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Orientifolds and Mirror Symmetry

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Abstract

We study parity symmetries and crosscap states in classes of $\mathcal{N}=2$ supersymmetric quantum field theories in $1+1$ dimensions, including non-linear sigma models, gauged WZW models, Landau-Ginzburg models, and linear sigma models. The parity anomaly and its cancellation play important roles in many of them. The case of the $\mathcal{N}=2$ minimal model are studied in complete detail, from all three realizations — gauged WZW model, abstract RCFT, and LG models. We also identify mirror pairs of orientifolds, extending the correspondence between symplectic geometry and algebraic geometry by including unorientable worldsheets. Through the analysis in various models and comparison in the overlapping regimes, we obtain a global picture of orientifolds and D-branes.

Contents

1 Introduction

String compactifications with $\mathcal{N} = 1$ supersymmetry in $3 + 1$ dimensions are theoretically very interesting and are believed to be important for real world physics. There are several approaches to the constructions of models, starting with Heterotic strings on Calabi–Yau manifolds [1]. The approach that has been attracting more recent attention is to consider Type II strings involving D-branes, which fill out the $(3 + 1)$-dimensional world. In such constructions, orientifolds [2–6], are indispensable elements in order to have consistent theories with finite Newton's constant and supersymmetry. Despite this importance, orientifolds are less studied compared to D-branes which have been investigated extensively in recent years. In particular, it is not well understood what kinds of orientifolds are possible in which kinds of models.

In this paper, we systematically study parity symmetries of $(2, 2)$ theories in $1 + 1$ dimensions commuting with one half of the worldsheet $(2, 2)$ supersymmetry, which are relevant for the construction of supersymmetric orientifolds. We particularly study general properties of parity symmetries and the associated crosscap states, such as the Witten index twisted by parity symmetry and the dependence of certain $\mathbb{RP}^2$ diagrams on the parameters of the theory. Our emphasis is on supersymmetry rather than superconformal invariance, and we do not limit ourselves to conformal field theories. This attitude allows us to treat a broader class of models and has proved to be useful in various other contexts. The general story is examined and illustrated in several important classes of theories including the non-linear sigma models on Kähler manifolds, gauged Wess–Zumino–Witten (WZW) models, Landau–Ginzburg (LG) models, and linear sigma models. All these models are related in one way or another and understanding relations between parities in these models will be very important.

As the primary example, we perform a complete study of parity symmetries and crosscap states in the $\mathcal{N} = 2$ minimal model. The minimal model [7, 8] is the simplest non-trivial theory with $(2, 2)$ superconformal invariance [9]; it has been playing a central role in the study of supersymmetric string compactification. In particular, it can be used as the building block of the Gepner model of critical supersymmetric string theory in $3+1$ dimensions [10, 11]. The model is realized in three different ways:

(i) as an abstract RCFT using modular matrices, S, T [10, 12, 13] and P,

(ii) as the $SU(2)$ mod $U(1)$ supersymmetric gauged WZW model [14, 15],

(iii) as the IR limit of the LG model with superpotential $W = \Phi^{k+2}$ [16–18].

D-branes in the minimal model are studied in these realizations in [19, 20],[21],[22, 23] respectively. Exact results on the crosscap states are obtained in the realization (i), following the general RCFT procedure [24–30]. However, the information obtained in this way is about the theory with a particular GSO projection, in which the $\mathcal{N} = 2$ supersymmetry is not manifest. The essential task required here is to entangle the GSO projection and obtain the information on the parities and crosscap states of the *full* $\mathcal{N} = 2$ theory before the GSO projection. This is done as one of the important achievements of the paper. The results are in complete agreement with the results from approach (ii) and (iii) whenever available.

We also study parity symmetries of linear sigma models. These are simple gauge theories that flow under renormalization group to the models of interest [31]. In many important cases, they are defined on the whole moduli space of theories, which interpolates the Gepner models and large volume Calabi–Yau sigma model, and provide a good understanding of the singularity of the worldsheet theory. Moreover they can be used to derive mirror symmetry [32]. Thus, by understanding parity symmetries of linear sigma models, one can first of all argue on the existence or absence of orientifolds on the moduli space, one can provide a relation between the Gepner model orientifold (that is obtained as the application of the orientifolds of $\mathcal{N} = 2$ minimal models) and the orientifolds of large volume sigma model, and one can find the mirror correspondence of orientifold models. D-branes are studied in the context of linear sigma models in [22, 33–41].

As for any other symmetry, one needs to check if the parity symmetry of the classical system is maintained in the quantum theory. In many of the examples studied in this paper, we do encounter anomalies of classical parity symmetries. They are anomalous because the path-integral measure is not invariant: in certain topologically non-trivial backgrounds, there is an odd number of fermion zero mode pairs that are exchanged under some of the parities. The anomaly can be cancelled by combining it with another anomalous symmetry. One possibility is to use $(-1)^{F_L}$ that flips the sign of the left-moving fermions. This works when the theory is conformal and is indeed applied in the $\mathcal{N} = 2$ minimal model in this paper. Another way is to turn on a B-field. We recall that the B-field term $\int_\Sigma \phi^* B$ flips its sign under the orientation reversal of Σ, and for this reason it can generate or cancel phase factors in the parity transformation of the path-integral measure.

We also present a number of new observations in this paper. For example, in specifying a parity of non-linear sigma models, in addition to the action τ on the target space X, one must specify its action on a complex line bundle on X whose first Chern class is $(\tau^*[B] + [B])/2\pi$. We also show (with the help of M. Kapranov and Y.-G. Oh) that the deformation theory of holomorphic Calabi-Yau orientifolds is *not* obstructed, namely, the classical moduli space of holomorphic orientifolds is smooth. This is in contrast to the case of holomorphic D-branes whose deformation *is* obstructed in general [20, 42].

This paper is organized as follows.

In Section 2, we describe general features of parity symmetries and crosscap states of theories with $\mathcal{N} = (2, 2)$ supersymmetry. In particular, we consider parities commuting with half of the $(2, 2)$ supersymmetry. As in the case with boundary conditions [43, 22], we will find essentially two types of parities and call them A-parities and B-parities, following [43]. We show that the overlap of the crosscap and the supersymmetric ground states obey certain differential equations with respect to the parameters of the theory. We also find the relation of these overlaps and the parity-twisted Witten indices, which will be called bilinear identities.

This general story is illustrated in Section 3, in the examples of non-linear sigma models with Kähler target spaces. A-parities are associated with anti-symplectic isometries, while B-parities correspond to holomorphic isometries. We determine the conditions on the complex structure and complexified Kähler parameters for the parity symmetry. We also compute the parity-twisted Witten index using supersymmetric localization applied to path-integrals and interpret the result from the canonical formalism. For A-type parities and branes, the index is the self-intersection number of the orientifold plane (O-plane) for closed string, while it is the intersection number of the O-plane and D-brane for open string. The overlaps with the RR ground states are period integrals, and the bilinear identity is nothing but the classical Riemann bilinear identity. For B-type objects, the index is the $\mathbb{Z}_2$-signature for closed string while it is the holomorphic Lefschetz number for the open string. The path-integral computation reproduces the $\mathbb{Z}_2$-signature theorem and the Lefschetz fixed-point theorem.

In Sections 4 and 5, we consider $\mathcal{N} = 2$ minimal model. We introduce the model as the gauged WZW model (realization (ii)) which can be regarded as the sigma model on the unit disk. We find A-parities that act on the disk as complex conjugation, folding along diameters, and B-parities that act as rotation around the center. We compute the parity-twisted partition functions (Klein bottle amplitudes). We next consider a non-chiral GSO projection that leads to the realization (i), and determine the crosscap states following the general procedure [25, ?, ?, 26–30]. (A part of the computation given here was also done in [44], and some earlier results in the context of Gepner models have already been obtained in [45, 46]) In the final subsection of Section 4, we entangle the GSO projection and determine the crosscap states of the original $\mathcal{N} = 2$ minimal model. We compute the Klein bottle amplitudes, including parity-twisted Witten index for the closed string, and the result matches with the one from the gauged WZW computation. Section 5 is devoted to the study of parity actions on the D-branes and stretched open strings. The D-branes we consider are A-branes (straight segments in the disk) and B-branes (concentric disks). The geometric picture allows us to read off how the A- and B-parities act on them, which is confirmed by the Möbius strip amplitudes. We also compute the parity-twisted open string Witten index, after entangling the GSO projection for the boundary states.

In Section 6, we consider Landau–Ginzburg models. We find that A-parities are antiholomorphic maps of the LG fields such that the superpotential is complex conjugated, and B-parities are holomorphic maps that reverse the sign of the superpotential. We show that the overlaps of the crosscap states and the RR ground states are given by a weighted period integral on the suitably modified orientifold planes, and the parity-twisted Witten indices are intersection numbers of suitably modified branes and O-planes. This general result is applied to the particular example of the LG model of a single field Φ with superpotential $W = \Phi^{k+2}$, which flows in the IR limit to the $\mathcal{N} = 2$ minimal model (realization (iii)). We compute the closed and open string parity-twisted Witten index as well as the overlaps with the RR ground states. The results are in complete agreement with the results from Sections 4 and 5.

In Sections 7 and 8, we study parity symmetries of linear sigma models. We determine the conditions on the parameters for the theory to be invariant under A-type and B-type parities. These conditions match the ones derived from the non-linear sigma model in the large volume limit, and the ones coming from the LG model at the Gepner point. We also determine the corresponding parity in the mirror Landau–Ginzburg model. In particular, we find that the information on the parity actions on the line bundle $\mathcal{L}_{\tau^* B+B}$ mentioned above has a natural counterpart in the mirror LG model in terms of the type of the orientifold planes. The results are applied to several specific examples where we find highly non-trivial agreement of the mirror models. In Section 8, we discuss orientifolds of the system including compact Calabi–Yau sigma models in its moduli space. We classify the possible orientifolds of the quintic hypersurface in $\mathbb{CP}^4$, at least those present for the Fermat type quintic. We find six of them, three A-type and three B-type. Using the linear sigma model, we identify the mirror parities in the mirror quintic. We also discuss issues concerning the spacetime physics of Type II orientifolds on Calabi–Yau manifolds. We count the number of light chiral multiplets and vector multiplets from the closed string. We also discuss spacetime superpotential. Especially, we argue that the moduli space of holomorphic orientifolds is smooth. This last section is a preparation for a more complete analysis of the full orientifold models of string compactification, which is now possible to do as an application of the present paper.

2 Parities and Crosscaps in $\mathcal{N}=2$ Theories

In this section, we describe general features of parity symmetry of theories with $\mathcal{N}=(2,2)$ supersymmetry (not necessarily with conformal invariance). In particular, we consider parities that preserve half of the $(2,2)$ supersymmetry. As in the case with boundaries, we will find essentially two types of parities, A-type and B-type. We will also study and describe the properties of the corresponding crosscap states. Especially, we show that the overlap of the crosscap and the supersymmetric ground states obey certain differential equations with respect to the parameters of the theory. We also find the relation of these overlaps and the parity-twisted Witten indices.

2.1 A-parity and B-parity

2.1.1 Parity of the (2,2) superspace

Let us first classify parities of the $(2,2)$ superspace in $1+1$ dimensions. The superspace has two bosonic coordinates x^0, x^1 (or $x^\pm = x^0 \pm x^1$) and four fermionic coordinates $\theta^\pm, \overline{\theta}^\pm = (\theta^\pm)^\dagger$. By definition, parity reverses the orientation of the space coordinates $x^1 \to -x^1+$ constant, and therefore exchanges the chirality. Let us consider the ones maintaining the holomorphy of $\mathcal{N}=2$ supersymmetry:

$$\Omega_A:\ (x^\pm,\theta^+,\theta^-,\overline{\theta}^+,\overline{\theta}^-)\longmapsto(x^\mp,-\overline{\theta}^-,-\overline{\theta}^+,-\theta^-,-\theta^+),$$
$$\Omega_B:\ (x^\pm,\theta^+,\theta^-,\overline{\theta}^+,\overline{\theta}^-)\longmapsto(x^\mp,\theta^-,\theta^+,\overline{\theta}^-,\overline{\theta}^+).$$

We shall call the former *A-parity*, and the latter *B-parity*. The supersymmetry generators $Q_\pm = \partial/\partial\theta^\pm + i\overline{\theta}^\pm\partial/\partial x^\pm$, $\overline{Q}_\pm = -\partial/\partial\overline{\theta}^\pm - i\theta^\pm\partial/\partial x^\pm$ are then transformed as

$$A:\quad Q_\pm \longrightarrow \overline{Q}_\mp,\quad \overline{Q}_\pm \longrightarrow Q_\mp, \tag{2.1}$$
$$B:\quad Q_\pm \longrightarrow Q_\mp,\quad \overline{Q}_\pm \longrightarrow \overline{Q}_\mp. \tag{2.2}$$

The vector and axial R-rotations, $U(1)_V$ and $U(1)_A$, are also transformed: A-parity reverses $U(1)_V$ and preserves $U(1)_A$ while B-parity preserves $U(1)_V$ and reverses $U(1)_A$. The differential operators $D_\pm = \partial/\partial\theta^\pm - i\overline{\theta}^\pm\partial/\partial x^\pm$, $\overline{D}_\pm = -\partial/\partial\overline{\theta}^\pm + i\theta^\pm\partial/\partial x^\pm$ are transformed as $A: D_\pm \leftrightarrow \overline{D}_\mp$ and $B: D_+ \leftrightarrow D_-$, $\overline{D}_+ \leftrightarrow \overline{D}_-$. Accordingly, chiral and twisted chiral superfields are mapped by A-parity and B-parity as

$$A:\quad \begin{array}{rcl} \text{chiral} & \longleftrightarrow & \text{antichiral} \\ \text{twisted chiral} & \longleftrightarrow & \text{twisted chiral,} \end{array}$$

$$B:\quad \begin{array}{rcl} \text{chiral} & \longleftrightarrow & \text{chiral} \\ \text{twisted chiral} & \longleftrightarrow & \text{twisted antichiral.} \end{array}$$

One can modify the above parities by the $U(1)$ R-rotations, $A_{\alpha,\beta} : \theta^{\pm} \mapsto -\,\mathrm{e}^{i\alpha\mp i\beta}\overline{\theta}^{\mp}$, and $B_{\alpha,\beta} : \theta^{\pm} \mapsto \mathrm{e}^{-i\alpha\pm i\beta}\theta^{\mp}$. They transform the supercharges as

$$A_{\alpha,\beta}: \quad Q_{\pm} \to \mathrm{e}^{-i\alpha\mp i\beta}\overline{Q}_{\mp},\ \ \overline{Q}_{\pm} \to \mathrm{e}^{i\alpha\pm i\beta}Q_{\mp}, \tag{2.3}$$

$$B_{\alpha,\beta}: \quad Q_{\pm} \to \mathrm{e}^{-i\alpha\mp i\beta}Q_{\mp},\ \ \overline{Q}_{\pm} \to \mathrm{e}^{i\alpha\pm i\beta}\overline{Q}_{\mp}. \tag{2.4}$$

The fixed-point set of Ω_A and Ω_B is $x^1 = 0$, $\theta^+ + \overline{\theta}^- = 0$, $\overline{\theta}^+ + \theta^- = 0$ and $x^1 = 0$, $\theta^+ = \theta^-$, $\overline{\theta}^+ = \overline{\theta}^-$, respectively. These are nothing but the A-boundary and B-boundary that are relevant to the superfield description of boundary $\mathcal{N} = 2$ theories [47, 35]. One can also consider parity actions on boundary superspace whose bosonic subspace is the strip $0 \le x^1 \le \pi$ preserved by $x^1 \leftrightarrow \pi - x^1$. One can consider A-boundaries $\theta^+ + \overline{\theta}^- = \overline{\theta}^+ + \theta^- = 0$ or B-boundaries $\theta^+ - \theta^- = \overline{\theta}^+ - \overline{\theta}^- = 0$ at $x^1 = 0$ and π. Under both A-parity and B-parity, an A(B)-boundary at $x^1 = 0$ is mapped to an A(B)-boundary at $x^1 = \pi$ and vice versa. Chiral superfields on an A-boundary at $x^1 = 0$ are mapped by A-parity (B-parity) to chiral (antichiral) superfields on an A-boundary at $x^1 = \pi$.

2.1.2 A-parity and B-parity in $(2, 2)$ theories

In any quantum field theory in $1 + 1$ dimensions, a parity symmetry takes the form $\tau \circ \Omega$, where Ω is the space inversion $x = (x^0, x^1) \to \widetilde{x} = (x^0, -x^1)$ and τ is an internal action of the fields. In general, only the combination $\tau \circ \Omega$ is a symmetry, not τ and Ω individually. The parity symmetry is realized as an operator P on the Hilbert space of states that commutes with the Hamiltonian but inverts the momentum. In a $(2, 2)$ supersymmetric theory, an A-parity and a B-parity take the form $P_A = \mathscr{T}_A \circ \Omega_A$ and $P_B = \mathscr{T}_B \circ \Omega_B$ respectively, where $\mathscr{T}_{A,B}$ are internal actions of the superfields. They are realized as operators on the Hilbert space that transform the supercharges as (2.1) and (2.2) respectively.

In particular, they transform the supercurrents $G^{\mu}_{\pm}$, $\overline{G}^{\mu}_{\pm}$ ($\mu = 0, 1$) as

$$P_A : G^{\mu}_{\pm}(x) \to (-1)^{\mu}\overline{G}^{\mu}_{\mp}(\widetilde{x}),\ \ \overline{G}^{\mu}_{\pm}(x) \to (-1)^{\mu}G^{\mu}_{\mp}(\widetilde{x}), \tag{2.5}$$

$$P_B : G^{\mu}_{\pm}(x) \to (-1)^{\mu}G^{\mu}_{\mp}(\widetilde{x}),\ \ \overline{G}^{\mu}_{\pm}(x) \to (-1)^{\mu}\overline{G}^{\mu}_{\mp}(\widetilde{x}). \tag{2.6}$$

If the system has vector and/or axial R-symmetry, and if the parity respects them, the R-currents are transformed as

$$P_A : J^{\mu}_{V}(x) \to -(-1)^{\mu}J^{\mu}_{V}(\widetilde{x}),\ \ J^{\mu}_{A}(x) \to (-1)^{\mu}J^{\mu}_{A}(\widetilde{x}), \tag{2.7}$$

$$P_B : J^{\mu}_{V}(x) \to (-1)^{\mu}J^{\mu}_{V}(\widetilde{x}),\ \ J^{\mu}_{A}(x) \to -(-1)^{\mu}J^{\mu}_{A}(\widetilde{x}). \tag{2.8}$$

For each A-parity P_A we obtain an $A_{\alpha,\beta}$-parity by combining it with the R-symmetry: $P_{A_{\alpha,\beta}} = \mathrm{e}^{-i\alpha F_V - i\beta F_A} P_A$. Similarly, a $B_{\alpha,\beta}$-parity can be obtained: $P_{B_{\alpha,\beta}} = \mathrm{e}^{-i\alpha F_V - i\beta F_A} P_B$. (We define the transformation of operators by a symmetry U by $\mathcal{O} \to U^{-1}\mathcal{O}U$.)

An A-parity in one theory is mapped to a B-parity of the mirror, since mirror symmetry exchanges Q_- and $\overline{Q}_-$. Also, mirror symmetry exchanges $A_{\alpha,\beta}$ and $B_{\beta,\alpha}$.

