-orientable if and only if*

$$\mathrm{dist}(\mathrm{J(g)}^1, \mathbb{Z}) < \mathrm{dist}(\mathrm{J(g)}^1, 2\mathbb{Z}). \tag{69}$$

On the other hand, if

$$\mathrm{dist}(\mathrm{J(g)}^1, 2\mathbb{Z}) < \mathrm{dist}(\mathrm{J(g)}^1, 2\mathbb{Z}+1) \tag{70}$$

then all the global energy minimizers are orientable.

Moreover, if

$$\mathrm{dist}(\mathrm{J(g)}^1, 2\mathbb{Z}) = \mathrm{dist}(\mathrm{J(g)}^1, 2\mathbb{Z}+1) \tag{71}$$

then there exist both an orientable and a non-orientable energy minimizer.

(ii) *Let M_δ be the domain defined in* (54). *Let $\bar{n}_\delta \in W^{1,2}(M_\delta, \mathbb{S}^1)$ be any global energy minimizer of $\mathcal{J}_\delta(n)$ in $W^{1,2}(M_\delta; \mathbb{S}^1)$ (subject to tangent vector-field boundary conditions on the outer boundary, as in Fig.* 3b). *Let $\bar{Q}_\delta \in W^{1,2}(M_\delta; \boldsymbol{\mathcal{Q}}_2)$ be any global energy minimizer of $\mathcal{I}_\delta(Q)$ in $W^{1,2}(M_\delta; \boldsymbol{\mathcal{Q}}_2)$ (subject to tangent line-field boundary conditions on the outer boundary, as in Fig.* 3a).

For any $\delta > 1$ we have

$$\mathcal{I}_\delta(\bar{Q}_\delta) < \mathcal{J}_\delta(\bar{n}_\delta).$$

(iii) *Let*

$$G_\delta \stackrel{\text{def}}{=} \{x = (x_1, x_2) \in \mathbb{R}^2 : \frac{1}{2} < x_1^2 + x_2^2 < 1,\, x_1^2 + (x_2 - \frac{3}{4})^2 > \delta\}$$

for $\delta < \frac{1}{4}$. Let $g : \{(x_1, x_2) \in \mathbb{R}^2 : x_1^2 + x_2^2 = 1\} \to \boldsymbol{\mathcal{Q}}_2$ be a smooth orientable line field.

There exists a δ_0 so that for any $\delta < \delta_0$ any global energy minimizer of $\mathcal{I}_\delta(Q)$ must necessarily be orientable.

(iv) *Let*

$$M_6^\rho \stackrel{\text{def}}{=} \{x = (x_1, x_2) \in \mathbb{R}^2 : x_1^2 + (x_2 - 2)^2 \le \rho\}$$

for $\rho < 1$. With M_i, $i = 1, \ldots, 5$ defined as in (53) (*where we take* $\delta = 2$) *we consider the domain:*

$$N_\rho \stackrel{\text{def}}{=} M_1 \cup M_2 \cup M_3 \setminus (M_5 \cup M_6^\rho)$$

We impose tangential line field boundary conditions on the outer boundary of N_ρ. Then there exists a $\rho \in (0, \frac{1}{2})$ so that there exist both an orientable and a non-orientable global energy minimizer of $\mathcal{I}_{N_\rho}(Q)$, subject to the imposed boundary conditions.

Proof. (i) Let $d_1, d_2 \in \mathbb{Z}$ be some arbitrary pair such that

$$d_1 + d_2 = -\deg(A(g), \partial\Omega). \tag{72}$$

Relation $d \cdot e = J(g) \cdot e$ (with $e = (1,1)$), shown in the proof of Theorem 3, gives that:

$$d_1 + d_2 = J(g)^1 + J(g)^2. \tag{73}$$

Corresponding to this domain G we have the symmetric matrix $D = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$ and $D(1,1)^t = 0$. Thus $b = -a$, $a = c$ and

$$D = a \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}.$$

So equation (59) becomes

$$Dc = a \begin{pmatrix} c_1 - c_2 \\ c_2 - c_1 \end{pmatrix} = \begin{pmatrix} d_1 \\ d_2 \end{pmatrix} - \begin{pmatrix} J(g)^1 \\ J(g)^2 \end{pmatrix}. \tag{74}$$