2.1.3 Parity actions on chiral superfields

For example, let us consider the theory of a single chiral superfield $\Phi(x,\theta) = \Phi(x^\pm, \theta^\pm, \overline{\theta}^\pm)$ with the Lagrangian

$$L = \int \mathrm{d}^4\theta \; \overline{\Phi}\Phi.$$

Since the measure $\mathrm{d}^4\theta = \mathrm{d}\theta^+ \mathrm{d}\theta^- \mathrm{d}\overline{\theta}^- \mathrm{d}\overline{\theta}^+$ is invariant under both Ω_A and Ω_B, the Lagrangian is invariant under

$$A: \; \Phi(x,\theta) \longrightarrow \overline{\Omega_A^* \Phi(x,\theta)} = \overline{\Phi(\Omega_A(x,\theta))}, \qquad (2.9)$$

$$B: \; \Phi(x,\theta) \longrightarrow \Omega_B^* \Phi(x,\theta) = \Phi(\Omega_B(x,\theta)). \qquad (2.10)$$

The right hand sides are both chiral superfields: $\Omega_B^*\Phi$ is chiral as we have seen, while $\Omega_A^*\Phi$ is antichiral and therefore its hermitian conjugate $\overline{\Omega_A^*\Phi}$ is chiral. In this way they determine consistent transformations of the field, $\Phi \to P_A^{-1}\Phi P_A$ and $\Phi \to P_B^{-1}\Phi P_B$. They realize an A-parity and a B-parity, $P_A^{-1} Q_\pm P_A = \overline{Q}_\mp$ and $P_B^{-1} Q_\pm P_B = Q_\mp$. For P_B this is because (2.2) says that $\Omega_B^*[Q_\pm, \Phi] = [Q_\mp, \Omega_B^*\Phi]$, which means that $P_B^{-1}[Q_\pm, \Phi] P_B = [Q_\mp, P_B^{-1}\Phi P_B]$. For P_A, (2.1) says $\Omega_A^*[\overline{Q}_\pm, \Phi] = [Q_\mp, \Omega_A^*\Phi]$ and its hermitian conjugate equation is $P_A^{-1}[Q_\pm, \Phi] P_A = [\overline{Q}_\mp, P_A^{-1}\Phi P_A]$. In terms of the component fields, $\Phi = \phi + \theta^+\psi_+ + \theta^-\psi_- + \theta^+\theta^- F + \cdots$, the actions are as follows:

$$P_A: \begin{cases} \phi(x) \longrightarrow \overline{\phi}(\widetilde{x}) \\ \psi_\pm(x) \longrightarrow \overline{\psi}_\mp(\widetilde{x}) \\ F(x) \longrightarrow \overline{F}(\widetilde{x}) \end{cases} \qquad (2.11)$$

$$P_B: \begin{cases} \phi(x) \longrightarrow \phi(\widetilde{x}) \\ \psi_\pm(x) \longrightarrow \psi_\mp(\widetilde{x}) \\ F(x) \longrightarrow -F(\widetilde{x}) \end{cases} \qquad (2.12)$$

The parity transformations (2.9) and (2.10) are essentially those used in more interesting systems described in terms of chiral superfields, such as non-linear sigma models (Section 3), Landau–Ginzburg models (Section 6) and linear sigma models (Sections 7 and 8). In the last example, we will also encounter parity actions on twisted chiral superfields.

2.1.4 Unbroken supersymmetry and the Witten index

Half of the (2, 2) supersymmetry is invariant under A-parity and B-parity. The invariant combinations are respectively

$$Q_A = \overline{Q}_+ + Q_-, \quad Q_A^\dagger = Q_+ + \overline{Q}_-, \tag{2.13}$$

$$Q_B = \overline{Q}_+ + \overline{Q}_-, \quad Q_B^\dagger = Q_+ + Q_-. \tag{2.14}$$

These are the same as the supercharges that are preserved by A-branes and B-branes [43, 22]. $Q = Q_A$ or Q_B obey

$$\{Q, Q^\dagger\} = 2H, \quad Q^2 = 0, \tag{2.15}$$

which are the relations of $\mathcal{N} = 2$ supersymmetric quantum mechanics. The symmetry generated by $Q_A, Q_A^\dagger$ and $Q_B, Q_B^\dagger$ shall be called $\mathcal{N} = 2_A$ and $\mathcal{N} = 2_B$ supersymmetries respectively.

One may consider the Witten index with a twist by $P = P_A$ or P_B

$$I_P = \operatorname*{Tr}_{\mathcal{H}_{\rm RR}}(P(-1)^F \,\mathrm{e}^{-\beta H}). \tag{2.16}$$

As a consequence of supersymmetry (2.15), it receives a contribution only from the ground states, and is invariant under supersymmetric deformations of the theory. In particular, it is independent of β and of the radius of the circle on which the system is quantized.

One can also consider the parity of an open string stretched between D-branes. Under both A-parity and B-parity, A(B)-branes are mapped to A(B)-branes. Furthermore an open string stretched between A(B)-branes preserves the $\mathcal{N} = 2$ supersymmetry generated by Q_A and $Q_A^\dagger$ (Q_B and $Q_B^\dagger$), which are invariant under an A(B)-parity. For an A(B)-brane a and its image Pa under A(B)-parity $P = P_A(P_B)$, one can also consider the Witten index

$$I_P(a, Pa) = \operatorname*{Tr}_{\mathcal{H}_{a,Pa}}(P(-1)^F \,\mathrm{e}^{-\beta H}), \tag{2.17}$$

where $\mathcal{H}_{a,Pa}$ is the space of states of the a-Pa string. It receives contributions only from the supersymmetric ground states and is a topological invariant of the open string system.

The modified versions $A_{\alpha,\beta}$ and $B_{\alpha,\beta}$ may or may not preserve half of the supersymmetry. $A_{\alpha,\beta}$-parity (resp. $B_{\alpha,\beta}$-parity) preserves an $\mathcal{N} = 2$ supersymmetry if and only if $\beta \in \pi\mathbb{Z}$ (resp. $\alpha \in \pi\mathbb{Z}$). The invariant combinations are

$$Q_{A_{\alpha,\beta}} = \overline{Q}_+ + \mathrm{e}^{i\alpha+i\beta} Q_-, \quad Q_{A_{\alpha,\beta}}^\dagger = Q_+ + \mathrm{e}^{-i\alpha-i\beta}\overline{Q}_-, \quad \beta \in \pi\mathbb{Z},$$

$$Q_{B_{\alpha,\beta}} = \overline{Q}_+ + \mathrm{e}^{i\alpha+i\beta}\overline{Q}_-, \quad Q_{B_{\alpha,\beta}}^\dagger = Q_+ + \mathrm{e}^{-i\alpha-i\beta} Q_-, \quad \alpha \in \pi\mathbb{Z}.$$

Thus for such values of β (resp. α), the twisted Witten indices are deformation invariants of the theory.

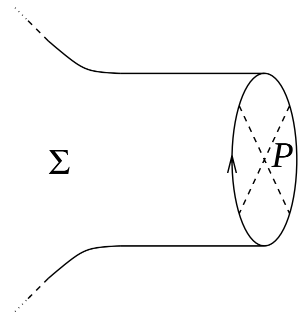

Figure 1: Crosscap $\langle \Sigma | \mathscr{C}_P \rangle$

2.2 Crosscap states

For each parity symmetry $P = \tau \circ \Omega$, there is a so-called 'crosscap state' $|\mathscr{C}_P\rangle$ which can be used to express partition functions on unorientable surfaces [48, 49]. Let Σ be an orientable or unorientable surface with an oriented boundary circle around which Σ is flat, as in Fig. 1. We choose a coordinate system (σ^1, σ^2), $\sigma^1 \equiv \sigma^1 + 2\pi$, $\sigma^2 \geq 0$, where the boundary circle is at $\sigma^2 = 0$ and is parametrized by σ^1. We glue Σ and its copy $\overline{\Sigma}$ along the boundary circles to make a double $\Sigma \# \overline{\Sigma}$ which has an involution Ω that extends $(\sigma^1, \sigma^2) \mapsto (\sigma^1 + \pi, -\sigma^2)$. Consider a path integral over the fields on this double $\Sigma \# \overline{\Sigma}$ obeying the condition $\mathcal{O} = \tau \Omega^* \mathcal{O}$. The crosscap state is defined by the property that the path-integral is expressed as $\langle \Sigma | \mathscr{C}_P \rangle$, where $\langle \Sigma |$ is the state at the boundary circle resulting from the path-integral over the fields on Σ. The fields are periodic along the circle if and only if P is involutive, $P^2 = \mathrm{id}$. If not, $|\mathscr{C}_P\rangle$ belongs to the sector in which the fields obey the twisted boundary condition $\mathcal{O}(\sigma^1, \sigma^2) = P^{-2} \mathcal{O}(\sigma^1 + 2\pi, \sigma^2) P^2$. In such a case, the pairing $\langle \Sigma | \mathscr{C}_P \rangle$ makes sense only if $\langle \Sigma |$ belongs to the sector with the same periodicity (which can be realized, say, by inserting a twist operator in the interior of Σ).

We study its properties when P is an A-parity or a B-parity, or their variants.

2.2.1 Current conditions

The transformation rule of the currents (2.5)-(2.7) or (2.6)-(2.8) yields current conditions on the crosscap states. We write them down using the 'tree-channel' coordinates (σ^1, σ^2) which are obtained from the 'loop-channel' Minkowski coordinates (x^0, x^1) via Wick rotation and 90°-rotation.[1] The crosscap state $|\mathscr{C}_{P_A}\rangle$ for an A-parity P_A obeys the following

[1] (σ^1, σ^2) here is $(ix^0, -x^1)$ there. The tree-channel supercurrents are related to those of the loop channel as $G_\pm^{\mathrm{loop}} = \mathrm{e}^{\pm \pi i/4} G_\pm^{\mathrm{tree}}$. The factors of i that appear in (2.18) or (2.19) have their origin in the phase factor $\mathrm{e}^{\pm \pi i/4}$ here.

condition for $z = (\sigma^1, \sigma^2) \to \widetilde{z} = (\sigma^1 + \pi, -\sigma^2)$:

$$
\begin{aligned}
&\overline{G}^\mu_+(z) - i(-1)^\mu G^\mu_-(\widetilde{z}) = G^\mu_+(z) - i(-1)^\mu \overline{G}^\mu_-(\widetilde{z}) \\
&= G^\mu_-(z) + i(-1)^\mu \overline{G}^\mu_+(\widetilde{z}) = \overline{G}^\mu_-(z) + i(-1)^\mu G^\mu_+(\widetilde{z}) = 0, \\
&J^\mu_V(z) - (-1)^\mu J^\mu_V(\widetilde{z}) = J^\mu_A(z) + (-1)^\mu J^\mu_A(\widetilde{z}) = 0.
\end{aligned}
\tag{2.18}
$$

where $\mu = 1, 2$. The crosscap state $|\mathscr{C}_{P_B}\rangle$ for a B-parity P_B obeys

$$
\begin{aligned}
&\overline{G}^\mu_+(z) - i(-1)^\mu \overline{G}^\mu_-(\widetilde{z}) = G_+(z) - i(-1)^\mu G_-(\widetilde{z}) \\
&= \overline{G}^\mu_-(z) + i(-1)^\mu \overline{G}_+(\widetilde{z}) = G_-(z) + i(-1)^\mu G_+(\widetilde{z}) = 0, \\
&J^\mu_V(z) + (-1)^\mu J^\mu_V(\widetilde{z}) = J^\mu_A(z) - (-1)^\mu J^\mu_A(\widetilde{z}) = 0.
\end{aligned}
\tag{2.19}
$$

The supercurrents are periodic along the circle since P_A^2 and P_B^2 act trivially on the supercurrent. Namely, *the crosscap states for A-parity and B-parity belong to sectors in which the supercurrents are periodic,* such as Ramond-Ramond sector. The R-charges $q_V = \int J_V^2(\sigma^1)\mathrm{d}\sigma^1$, $q_A = \int J_A^2(\sigma^1)\mathrm{d}\sigma^1$ of the crosscap states are also constrained; *The crosscap state for A-parity (B-parity) has vanishing axial (vector) R-charge* $q_A = 0$ *(*$q_V = 0$*).*

Let us next consider the 'bra-crosscap' which is defined as the dagger of the 'ket-crosscap'

$$\langle \mathscr{C}_P| := |\mathscr{C}_P\rangle^\dagger.$$

For $P = P_A$ or P_B, the condition obeyed by this state is obtained by taking the dagger of (2.18) or (2.19). Note that each factor of i receives a minus sign under dagger. Thus, the bra-crosscap $\langle \mathscr{C}_P|$ fulfills the same condition as the ket-crosscap $|\mathscr{C}_{(-1)^F P}\rangle$ for the parity $(-1)^F P$. In other words, the fields are subject to the condition $\mathcal{O} = (-1)^{|\mathcal{O}|}\tau\Omega^*\mathcal{O}$ at the state $\langle \mathscr{C}_P|$ where $|\mathcal{O}|$ is the mod 2 fermion number of $\mathcal{O}$. [2] See Fig. 2.

The crosscap states $|\mathscr{C}_{P_{A_{\alpha,\beta}}}\rangle$ and $|\mathscr{C}_{P_{B_{\alpha,\beta}}}\rangle$ for $A_{\alpha,\beta}$ and $B_{\alpha,\beta}$-parities obey the same R-current conditions as above but the supercurrent conditions are modified as

$$A_{\alpha,\beta}: \quad \overline{G}^\mu_\pm(z) \mp i(-1)^\mu\, \mathrm{e}^{i\alpha \pm i\beta} G^\mu_\mp(\widetilde{z}) = G^\mu_\pm(z) \mp i(-1)^\mu\, \mathrm{e}^{-i\alpha \mp i\beta} \overline{G}^\mu_\mp(\widetilde{z}) = 0, \tag{2.20}$$

$$B_{\alpha,\beta}: \quad \overline{G}^\mu_\pm(z) \mp i(-1)^\mu\, \mathrm{e}^{i\alpha \pm i\beta} \overline{G}^\mu_\mp(\widetilde{z}) = G_\pm(z) \mp i(-1)^\mu\, \mathrm{e}^{-i\alpha \mp i\beta} G_\mp(\widetilde{z}) = 0. \tag{2.21}$$

The supercurrents fulfill the boundary condition $G^\mu_\pm(\sigma^1) = \mathrm{e}^{\mp 2i\beta} G^\mu_\pm(\sigma^1 + 2\pi)$ (for $|\mathscr{C}_{P_{A_{\alpha,\beta}}}\rangle$) and $G^\mu_\pm(\sigma^1) = \mathrm{e}^{-2i\alpha} G^\mu_\pm(\sigma^1 + 2\pi)$ (for $|\mathscr{C}_{P_{B_{\alpha,\beta}}}\rangle$). Note that they are periodic if and only if the parity preserves an $\mathcal{N} = 2$ supersymmetry. We often call those parities $\widetilde{A}$-parity or $\widetilde{B}$-parity if the crosscap states belong to sectors in which the supercurrents are anti-periodic,

[2] To be more precise, $|\mathcal{O}|$ is twice the mod $\mathbb{Z}$ spin of $\mathcal{O}$. However, we only consider theories in which the spin-statistics correlation holds and the two definitions of $|\mathcal{O}|$ agree.

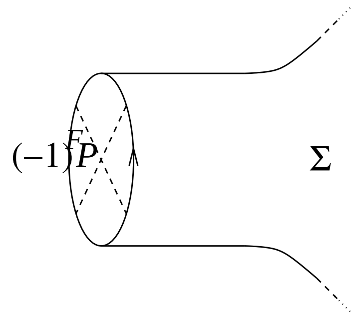

Figure 2: Crosscap $\langle \mathscr{C}_P | \Sigma \rangle$

such as the Neveu-Schwarz-Neveu-Schwarz sector. ($A_{\alpha,\beta}$ is an $\widetilde{A}$-parity iff $\beta \in \pi(\mathbb{Z}+\frac{1}{2})$, and $B_{\alpha,\beta}$ is a $\widetilde{B}$-parity iff $\alpha \in \pi(\mathbb{Z}+\frac{1}{2})$.) Taking the dagger, one realizes that the bra-crosscap state $\langle \mathscr{C}_P|$ obeys the same condition as the ket-crosscap state for the parity $(-1)^F P$ (see Fig. 2).

2.2.2 Partition functions and crosscaps

Let us consider the pairing of the crosscap states $\langle \mathscr{C}_{P_1}|q_t^H|\mathscr{C}_{P_2}\rangle$ for two parities $P_i = \tau_i \circ \Omega$. This can be identified as the partition function on the Klein bottle $(x,y) \equiv (x+2,y) \equiv (-x,y+1)$ with a suitable metric, where the fields fulfill the following boundary conditions

$$\mathcal{O}(x,y) = \tau_2 \mathcal{O}(2-x,y+1) = (-1)^{|\mathcal{O}|}\tau_1 \mathcal{O}(-x,y+1).$$

Here $(-1)^{|\mathcal{O}|}$ is the mod 2 fermion number of $\mathcal{O}$ whose appearance here is explained above. It follows that $\mathcal{O}(x,y) = (-1)^{|\mathcal{O}|}\tau_1\mathcal{O}(2-(x+2),y+1) = (-1)^{|\mathcal{O}|}\tau_1\tau_2^{-1}\mathcal{O}(x+2,y)$. Namely, the fields obey the boundary condition

$$\mathcal{O}(x,y) = U^{-1}\mathcal{O}(x+2,y)U, \quad \text{where } U = (-1)^F P_1 P_2^{-1}. \tag{2.22}$$

Thus, the pairing can be identified as the twisted partition function

$$\langle \mathscr{C}_{P_1}|q_t^H|\mathscr{C}_{P_2}\rangle = \mathrm{Tr}_{\mathcal{H}_{(-1)^F P_1 P_2^{-1}}} (-1)^F P_2 q_l^H, \tag{2.23}$$

where $\mathcal{H}_{(-1)^F P_1 P_2^{-1}}$ is the space of states with the twisted boundary condition (2.22). Using this, one can express various twisted partition functions with the help of the crosscap states. For example, the partition function in the NSNS sector can be written as

$$\mathrm{Tr}_{\mathcal{H}_{\mathrm{NSNS}}} P q^H = \langle \mathscr{C}_{(-1)^F P}|q_t^H|\mathscr{C}_{(-1)^F P}\rangle. \tag{2.24}$$

Also, the twisted Witten index can be expressed as

$$I_P = \operatorname*{Tr}_{\mathcal{H}_{\rm RR}} (-1)^F P q^H = \langle \mathscr{C}_{(-1)^F P} | q_t^H | \mathscr{C}_P \rangle. \tag{2.25}$$

The twisted Witten index for a supersymmetric open string can be obtained in terms of crosscap and boundary states

$$I_P(a, Pa) = \operatorname*{Tr}_{\mathcal{H}_{a,Pa}} (P(-1)^F\, \mathrm{e}^{-\beta H}) = \langle \mathscr{B}_a | q_t^H | \mathscr{C}_P \rangle. \tag{2.26}$$

Here it is important that the boundary state $\langle \mathscr{B}_a|$ is chosen in such a way that it preserves the same supersymmetry as $|\mathscr{C}_P\rangle$ does.

2.3 Overlap with supersymmetric ground states

For D-branes, the overlaps of the boundary states and the RR ground states are their important characteristics — they obey certain differential equations with respect to the parameters of the theory, and also carry information on the RR charge and tension [22]. Here we study the analogs for orientifolds. Let P be an A-parity or a B-parity. One may also consider a variant that preserves an $\mathcal{N} = 2$ supersymmetry (namely $A_{\alpha,\beta}$ with $\beta \in \pi\mathbb{Z}$ or $B_{\alpha,\beta}$ with $\alpha \in \pi\mathbb{Z}$). In such cases, the crosscap states $|\mathscr{C}_P\rangle$ and $\langle \mathscr{C}_{(-1)^F P}|$ are in the sector in which the supercurrents are periodic, that is, a sector with $(2, 2)$ supersymmetry. Therefore, one can consider the overlaps with the supersymmetric ground states $|i\rangle$ in that sector:

$$\begin{aligned} \Pi_i^P &= \langle \mathscr{C}_{(-1)^F P} | i \rangle, \\ \widetilde{\Pi}_i^P &= \langle i | \mathscr{C}_P \rangle. \end{aligned} \tag{2.27}$$

We study the properties of such overlaps.

2.3.1 Dependence on parameters

Let us study the dependence of the overlaps on the parameters of the theory. Let P be an A-parity (or an $A_{\alpha,\beta}$-parity with $\beta \in \pi\mathbb{Z}$) in a theory that admits a B-twist. As the ground states $|i\rangle$, we use those corresponding to *cc* ring elements ϕ_i. One important point is that parity symmetry imposes constraints on the allowed deformations of the theory. Thus the parameter space is generally reduced. For A-parities, the constraints are holomorphic for twisted chiral parameters and antiholomorphic for the chiral parameters. This will be explained in several examples in later sections.

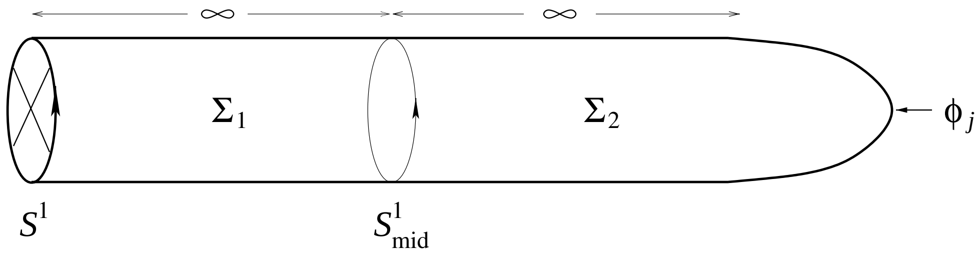

Figure 3: The semi-infinite $\mathbb{RP}^2$

Let us first consider twisted F-term deformations. Since the constraints are holomorphic, the allowed twisted chiral parameters are complex. It is easy to see, using the standard techniques as in [22] that the overlaps are invariant under the allowed twisted F-term deformations

$$\frac{\partial \Pi_i^P}{\partial t_{ac}} = 0, \quad \frac{\partial \Pi_i^P}{\partial \bar{t}_{ac}} = 0. \tag{2.28}$$

Let us next study the F-term deformations. Since the constraints are antiholomorphic, the allowed chiral parameters are real. To be more precise, the allowed moduli space is a middle dimensional real subspace of the (complex) moduli space of all chiral parameters t^i. Let $t^i = x^i + iy^i$ be the decomposition into the tangent direction x^i and orthogonal directions y^i. It can then be shown that the overlaps obey the following differential equations:

$$\begin{aligned} (\nabla_{x^i}\Pi^P)_j &= (D_{x^i}\delta_j^k + \beta C_{y^i j}^k)\Pi_k^P = 0, \\ (\nabla_{x^i}\Pi^P)_{\bar{j}} &= (D_{x^i}\delta_{\bar{j}}^{\bar{k}} + \beta C_{y^i\bar{j}}^{\bar{k}})\Pi_{\bar{k}}^P = 0, \end{aligned} \tag{2.29}$$

where β is the circumference of the boundary circle S^1. Here D_{x^i} is the covariant derivative of the vacuum bundle [50] in the direction of x^i, and

$$C_{y^i j}^k = iC_{ij}^k - iC_{\bar{\imath}j}^k,$$

where C_{ij}^k are the structure constants of the chiral ring and $C_{\bar{\imath}j}^k = g^{k\bar{l}} g_{j\overline{m}} C_{\bar{\imath}\bar{l}}^{\overline{m}}$. The relations (2.28)-(2.29) can be shown by the standard gymnastics in tt^* equation, using the worldsheet in Figure 3, just as in the derivation of the similar equation for overlaps with boundary states. The essential point is that the contour integral of the supercurrent bounces back at the boundary of the cigar, with $\overline{G}_\pm$ turned into $\pm iG_\mp$ via the supercurrent condition at the crosscap shown in Eq. (2.18). In the derivation of (2.29), we consider t^i and $\bar{t}^{\bar{\imath}}$ variations in the combination of $\partial/\partial x^i = \partial/\partial t^i + \partial/\partial \bar{t}^{\bar{\imath}}$. From the t^i-variation we obtain the term $-i\beta C_{ij}^k \phi_k$ and from the $\bar{t}^{\bar{\imath}}$-variation we obtain the term $+i\beta C_{\bar{\imath}j}^k \phi_k$. The sum is $-\beta C_{y^i j}^k \phi_k$ which is the origin of the second term in (2.29). Essentially the same relation holds for the other overlaps $\widetilde{\Pi}_i^P = \langle i|P\rangle$: they do not depend on the twisted

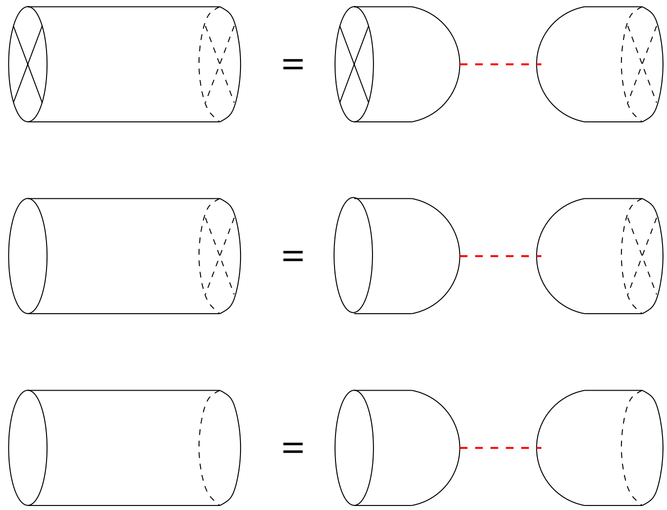

Figure 4: The Bilinear Identities (2.31), (2.32) and (2.34).

F-term deformations, and satisfy the follwing equation for the F-term deformations

$$(\nabla_{x^i}\widetilde{\Pi}^P)_j = (D_{x^i}\delta_j^k - \beta C^k_{y^i j})\widetilde{\Pi}^P_k = 0,$$
$$(\nabla_{x^i}\widetilde{\Pi}^P)_{\bar{j}} = (D_{x^i}\delta_{\bar{j}}^{\bar{k}} - \beta C^{\bar{k}}_{y^i \bar{j}})\widetilde{\Pi}^P_{\bar{k}} = 0. \tag{2.30}$$

2.3.2 Bilinear identities

Let P be a supersymmetric parity (an A-parity or a B-parity or their variant preserving an $\mathcal{N} = 2$ supersymmetry). We have seen that the twisted Witten index is expressed as the pairing $I_P = \langle \mathscr{C}_{(-1)^F P}|\,\mathrm{e}^{-TH}|\mathscr{C}_P\rangle$. Using a complete basis $|N\rangle$ of the closed string states, this can be rewritten as $\sum_N \langle \mathscr{C}_{(-1)^F P}|N\rangle\,\mathrm{e}^{-TE_N}\langle N|\mathscr{C}_P\rangle$. Note that $|\mathscr{C}_P\rangle$ belongs to a sector in which there is a $(2,2)$ supersymmetry and hence the intermediate energies are non-negative, $E_N \geq 0$, with $E_N = 0$ corresponding to the supersymmetric ground states. Now we use the fact that the Witten index is independent of the deformation parameters, in particular T. The limit $T \to \infty$ projects out the positive energy states and we are left with $I_P = \sum_{i,\bar{j}} \langle \mathscr{C}_{(-1)^F P}|i\rangle g^{i\bar{j}}\langle \bar{j}|\mathscr{C}_P\rangle$, or

$$I_P = \Pi_i^P g^{i\bar{j}}\widetilde{\Pi}^P_{\bar{j}}, \tag{2.31}$$

where $g^{i\bar{j}}$ is the inverse of $g_{\bar{\imath}j} = \langle \bar{\imath}|j\rangle$. Similarly, for the twisted Witten index for the a–Pa open string we have

$$I_P(a, Pa) = \Pi_i^a g^{i\bar{j}}\widetilde{\Pi}^P_{\bar{j}}. \tag{2.32}$$

The following must also hold

$$I_P(Pa, a) = \Pi_i^P g^{i\bar{j}}\widetilde{\Pi}^a_{\bar{j}}. \tag{2.33}$$

These generalize the more standard expression for the open string Witten index

$$I(a,b) = \Pi_i^a g^{i\bar{\jmath}} \widetilde{\Pi}_{\bar{\jmath}}^b. \tag{2.34}$$

These 'bilinear identities' are summarized in Fig. 4.

2.3.3 Other identities

Applying the parity symmetry to partition and correlation functions, one can derive several identities of different type. The first identity is obtained by applying the parity symmetry to the path-integral on the cylinder. As is evident from Fig. 5, we find the

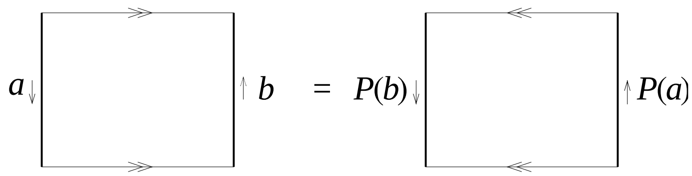

Figure 5: Applying parity to the cylinder

relation between the open string Witten indices

$$I(a,b) = I(Pb, Pa). \tag{2.35}$$

By applying the parity to semi-infinite cigars, we obtain

$$\Pi_i^a = \widetilde{\Pi}_{P(i)}^{Pa}, \qquad \widetilde{\Pi}_{\bar{\imath}}^a = \Pi_{P(\bar{\imath})}^{Pa}, \tag{2.36}$$

where $P(i)$ is the label for the ground state obtained by applying P to the ground state labelled by i. (Similarly for $P(\bar{\jmath})$.) Note that (2.35) also follows from these relations applied to the bilinear identity (2.34), with the help of unitarity of the parity operator P. Application of parity to the semi-infinite $\mathbb{RP}^2$ yields

$$\Pi_i^P = \widetilde{\Pi}_{P(i)}^P, \qquad \widetilde{\Pi}_{P(\bar{\imath})}^P = \Pi_{P(\bar{\imath})}^P. \tag{2.37}$$

It follows from (2.36) and (2.37) and from the unitarity of P that

$$\Pi_i^a g^{i\bar{\jmath}} \widetilde{\Pi}_{\bar{\jmath}}^P = \Pi_i^P g^{i\bar{\jmath}} \widetilde{\Pi}_{\bar{\jmath}}^{Pa}. \tag{2.38}$$

This relation ensures the consistency of the two bilinear identities for boundary and crosscap states, (2.32) and (2.33).

3 Geometric Picture

In this section, the general theory of parity symmetry and crosscap states developed in the previous section is applied to and illustrated by the examples of non-linear sigma models on Kähler manifolds. The classical action of the sigma model on a Kähler manifold (X, g) is

$$L = \int K(\Phi, \overline{\Phi})\, \mathrm{d}^4\theta.$$

$K(z, \overline{z})$ is a Kähler potential in a coordinate patch on which the metric is expressed as $g_{i\overline{j}} = \partial_i \partial_{\overline{j}} K$. We start by studying the parity invariance of this classical action.

3.1 Antiholomorphic and holomorphic involutions

We recall the basic parity actions on a chiral superfield, (2.9), (2.10),

$$A : \ \Phi \longrightarrow \overline{\Omega_A^* \Phi},$$
$$B : \ \Phi \longrightarrow \Omega_B^* \Phi,$$

which are A- and B-parities respectively. Similarly, for a holomorphic coordinate transformation $f^i(z) = f^i(z^1, ..., z^n)$ $(i = 1, ..., n)$, the transformations of the form $\Phi^i \to \overline{f^i(\Omega_A^* \Phi)}$ and $\Phi^i \to f^i(\Omega_B^* \Phi)$ are A- and a B-parities. We would like to find such actions that leave the Lagrangian invariant. Since the measure $\mathrm{d}^4\theta$ is invariant under both Ω_A and Ω_B, what we need to find is a transformation that leaves the Kähler potential invariant, up to a Kähler transformation $K(z, \overline{z}) \to K(z, \overline{z}) + g(z) + \overline{g(z)}$.

Let $f : X \to X$ be a holomorphic and isometric diffeomorphism. Using complex coordinates, it can be represented as $f : z^i \to f^i(z)$ where $f^i(z)$ are holomorphic functions of $z = (z^1, ..., z^n)$ obeying $K(f(z), \overline{f(z)}) = K(z, \overline{z})$, up to a Kähler transformation. Thus, the sigma model action is invariant under a B-parity

$$\Phi^i \to f^i(\Omega_B^* \Phi).$$

Let $f : X \to X$ be an antiholomorphic and isometric diffeomorphism. It can be represented as $f : z^i \to \overline{h^{\overline{i}}(z)}$ where $h^{\overline{i}}(z)$ are holomorphic functions of z obeying $K(\overline{h(z)}, h(z)) = K(z, \overline{z})$, up to a Kähler transformation. Thus, the sigma model action is invariant under an A-parity

$$\Phi^i \to \overline{h^{\overline{i}}(\Omega_A^* \Phi)}.$$

Thus, classically *the sigma model has an A-parity symmetry for each antiholomorphic isometry and a B-parity symmetry for each holomorphic isometry.* Note that both of the above can be regarded as the action $f \circ \Omega$ on the component fields, $\phi^i, \psi_\pm^i$.

One can state (anti-)holomorphicity of a map in terms of the Kähler form $\omega = \frac{i}{2} g_{i\bar{j}} \mathrm{d}z^i \wedge \mathrm{d}\bar{z}^{\bar{j}}$. An isometry $f : X \to X$ is holomorphic if and only if it preserves the Kähler form,

$$f^*\omega = \omega.$$

It is antiholomorphic if and only if it reverses the Kähler form,

$$f^*\omega = -\omega.$$

Thus, holomorphic and antiholomorphic isometries can be regarded respectively as symplectic and anti-symplectic maps with respect to the Kähler form ω.

3.2 Parity anomaly

So far our considerations have been exclusively in the classical system. In the quantum theory, we always have to check the potential anomaly. It turns out that the B-parity is always anomaly-free but the A-parity is potentially in danger. Let $\phi : \Sigma \to X$ be a map of the worldsheet into the target space. For this bosonic background the path-integral measure of the fermions $\Psi = (\psi_\pm, \overline{\psi}_\pm)$ changes under A-parity (that acts on the bosonic part as $\phi \to \phi' = f \circ \phi \circ \Omega$ where Ω is the worldsheet orientation reversal) as

$$\mathcal{D}_\phi \Psi \longrightarrow (-1)^{\int_\Sigma \phi^* c_1(X)} \mathcal{D}_{\phi'} \Psi.$$

This is seen by looking at the action on the fermion zero modes. The detail will be discussed in Section 4.2.1 where the parity anomaly of supersymmetric gauged WZW models is considered (in which 'A' and 'B' are exchanged). The sign $(-1)^{\int_\Sigma \phi^* c_1(X)}$ is the anomaly. It is always trivial when X is spin so that $c_1(X)$ is even. (Recall that the second Stiefel-Whitney class $w_2(X)$, the obstruction against spin structures, is the mod 2 reduction of $c_1(X)$.) In particular, the A-parity is anomaly-free if X is Calabi–Yau.

The anomaly can be cancelled by combining the parity action with $(-1)^{F_R}$ which flips the sign of ψ_- and $\overline{\psi}_-$. This is the option that will be used in the gauged WZW model. There is an alternative way using B-fields which we discuss now.

3.2.1 Anomaly cancellation by B-field

The B-field term of the sigma model $\int_\Sigma \phi^* B$ flips by sign under the worldsheet orientation reversal Ω. The only way to make the term invariant is to combine it with a diffeomorphism $f : X \to X$ such that $f^*[B] = -[B]$, where $[B] \in H^2(X, \mathbb{R})$ is the cohomology class represented by B. However, since the B-field enters into the path-integral weight in the form $e^{i\int_\Sigma \phi^* B}$, one may have a shift of $[B]$ by 2π times an integral class. Thus the condition of invariance is $f^*[B] = -[B] \bmod 2\pi H^2(X,\mathbb{Z})$. This is the whole story for a B-parity which is always anomaly-free. In the case of an A-parity, one may use this freedom of choosing the B-field to cancel the anomaly $(-1)^{\int_\Sigma \phi^* c_1(X)}$. This works out if the B-field is chosen such that $f^*[B] = -[B] + \pi c_1(X) \bmod 2\pi H^2(X,\mathbb{Z})$. Recalling the action on the Kähler form of holomorphic and antiholomorphic isometries, we find the following: *An A-parity is a symmetry if it acts on the complexified Kähler form as*

$$A: \quad f^*[\omega - iB] = -[\omega - iB] + \pi i c_1(X) \mod 2\pi i H^2(X,\mathbb{Z}),$$

whereas a B-parity is a symmetry if it acts as the complex conjugation

$$B: \quad f^*[\omega - iB] = \overline{[\omega - iB]} \mod 2\pi i H^2(X,\mathbb{Z}).$$

Let us choose an integral basis $\{\omega_a\}$ of $H^2(X,\mathbb{R})$ and express the complexified Kähler class as $[\omega - iB] = \sum_a \omega_a t^a$. If the action of f on $H^2(X)$ is $f^*\omega_a = \sum_b f_a^{\ b}\omega_b$, the condition of unbroken symmetry is written as

$$A: \quad t^b f_b^{\ a} = -t^a + \pi i c_1(X)^a + 2\pi i n^a, \quad n^a \in \mathbb{Z}, \tag{3.1}$$

$$B: \quad t^b f_b^{\ a} = \overline{t^a} + 2\pi i m^a, \quad m^a \in \mathbb{Z}, \tag{3.2}$$

where $c_1(X) = \sum_a \omega_a c_1(X)^a$. We find holomorphic constraints on t^a for A-parities but antiholomorphic constraints for B-parities. For B-parities, we loose a half of the complexified Kähler moduli.

Remark. If $\dim H^2(X) = 1$ (as is the case for $\mathbb{CP}^n$, Grassmannian, and submanifolds therein of dimensions > 2), an antiholomorphic map f acts on $H^2(X)$ by a sign flip. Then, the equation (3.1) is satisfied if and only if $c_1(X)$ is even, i.e. X is spin. Thus, for a non-spin manifold X with $b^2(X) = 1$ (such as $\mathbb{CP}^{even}$), the A-parity anomaly cannot be canceled by a B-field. However, if $\dim H^2(X) > 1$, there are cases in which this works. For example consider $X = \mathbb{CP}^{2m} \times \mathbb{CP}^{2m}$ and the map $(z_1, z_2) \mapsto (\overline{z_2}, \overline{z_1})$, where $z \mapsto \overline{z}$ is an antiholomorphic map of $\mathbb{CP}^{2m}$. Since $c_1(\mathbb{CP}^{2m})$ is $(2m+1)$ times the integral generator, the equation (3.1) reads

$$\begin{pmatrix} -t^2 \\ -t^1 \end{pmatrix} = -\begin{pmatrix} t^1 \\ t^2 \end{pmatrix} + \pi i \begin{pmatrix} 2m+1 \\ 2m+1 \end{pmatrix} + 2\pi i \begin{pmatrix} n^1 \\ n^2 \end{pmatrix}.$$

This has a solution $(t^1, t^2) = (t, t - \pi i)$.

3.3 Witten index

We compute the parity-twisted Witten indices in the non-linear sigma model. Throughout this subsection we consider involutive parities. We start with the case without B-field.

3.3.1 General formula

We first present the index formula that applies to a general Riemannian manifold X for which the sigma model has $\mathcal{N} = (1,1)$ supersymmetry. We consider a parity of the form $P = \tau \circ \Omega$ where $\tau : X \to X$ is an involution of X, which acts on the fields as $\phi^I(x) \to \tau^I(\phi(\widetilde{x}))$, $\psi_\pm^I(x) \to \tau_{*J}^I \psi_\mp^J(\widetilde{x})$, in the notation using real coordinates of X. This preserves the diagonal $\mathcal{N} = 1$ supersymmetry.

The closed string twisted Witten index is the partition function on the Klein bottle $(x_1, x_2) \equiv (x_1 + L_1, x_2) \equiv (-x_1, x_2 + L_2)$ with periodic boundary conditions along x_1, $x_1 \to x_1 + L_1$, but with the twisted boundary condition along x_2:

$$\phi^I(x_1, x_2) = \tau^I(\phi(-x_1, x_2 + L_2)),$$
$$\psi_\pm^I(x_1, x_2) = \tau_{*J}^I \psi_\mp^J(-x_1, x_2 + L_2).$$

Because of the $\mathcal{N} = 1$ supersymmetry, the computation localizes on the zero modes, which are the constant maps to the submanifold $X^\tau \subset X$ of τ-fixed points. The relevant computation is performed in [51] (following [52, 53]) and the result is

$$I_{\tau\Omega} = \int_{X^\tau} \frac{L(T(X^\tau))}{L(N(X^\tau))} e(N(X^\tau)), \tag{3.3}$$

where $T(X^\tau)$ and $N(X^\tau)$ are the tangent and normal bundles of X^τ (in X), which has an orientation determined by the type of parity action. $L(V)$ and $e(V)$ are the Hirzebruch L-genus and the Euler class. An outline of the derivation is recorded in Appendix A. Note that we allow X^τ to have many components — the above formula is understood as the sum of the integrals over all the components of X^τ. The same remark applies to the formulae below.

We next consider an open string with one end on a D-brane wrapped on $W \subset X$ and supporting a vector bundle E and the other end on the parity image $(\tau W, \tau E)$. The image bundle τE is topologically $\tau^* \overline{E}$, where the complex conjugation is involved because the left and the right boundaries of the string worldsheet have opposite orientation. The $\mathcal{N} = 1$ supersymmetry survives the D-brane boundary condition as well as $\tau\Omega$, and one can consider the $\tau\Omega$-twisted Witten index. It is represented as the partition function on the Möbius strip $(x_1, x_2) \equiv (L_1 - x_1, x_2 + L_2)$, $0 \leq x_1 \leq L_1$, with the standard Dirichlet/Neumann boundary conditions on $x_1 = 0$ and $x_1 = L_1$, and the periodicity along x_2 as above (where $-x_1$ there replaced by $L_1 - x_1$). This also localizes on the zero modes, which are constant maps to $W \cap X^\tau$. By a computation similar to [51] we find the following expression

$$\begin{aligned} & I_{\tau\Omega}((W,E),(\tau W,\tau E)) \\ & = \int_{W\cap X^\tau} 2^{\dim_r X^\tau - \frac{1}{2}\dim_r X} \mathrm{ch}(\overline{E}) \sqrt{\frac{\widehat{A}(T(W))}{\widehat{A}(N(W))}} \sqrt{\frac{L(\frac{1}{4}T(X^\tau))}{L(\frac{1}{4}N(X^\tau))}} e(N(W)\cap N(X^\tau)), \end{aligned} \tag{3.4}$$

where $\dim_r X^\tau$ and $\dim_r X$ are real dimensions of X^τ and X. See Appendix A for the derivation. $\widehat{A}(V)$ is the A-roof genus. We note that the formula could be changed by an overall sign, $I \to -I$, depending on the type of the parity action.