We claim now that $a \neq 0$. Assuming for contradiction that $a = 0$ equation (74) (which always has a solution by the arguments in the proof of Proposition 3) implies $d_1 = J(g)^1$ and $d_2 = J(g)^2$. However if we replace d_1 by $d_1 + 1$ and d_2 by $d_2 - 1$ (so that their sum is still $-\deg(A(g), \partial\Omega)$) then (74) no longer has a solution. Thus our claim is proved.

Moreover, recalling that $D_{ij} = \frac{1}{2\pi} \int_G \nabla h_i(x) \nabla h_j(x)\, dx$ one can easily see that D is non-negative definite, so $a > 0$. Hence

$$\begin{aligned} c \cdot Dc &= (d_1 - J(g)^1)c_1 + (d_2 - J(g)^2)c_2 \\ &\stackrel{(73)}{=} (d_1 - J(g)^1)(c_1 - c_2) \stackrel{(74)}{=} \frac{1}{a}(d_1 - J(g)^1)^2. \end{aligned}$$

Thus Proposition 3 implies that all minimizers are non-orientable if and only if (69) holds. Relations (72) and (73) together with the assumption that g is orientable, hence $\deg(A(g), \partial\Omega)$ is even means that we always have $\mathrm{dist}(\mathrm{J(g)}^1, 2\mathbb{Z}) = \mathrm{dist}(\mathrm{J(g)}^2, 2\mathbb{Z})$ and $\mathrm{dist}(\mathrm{J(g)}^1, 2\mathbb{Z}+1) = \mathrm{dist}(\mathrm{J(g)}^2, 2\mathbb{Z}+1)$.

The claimed criteria are now a consequence of Remark 7.

(ii) We claim that the symmetry of the domain and that of the boundary data imply that $h(g)(x_1, x_2) = h(g)(x_1, -x_2)$. Indeed, let $\tilde{h}(g)(x_1, x_2) \stackrel{\text{def}}{=} h(g)(x_1, -x_2)$.

One can check that

$$A(g) \times \frac{\partial A(g)}{\partial \tau}(x_1, x_2) = \begin{cases} 2 & \text{if } (x_1, x_2) \in \partial(\cup_{i=1}^3 M_i),\ |x_2| \geq \delta \\ 0 & \text{if } (x_1, x_2) \in \partial(\cup_{i=1}^3 M_i),\ |x_2| < \delta \end{cases}$$

and thus the boundary data is symmetric with respect to the $x_2 = 0$ axis. Then $h(g)$ and $\tilde{h}(g)$ are both functions that solve problem (58), that has a unique solution. Thus $h(g) = \tilde{h}(g)$ and our claim is proved.

Let us observe that the line-field example in the proof of Lemma 12 shows that $W_g^{1,2} \neq \emptyset$. Taking this into account, together with $g \in W^{1,2}(\partial\Omega)$, we can use the first part of the Proposition (in our case the domain in only C^1 and not smooth but one can check that C^1 regularity suffices for using the Theorem 3 and thus the first part of the Proposition). The symmetry of $h(g)$ implies that $J(g)^1 = J(g)^2$ and relations (72), (73) together with $\deg(A(g), \partial(M_1 \cup M_2 \cup M_3)) = 2$ imply $J(g)^1 = J(g)^2 = -1$. Hence the criterion (69) holds for any $\delta > 1$.

(iii) Let us first observe that since we took the boundary data to be smooth the function space $W_g^{1,2}(G_\delta, \boldsymbol{\mathcal{Q}}_2)$ is non-empty (see Remark 8). Let $\Xi_\delta \stackrel{\text{def}}{=} \{x = (x_1, x_2) \in \mathbb{R}^2 : x_1^2 + (x_2 - \frac{3}{4})^2 < \delta^2\}$ which we regard as the hole ω_1 and denote by $h^\delta(g)$ the solution of (58) for the domain G_δ. We claim that

$$\int_{\partial \Xi_{\frac{1}{8}}} \frac{\partial h^\delta(g)}{\partial \nu}\, d\sigma \to 0 \ \text{ as } \delta \to 0. \tag{75}$$

On the other hand the divergence theorem shows that (for $\delta < \frac{1}{8}$):

$$0 = \int_{\Xi_{\frac{1}{8}} \setminus \Xi_\delta} \Delta h^\delta(g)\, dx = \int_{\partial \Xi_{\frac{1}{8}}} \frac{\partial h^\delta(g)}{\partial \nu}\, d\sigma - \underbrace{\int_{\partial \Xi_\delta} \frac{\partial h^\delta(g)}{\partial \nu}\, d\sigma}_{=2\pi \cdot J(g)^1}.$$

The last relation, together with our previous claim and the criterion (70) finish the proof.

It remains to prove our claim (75) and to this end let us consider the solution H of the pure Neumann problem:

$$\begin{cases} \Delta H = 0 & \text{on } B_1(0) \setminus B_{1/2}(0) \\ \frac{\partial H}{\partial \nu} = A(g) \times \frac{\partial A(g)}{\partial \tau} & \text{on } \partial B_1(0) \\ \frac{\partial H}{\partial \nu}(\frac{1}{2}, \theta) = -2\Big(A(g) \times \frac{\partial A(g)}{\partial \tau}\Big)(1, \theta) & \text{for } \theta \in [0, 2\pi] \\ \int_{B_1(0) \setminus B_{\frac{1}{2}}(0)} H(x)\, dx = 0 \end{cases} \tag{76}$$

where on the third line of the system above we used polar coordinates $(r, \theta) \in [\frac{1}{2}, 1] \times [0, 2\pi]$.

The solution is smooth on $B_1(0) \setminus B_{1/2}(0)$ and continuous on the closure, [27], thus there exist $c_1, c_2 > 0$ so that $c_2 > H(x) > -c_1$ for all $x \in \overline{B_1(0) \setminus B_{1/2}(0)}$. Let $w(x) \stackrel{\text{def}}{=} h^\delta(g)(x) - c_1 - H(x)$. Then $\Delta w = 0$ on the set G_δ. As $\frac{\partial w}{\partial \nu} = 0$ on $\partial B_1(0)$ Hopf's lemma shows that w cannot attain its maximum on $\partial B_1(0)$. Hence, by the maximum principle, it attains its maximum on $\partial G_\delta \setminus \partial B_1(0)$ where, by our construction, $w \le 0$. Thus $w \le 0$ on $\overline{G_\delta}$ and hence $h^\delta(g) \le c_1 + H$ on $\overline{G_\delta} \subset \overline{B_1(0) \setminus B_{1/2}(0)}$. Similarily, taking the function $v = H - c_2 - h^\delta(g)$ and reasoning analogously we obtain that $H - c_2 \le h^\delta(g)$ on $\overline{G_\delta} \subset \overline{B_1(0) \setminus B_{1/2}(0)}$. Thus

$$H - c_2 \le h^\delta(g) \le c_1 + H, \quad \text{on } \overline{G_\delta} \subset \overline{B_1(0) \setminus B_{1/2}(0)} \tag{77}$$