Let us now specialize to the parity symmetries that preserve an $\mathcal{N} = 2$ supersymmetry. X is thus assumed to be a Kähler manifold.

3.3.2 A-parity

Let us consider an A-parity $\tau\Omega$ that is associated with an antiholomorphic and isometric involution $\tau : X \to X$. In such a case, the fixed-point set X^τ is a middle dimensional Lagrangian submanifold of X. Then, $\mathrm{rank} N(X^\tau) = \mathrm{rank} T(X^\tau)$ and the Euler class $e(N(X^\tau))$ in the index formula (3.3) saturates the dimension of X^τ. The index is thus the integral of just the Euler class of the normal bundle. The Euler class is the obstruction against trivialization and counts the number of zeroes of a generic section of the bundle. On the other hand, a section of the normal bundle simply corresponds to a deformation of X^τ inside X, and its zero corresponds to the intersection of X^τ and its deformation. Thus, the index is nothing but the self-intersection number

$$I_{\tau\Omega} = \int_{X^\tau} e(N(X^\tau)) = \#(X^\tau \cap X^\tau). \tag{3.5}$$

Let us next consider an A-brane wrapped on a Lagrangian submanifold $L \subset X$. Since L and X^τ are both middle dimensional, we have $\mathrm{rank}(N(L) \cap N(X^\tau)) = \dim(L \cap X^\tau)$. The Euler class in the formula (3.4) again saturates, and we find

$$I_{\tau\Omega}(L, \tau L) = \int_{L\cap X^\tau} e(N(L) \cap N(X^\tau)) = \#(L \cap X^\tau). \tag{3.6}$$

3.3.3 B-parity

We next consider the B-parity associated with a holomorphic involution $\tau : X \to X$. The general formula from the path-integral is compared with the consideration from the canonical formalism. This reproduces the various signature and fixed-point theorems.

Closed string

The supersymmetric ground states of the sigma model are in one to one correspondence with the Harmonic forms or de Rham cohomology classes where the correspondence is given by $\psi_-^i \sim \mathrm{d}z^i$, $\overline{\psi}_+^{\overline{\jmath}} \sim \mathrm{d}\overline{z}^{\overline{\jmath}}$, $\overline{\psi}_-^{\overline{\jmath}} \sim g^{i\overline{\jmath}} i_{\partial/\partial z^i}$ and $\psi_+^i \sim g^{i\overline{\jmath}} i_{\partial/\partial \overline{z}^{\overline{\jmath}}}$. The parity action $\Omega : \psi_-^i \leftrightarrow \psi_+^i, \overline{\psi}_-^{\overline{\jmath}} \leftrightarrow \overline{\psi}_+^{\overline{\jmath}}$ therefore corresponds to the Hodge $*$-operator that sends $H^{p,q}(X)$ to $H^{n-q,n-p}(X)$. Thus, the twisted Witten index, which receives contribution only from the ground states is identified as the *signature* of X;

$$I_\Omega = \sum_{p+q=n} (-1)^n \mathrm{tr}_{H^{p,q}(X)}(*) = \mathrm{Sign}(X). \tag{3.7}$$

Since the fixed-point set X^τ is X itself the formula (3.3) tells

$$\mathrm{Sign}(X) = \int_X L(T(X)), \tag{3.8}$$

which is nothing but the Hirzebruch signature formula. If τ is a non-trivial (holomorphic) involution, the twisted Witten index is identified as the $\mathbb{Z}_2$-signature,

$$I_{\tau\Omega} = \sum_{p+q=n} (-1)^n \mathrm{tr}_{H^{p,q}(X)}(\tau^* \circ *) = \mathrm{Sign}(\tau, X). \tag{3.9}$$

The formula (3.3) is nothing but the G-signature formula for the case of $G = \mathbb{Z}_2$.

Open string

Let us next consider the twisted Witten index for open string stretched between B-branes. Here we restrict our attention to B-branes wrapped totally on the target space, $W = X$, and supporting a holomorphic vector bundle E over X. We note the standard subtlety in the normal coordinate expansion in the evaluation of the index, and here we take the one natural for the holomorphic category, resulting in a replacement of the A-roof genus in the formula by the Todd class. Then the index formula is

$$I_{\tau\Omega}(E, \tau^*\overline{E}) = \int_{X^\tau} 2^{\dim_r X^\tau - \frac{1}{2}\dim_r X} \mathrm{ch}(\overline{E}) \sqrt{\mathrm{td}(X)} \sqrt{\frac{L(\frac{1}{4}T(X^\tau))}{L(\frac{1}{4}N(X^\tau))}}$$

$$= \int_{X^\tau} \mathrm{ch}(2\overline{E}) \frac{\mathrm{td}(X^\tau)}{\mathrm{ch}(\bigwedge \overline{N}_{X^\tau})}. \tag{3.10}$$

The second equality is an algebraic identity, where $\overline{N}_{X^\tau}$ is the antiholomorphic part of the complexified normal bundle (see Appendix A for a proof).

Let us now study the index in the canonical formalism. In the zero mode approximation (which gives the exact answer for the index), the open string states are antiholomorphic forms on X with values in $\overline{E} \otimes \tau^*\overline{E}$, where the identification is based on $\overline{\psi}_-^{\bar\imath} + \overline{\psi}_+^{\bar\imath} \sim \mathrm{d}\overline{z}^{\bar\imath}$ and $\psi_-^i + \psi_+^i \sim g^{i\bar\jmath} i_{\partial/\partial\overline{z}^{\bar\jmath}}$. The supercharges are identified as the Dolbeault operator $\overline{\partial}$ and the supersymmetric ground states are the Dolbeault cohomology classes

$$\mathcal{H}_{\mathrm{SUSY}}^{\text{zero mode}} = \bigoplus_{p=1}^{n} H^{0,p}(X, \overline{E} \otimes \tau^*\overline{E}).$$

The parity $\tau\Omega$ acts naturally on antiholomorphic forms as $\mathrm{d}\overline{z}^{\bar\imath} \cdots \to \tau^*(\mathrm{d}\overline{z}^{\bar\imath} \cdots)$ since Ω simply exchanges $\overline{\psi}_-^{\bar\imath} \leftrightarrow \overline{\psi}_+^{\bar\imath}$, leaving $\overline{\psi}_-^{\bar\imath} + \overline{\psi}_+^{\bar\imath}$ fixed. The action τ_{CP} of the parity on the Chan-Paton factor $\overline{E} \otimes \tau^*\overline{E}$ can be of various types [54–56], although it is basically the exchange of the left and the right factors. τ^* combined with such an action τ_{CP} on the Chan-Paton bundle defines a map τ of $H^{0,p}(X, \overline{E} \otimes \tau^*\overline{E})$ into itself. This is the action of parity on the ground states in the zero mode approximation. Thus, the index is identified as

$$I_{\tau\Omega}(E, \tau^*\overline{E}) = \sum_{p=1}^{n} (-1)^p \operatorname{tr}_{H^{0,p}(X, \overline{E} \otimes \tau^*\overline{E})} (\, \tau \,) =: L(\tau, \mathscr{E}^\vee \otimes \tau^*\mathscr{E}^\vee). \tag{3.11}$$

This number is known as the *holomorphic Lefschetz number*.

We obtained two representations of the Witten index, one (3.10) from the path-integral and another (3.11) from the canonical formalism. The two must agree. Here we quote the Lefschetz fixed-point theorem [57] which expresses the holomorphic Lefschetz number by topological data. Let $g : V \to V$ be a holomorphic bundle isomorphism covering a holomorphic automorphism $g : X \to X$. The theorem states

$$L(g, \mathscr{V}) := \sum_{p=0}^{n} (-1)^p \operatorname{tr}_{H^{0,p}(X,V)} (\, g \,) = \int_{X^g} \mathrm{ch}_g(V|_{X^g}) \frac{\mathrm{td}(X^g)}{\mathrm{ch}_g(\wedge_{-1} \overline{N}_{X^g})}. \tag{3.12}$$

Here ch_g is the g-twisted Chern character and $\wedge_{-1}\overline{N}_{X^g} := \wedge^{\mathrm{even}}\overline{N}_{X^g} - \wedge^{\mathrm{odd}}\overline{N}_{X^g}$. In the present case, since τ is involutive and hence is just a (-1) on the normal bundle, we find $\mathrm{ch}_\tau(\wedge_{-1}\overline{N}_{X^\tau}) = \mathrm{ch}(\wedge \overline{N}_{X^\tau})$. Now, with this fixed-point theorem (3.12), the canonical formula (3.11) looks very much close to the path-integral formula (3.10). Indeed, they agree as we now see in several cases (up to an important sign difference in certain cases). In other words, the path-integral result reproduces the fixed-point theorem.

Case I: $E = \mathscr{O}$, $\tau_{\rm CP}$ = simple exchange.
In this case the action on the Chan-Paton factor $\overline{\mathscr{O}} \otimes \tau^* \overline{\mathscr{O}} \cong \mathscr{O}$ is trivial. Thus, the twisted index can be identified as

$$I_{\tau\Omega}(\mathscr{O}, \mathscr{O}) = \sum_{p=1}^{n} (-1)^p \operatorname*{tr}_{H^{0,p}(X)}(\tau^*) = L(\tau, \mathscr{O}), \tag{3.13}$$

the original holomorphic Lefschetz number of the map τ. It is evident that the two index formulae agree.

Case II: E general, $\tau_{\rm CP}$ = simple exchange.
On the fixed-point locus X^τ, the Chan-Paton factor is $\overline{E} \otimes \overline{E}$. $\tau_{\rm CP}$ acts trivially on the symmetric part $\mathrm{Sym}^2\overline{E}$ but as (-1)-multiplication on the anti-symmetric part $\wedge^2\overline{E}$. Thus, we find

$$\mathrm{ch}_\tau(\overline{E} \otimes \tau^* \overline{E}|_{X^\tau}) = \mathrm{ch}(\mathrm{Sym}^2\overline{E}) - \mathrm{ch}(\wedge^2\overline{E}) = \mathrm{ch}(2\overline{E}).$$

The last equality is a simple algebraic identity. Thus, the two index formulae agree in these cases as well.

Case III: $E = F \oplus F$, $\tau_{\rm CP}$ = exchange with symplectic action.
One may also consider combining the exchange with an internal action γ_{ij},

$$|i,j\rangle \mapsto \sum_{i',j'} \gamma_{ii'} |j', i'\rangle \gamma^{-1}_{j'j}.$$

Here we consider the case where $E = F \oplus F$, with F a rank r bundle, and γ_{ij} is of the form

$$\left(\gamma_{ij}\right) = \begin{pmatrix} \mathbf{0} & -\mathbf{1}_r \\ \mathbf{1}_r & \mathbf{0} \end{pmatrix}$$

We focus on the fixed-point locus X^τ on which the Chan-Paton bundle is $\overline{E} \otimes \overline{E}$. The Chan-Paton factors of the forms $|i_{(1)}, j_{(1)}\rangle \pm |j_{(2)}, i_{(2)}\rangle$, $|i_{(1)}, j_{(2)}\rangle \mp |j_{(1)}, i_{(2)}\rangle$, $|i_{(2)}, j_{(1)}\rangle \mp |j_{(2)}, i_{(1)}\rangle$ have eigenvalue ± 1 under $\tau_{\rm CP}$, where $i_{(1)}$ and $i_{(2)}$ are the index for the first and the second factor of $\overline{E} = \overline{F} \oplus \overline{F}$. These vectors are the basis of bundles isomorphic to $\overline{F} \otimes \overline{F}$, $\overline{F} \otimes \overline{F}$, $\wedge^2\overline{F}$, $\mathrm{Sym}^2\overline{F}$, $\wedge^2\overline{F}$, $\mathrm{Sym}^2\overline{F}$, respectively. On the other hand, it is easy to see that

$$\begin{aligned} \mathrm{Sym}^2\overline{E} &\cong (\overline{F} \otimes \overline{F}) \oplus (\mathrm{Sym}^2\overline{F}) \oplus (\mathrm{Sym}^2\overline{F}), \\ \wedge^2\overline{E} &\cong (\overline{F} \otimes \overline{F}) \oplus (\wedge^2\overline{F}) \oplus (\wedge^2\overline{F}). \end{aligned}$$

Thus, we find

$$\mathrm{ch}_\tau(\overline{E} \otimes \tau^* \overline{E}|_{X^\tau}) = \mathrm{ch}(\wedge^2\overline{E}) - \mathrm{ch}(\mathrm{Sym}^2\overline{E}) = -\mathrm{ch}(2\overline{E}).$$

The two index formulae differ by just an overall sign. We see that we must have minus sign -1 in the path-integral formula (3.10) in this case. This corresponds to the sign flip of the crosscap state, which is the standard feature for Sp-type orientifolds.

3.3.4 Inclusion of the B-field

Let us now turn on a B-field. As we have seen, we are allowed to have discrete values of B-field. In some cases for A-parity, a non-zero B-field is enforced to cancel parity anomaly. See the general conditions of parity symmetry (3.1) and (3.2).

The inclusion of a B-field does not affect the closed string Witten index. This is because the index computation localizes on constant maps, for which the B-field has no effect. Thus, the formulae (3.3), (3.5) and (3.7) or (3.9) remain valid.

The B-field does affect the open string indices. The effect is to shift the first Chern class of the gauge bundle on the brane by $-B/2\pi$. Thus, the Chan-Paton bundle is effectively replaced as $E \to E \otimes \mathcal{L}_B^{-1}$, where '$\mathcal{L}_B$' is the 'line bundle whose first Chern-class is $B/2\pi$'. This affects the parity transformation of the Chan-Paton bundle, which would be

$$\tau\Omega : (W, E) \longrightarrow (\tau W, \tau^* \overline{E})$$

if B were zero. For simplicity let us consider how $\Omega = \mathrm{id} \circ \Omega$ acts on the open string stretched from the brane (X, E) to the brane (X, F). The effective Chan-Paton factor is $E \otimes \mathcal{L}_B^{-1}$ backward in time on the left boundary and $F \otimes \mathcal{L}_B^{-1}$ forward in time on the right boundary. After the parity action which swaps the left and the right boundary, the Chan-Paton factor is $F \otimes \mathcal{L}_B^{-1}$ forward on the left and $E \otimes \mathcal{L}_B^{-1}$ backward on the right, or equivalently $\overline{F \otimes \mathcal{L}_B^{-1}}$ backward on the left and $\overline{E \otimes \mathcal{L}_B^{-1}}$ forward on the right. Since

$$\overline{E \otimes \mathcal{L}_B^{-1}} \cong \overline{E} \otimes \mathcal{L}_B = (\overline{E} \otimes \mathcal{L}_B^{\otimes 2}) \otimes \mathcal{L}_B^{-1},$$

we see that the parity Ω transforms the bundle E to the bundle $\overline{E} \otimes \mathcal{L}_B^{\otimes 2}$. More generally the transformation rule becomes

$$\tau\Omega : (W, E) \longrightarrow (\tau W, \tau^*(\overline{E} \otimes \mathcal{L}_B) \otimes \mathcal{L}_B) \tag{3.14}$$

It differs from $\tau^* \overline{E}$ by the factor $\tau^* \mathcal{L}_B \otimes \mathcal{L}_B$ whose 'first Chern class' is

$$c_1(\tau^* \mathcal{L}_B \otimes \mathcal{L}_B) = \frac{1}{2\pi}(\tau^*[B] + [B])\bigg|_{\tau W}. \tag{3.15}$$

The parity symmetry condition (3.1) and (3.2) includes

$$\frac{1}{2\pi}(\tau^*[B] + [B]) = \begin{cases} \frac{1}{2} c_1(M) \bmod H^2(X, \mathbb{Z}) & \text{for A-parity} \\ 0 \bmod H^2(X, \mathbb{Z}) & \text{for B-parity} \end{cases}$$

Thus the class (3.15) is evidently integral, except in the cases of A-parity on non-spin manifolds. In the latter case, however, if we assume W to be an A-brane wrapped on

Lagrangian submanifold supporting a flat bundle, the B-field has to be zero on W [22].[1] Since τ is anti-symplectic, τW is also Lagrangian and $B = \tau^* B = 0$ on τW. Thus, the class (3.15) is zero and hence integral also in this case. In any case, $\tau^* \mathcal{L}_B \otimes \mathcal{L}_B$ is a well-defined complex line bundle $\mathcal{L}_{\tau^* B + B}$ on τW. Then, the Chan-Paton factor for the stretched open string is the bundle $\mathcal{L}_{\tau^* B+B} \otimes \overline{E} \otimes \tau^* \overline{E}$. To specify the system, *one needs to specify a hermitian connection and a parity action on* $\mathcal{L}_{\tau^* B+B}$ *as well.* In the discussion below, we suppose that a choice has been made.

Now it is straightforward to generalize the index formulae obtained above to the present situation. The computation again localizes on the fixed-point set X^τ. On this set, the parity action on the bundle $\mathcal{L}_{\tau^* B+B} = \mathcal{L}_{2B}$ is just a bundle map which is $+1$ or -1 at each connected component of X^τ. Let

$$\varepsilon_B : X^\tau \to \{\pm 1\} \tag{3.16}$$

be the locally constant function determined by this sign. Then, the general index formula is given by

$$\begin{aligned} &I^B_{\tau\Omega}((W,E),(\tau W, \tau E)) \\ &\quad = \int_{W \cap X^\tau} 2^{\dim_r X^\tau - \frac{1}{2}\dim_r X} \mathrm{ch}(\overline{E}) \varepsilon_B \, \mathrm{e}^{B/2\pi} \sqrt{\frac{\widehat{A}(T(W))}{\widehat{A}(N(W))}} \sqrt{\frac{L(\frac{1}{4}T(X^\tau))}{L(\frac{1}{4}N(X^\tau))}} e(N(W) \cap N(X^\tau)), \end{aligned}$$

where τE is now $\tau^* \overline{E} \otimes \mathcal{L}_{\tau^* B+B}$.

Let us consider an A-parity associated with an antiholomorphic isometry $\tau : X \to X$ and a Lagrangian A-brane L with trivial $U(1)$ connection. The only effect of the B-field is a possible non-trivial parity action on the trivial bundle $\mathcal{L}_{\tau^* B+B}$. This modifies the index formula by the sign ε_B. Let $X^\tau = \sum_i X^\tau_i$ be the decomposition into the connected components. The B-field modifies it to $X^\tau_B = \sum_i \varepsilon_B(i) X^\tau_i$ in the index formula

$$I^B_{\tau\Omega}(L, \tau L) = \#(L \cap X^\tau_B) \tag{3.17}$$

Let us now consider a B-parity associated with a holomorphic isometry $\tau : X \to X$ and a B-brane wrapped on X and supporting a holomorphic vector bundle E. The general index formula, with the replacement $\hat{A} \to \mathrm{td}$, is given by

$$I^B_{\tau\Omega}(E, \tau E) = \int_{X^\tau} \mathrm{ch}(2\overline{E}) \varepsilon_B \, \mathrm{e}^{B/\pi} \frac{\mathrm{td}(X^\tau)}{\mathrm{ch}(\wedge \overline{N}_{X^\tau})}. \tag{3.18}$$

[1] There are non-Lagrangian A-branes with non-flat connection or non-zero B-field [43, 58]. We do not include this in our paper. It would be interesting to see the consistency of parity with such A-branes in of non-spin manifolds.

One the other hand, the canonical formalism identifies the twisted index as the holomorphic Lefschetz number

$$\begin{aligned} I^B_{\tau\Omega}(E,\tau E) &= L(\tau, \mathscr{L}_{\tau^*B+B}\otimes\mathscr{E}^\vee\otimes\tau^*\mathscr{E}^\vee) \\ &= \int_{X^\tau}\mathrm{ch}_\tau(\mathcal{L}_{\tau^*B+B}\otimes\overline{E}\otimes\tau^*\overline{E}|_{X^\tau})\frac{\mathrm{td}(X^\tau)}{\mathrm{ch}(\wedge\overline{N}_{X^\tau})}. \end{aligned} \tag{3.19}$$

Here $\mathscr{L}_{\tau^*B+B}$ is the holomorphic line bundle associated with the hermitian connection of $\mathcal{L}_{\tau^*B+B}$ which we assumed to have specified. By definition of ε_B, we have

$$\mathrm{ch}_\tau(\mathcal{L}_{\tau^*B+B}|_{X^\tau}) = \varepsilon_B\mathrm{ch}(\mathcal{L}_{2B}) = \varepsilon_B\,\mathrm{e}^{B/\pi}$$

Thus, the two index formula agree with each other (except the overall sign for the symplectic orientifolds).

3.4 Overlaps with supersymmetric ground states

We now compute the overlaps of the crosscap states and the RR ground states. For A-parity, we consider Calabi–Yau sigma model which is B-twistable. For B-parity, we consider general sigma model which can always be A-twisted. In both cases, we examine or use the bilinear identities (2.31) (2.32), (2.33).

3.4.1 A-parity in Calabi–Yau sigma model

The overlap $\Pi_i^{\tau\Omega}$ for an A-parity in Calabi–Yau sigma model can be computed exactly. The point is that they are independent of the twisted chiral parameters. In particular, they are constant along the moduli space of complexified Kähler class and the computation in the large volume limit is exact. In this limit, the theory reduces to the quantum mechanics, and the overlaps are simply the integration of the ground state wavefunctions ω_i over the orientifold plane,

$$\Pi_i^{\tau\Omega} = \widetilde{\Pi}_i^{\tau\Omega} = \int_{X^\tau}\varepsilon_B\omega_i. \tag{3.20}$$

ε_B is the sign function on X^τ associated with the B-field. The bilinear identity can be shown to hold with this factor. We recall the overlaps for the A-branes

$$\Pi_i^L = \widetilde{\Pi}_i^L = \int_L \omega_i.$$

Then, the bilinear identity is nothing but Riemann's bilinear identity. For example,

$$\#(L\cap X^\tau_B) = \left(\int_L\omega_i\right)\eta^{ij}\left(\int_{X^\tau}\varepsilon_B\omega_j\right).$$

Note that the overlap is non-vanishing only if $\omega_i \in H^n(X)$. This is consistent with the selection rule — the crosscap state for an A-parity has vanishing axial R-charge and thus have non-zero overlaps only with ground states of zero axial R-charge (which are middle dimensional forms). The most important of them is the overlap with the ground states of minimum vector R-charge. This is simply the period integral over the orientifold plane

$$\Pi_0^{\tau\Omega} = \int_{X_B^\tau} \varOmega, \tag{3.21}$$

where $\varOmega$ is the holomorphic volume form of the Calabi–Yau manifold X. This has an interpretation of the tension of the orientifold plane in superstring theory.

3.4.2 B-parity in general model

The overlaps of the crosscap states for a B-parity $\tau\Omega$ depends on the Kähler class parameter and the exact result is hard to obtain. However, an approximate formulae valid at large volume is obtained by requiring the bilinear identities and by the differential equation. We recall that the overlaps of the B-brane (X, E) and the RR ground states are

$$\Pi_i^E = \int_X \mathrm{ch}(\overline{E})\, \mathrm{e}^{B+i\omega} \sqrt{\mathrm{td}(X)}\, \omega_i + \cdots,$$
$$\widetilde{\Pi}_i^E = \int_X \mathrm{ch}(E)\, \mathrm{e}^{-B-i\omega} \sqrt{\mathrm{td}(X)}\, \omega_i + \cdots,$$

where ω is the Kähler class and $+\cdots$ are corrections that vanish in the large volume limit. Requiring the bilinear identity, we find that the overlap of the crosscap states and the RR ground states are

$$\Pi_i^{\tau\Omega} = 2^{\dim_r X^\tau - \frac{1}{2}\dim_r X} \int_{X^\tau} \varepsilon_B\, \mathrm{e}^{i\omega} \sqrt{\frac{L(\frac{1}{4}T(X^\tau))}{L(\frac{1}{4}N(X^\tau))}}\omega_i + \cdots, \tag{3.22}$$

$$\widetilde{\Pi}_i^{\tau\Omega} = 2^{\dim_r X^\tau - \frac{1}{2}\dim_r X} \int_{X^\tau} \varepsilon_B\, \mathrm{e}^{-i\omega} \sqrt{\frac{L(\frac{1}{4}T(X^\tau))}{L(\frac{1}{4}N(X^\tau))}}\omega_i + \cdots. \tag{3.23}$$

3.5 D-Branes from parity

One can sometimes associate a D-brane to a parity symmetry. For example, let us consider a bosonic sigma model on the real line described by a scalar field $X(x^0, x^1)$, and its parity symmetries $P_\pm : X(x^0, x^1) \to \pm X(x^0, -x^1)$. Consider a smooth field configuration invariant under P_+: $X(x^0, -x^1) = X(x^0, x^1)$. It can be regarded as the extension of a configuration on the left half-plane $x^1 \leq 0$ which obey Neumann boundary conditions at the boundary, $\partial_1 X|_{x^1=0} = 0$. Likewise, a P_- invariant configuration can be regarded as the extension of a configuration on the left half-plane obeying Dirichlet boundary conditions $X|_{x^1=0} = 0$. In other words, D1-brane is associated with the parity P_+ and D0-brane at $X = 0$ is associated with P_-. It should be noted, however, that it is not always possible to associate a D-brane to a parity. For instance, $P'_+ : X(x^0, x^1) \to X(x^0, -x^1) + \Delta X$ is also a parity symmetry, but it is impossible to have a P'_+-invariant configuration if $\Delta X \neq 0$.

This can be generalized to systems with $(2,2)$ supersymmetry. Suppose a $(2,2)$ theory has an A-parity P_A. Since it acts on the supercurrents as (2.5), a field configuration invariant under P_A obeys in particular

$$G^1_+(x^0, x^1) + \overline{G}^1_-(x^0, -x^1) = 0.$$

Such a configuration can be considered as an extension of a configuration on the left half-plane obeying a boundary condition at $x^1 = 0$ such that

$$(G^1_+ + \overline{G}^1_-)|_{x^1=0} = 0.$$

The latter is the condition on D-branes to preserve an A-type supersymmetry. Thus, *a D-brane associated with an A-parity is an A-brane.* Likewise, *a D-brane associated with a B-parity is a B-brane.*

For example, let us consider a supersymmetric sigma model on the complex plane $\mathbb{C}$, described by a chiral superfield Φ. The transformation $\Phi \longrightarrow \overline{\Omega^*_A \Phi}$ is an A-parity. The condition of invariance under this parity is

$$\Phi = \overline{\Omega^*_A \Phi}.$$

Such a configuration can be regarded as the smooth extension of the configuration on the left (super) half-plane which obeys the boundary condition

$$\Phi = \overline{\Phi} \qquad \text{at A-boundary.}$$

This is nothing but the $\mathcal{N} = 2_A$ preserving boundary condition corresponding to the D1-brane at the real line $\phi = \overline{\phi}$.

Similarly, the transformations $\Phi \longrightarrow \pm\Omega^*_B \Phi$ are B-parities. Invariant configurations

$$\Phi = \pm\Omega^*_B \Phi$$

can be regarded as the smooth extensions of the configuration on the left (super) half-plane which obey the boundary condition

$$\left\{ \begin{array}{l} D_+\Phi = D_-\Phi \\ \Phi = 0 \end{array} \right. \qquad \text{at B-boundary.}$$

They are nothing but the $\mathcal{N} = 2_B$ preserving boundary condition corresponding to D2-brane filling $\mathbb{C}$ or D0-brane at $\phi = 0$ respectively.

These examples generalize to any parity symmetry of the form $\tau\Omega$ where $\tau : X \to X$ is an antiholomorphic or holomorphic isometry of the target Kähler manifold X. If τ has fixed points, the corresponding boundary condition is the one for the D-brane wrapped on the fixed-point set X^τ. If τ has no fixed point, there is no invariant configurations and therefore no associated boundary condition.

It is important that $\tau\Omega$ is a symmetry of the theory. If that is anomalous, the corresponding D-brane boundary condition is expected to suffer from some pathology. Let us consider the example $\tau : (z_1, z_2) \mapsto (\overline{z}_2, \overline{z}_1)$ for $X = \mathbb{CP}^n \times \mathbb{CP}^n$, where the two $\mathbb{CP}^n$ have the same size and no B-field $t_1 = t_2 = r$. The fixed-point set is

$$X^\tau = \Big\{ (z_1, z_2) \in \mathbb{CP}^n \times \mathbb{CP}^n \,|\, z_2 = \overline{z}_1 \Big\} \cong \mathbb{CP}^n.$$

If n is odd, $\tau\Omega$ is a symmetry as long as $t_1 = t_2$, and the D-brane boundary condition for X^τ is expected to be good. If n is even, however, we have seen that $\tau\Omega$ is anomalous if $t_1 = t_2$. Then, the D-brane boundary condition for X^τ is expected to be bad. In fact, the quantization of open string in such examples are studied from the point of view of symplectic geometry [59], and it was found that definition of (a finite dimensional model of) the open string Hilbert space suffers from a problem if n is even.[1] However, if we turn on a B-field of period π on one of the $\mathbb{CP}^n$ in $X = \mathbb{CP}^n \times \mathbb{CP}^n$, the parity anomaly is cancelled and the D-brane at X^τ will not suffer from pathology. Pathology of X^τ for the case n even, $t_1 = t_2$, as well as the remedy by taking $t_1 = t_2 + \pi i$ will be explicitly seen in the mirror description in Section 7.2.1.

4 Orientifolds of $\mathcal{N} = 2$ Minimal Models I

In this section, we study parity symmetries of the $\mathcal{N} = 2$ minimal model. The discussion crucially depends on whether or not, and how, GSO projection is imposed. We introduce the model as a supersymmetric gauged WZW model where the GSO projection is (of course) not imposed. We find the A and B parities of the system and compute the twisted Witten indices for the closed string. We then move on to the model with non-chiral GSO projection. The procedure proposed by Pradisi, Sagnotti and Stanev (PSS)

[1] Roughly speaking, the definition involves a study of the moduli space of holomorphic discs, and the moduli space is required to be orientable. If n is even the moduli space is unorientable. We thank Y.-G. Oh for explanation of this point.

[25] (and established by later works [26–30]) gives a prescription to construct the crosscap states of the GSO projected model, and we will compute the Klein bottle amplitudes. Then, we determine the crosscap states of the model before the GSO projection, using the PSS procedure combined with the supercurrent condition. This enables us to compute the Witten indices as well as the overlap of the crosscap states and supersymmetric ground states.

Sections 4.1 through 4.3 deal with the system before GSO projection, Section 4.4 discusses the GSO projected model, and Section 4.5 provides the relation between them. In Section 5, we extend the study by including D-branes and open strings.

4.1 The minimal model

The $\mathcal{N} = 2$ minimal model is realized as the $SU(2)$ WZW model and a Dirac fermion system, which are coupled through a $U(1)$ gauge field. Let $G = SU(2)$ and $H = U(1) \subset SO(3)$. Let us consider a Dirac fermion arranged into the hermitian 2×2 matrix

$$\Psi_\pm = \begin{pmatrix} 0 & \overline{\psi}_\pm \\ \psi_\pm & 0 \end{pmatrix}.$$

It preserves its form under the H-action $\Psi_\pm \to h^{-1}\Psi_\pm h$: the component $\psi_\pm$ transforms as a charge 1 field. We consider the system with the action

$$S(A, g, \Psi) = kS(A, g) + \frac{i}{2\pi}\int_\Sigma \mathrm{d}^2x \, \mathrm{tr}\left(\Psi_- D_+ \Psi_- + \Psi_+ D_- \Psi_+\right),$$

where $kS(A, g)$ is the WZW action [60] and $D_\mu \Psi := \partial_\mu \Psi + [A_\mu, \Psi]$. The action is invariant under the H-valued gauge transformation $A \to h^{-1}Ah + h^{-1}\mathrm{d}h$, $g \to h^{-1}gh$, $\Psi_\pm \to h^{-1}\Psi_\pm h$. Under the H-valued constant chiral gauge transformation

$$g \to h_1^{-1} g h_2, \quad \Psi_- \to h_1^{-1}\Psi_- h_1, \quad \Psi_+ \to h_2^{-1}\Psi_+ h_2, \tag{4.1}$$

the path-integral measure changes by the factor [61]

$$\exp\left[(k+2)\frac{i}{2\pi}\int \mathrm{tr}\Big(F_A \log(h_1 h_2^{-1})\Big)\right].$$

The origin of 'k' in the exponent is the change in the action $kS(A, g)$ while '+2' comes from the chiral anomaly of the charged fermion. Supersymmetric gauged WZW models have been studied in [63, 15] (see also [62, 64] for bosonic models).

4.1.1 $(2,2)$ Superconformal symmetry

The important property of the system is $(2,2)$ supersymmetry. Using

$$\eta_\pm = \frac{1}{\sqrt{k}}\psi_\pm\sigma_- = \frac{1}{\sqrt{k}}\begin{pmatrix} 0 & 0 \\ \psi_\pm & 0 \end{pmatrix}, \quad \overline{\eta}_\pm = \frac{1}{\sqrt{k}}\overline{\psi}_\pm\sigma_+ = \frac{1}{\sqrt{k}}\begin{pmatrix} 0 & \overline{\psi}_\pm \\ 0 & 0 \end{pmatrix},$$

the supersymmetry transformations are written as

$$\begin{aligned} \delta g &= -i\sqrt{2}\left(\epsilon_+\eta_- g - \overline{\epsilon}_+\overline{\eta}_- g - \overline{\epsilon}_- g\eta_+ + \epsilon_- g\overline{\eta}_+\right), \\ \delta\eta_- &= \sqrt{2}\overline{\epsilon}_+\left[D_- g g^{-1}\right]_-, \quad \delta\eta_+ = -\sqrt{2}\epsilon_-\left[g^{-1}D_+ g\right]_-, \\ \delta\overline{\eta}_- &= -\sqrt{2}\epsilon_+\left[D_- g g^{-1}\right]_+, \quad \delta\overline{\eta}_+ = \sqrt{2}\overline{\epsilon}_-\left[g^{-1}D_+ g\right]_+, \end{aligned} \tag{4.2}$$

where $[X]_\pm$ is the projection to the upper-right/lower-left entry of X. The naive chiral R-symmetry $\psi_\mp \to \mathrm{e}^{\mp i\alpha_\mp}\psi_\mp$ has an anomaly, but it can be cancelled using the 'anomalous' chiral gauge transformations (4.1). The following combination is anomaly-free:

$$U(1)_R : \begin{cases} g \to h_{\alpha_-}^{-1} g, \\ \eta_- \to \mathrm{e}^{-i\alpha_-} h_{\alpha_-}^{-1}\eta_- h_{\alpha_-}, \\ \overline{\eta}_- \to \mathrm{e}^{i\alpha_-} h_{\alpha_-}^{-1}\overline{\eta}_- h_{\alpha_-}, \end{cases} \quad U(1)_L : \begin{cases} g \to g h_{\alpha_+}^{-1}, \\ \eta_+ \to \mathrm{e}^{i\alpha_+} h_{\alpha_+}\eta_+ h_{\alpha_+}^{-1}, \\ \overline{\eta}_+ \to \mathrm{e}^{-i\alpha_+} h_{\alpha_+}\overline{\eta}_+ h_{\alpha_+}^{-1}, \end{cases} \tag{4.3}$$

where $h_\alpha := \exp(i\alpha\sigma_3/(k+2))$. The R-symmetries are trivial at $\alpha_\pm = 2\pi m_\pm(k+2)$ with $m_\pm \in \mathbb{Z}$ since $h_{2\pi m_\pm(k+2)} = 1$. Note also that $(\alpha_-, \alpha_+) = (2\pi m, -2\pi m)$ with $m \in \mathbb{Z}$ is gauge-equivalent to the trivial transformation.

The supercurrents and the R-currents are found by the Noether procedure. In fact, the left and right moving currents decouple as $G^0_- = G^1_- = G$, $G^0_+ = -G^1_+ = \widetilde{G}$, and $J^0_R = J^1_R = J$, $J^0_L = -J^1_L = \widetilde{J}$, reflecting the $(2,2)$ super*conformal* symmetry that the system actually has. The expressions of these currents are

$$\overline{G} = i\sqrt{2}k\,\mathrm{tr}(\overline{\eta}_- D_- g g^{-1}) = \sqrt{\frac{2}{k}}\overline{\psi}_- J^G(\sigma_+), \tag{4.4}$$

$$G = i\sqrt{2}k\,\mathrm{tr}(\eta_- D_- g g^{-1}) = \sqrt{\frac{2}{k}}\psi_- J^G(\sigma_-), \tag{4.5}$$

$$\overline{\widetilde{G}} = -i\sqrt{2}k\,\mathrm{tr}(\eta_+ g^{-1} D_+ g) = \sqrt{\frac{2}{k}}\psi_+ \widetilde{J}^G(\sigma_-), \tag{4.6}$$

$$\widetilde{G} = -i\sqrt{2}k\,\mathrm{tr}(\overline{\eta}_+ g^{-1} D_+ g) = \sqrt{\frac{2}{k}}\overline{\psi}_+ \widetilde{J}^G(\sigma_+), \tag{4.7}$$

$$J = k\,\mathrm{tr}(\overline{\eta}_-\eta_- + \frac{i}{k+2}\sigma_3(D_- g g^{-1} + i[\overline{\eta}_-, \eta_-])) = \overline{\psi}_-\psi_- + \frac{1}{k+2}J^H, \tag{4.8}$$

$$\widetilde{J} = -k\,\mathrm{tr}(\overline{\eta}_+\eta_+ + \frac{i}{k+2}\sigma_3(D_- g g^{-1} + i[\overline{\eta}_+, \eta_-])) = -\overline{\psi}_+\psi_+ - \frac{1}{k+2}\widetilde{J}^H. \tag{4.9}$$

Here $J^G(X)$ and $\widetilde{J}^G(X)$ are the currents of the G-WZW sector

$$J^G(X) = ik\,\mathrm{tr}(D_-gg^{-1}X), \quad \widetilde{J}^G(X) = -ik\,\mathrm{tr}(g^{-1}D_+gX),$$

while J^H and $\widetilde{J}^H$ are (2 times) the right and left components of the gauge current:

$$J^H = 2(J^G(\sigma_3/2) + \psi_-\overline{\psi}_-), \quad \widetilde{J}^H = 2(\widetilde{J}^G(\sigma_3/2) + \psi_+\overline{\psi}_+). \tag{4.10}$$

By definition, R-symmetries transform the supercurrents as $G \to \mathrm{e}^{-i\alpha_-}G, \overline{G} \to \mathrm{e}^{i\alpha_-}\overline{G}$, $\widetilde{G} \to \mathrm{e}^{-i\alpha_+}\widetilde{G}, \overline{\widetilde{G}} \to \mathrm{e}^{i\alpha_+}\overline{\widetilde{G}}$. Those that transform them only by sign form the subgroup $\mathbb{Z}_{2(k+2)} \times \mathbb{Z}_2$ generated by $(\alpha_-, \alpha_+) = (\pi, 0)$ and by $(\alpha_-, \alpha_+) = (\pi, -\pi)$. Up to gauge transformations, the former is equivalent to the axial rotation of order $2(k+2)$:

$$a = \mathrm{e}^{-\pi i J_0} : A \to A, \;\; g \to h_{\frac{\pi}{2}}^{-1} g h_{\frac{\pi}{2}}^{-1}, \;\; \Psi_- \to -h_{\frac{\pi}{2}}^{-1}\Psi_- h_{\frac{\pi}{2}}, \;\; \Psi_+ \to h_{\frac{\pi}{2}}\Psi_+ h_{\frac{\pi}{2}}^{-1}, \tag{4.11}$$

and the latter is equivalent to the fermion number

$$(-1)^F = \mathrm{e}^{-\pi i(J_0 - \widetilde{J}_0)} : (A, g, \Psi) \to (A, g, -\Psi). \tag{4.12}$$

We will later consider the model where $(-1)^F$ is gauged — the model with non-chiral GSO projection.