and since the sequence of harmonic functions $h^\delta(g)$ is uniformly bounded on a sequence of domains shrinking into the annulus $B_1(0) \setminus B_{1/2}(0)$, we obtain [17], p.23, by taking a diagonal sequence, that there exists a function f so that $h^{\delta_j}(g)$ converges uniformly on compact subsets of $\Big(B_1(0) \setminus B_{1/2}(0)\Big) \setminus \{(0, \frac{3}{4})\}$ to f. Then f is harmonic [17], p.23 on $\Big(B_1(0) \setminus B_{1/2}(0)\Big) \setminus \{(0, \frac{3}{4})\}$ and bounded (by (77)) and hence it has a removable singularity at $(0, \frac{3}{4})$ and is harmonic on $B_1(0) \setminus B_{1/2}(0)$. Then $\int_{\partial \Xi_{\frac{1}{8}}} \frac{\partial f}{\partial \nu}\, d\sigma = \int_{\Xi_{\frac{1}{8}}} \Delta f(x)\, dx = 0$ and since $h(g)^{(\delta)}$ converges uniformly (and thus in C^∞ since $h(g)^{(\delta)}$ are harmonic) on compact sets to f we obtain the claimed relation (75).

(iv) Let g correspond to tangential boundary conditions on the outer boundary of N_ρ. The example constructed in Lemma 12 can be easily modified to show that $W^{1,2}_g(N_\rho) \neq \emptyset$, for all $\rho \in (0,1)$. We denote by $S \overset{\text{def}}{=} \cup_{i=1}^3 M_i$ the stadium without holes.

Let $\tilde{H}^1_0(N_\rho) \overset{\text{def}}{=} \{u \in H^1(N_\rho); \operatorname{Tr} u\Big|_{\partial M_5 \cup \partial M_6^\rho} = 0\}$. Let $\varphi \in C^\infty(N_\rho) \cap H^1_0(N_\rho)$ be a function vanishing in a neighbourhood of $\partial M_5 \cup \partial M_6^\rho$ and denote by $\tilde{\varphi}$ its extension by zero to a function on S. Denoting by $h(g)^\rho \in \tilde{H}^1_0(N_\rho)$ the solution of problem (58) on N_ρ, we have:

$$\int_{N_\rho} \nabla h(g)^\rho \cdot \nabla \varphi\, dx = \int_{\partial N_\rho} \frac{\partial h(g)^\rho}{\partial \nu} \varphi\, d\sigma = \int_{\partial S} A(g) \times \frac{\partial A(g)}{\partial \tau} \varphi\, d\sigma \tag{78}$$

and then

$$\begin{aligned} &\left|\int_{N_\rho} \nabla h(g)^\rho \cdot \nabla \varphi\, dx\right| = \left|\int_{\partial S} \frac{\partial h(g)^\rho}{\partial \nu} \varphi\, d\sigma\right| \\ &\le \|\frac{\partial h(g)^\rho}{\partial \nu}\|_{H^{-1/2}(\partial S)} \|\varphi\|_{H^{1/2}(\partial S)} \le C_1 \|\frac{\partial h(g)^\rho}{\partial \nu}\|_{H^{-1/2}(\partial S)} \|\tilde{\varphi}\|_{H^1(S)}. \end{aligned}$$

In the last inequality we can assume without loss of generality that C_1 is a constant independent of ρ, because $\varphi = \tilde{\varphi}$ on ∂S and the last inequality expresses the continuity of the trace operator in $H^1(S)$.

We denote by $\tilde{h}(g)^\rho$ the extension by zero of $h(g)^\rho$ to a function on S and then the last inequality implies

$$\left|\int_S \nabla \tilde{h}(g)^\rho \cdot \nabla \tilde{\varphi}\, dx\right| \le C_1 \|\frac{\partial h(g)^\rho}{\partial \nu}\|_{H^{-1/2}(\partial S)} \|\tilde{\varphi}\|_{H^1(S)}. \tag{79}$$

Replacing $\tilde{\varphi}$ in the inequality by $\tilde{\varphi}_k$ with $\tilde{\varphi}_k \to \tilde{h}(g)^\rho$ in $H^1(S)$ as $k \to \infty$ the last inequality implies:

$$\|\nabla \tilde{h}(g)^\rho\|^2_{L^2(S)} \leq C_1 \|A(g) \times \frac{\partial A(g)}{\partial \tau}\|_{H^{-1/2}(\partial S)} \|\tilde{h}(g)^\rho\|_{H^1(S)}. \tag{80}$$