4.1.2 Geometric picture

The model has a geometrical interpretation. We parametrize the group element by

$$g = \mathrm{e}^{i(\phi+t)\sigma_3/3}\,\mathrm{e}^{i\theta\sigma_1}\,\mathrm{e}^{i(\phi-t)\sigma_3/2} = \begin{pmatrix} \mathrm{e}^{i\phi}\cos\theta & i\,\mathrm{e}^{it}\sin\theta \\ i\,\mathrm{e}^{-it}\sin\theta & \mathrm{e}^{-i\phi}\cos\theta \end{pmatrix},$$

and the fermions by

$$\overline{\chi}_+ = -\sqrt{\frac{2}{k}}\,\psi_+\,\mathrm{e}^{it}\sin\theta, \;\; \chi_+ = -\sqrt{\frac{2}{k}}\,\overline{\psi}_+\,\mathrm{e}^{-it}\sin\theta,$$
$$\overline{\chi}_- = \sqrt{\frac{2}{k}}\,\overline{\psi}_-\,\mathrm{e}^{-it}\sin\theta, \;\; \chi_- = \sqrt{\frac{2}{k}}\,\psi_-\,\mathrm{e}^{it}\sin\theta,$$

After integrating out the gauge field, we obtain the following action which involves (θ, ϕ) or $z = \mathrm{e}^{-i\phi}\cos\theta$ but does not contain t:

$$S = \frac{1}{2\pi}\int_\Sigma \mathrm{d}^2x \Big\{ -g_{z\overline{z}}\partial^\mu z\partial_\mu\overline{z} + 2ig_{z\overline{z}}(\overline{\chi}_-\mathcal{D}_+\chi_- + \overline{\chi}_+\mathcal{D}_-\chi_+) + R_{z\overline{z}z\overline{z}}\chi_+\chi_-\overline{\chi}_-\overline{\chi}_+ \Big\}.$$

Here $g_{z\overline{z}} = \frac{k}{2(1-|z|^2)}$ and $\mathcal{D}_\mu := \partial_\mu + \Gamma^z_{zz}\partial_\mu z$. This is the action for the supersymmetric sigma model whose target space is the disc $|z| \leq 1$ with metric

$$\mathrm{d}s^2 = k\Big[\,(\mathrm{d}\theta)^2 + \cot^2\theta(\mathrm{d}\phi)^2\,\Big] = k\frac{|\mathrm{d}z|^2}{1-|z|^2}. \tag{4.13}$$

The supersymmetry variation (4.2) transforms the complex coordinate as $\delta z = \epsilon_+\chi_- - \epsilon_-\chi_+$, which shows that z and $\chi_\pm$ form a chiral multiplet. The R-symmetry acts on z as

$$z \longrightarrow \mathrm{e}^{i(\alpha_- + \alpha_+)/(k+2)} z.$$

4.2 Parity symmetry

Let us study the parity symmetry of the system. The coordinate transformation $(x^0, x^1) \to (x^0, -x^1)$ that reverses the orientation of the worldsheet is denoted by Ω. We have seen in [30] that the bosonic gauged WZW model has two types of parity invariance: Ω combined with $(A, g) \to (A, g^{-1})$ or $(A, g) \to (g_*^{-1} A g_*, g_*^{-1} g^{-1} g_*)$, where

$$g_* := i\sigma_1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad \text{modulo } H\text{-action,}$$

extending the parity symmetries of the ordinary WZW model [65–68] to the gauged case. Analogs of these in the supersymmetric model are Ω combined with

$$\mathcal{I}_A : (A, g, \Psi) \to (A, g^{-1}, \Psi) \tag{4.14}$$

and

$$\mathcal{I}_B : (A, g, \Psi) \longrightarrow (g_*^{-1} A g_*, g_*^{-1} g^{-1} g_*, g_*^{-1} \Psi g_*). \tag{4.15}$$

Note that Ω reverses the worldsheet chirality, and hence it exchanges the left and right components of the fermion Ψ: $(\Omega\Psi)_\pm(x^0, x^1) = \Psi_\mp(x^0, -x^1)$. (It also maps the gauge field as $(\Omega A)_\pm(x^0, x^1) = A_\mp(x^0, -x^1)$, just as in the bosonic case.) In particular, $\mathcal{I}_A\Omega$ and $\mathcal{I}_B\Omega$ act on the components $\psi_\pm, \overline{\psi}_\pm$ as

$$\mathcal{I}_A\Omega : \psi_\pm(x^0, x^1) \to \psi_\mp(x^0, -x^1), \;\; \overline{\psi}_\pm(x^0, x^1) \to \overline{\psi}_\mp(x^0, -x^1), \tag{4.16}$$

$$\mathcal{I}_B\Omega : \psi_\pm(x^0, x^1) \to \overline{\psi}_\mp(x^0, -x^1), \;\; \overline{\psi}_\pm(x^0, x^1) \to \psi_\mp(x^0, -x^1). \tag{4.17}$$

The classical action $S(A, g, \Psi)$ is invariant under both $\mathcal{I}_A\Omega$ and $\mathcal{I}_B\Omega$. Thus, these are the candidates for parity symmetry of the supersymmetric model.

4.2.1 Parity anomaly

Let us examine whether there is an anomaly from the fermionic sector. We recall that there are fermionic zero modes in a background A in which the first Chern class

$$c_1 = \frac{i}{2\pi} \int_\Sigma \mathrm{tr}\left(F_A(\tfrac{-1}{2}\sigma_3)\right)$$

is non-zero. If it is positive, there are generically c_1 zero modes for both ψ_- and $\overline{\psi}_+$ and none for $\overline{\psi}_-$ and ψ_+. Thus the path-integral measure contains a factor

$$\mathcal{D}_A^{(0)}\Psi = \prod_{i=1}^{c_1} \mathrm{d}\psi_-^{(0)i}\mathrm{d}\overline{\psi}_+^{(0)i}, \tag{4.18}$$

where $\psi_-^{(0)i}$ and $\overline{\psi}_+^{(0)i}$ are the zero modes, which are complex conjugates of each other. If c_1 is negative, the zero modes originate from $\overline{\psi}_-$ and ψ_+. Observe that c_1 flips its sign under the worldsheet orientation reversal Ω. It also flips under the g_*-conjugation since $g_*\sigma_3 g_*^{-1} = -\sigma_3$. Thus, the first Chern class flips its sign under $\mathcal{I}_A\Omega$ but remains invariant under $\mathcal{I}_B\Omega$:

$$\mathcal{I}_A\Omega : c_1 \to -c_1, \tag{4.19}$$

$$\mathcal{I}_B\Omega : c_1 \to c_1. \tag{4.20}$$

Let us first examine $\mathcal{I}_B\Omega$, which preserves the topology of the $U(1)$ gauge bundle by (4.20). By (4.17), $\mathcal{I}_B\Omega$ sends the $(\psi_-, \overline{\psi}_+)$ zero modes in the background A to the $(\overline{\psi}_+, \psi_-)$ zero modes in the background $-\Omega^*A$. Because of the Fermi statistics $\mathrm{d}\overline{\psi}_+^{(0)i}\mathrm{d}\psi_-^{(0)i} = -\mathrm{d}\psi_-^{(0)i}\mathrm{d}\overline{\psi}_+^{(0)i}$, the measure (4.18) is transformed as

$$\mathcal{D}_A^{(0)}\Psi \longrightarrow (-1)^{c_1}\mathcal{D}_{-\Omega^*A}^{(0)}\Psi. \tag{4.21}$$

There is no extra sign from the measure of the non-zero modes. Since the field configurations A and $-\Omega^*A$ are smoothly connected, there is no way to define the measure so that it is invariant under $\mathcal{I}_B\Omega$. Thus, the parity $\mathcal{I}_B\Omega$ suffers from an anomaly. Let us next consider $\mathcal{I}_A\Omega$. By (4.19), it changes the topology of the gauge bundle, except for the trivial one $c_1 = 0$. In the latter case there is no net fermionic zero mode and therefore it is $\mathcal{I}_A\Omega$-invariant. For $c_1 \neq 0$, one can choose the phase of the measure, first for $c_1 > 0$ and then for $c_1 < 0$, so that it is invariant under $\mathcal{I}_A\Omega$,which sends c_1 to $-c_1$. Thus, the parity $\mathcal{I}_A\Omega$ is anomaly-free.

4.2.2 Cancellation of the anomaly

We have seen that

$$P_A := \mathcal{I}_A\Omega \tag{4.22}$$

is anomaly-free but $\mathcal{I}_B\Omega$ suffers from a $\mathbb{Z}_2$ anomaly originating from the sign $\mathrm{d}\overline{\psi}_+^{(0)}\mathrm{d}\psi_-^{(0)} = -\mathrm{d}\psi_-^{(0)}\mathrm{d}\overline{\psi}_+^{(0)}$. It is obviously cancelled by combination with the transformation

$$(-1)^{F_R} : (A, g, \Psi_-, \Psi_+) \to (A, g, -\Psi_-, \Psi_+).$$

In other words, $(-1)^{F_R}$ has the same anomaly (4.21) as $\mathcal{I}_B\Omega$. Thus,

$$P_B := (-1)^{F_R}\mathcal{I}_B\Omega \tag{4.23}$$

is an anomaly-free parity symmetry of the system.

4.2.3 Action on the supercurrents

We determine how the parity symmetries transform the supercurrents of the system. Let us start with $P_A : (A, g, \Psi) \to \Omega(A, g^{-1}, \Psi)$, which acts in the following way $\psi_\pm(x) \to \psi_\mp(\widetilde{x})$, $D_-gg^{-1}(x) \to D_+g^{-1}g(\widetilde{x}) = -g^{-1}D_+g(\widetilde{x})$, and $g^{-1}D_+g(x) \to gD_-g^{-1}(\widetilde{x}) = -D_-gg^{-1}(\widetilde{x})$. It follows that P_A transforms the currents as

$$\begin{array}{c} G(x) \to \overline{\widetilde{G}}(\widetilde{x}),\ \ \overline{G}(x) \to \widetilde{G}(\widetilde{x}),\ \ J(x) \to -\widetilde{J}(\widetilde{x}) \\ \widetilde{G}(x) \to \overline{G}(\widetilde{x}),\ \ \overline{\widetilde{G}}(x) \to G(\widetilde{x}),\ \ \widetilde{J}(x) \to -J(\widetilde{x}) \end{array}$$

and therefore P_A is an A-parity. One may also consider the combination

$$P_A^{\alpha,\beta} := \mathrm{e}^{-i\alpha F_V - i\beta F_A} P_A. \tag{4.24}$$

This is an $\mathrm{A}_{\alpha,\beta}$-parity of the system. It preserves an $\mathcal{N} = 2$ supersymmetry if $\beta \in \pi\mathbb{Z}$. Note that $a^\ell P_A = P_A^{\frac{\pi\ell}{2},-\frac{\pi\ell}{2}}$ and $(-1)^F a^\ell P_A = P_A^{\frac{\pi\ell}{2},-\frac{\pi(\ell-2)}{2}}$ preserve an $\mathcal{N} = 2$ supersymmetry if ℓ is even, while they are $\widetilde{A}$-parities if ℓ is odd.

We next consider $P_B = (-1)^{F_R} \circ \mathrm{ad}g_*^{-1} \circ P_A$. Note that $\mathrm{ad}g_*^{-1}$ exchanges $G \leftrightarrow \overline{G}$, $\widetilde{G} \leftrightarrow \overline{\widetilde{G}}$, $J \leftrightarrow -J$, and $\widetilde{J} \leftrightarrow -\widetilde{J}$. Thus P_B transforms the currents as

$$\begin{array}{c} G(x) \to -\widetilde{G}(\widetilde{x}),\ \ \overline{G}(x) \to -\overline{\widetilde{G}}(\widetilde{x}),\ \ J(x) \to \widetilde{J}(\widetilde{x}) \\ \widetilde{G}(x) \to G(\widetilde{x}),\ \ \overline{\widetilde{G}}(x) \to \overline{G}(\widetilde{x}),\ \ \widetilde{J}(x) \to J(\widetilde{x}). \end{array}$$

Thus, P_B is a $\widetilde{\mathrm{B}}$-parity. It can also be modified by the R-symmetry:

$$P_B^{\alpha,\beta} := \mathrm{e}^{-i\alpha F_V - i\beta F_A} P_B. \tag{4.25}$$

This is a $\mathrm{B}_{\alpha+\frac{\pi}{2},\beta-\frac{\pi}{2}}$-parity of the system. It preserves an $\mathcal{N} = 2$ supersymmetry if $\alpha + \frac{\pi}{2} \in \pi\mathbb{Z}$. Note that $a^\ell P_B = P_B^{\frac{\pi\ell}{2},-\frac{\pi\ell}{2}}$ and $(-1)^F a^\ell P_B = P_B^{\frac{\pi\ell}{2},-\frac{\pi(\ell-2)}{2}}$ preserve an $\mathcal{N} = 2$ supersymmetry if ℓ is odd, while they are $\widetilde{B}$-parities if ℓ is even.

4.2.4 The square of the parity

In order to determine the sector to which the crosscap states belong, we compute the square of the above parities.

The basic A-parity $P_A = \mathcal{I}_A\Omega$ is clearly involutive, $P_A^2 : (A, g, \Psi) \to \Omega(A, g^{-1}, \Psi) \to \Omega^2(A, (g^{-1})^{-1}, \Psi) = (A, g, \Psi)$. To find the square of the modified parities $P_A^{\alpha,\beta}$, we note that conjugation by P_A flips the sign of $F_V = J_0 + \widetilde{J}_0$, while $F_A = -J_0 + \widetilde{J}_0$ is invariant, thus we see that

$$(P_A^{\alpha,\beta})^2 = \mathrm{e}^{-2i\beta F_A}.$$

Since $\mathrm{e}^{-2i\beta F_A}$ is gauge-equivalent to the fermion phase rotation $\psi_\pm \to \mathrm{e}^{-2i\beta}\psi_\pm$, $P_A^{\alpha,\beta}$ is involutive as long as $\beta \in \pi\mathbb{Z}$, that is, if and only if it preserves an $\mathcal{N} = 2$ supersymmetry. Namely, $|\mathscr{C}_{P_A^{\alpha,\beta}}\rangle$ belongs to the RR-sector if and only if $P_A^{\alpha,\beta}$ is supersymmetric. Both $a^\ell P_A$ and $(-1)^F a^\ell P_A$ square to $(-1)^{\ell F}$, and their crosscaps therefore belong to the RR (resp. NSNS) sector if ℓ is even (resp. odd),

Let us compute the square of $P_B = (-1)^{F_R}\mathcal{I}_B\Omega$. The transformation $\mathcal{I}_B\Omega$ is involutive since $g_*^2 = -1$ is the central element of $SU(2)$. On the other hand, $(-1)^{F_R}$ is transformed to $(-1)^{F_L}$ under conjugation by $\mathcal{I}_B\Omega$. Thus, we find $P_B^2 = (-1)^F$. To find the square of $P_B^{\alpha,\beta}$, we note that conjugation by P_B keeps F_V invariant but flips the sign of F_A. Thus, we see that

$$(P_B^{\alpha,\beta})^2 = \mathrm{e}^{-2i\alpha F_V}(-1)^F.$$

$\mathrm{e}^{-2i\alpha F_V}$ is the fermion phase rotation $\psi_\pm \to \mathrm{e}^{\mp 2i\alpha}\psi_\pm$ combined with the axial rotation by $h_{-2\alpha}$. This cancels $(-1)^F$ if and only if $2\alpha \in \pi(k+2)\mathbb{Z}$ as well as $2\alpha \in \pi(2\mathbb{Z}+1)$. This is possible only if k is odd, in which case one may take $\alpha = \pi\frac{k+2}{2}$. Thus, if k is even, none of the Parities $P_B^{\alpha,\beta}$ is involutive. If k is odd, $P_B^{\pi(k+2)/2,\beta}$ is involutive. Both of $a^\ell P_B$ and $(-1)^F a^\ell P_B$ square to $\mathrm{e}^{-\pi i\ell F_V}(-1)^F = (-1)^{(\ell+1)F}a^{2\ell}$ and hence the crosscap states belong to the sector twisted by this symmetry. If k is odd, $a^{k+2}P_B$ and $(-1)^F a^{k+2}P_B$ are involutive.

4.2.5 Geometric picture

The above parities yield transformations of the disc $z \mapsto P^{-1}zP$, which are isometries of the metric (4.13). The A-parities act on the coordinate z as

$$P_A^{\alpha,\beta} : z \to \mathrm{e}^{i\frac{2\alpha}{k+2}}\overline{z}, \tag{4.26}$$

which is the reflection with respect to the line $\mathrm{e}^{i\frac{\alpha}{k+2}}\mathbb{R}$. The action of the B-parities are

$$P_B^{\alpha,\beta} : z \to \mathrm{e}^{i\frac{2\alpha}{k+2}}z, \tag{4.27}$$

which are rotations of the disc. The involutive parity $P_B^{\pi(k+2)/2,\beta}$ acts on the disc as a rotation by π, or an inversion $z \to -z$.

4.3 Description in the operator formalism

In this subsection, we see how the parity symmetries act on the states.

4.3.1 The spectrum

The space of states of the theory are the gauge invariant states in the parent theory, $SU(2)$ WZW model plus the free Dirac fermion system, modulo the large gauge transformations. We formulate the system on a circle of radius 2π. The space of states of the WZW model is

$$\mathcal{H}^{G,k} = \bigoplus_{j\in \mathrm{P}_k} \widehat{V}_j \otimes \widehat{V}_j,$$

where $\widehat{V}_j$ is the spin j representation of the $SU(2)$ current algebra at level k and $\mathrm{P}_k = \{0, \frac{1}{2}, 1, ..., \frac{k}{2}\}$ is the set of integrable spins at level k. The space of states of the Dirac fermion system is composed of the Fock space, which depends on the choice of periodicity along the circle. For the boundary condition $\psi_-(\sigma + 2\pi) = \mathrm{e}^{2\pi i a}\psi_-(\sigma)$ and $\psi_+(\sigma + 2\pi) = \mathrm{e}^{-2\pi i \widetilde{a}}\psi_+(\sigma)$, the space of states is

$$\mathcal{H}^f_{a,\widetilde{a}} = \mathcal{F}_a \otimes \mathcal{F}_{\widetilde{a}}.$$

Here $\mathcal{F}_a$ is the Fock space for the fermion oscillator algebra $\{\psi_{r_1}, \psi_{r_2}\} = \{\overline{\psi}_{r'_1}, \overline{\psi}_{r'_2}\} = 0$, $\{\psi_r, \overline{\psi}_{r'}\} = \delta_{r+r',0}$ where, $r \in \mathbb{Z} + a$ and $r' \in \mathbb{Z} - a$. It is built on the vacuum state $|0\rangle_a$, which is annihilated by ψ_r $(r \geq 0)$ and $\overline{\psi}_{r'}$ $(r' > 0)$. The fermion number operator is given by

$$J^f_0 = \sum_{r\in\mathbb{Z}+a} :\overline{\psi}_{-r}\psi_r: + a - [a] - \frac{1}{2}. \tag{4.28}$$

Here, $a = 0$ and $a = \frac{1}{2}$ are the R and NS sectors respectively. The space of states of the total parent theory is the tensor product

$$\mathcal{H}^{\text{parent}}_{a,\widetilde{a}} = \mathcal{H}^{G,k} \otimes \mathcal{H}^f_{a,\widetilde{a}} = \bigoplus_{j\in \mathrm{P}_k} \bigoplus_{\substack{s\in 2\mathbb{Z}+2a-1 \\ \widetilde{s}\in 2\mathbb{Z}+2\widetilde{a}-1}} \widehat{V}_j \otimes \mathcal{F}_a|_s \otimes \widehat{V}_j \otimes \mathcal{F}_{\widetilde{a}}|_{\widetilde{s}},$$

where $\mathcal{F}_a|_s$ is the subspace of $\mathcal{F}_a$ in which $J^f_0 = s/2$ (s takes values in $2(\mathbb{Z} + (a - 1/2))$ by (4.28)). The gauge invariant states are those obeying the condition

$$J^H_n = \widetilde{J}^H_n = 0, \quad n \geq 1,$$
$$J^H_0 + \widetilde{J}^H_0 + 2(a + \widetilde{a}) = 0,$$

where $J_n^H = J_n^3 - 2J_n^f$ and $\widetilde{J}_n^H = \widetilde{J}_n^3 - 2\widetilde{J}_n^f$ according to (4.10). J_n^H and $\widetilde{J}_n^H$ generate the $U(1)$ current algebra at level $k+2$. Let us denote by $B_{j,n,s}$ the subspace of $\widehat{V}_j \otimes \mathcal{F}_a|_s$ in which $J_n^H = 0$ $(n \geq 1)$ and $J_0^H = -n$, so that

$$\widehat{V}_j^{G,k} \otimes \mathcal{F}_a|_s = \bigoplus_{n \in 2\mathbb{Z}+2j+s} B_{j,n,s} \otimes \widehat{V}_{-n}^{H,k+2}.$$