We denote $J(g)^2_\rho \stackrel{\text{def}}{=} \frac{1}{2\pi}\int_{\partial M_5} \frac{\partial h(g)^\rho}{\partial \nu}\, ds$ and claim that

$$\rho \mapsto J(g)^2_\rho,\ (\rho \in (0, \frac{1}{2}]),\ \text{is a continuous function.} \tag{81}$$

In order to prove the claim we argue by contradiction and assume that there exists a $\bar{\rho} \in (0, \frac{1}{2}], \varepsilon_0 > 0$ and a sequence ρ_k with $\rho_k \to \bar{\rho}$ and

$$|J(g)^2_{\rho_k} - J(g)^2_{\bar{\rho}}| > \varepsilon_0,\ \text{for all } k. \tag{82}$$

First let us observe that for $\rho \in (\frac{\bar{\rho}}{2}, \frac{1}{2})$ the functions $\tilde{h}(g)^\rho$ are zero on a common set of non-zero measure $M_6^{\frac{\bar{\rho}}{2}}$. Then one has a Poincaré inequality:

$$\|\tilde{h}(g)^\rho\|_{L^2(S)} \leq C_2 \|\nabla \tilde{h}(g)^\rho\|_{L^2(S)} \tag{83}$$

for $\rho \in (\frac{\bar{\rho}}{2}, \frac{1}{2})$, with C_2 depending on S and $\bar{\rho}$ (see the argument in [38], p. 177 that can be checked to hold even for $p = n = 2$).

Relations (80) and (83) imply that there exists a subsequence of ρ_k, relabelled as the initial sequence, such that

$$\tilde{h}(g)^{\rho_k} \rightharpoonup L \ \text{ in } H^1(S) \tag{84}$$

for some function $L \in H^1(S)$. We claim that $L \equiv \tilde{h}(g)^{\bar{\rho}}$. To this end it suffices to show that for any smooth $\hat{\varphi} \in H^1(N_{\bar{\rho}})$ vanishing in a neighbourhood of $M_6^{\bar{\rho}} \cup M_5$ we have:

$$\int_S \nabla L \cdot \nabla \hat{\varphi}\, dx = \int_{\partial S} A(g) \times \frac{\partial A(g)}{\partial \tau} \hat{\varphi}\, d\sigma \tag{85}$$

which, by the uniqueness of the weak solution for the problem (58), implies the claim.

In order to prove the last equality let us take a sequence $\varphi^k \in H^1(S) \cap C^\infty(S)$, s upported in N_{ρ_k} and vanishing near $\partial M_6^{\rho_k} \cup \partial M_5$, so that $\varphi^k \to \hat{\varphi}$ in $H^1(S)$. Then replacing N_ρ by S, $h(g)^\rho$ by $\tilde{h}(g)^{\rho_k}$, φ by φ^k in (78) and passing to the limit $k \to \infty$ by using (84), we obtain relation (85).

Using that $\nabla h(g)^\rho \in H(W, \text{div})$ with $W \subset S \setminus (M_5 \cup M_6^{2\bar{\rho}})$ an open set such that $\partial M_5 \subset \partial W$, relation (84) and the continuity of the normal part of the trace in the space $H(W, \text{div})$ (see [27]) we have that $\frac{\partial h(g)^{\rho_k}}{\partial \nu} \rightharpoonup \frac{\partial h(g)^{\bar{\rho}}}{\partial \nu}$ in $H^{-\frac{1}{2}}(\partial M_5)$. On the other hand we can write $J(g)^2_{\rho_k} = \frac{1}{2\pi}\int_{\partial M_5} \frac{\partial h(g)^{\rho_k}}{\partial \nu}\, d\sigma = \frac{1}{2\pi}\int_{\partial M_5} \langle \frac{\partial h(g)^{\rho_k}}{\partial \nu}, 1\rangle\, d\sigma$, with $\langle \cdot, \cdot \rangle$ denoting the duality between $H^{-\frac{1}{2}}(\partial M_5)$ and $H^{\frac{1}{2}}(\partial M_5)$, and the previously proved weak convergence implies $J(g)^2_{\rho_k} \to$

$J(g)^2_{\bar\rho}$ thus contradicting (82). The contradiction we have reached proves our earlier claim (81).

The argument in part (*iii*) can be easily adapted to show that $J(g)^2_\rho \to 0$ as $\rho \to 0$ and the argument in part (*ii*) shows that $J(g)^2_{\frac{1}{2}} = -1$. Thus (81) shows there exists some $\rho_0 \in (0, \frac{1}{2})$ so that $J(g)^2_{\rho_0} = -\frac{1}{2}$. Then the criterion (71) shows that on N_{ρ_0} there exist both an orientable and a non-orientable energy minimizer. □

Remark 9. Part (ii) of Proposition 8 shows that in Lemma 12, one does not in fact need the assumption that δ is large enough.

Remark 10. The proof of Part (iv) of Proposition 8 can be easily adapted to show that for any domain with two holes, if one shrinks enough one of the holes (while keeping the other hole and the orientable boundary data unchanged) then the global energy minimizer will necessarily be orientable.

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Appendix A. The relation between sets of class C^k (Lipschitz) and C^k (Lipschitz) manifolds with boundary

We recall (see [30], p.12, [24], pp. 29-30) that a subset $X \subset \mathbb{R}^m$ is an m-dimensional C^k(Lipschitz) manifold with boundary, for $k \geq 0$, if each $x \in X$ has a neighbourhood U in $\mathbb{R}^m$ such that $U \cap X$ is C^k-diffeomorphic (homeomorphic for $k = 0$, or bi-Lipschitz homeomorphic in the case of Lipschitz manifolds) with a set $V \cap H^m$ in H^m, where V is an open set in $\mathbb{R}^m$ and

$$H^m = \{(x_1, \dots, x_m) \in \mathbb{R}^m \,|x_m \geq 0\}.$$

The *boundary* of the manifold X is the set of all points in X which correspond to points of $\partial H \stackrel{\text{def}}{=} \mathbb{R}^{m-1} \times \{0\} \subset \mathbb{R}^m$ under such a diffeomorphism. It can be shown that the condition for a point in X to be on the boundary is independent of the chart chosen (see [24]).

In order to show that a domain Ω of class C^k (Lipschitz) is a manifold with boundary and the topological boundary coincides with the boundary as a manifold it suffices to show that, for $P \in \partial\Omega$ and $\delta > 0$ such that $B_\delta(P) \cap \partial\Omega$ can be represented as the graph of a function, there exists a C^k-diffeomorphism F from $B_\delta(P) \cap \overline{\Omega}$ to $V \cap H^m$ which carries $B_\delta(P) \cap \partial\Omega$ into $\partial H^m \cap V$.

To this end we let $f : \mathbb{R}^{m-1} \times \{0\} \subset \mathbb{R}^m \to \mathbb{R}$ be such that

$$B_\delta(P) \cap \Omega = \{y = (y', y_m) \in \mathbb{R}^m : \, y_m > f(y', 0), \, |y'| < \delta\}$$

where f is of class C^k. Then it can be easily checked that $F : B_{\frac{\delta}{2}}(P) \cap \overline{\Omega} \to V \cap H^m$ defined by $F(y', y_m) = (y', y_m - f(y', 0))$ is injective, onto the image (we take V so that F is onto) and a C^k-diffeomorphism (respectively bi-Lipschitz) with inverse $F^{-1}(y', y_m) = (y', y_m + f(y', 0))$. Moreover F carries $B_\delta(P) \cap \partial\Omega$ bijectively into $V \cap \partial H_m$.

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