Then, the space of gauge invariant states is

$$\left(\mathcal{H}_{a,\widetilde{a}}^{\text{parent}}\right)^{H-\text{inv}} = \bigoplus_{j \in \mathrm{P}_k} \bigoplus_{\substack{s \in 2\mathbb{Z}+2a-1 \\ \widetilde{s} \in 2\mathbb{Z}+2\widetilde{a}-1}} \bigoplus_{n \in 2\mathbb{Z}+2j+s} B_{j,n,s} \otimes B_{j,-n+2(a+\widetilde{a}),\widetilde{s}}.$$

The topology of the gauge group $\pi_1(H) = \pi_1(U(1)) = \mathbb{Z}$ allows large gauge transformations, which act on the labels as $(j,n,s,\tilde{s}) \to (\frac{k}{2} - j, n \pm (k+2), s \pm 2, \tilde{s} \mp 2) \to (j, n \pm 2(k+2), s \pm 4, \tilde{s} \mp 4) \to \cdots$. This induces the so-called "field identification" [69–71]. Namely, the space of states of our gauge system is given by

$$\mathcal{H}_{a,\widetilde{a}} = \bigoplus_{(j,n,s,\tilde{s}) \in \widetilde{\mathrm{M}}_k(a,\widetilde{a})} B_{j,n,s} \otimes B_{j,-n+2(a+\widetilde{a}),\widetilde{s}} \tag{4.29}$$

where $\widetilde{\mathrm{M}}_k(a,\widetilde{a})$ is the infinite set

$$\widetilde{\mathrm{M}}_k(a,\widetilde{a}) = \left\{ (j,n,s,\tilde{s}) \,\middle|\, \begin{array}{l} j \in \mathrm{P}_k, \quad 2j - s + n \text{ even} \\ s \in 2\mathbb{Z}+2a-1, \ \widetilde{s} \in 2\mathbb{Z}+2\widetilde{a}-1 \end{array} \right\} \Big/ \pi_1(H).$$

The R-currents (4.8) and (4.9) are expressed as

$$J_n = \frac{1}{k+2} J_n^H + J_n^f, \quad \widetilde{J}_n = -\frac{1}{k+2} \widetilde{J}_n^H - \widetilde{J}_n^f,$$

and hence the R-charges ($(J_0, \widetilde{J}_0)$-eigenvalues) are given by

$$(q, \widetilde{q}) = \left(-\frac{n}{k+2} + \frac{s}{2}, \frac{\widetilde{n}}{k+2} - \frac{\widetilde{s}}{2} \right) \quad \text{on } B_{j,n,s} \otimes B_{j,\widetilde{n},\widetilde{s}}. \tag{4.30}$$

The supercharges (4.4)-(4.7) are expressed, after the standard shift $k \to k+2$, as

$$\overline{G}_{r'} = \sqrt{\frac{2}{k+2}} \sum_{n \in \mathbb{Z}} \overline{\psi}_{r'-n} J_n^+, \quad G_r = \sqrt{\frac{2}{k+2}} \sum_{n \in \mathbb{Z}} \psi_{r-n} J_n^-, \quad (r, -r' \in \mathbb{Z} + a),$$
$$\overline{\widetilde{G}}_{\widetilde{r}} = \sqrt{\frac{2}{k+2}} \sum_{n \in \mathbb{Z}} \widetilde{\psi}_{\widetilde{r}-n} \widetilde{J}_n^-, \quad \widetilde{G}_{\widetilde{r}'} = \sqrt{\frac{2}{k+2}} \sum_{n \in \mathbb{Z}} \overline{\widetilde{\psi}}_{\widetilde{r}'-n} \widetilde{J}_n^+, \quad (\widetilde{r}, -\widetilde{r}' \in \mathbb{Z} + \widetilde{a}).$$

Remark. We regard the $(a,\widetilde{a})$ sector to be the sector in which the fields obey the boundary condition

$$\Phi(\sigma) = U_{a,\widetilde{a}}^{-1} \Phi(\sigma + 2\pi) U_{a,\widetilde{a}},$$
$$U_{a,\widetilde{a}} := \mathrm{e}^{-2\pi i (a J_0 + \widetilde{a} \widetilde{J}_0)} = \mathrm{e}^{-2\pi i (a - \frac{1}{2}) J_0 - 2\pi i (\widetilde{a} + \frac{1}{2}) \widetilde{J}_0} (-1)^F.$$

Note that $(-1)^F := e^{\pi i(-J_0+\widetilde{J}_0)}$ is the mod 2 fermion number (4.12). In particular the standard NSNS sector is $(a,\widetilde{a}) = (\frac{1}{2},-\frac{1}{2})$, and it indeed includes the $SL(2,\mathbb{C})$-invariant ground state $[|0;0\rangle\otimes|0\rangle_{\frac{1}{2}}]\otimes[|0;0\rangle\otimes|0\rangle_{-\frac{1}{2}}] \in B_{0,0,0}\otimes B_{0,0,0} \subset \mathcal{H}_{\frac{1}{2},-\frac{1}{2}}$. The $(a,\widetilde{a})$ sector is the spectral flow [72] from this by $(a-\frac{1}{2},\widetilde{a}+\frac{1}{2})$. Because of the periodicity of the R-symmetries, we have the identifications $(a,\widetilde{a}) \equiv (a+(k+2),\widetilde{a}) \equiv (a,\widetilde{a}+(k+2)) \equiv (a+1,\widetilde{a}-1)$.

Chiral primaries

The local operators of the system can be mapped one-to-one to the states in the NSNS sector ($a = -\widetilde{a} = \frac{1}{2}$). Chiral primary fields correspond to those obeying the conditions $\overline{G}_{-\frac{1}{2}} = \overline{\widetilde{G}}_{-\frac{1}{2}} = 0$ and $G_{\frac{1}{2}} = \widetilde{G}_{\frac{1}{2}} = 0$. One can show that they are given by

$$|j\rangle_{cc} = \Big[|j,j\rangle\otimes|0\rangle_{\rm NS}\Big]\otimes\Big[|j,-j\rangle\otimes|0\rangle_{\rm NS}\Big],\quad j = 0,\frac{1}{2},1,\ldots,\frac{k}{2},$$

which belongs to $B_{j,-2j,0}\otimes B_{j,2j,0}$. The corresponding chiral primary $\mathcal{O}_j$ has R-charges $(q,\widetilde{q}) = (\frac{2j}{k+2},\frac{2j}{k+2})$. There are also antichiral primaries $\overline{\mathcal{O}}_j$ corresponding to $|j\rangle_{aa} = [|j,-j\rangle\otimes|0\rangle_{\rm NS}]\otimes[|j,j\rangle\otimes|0\rangle_{\rm NS}]$ in $B_{j,2j,0}\otimes B_{j,-2j,0}$ with charge $(q,\widetilde{q}) = (-\frac{2j}{k+2},-\frac{2j}{k+2})$. There are no twisted (anti)chiral primaries, except for the identity operator.

Supersymmetric ground states

The supersymmetry of the sector with $a,\widetilde{a}\in\mathbb{Z}$ is generated by $G_0,\overline{G}_0,\widetilde{G}_0,\overline{\widetilde{G}}_0$. We would like to find the supersymmetric ground states, namely the states annihilated by all of $G_0,\overline{G}_0,\widetilde{G}_0,\overline{\widetilde{G}}_0$.

We start with the RR sector ($a=\widetilde{a}=0$). The supersymmetric ground states are

$$|j\rangle_{\rm RR} = \Big[|j;j\rangle\otimes|0\rangle_{\rm R}\Big]\otimes\Big[|j;-j\rangle\otimes\overline{\psi}_0|0\rangle_{\rm R}\Big],\quad j = 0,\frac{1}{2},1,\ldots,\frac{k}{2},$$

where $|0\rangle_{\rm R}$ is the vacuum state $|0\rangle_0\in\mathcal{F}_0$, which is annihilated by ψ_0 and has $J_0^f = -1/2$. The state $|j\rangle_{\rm RR}$ belongs to $B_{j,-(2j+1),-1}\otimes B_{j,2j+1,1}$ and has R-charges $(q,\widetilde{q}) = (\frac{2j+1}{k+2}-\frac{1}{2},\frac{2j+1}{k+2}-\frac{1}{2})$. Another representation of the same state (up to a phase) is

$$|j\rangle'_{\rm RR} = \Big[|\frac{k}{2}-j;-\frac{k}{2}+j\rangle\otimes\overline{\psi}_0|0\rangle_{\rm R}\Big]\otimes\Big[|\frac{k}{2}-j;\frac{k}{2}-j\rangle\otimes|0\rangle_{\rm R}\Big],$$

which belongs to $B_{\frac{k}{2}-j,(k+2)-(2j+1),1}\otimes B_{\frac{k}{2}-j,-(k+2)+(2j+1),-1}$. The state $|j\rangle_{\rm RR}\propto|j\rangle'_{\rm RR}$ corresponds to the chiral primary field $\mathcal{O}_j$.

We next consider the sectors with twisted boundary conditions $(a,\widetilde{a})\not\equiv(0,0)$. (There are $k+1$ such sectors labelled by $a+\widetilde{a}\in\{1,2,\ldots,k+1\}$.) For each such sector, there is a unique supersymmetric ground state

$$|\mathrm{G}\rangle_{a,\widetilde{a}}=\Big[|j_*;j_*\rangle\otimes|0\rangle_{\mathrm{R}}\Big]\otimes\Big[|j_*;j_*\rangle\otimes|0\rangle_{\mathrm{R}}\Big]\in B_{j_*,-2j_*-1,-1}\otimes B_{j_*,-2j_*-1,-1},$$

where $j_*\in\mathrm{P}_k$ is defined by $2j_*+1\equiv -a-\widetilde{a}$ mod $(k+2)$. It has R-charge $(q,\widetilde{q})=(\frac{2j_*+1}{k+2}-\frac{1}{2},-\frac{2j_*+1}{k+2}+\frac{1}{2})$.

The above results are consistent with the equivariant Witten indices. Let us consider the partition function on the torus $(x,y)\equiv(x+1,y)\equiv(x,y+1)$ with the (twisted) boundary condition $\Phi(x,y)=\Phi(x+1,y)=U_{a,\widetilde{a}}^{-1}\Phi(x,y+1)U_{a,\widetilde{a}}$. There is a supersymmetry as long as $a,\widetilde{a}\in\mathbb{Z}$, and the partition function is regarded as the Witten index. If we consider x as the space and y as the time coordinate, this can be regarded as the trace over the RR sector of the operator $U_{a,\widetilde{a}}(-1)^F\,\mathrm{e}^{-\beta H}$. If, on the other hand, we consider x as the time and y as the space coordinate, the partition function is identified as the trace over the $(a,\widetilde{a})$-sector of the operator $(-1)^F\,\mathrm{e}^{-\beta' H}$. We thus find the identity

$$\mathop{\mathrm{Tr}}_{\mathcal{H}_{\mathrm{RR}}}\mathrm{e}^{2\pi i(aJ_0+\widetilde{a}\widetilde{J}_0)}(-1)^F\,\mathrm{e}^{-\beta H}=\mathop{\mathrm{Tr}}_{\mathcal{H}_{a,\widetilde{a}}}(-1)^F\,\mathrm{e}^{-\beta' H}.$$

The left hand side is the equivariant Witten index and can be computed, using our knowledge of the supersymmetric ground states in the RR sector, as

$$\begin{aligned}\mathrm{LHS}&=\sum_{j\in\mathrm{P}_k}\mathrm{e}^{2\pi i(a-\frac{1}{2})q_j+2\pi i(\widetilde{a}+\frac{1}{2})\widetilde{q}_j}=\sum_{2j=0,1,\ldots,k}\mathrm{e}^{2\pi i(a+\widetilde{a})(\frac{2j+1}{k+2}-\frac{1}{2})}\\&=-\,\mathrm{e}^{-\pi i(a+\widetilde{a})}=\pm1\qquad\text{if } a+\widetilde{a}\not\equiv 0 \text{ mod } (k+2).\end{aligned}$$

On the other hand, the right hand side is the ordinary Witten index for the system twisted by $U_{a,\widetilde{a}}$, which computes the number of bosonic supersymmetric ground states minus the number of fermionic ones. That it is equal to LHS$=\pm1$ if $a+\widetilde{a}\not\equiv 0$ is consistent with the above conclusion that there is a unique supersymmetric ground states in $\mathcal{H}_{a,\widetilde{a}}$ with $a+\widetilde{a}\not\equiv 0$.

4.3.2 The parity action

Let us now see how parity acts on the states. We will also compute the twisted partition function for the NSNS and RR sectors.

A-Parity

We start with the basic parity symmetry $P_A = \mathcal{I}_A\Omega$. Since the space of states is basically the subspace of the tensor product $\mathcal{H}^{G,k} \otimes \mathcal{H}^f_{a,\widetilde{a}}$, we separate the discussion into bosonic and fermionic sectors. The action on the bosonic part is determined in [30]:

$$u_b \otimes \widetilde{v}_b \in \widehat{V}_j \otimes \widehat{V}_j \;\longmapsto\; (-1)^{2j} v_b \otimes \widetilde{u}_b \in \widehat{V}_j \otimes \widehat{V}_j. \tag{4.31}$$

For the fermionic sector it exchanges the periodicity parameter a, $\widetilde{a}$ of the left and right movers, $\mathcal{H}^f_{a,\widetilde{a}} \to \mathcal{H}^f_{\widetilde{a},a}$, mapping the oscillators as

$$\psi_r,\ \overline{\psi}_{r'},\ \widetilde{\psi}_{\widetilde{r}},\ \overline{\widetilde{\psi}}_{\widetilde{r}'} \longrightarrow \widetilde{\psi}_r,\ \overline{\widetilde{\psi}}_{r'},\ \psi_{\widetilde{r}},\ \overline{\psi}_{\widetilde{r}'}, \tag{4.32}$$

where $r, -r' \in \mathbb{Z} + a$ and $\widetilde{r}, -\widetilde{r}' \in \mathbb{Z} + \widetilde{a}$. It follows that the ground state $|0\rangle_{a,\widetilde{a}} = |0\rangle_a \otimes |0\rangle_{\widetilde{a}}$ (annihilated by $\psi_{r\geq 0}$, $\overline{\psi}_{r'>0}$, $\widetilde{\psi}_{\widetilde{r}\geq 0}$ and $\overline{\widetilde{\psi}}_{\widetilde{r}'>0}$) is mapped to the ground state $|0\rangle_{\widetilde{a},a}$ (annihilated by $\psi_{\widetilde{r}\geq 0}$, $\overline{\psi}_{\widetilde{r}'>0}$, $\widetilde{\psi}_{r\geq 0}$ and $\overline{\widetilde{\psi}}_{r'>0}$) up to a phase, $|0\rangle_{a,\widetilde{a}} \mapsto \epsilon_{a,\widetilde{a}}|0\rangle_{\widetilde{a},a}$. More general states are mapped as

$$\mathcal{O}_1\widetilde{\mathcal{O}_2}|0\rangle_{a,\widetilde{a}} \;\longmapsto\; \epsilon_{a,\widetilde{a}}\widetilde{\mathcal{O}_1}\mathcal{O}_2|0\rangle_{\widetilde{a},a} = \epsilon_{a,\widetilde{a}}(-1)^{|\mathcal{O}_1||\mathcal{O}_2|}\mathcal{O}_2\widetilde{\mathcal{O}_1}|0\rangle_{\widetilde{a},a}. \tag{4.33}$$

Here, $\mathcal{O}_i$ are polynomials of the fermion oscillators $\psi_\bullet$, $\overline{\psi}_\bullet$, and $\widetilde{\mathcal{O}_i}$ are the ones where $\psi_\bullet$, $\overline{\psi}_\bullet$ are replaced by $\widetilde{\psi}_\bullet$, $\overline{\widetilde{\psi}}_\bullet$. $|\mathcal{O}_i|$ is the fermion number of $\mathcal{O}_i$. Thus, we find that the states of the combined system are mapped as follows

$$\begin{aligned} P_A \quad : \quad & u_a \otimes \widetilde{v}_b \otimes \mathcal{O}_1\widetilde{\mathcal{O}_2}|0\rangle_{a,\widetilde{a}} \in B_{j,n,s} \otimes B_{j,-n+2(a+\widetilde{a}),\widetilde{s}} \\ & \longmapsto\; \epsilon_{a,\widetilde{a}}(-1)^{2j+|\mathcal{O}_1||\mathcal{O}_2|} v_b \otimes \widetilde{u}_b \otimes \mathcal{O}_2\widetilde{\mathcal{O}_1}|0\rangle_{\widetilde{a},a} \in B_{j,-n+2(a+\widetilde{a}),\widetilde{s}} \otimes B_{j,n,s}, \end{aligned} \tag{4.34}$$

where

$$|\mathcal{O}_1| = \frac{s}{2} - (a - [a] - \frac{1}{2}), \quad |\mathcal{O}_2| = \frac{\widetilde{s}}{2} - (\widetilde{a} - [\widetilde{a}] - \frac{1}{2}).$$

One can check that this action is compatible with the field identification. Let us show this for the action on the RR ground states, $|j\rangle_{\rm RR} = |j;j\rangle \otimes |j;-j\rangle \otimes \overline{\widetilde{\psi}}_0|0\rangle_{0,0} \propto |j\rangle'_{\rm RR} = |\frac{k}{2}-j;-(\frac{k}{2}-j)\rangle \otimes |\frac{k}{2}-j;\frac{k}{2}-j\rangle \otimes \overline{\psi}_0|0\rangle_{0,0}$ Using (4.34), we find that they are mapped as

$$|j\rangle_{\rm RR} \mapsto \epsilon_{0,0}(-1)^{2j}|\frac{k}{2}-j\rangle'_{\rm RR}, \quad |j\rangle'_{\rm RR} \mapsto \epsilon_{0,0}(-1)^{k-2j}|\frac{k}{2}-j\rangle_{\rm RR}.$$

This is consistent with the field identification $|j\rangle_{\rm RR} \propto |j\rangle'_{\rm RR}$, provided that

$$|\frac{k}{4}\rangle_{\rm RR} = \pm|\frac{k}{4}\rangle'_{\rm RR}, \tag{4.35}$$

which is non-vacuous only if k is even. This also shows that P_A is involutive only if $\epsilon_{0,0}$ is 1 or -1.

Let us compute the twisted partition function for the NSNS and RR sectors. The subspaces $B_{j,n,s}\otimes B_{j,-n,\widetilde{s}}$ that contribute to it are such that it is equivalent to $B_{j,-n,\widetilde{s}}\otimes B_{j,n,s}$ up to field identification. This is so for $B_{j,0,s}\otimes B_{j,0,s}$ and $B_{\frac{k}{4},-\frac{k+2}{2},s}\otimes B_{\frac{k}{4},\frac{k+2}{2},s+2}$ (k even). It is then straightforward to compute the partition functions. We present the result for the more general parity $P_A^{\alpha,\beta}=\mathrm{e}^{-i\alpha F_V-i\beta F_A}P_A$. For the NSNS sector it is

$$\begin{aligned}\mathop{\mathrm{Tr}}_{\mathcal{H}_{\rm NSNS}}(P_A^{\alpha,\beta}q^H) &= \epsilon_{\rm NSNS}\sum_{2j,s\,\text{even}}(-1)^{\frac{s}{2}}\,\mathrm{e}^{i\beta s}\mathrm{ch}_{j,0,s}(2\tau)\\ &= \epsilon_{\rm NSNS}\sum_{2j\,\text{even}}(\chi_{j,0,0}-\chi_{j,0,2})(2\tau,\tfrac{\beta}{\pi}),\end{aligned}\tag{4.36}$$

and for the RR sector

$$\begin{aligned}&\mathop{\mathrm{Tr}}_{\mathcal{H}_{\rm RR}}(P_A^{\alpha,\beta}q^H)\\ &= -\epsilon_{\rm RR}\sum_{2j,s\,\text{odd}}(-1)^{\frac{s+1}{2}}\,\mathrm{e}^{i\beta s}\mathrm{ch}_{j,o,s}(2\tau)\pm\epsilon_{\rm RR}(-1)^{\frac{k}{2}}\sum_{s\,\text{odd}}(-1)^{\frac{s+1}{2}}\,\mathrm{e}^{2i\beta\frac{s+1}{2}}\mathrm{ch}_{\frac{k}{2},-\frac{k+2}{2},s}(2\tau)\\ &= -\epsilon_{\rm RR}\sum_{2j\,\text{odd}}(\chi_{j,0,-1}-\chi_{j,0,1})(2\tau,\tfrac{\beta}{\pi})\pm\epsilon_{\rm RR}(-1)^{\frac{k}{2}}(\chi_{\frac{k}{2},-\frac{k+2}{2},-1}-\chi_{\frac{k}{2},-\frac{k+2}{2},1})(2\tau,\tfrac{\beta}{\pi}).\end{aligned}\tag{4.37}$$

In the above expressions, $\mathrm{ch}_{j,n,s}$ is the character

$$\mathrm{ch}_{j,n,s}(\tau)=\mathop{\mathrm{Tr}}_{B_{j,n,s}}q^{L_0-\frac{c}{24}},\tag{4.38}$$

and $\chi_{j,n,s}$, for $s\in\mathbb{Z}/4\mathbb{Z}$, is

$$\chi_{j,n,s}(\tau,u)=\sum_{p\in\mathbb{Z}}\mathop{\mathrm{Tr}}_{B_{j,n,s+4p}}q^{L_0-\frac{c}{24}}\,\mathrm{e}^{2\pi iJ_0u}=\sum_{p\in\mathbb{Z}}\mathrm{e}^{2\pi iu(-\frac{n}{k+2}+\frac{s+4p}{2})}\mathrm{ch}_{j,n,s+4p}(\tau),$$

The sign $\pm$ of the second term on the right hand side of (4.37) is the same as the one that appears in the Field Identification (4.35).

The special cases are the ones with $(-1)^{\nu F}a^\ell P_A$-twists ($\nu=0,1$):

$$\mathop{\mathrm{Tr}}_{\mathcal{H}_{\rm NSNS}}((-1)^{\nu F}a^\ell P_Aq^H)=\epsilon_{\rm NSNS}\sum_{2j\,\text{even}}(\chi_{j,0,0}-(-1)^\ell\chi_{j,0,2})(2\tau),\tag{4.39}$$

$$\begin{aligned}\mathop{\mathrm{Tr}}_{\mathcal{H}_{\rm RR}}((-1)^{\nu F}a^\ell P_Aq^H) &= -\,\mathrm{e}^{\frac{\pi i\ell}{2}}(-1)^\nu\epsilon_{\rm RR}\sum_{2j\,\text{odd}}(\chi_{j,0,-1}-(-1)^\ell\chi_{j,0,1})(2\tau)\\ &\quad\pm\epsilon_{\rm RR}\delta_k^{(2)}(-1)^{\frac{k}{2}}(\chi_{\frac{k}{4},-\frac{k+2}{2},-1}-(-1)^\ell\chi_{\frac{k}{4},-\frac{k+2}{2},1})(2\tau).\end{aligned}\tag{4.40}$$

We recall that $(-1)^{\nu F}a^\ell P_A$ with even ℓ preserves an $\mathcal{N}=2$ supersymmetry and the twisted partition function in the RR-sector can be regarded as the Witten index. Indeed, since $\chi_{j,n,-1}-\chi_{j,n,1}=\pm\delta_{n,\mp(2j+1)}$, we find

$$I_{a^{\rm even}P_A}=\pm\epsilon_{\rm RR}(-1)^{\frac{k}{2}}\delta_k^{(2)}.\tag{4.41}$$

For odd ℓ, $(-1)^{\nu F}a^\ell P_A$ breaks all supersymmetry and the partition function is indeed a non-trivial function of τ.

B-Parity

We next consider B-parity $P_B = (-1)^{F_R}\mathcal{I}_B\Omega$, which is the same as P_A followed by $(-1)^{J_0^f}\mathrm{ad}g_*^{-1}$. The action of $\mathrm{ad}g_*^{-1}$ on the bosonic sector is the usual one, $u_b\otimes\widetilde{v}_b \mapsto g_*u_b\otimes\widetilde{g_*v_b}$, where g_* acts on the ground states of $\widehat{V}_j$ as $g_*|j;m\rangle = i^{2j}|j;-m\rangle$ [30]. Let us next see how $\mathrm{ad}g_*^{-1}$ acts on the fermionic sector. Since it exchanges $\psi_\pm$ and $\overline{\psi}_\pm$, it flips the sign of the periodicity parameter $(a,\widetilde{a})$, $\mathcal{H}^f_{a,\widetilde{a}} \to \mathcal{H}^f_{-a,-\widetilde{a}}$, and maps the oscillators as

$$\mathrm{ad}g_*^{-1}\ :\ \psi_r,\ \overline{\psi}_{r'},\ \widetilde{\psi}_{\widetilde{r}},\ \widetilde{\overline{\psi}}_{\widetilde{r}'} \longrightarrow \overline{\psi}_r,\ \psi_{r'},\ \widetilde{\overline{\psi}}_{\widetilde{r}},\ \widetilde{\psi}_{\widetilde{r}'}. \tag{4.42}$$

It follows from this that the ground state $|0\rangle_{a,\widetilde{a}}$ (annihilated by $\psi_{r\geq 0}$, $\overline{\psi}_{r'>0}$, $\widetilde{\psi}_{\widetilde{r}\geq 0}$ and $\widetilde{\overline{\psi}}_{\widetilde{r}'>0}$) is mapped to the ground state $|0\rangle'_{-a,-\widetilde{a}}$ (annihilated by $\psi_{r'\geq 0}$, $\overline{\psi}_{r>0}$, $\widetilde{\psi}_{\widetilde{r}'\geq 0}$ and $\widetilde{\overline{\psi}}_{\widetilde{r}>0}$) up to a phase, $|0\rangle_{a,\widetilde{a}} \mapsto \eta_{a,\widetilde{a}}|0\rangle'_{-a,-\widetilde{a}}$, Note that one may set $|0\rangle'_{-a,-\widetilde{a}} = |0\rangle_{-a,-\widetilde{a}}$, as long as $a,\widetilde{a}\notin\mathbb{Z}$. If a or $\widetilde{a}$ is an integer, they are not proportional to each other. For example $|0\rangle_{0,0}$ is annihilated by ψ_0, $\widetilde{\psi}_0$, while $|0\rangle'_{0,0}$ is annihilated by $\overline{\psi}_0$, $\widetilde{\overline{\psi}}_0$, and therefore one may set $|0\rangle'_{0,0} = \overline{\psi}_0\widetilde{\overline{\psi}}_0|0\rangle_{0,0}$. More general states are mapped as

$$\mathrm{ad}g_*^{-1}\ :\ \mathcal{O}_1\widetilde{\mathcal{O}_2}|0\rangle_{a,\widetilde{a}} \ \longmapsto\ \eta_{a,\widetilde{a}}\overline{\mathcal{O}}_1\widetilde{\overline{\mathcal{O}}_2}|0\rangle'_{a,\widetilde{a}}.$$

Combining with the action of P_A and $(-1)^{F_R} = \mathrm{e}^{\pi i J_0^f}$, we find that $P_B = \mathrm{e}^{\pi i J_0^f}\mathrm{ad}g_*^{-1}P_A$ acts on the states as

$$\begin{aligned} P_B\ \ :\ \ & u_a\otimes\widetilde{v}_b\otimes\mathcal{O}_1\widetilde{\mathcal{O}_2}|0\rangle_{a,\widetilde{a}} \in B_{j,n,s}\otimes B_{j,-n+2(a+\widetilde{a}),\widetilde{s}} \\ \longmapsto\ & \varepsilon_{a,\widetilde{a}}(-1)^{2j+|\mathcal{O}_1||\mathcal{O}_2|}\,\mathrm{e}^{-\pi i\frac{\widetilde{s}}{2}}g_*(v_b)\otimes\widetilde{g_*(u_b)}\otimes\overline{\mathcal{O}}_2\widetilde{\overline{\mathcal{O}}_1}|0\rangle'_{-\widetilde{a},-a} \in B_{j,n-2(a+\widetilde{a}),-\widetilde{s}}\otimes B_{j,-n,-s}, \end{aligned} \tag{4.43}$$

where $\varepsilon_{a,\widetilde{a}} = \epsilon_{a,\widetilde{a}}\eta_{\widetilde{a},a}$. Let us see if it is compatible with the field identification. We examine it in the action on the RR ground states. Using (4.43), we find the following P_B action

$$\begin{aligned} |j\rangle_{\mathrm{RR}} = |j;j\rangle\otimes|j;-j\rangle\otimes\widetilde{\overline{\psi}}_0|0\rangle_{0,0} &\longmapsto \varepsilon_{0,0}\,\mathrm{e}^{-\frac{\pi i}{2}}|j;j\rangle\otimes|j;-j\rangle\otimes\psi_0|0\rangle'_{0,0} = \\ &= \varepsilon_{0,0}\,\mathrm{e}^{-\frac{\pi i}{2}}|j;j\rangle\otimes|j;-j\rangle\otimes\psi_0\overline{\psi}_0\widetilde{\overline{\psi}}_0|0\rangle_{0,0} = \varepsilon_{0,0}\,\mathrm{e}^{-\frac{\pi i}{2}}|j\rangle_{\mathrm{RR}} \\ |j\rangle'_{\mathrm{RR}} = |j';-j'\rangle\otimes|j';j'\rangle\otimes\overline{\psi}_0|0\rangle_{0,0} &\longmapsto \varepsilon_{0,0}\,\mathrm{e}^{\frac{\pi i}{2}}|j';-j'\rangle\otimes|j';j'\rangle\otimes\widetilde{\psi}_0|0\rangle'_{0,0} = \\ &= \varepsilon_{0,0}\,\mathrm{e}^{\frac{\pi i}{2}}|j';-j'\rangle\otimes|j';j'\rangle\otimes\widetilde{\psi}_0\overline{\psi}_0\widetilde{\overline{\psi}}_0|0\rangle_{0,0} = -\varepsilon_{0,0}\,\mathrm{e}^{\frac{\pi i}{2}}|j\rangle'_{\mathrm{RR}}, \end{aligned}$$

where $j' := \frac{k}{2} - j$. We see that the action is compatible with the field identification $|j\rangle_{\rm RR} \propto |j\rangle'_{\rm RR}$. Note that it would have been incompatible without the $(-1)^{F_R} = {\rm e}^{\pi i J_0^f}$ factor. This corresponds to the anomaly of $\mathcal{I}_B\Omega = {\rm ad}g_*^{-1}P_A$ and its cancellation by $(-1)^{F_R}$.

Let us now compute the twisted partition function in the NSNS and RR sectors. A non-zero contribution comes from the subspaces $B_{j,n,s} \otimes B_{j,-n,-s}$. The result is

$$\begin{aligned}
\mathop{\rm Tr}_{\mathcal{H}_{\rm NSNS}}(P_B^{\alpha,\beta}q^H) &= \varepsilon_{\rm NSNS} \sum_{2j+n,\, s\,{\rm even}} {\rm e}^{-2i\alpha(-\frac{n}{k+2}+\frac{s}{2})}{\rm ch}_{j,n,s}(2\tau) \\
&= \varepsilon_{\rm NSNS} \sum_{\substack{2j+n\,{\rm even}\\ s=0,2}} \chi_{j,n,s}(2\tau, -\tfrac{\alpha}{\pi}),
\end{aligned} \tag{4.44}$$

$$\begin{aligned}
\mathop{\rm Tr}_{\mathcal{H}_{\rm RR}}(P_B^{\alpha,\beta}q^H) &= \varepsilon_{\rm RR} \sum_{2j+n,\, s\,{\rm odd}} {\rm e}^{-2i\alpha(-\frac{n}{k+2}+\frac{s}{2})}{\rm ch}_{j,n,s}(2\tau) \\
&= \varepsilon_{\rm RR} \sum_{\substack{2j+n\,{\rm odd}\\ s=\pm 1}} \chi_{j,n,s}(2\tau, -\tfrac{\alpha}{\pi}),
\end{aligned} \tag{4.45}$$

where $\varepsilon_{\rm RR} = \varepsilon_{0,0}\,{\rm e}^{-\frac{\pi i}{2}}$.

Let us consider the special cases with $(-1)^{\nu F}a^\ell P_B$-twists ($\nu = 0, 1$):

$$\mathop{\rm Tr}_{\mathcal{H}_{\rm NSNS}}((-1)^{\nu F}a^\ell P_B q^H) = \varepsilon_{\rm NSNS} \sum_{\substack{2j+n\,{\rm even}\\ s=0,2}} {\rm e}^{\pi i\ell(\frac{n}{k+2}-\frac{s}{2})}\chi_{j,n,s}(2\tau) \tag{4.46}$$

$$\mathop{\rm Tr}_{\mathcal{H}_{\rm RR}}((-1)^{\nu F}a^\ell P_B q^H) = \varepsilon_{\rm RR} \sum_{\substack{2j+n\,{\rm odd}\\ s=-1,1}} {\rm e}^{\pi i\ell(\frac{n}{k+2}-\frac{s}{2})}\chi_{j,n,s}(2\tau) \tag{4.47}$$

$(-1)^{\nu F}a^\ell P_B$ with odd ℓ preserves an $\mathcal{N} = 2$ supersymmetry and the twisted partition function in the RR-sector can be regarded as the Witten index. Indeed, it is just a number

$$\begin{aligned}
I_{a^\ell P_B} &= \varepsilon_{\rm RR} \sum_{j\in {\rm P}_k} {\rm e}^{\pi i\ell(\frac{2j+1}{k+2}-\frac{1}{2})} = \varepsilon_{\rm RR}\,{\rm e}^{-\pi i\frac{\ell}{2}}(z + z^2 + \cdots + z^{k+1})|_{z={\rm e}^{\pi i\ell/(k+2)}} \\
&= \varepsilon_{\rm RR}\,{\rm e}^{-\pi i\frac{\ell}{2}}\frac{z - z^{k+2}}{1-z} = i\varepsilon_{\rm RR}\,{\rm e}^{-\frac{\pi i\ell}{2}}\cot\left[\frac{\pi\ell}{2(k+2)}\right], \quad \ell\ {\rm odd}.
\end{aligned} \tag{4.48}$$

For even ℓ, $(-1)^{\nu F}a^\ell P_B$ breaks all supersymmetry and the partition function is indeed a non-trivial function of τ.

4.4 RCFT point of view

We next study the system in which a certain GSO projection is imposed. The system can be regarded as a rational conformal field theory, the crosscaps can be studied using the standard procedure of Pradisi–Sagnotti–Stanev [25] which is reviewed (along with more recent developments such as [27, 28]) and extended in [30].

GSO projection

We perform the GSO projection with respect to the operator

$$(-1)^F = \mathrm{e}^{-\pi i(J_0-\widetilde{J}_0)}.$$

This is a non-chiral projection and the projected theory consists of NSNS as well as RR sectors, in each of which only the states with $(-1)^F = 1$ are kept. RR and NSNS sectors correspond to the twist parameters $(a, \widetilde{a}) = (0, 0)$ and $(\frac{1}{2}, -\frac{1}{2})$ respectively, and the GSO operator $(-1)^F$ is $\mathrm{e}^{-\pi i(s+\widetilde{s})/2}$ on the subspace $B_{j,n,s} \otimes B_{j,-n,\widetilde{s}}$. We therefore keep only the subspaces

$$B_{j,n,s} \otimes B_{j,-n,\widetilde{s}}, \qquad \text{with } s, \widetilde{s} \in \mathbb{Z},\ \ s + \widetilde{s} = 0 \bmod 4,\ \text{ and } 2j + n - s \text{ even.}$$

Note that $(-1)^F$ acts only on the Dirac fermions $\Psi = (\psi_\pm, \overline{\psi}_\pm)$ and the GSO projection of the latter system is equivalent to the rational $U(1)$ at level 2, the the circle sigma model of radius $R = \sqrt{2}$. Thus the GSO projection of the full minimal model can be regarded as the $SU(2)_k \times U(1)_2$ mod $U(1)$ gauged WZW model. From this point of view, it is natural to group the spaces as

$$\mathscr{H}_{j,n,s} = \bigoplus_{p\in\mathbb{Z}} B_{j,n,s+4p},$$

where s is now regarded as a mod 4 integer. The character $\chi_{jns}(\tau, u)$ that appears in (4.38) is simply the trace on this space. The Hilbert space of states of the GSO projected theory is expressed as

$$\mathcal{H}^{\mathrm{GSO}} = \bigoplus_{(j,n,s)\in \mathrm{M}_k} \mathscr{H}_{j,n,s} \otimes \mathscr{H}_{j,-n,-s}, \tag{4.49}$$

where M_k is the set of $(j, n, s) \in \mathrm{P}_k \times \mathbb{Z} \times \mathbb{Z}_4$ modulo $\pi_1(H) \cong \mathbb{Z}$, or more explicitly

$$\mathrm{M}_k = \frac{\left\{ (j,n,s) \in \mathrm{P}_k \times \mathbb{Z}_{2(k+2)} \times \mathbb{Z}_4 \;\middle|\; 2j+n+s \text{ even} \right\}}{(j,n,s) \sim (\frac{k}{2} - j, n + (k+2), s+2)}.$$

In this subsection n and s are thus mod $2(k + 2)$ and mod 4 integers that label $U(1)_{k+2}$ and $U(1)_2$ RCFT. We usually assume them to be in the standard range $-k-1, \ldots, k+2$ and $-1, 0, 1, 2$, and addition modulo $2k+4$ (or modulo 4 for s) is denoted by the symbol $\hat{+}$. We often put hat $\widehat{n}$, $\widehat{s}$ to the $U(1)_{k+2}$ and $U(1)_2$ labels n, s, to stress that they are brought into the respective standard ranges.

4.4.1 RCFT aspects of the theory

Modular matrices

The S- and T-matrices of the coset model have the factorized form

$$S_{(j,n,s)(j',n',s')} = 2\ S_{jj'}\ S^*_{nn'}\ S_{ss'}, \quad T_{(j,n,s),(j',n',s')} = T_{jj'}T^*_{nn'}T_{ss'}, \tag{4.50}$$

where it is understood that the matrices with pure j labels are those of the $SU(2)_k$ WZW model, matrices with pure n or pure s labels are those of $U(1)_{k+2}$ or $U(1)_2$. Using this factorization property, we find

$$N_{(j,n,s)(j',n',s')}^{\ (j'',n'',s'')} = N_{jj'}^{\ j''}\delta^{(2k+4)}_{n+n',n''}\delta^{(4)}_{s+s',s''} + N_{jj'}^{\ \frac{k}{2}-j''}\delta^{(2k+4)}_{n+n',n''+k+2}\delta^{(4)}_{s+s',s''+2}. \tag{4.51}$$

We also need to have expressions for $P = \sqrt{T}ST^2S\sqrt{T}$ and $Y^c_{ab} = \sum_d S_{ad}P_{bd}P^*_{cd}/S_{0d}$ [3, 25]. P relates the open and closed string channel of the Möbius strip and Y appears in the loop channel of the Möbius strip and Klein bottle. For the computation, it is useful to consider

$$Q = ST^2S, \quad \tilde{Y}^c_{ab} = \sum_d \frac{S_{ab}Q_{bd}Q^*_{cd}}{S_{0d}} = \sqrt{\frac{T_c}{T_b}}\ Y^c_{ab}.$$

Thanks to the factorization of S and T, we find

$$Q_{(j,n,s)(j',n',s')} = Q_{jj'}Q^*_{nn'}Q_{ss'} + Q_{\frac{k}{2}-j,j'}Q^*_{n\hat{+}(k+2),n'}Q_{s\hat{+}2,s'} \tag{4.52}$$

$$\tilde{Y}^{(j'',n'',s'')}_{(j,n,s)(j',n',s')} = \tilde{Y}^{j''}_{jj'}\ \overline{\tilde{Y}}^{n''}_{nn'}\tilde{Y}^{s''}_{ss'} + \tilde{Y}^{\frac{k}{2}-j''}_{jj'}\ \overline{\tilde{Y}}^{n''+k+2}_{nn'}\tilde{Y}^{s''+2}_{ss'}. \tag{4.53}$$

From this, one can compute $P = \sqrt{T}Q\sqrt{T}$ and $Y^c_{ab} = \sqrt{T_b/T_c}\tilde{Y}^c_{ab}$, using the following expressions for $\sqrt{T}$ in the coset model

$$\sqrt{T_{j,n,s}} = \sigma_{j,n,s}\sqrt{T_jT^*_nT_s}, \tag{4.54}$$

where σ is a sign factor defined by this equation and explicitly computed in Appendix B.

Discrete symmetries

The group of simple currents is given by the primaries $(0, n, s)$. For odd k the symmetry group is $\mathbb{Z}_{4k+8}$ and is generated by $(0, 1, 1)$. For even k it is $\mathbb{Z}_{2k+4} \times \mathbb{Z}_2$, generated by $(0, 1, 1)$ and $(0, 0, 2)$. The monodromy charge of the field (j, n, s) under the simple currents is

$$Q_{n,s}(j', n', s') = \frac{nn'}{2(k+2)} - \frac{ss'}{4} \mod 1\ . \tag{4.55}$$

Accordingly, there is a symmetry action on states such that

$$g_{n,s} = \mathrm{e}^{\pi i\left(\frac{nn'}{k+2}-\frac{ss'}{2}\right)} \quad \text{on } \mathscr{H}_{j',n',s'} \otimes \mathscr{H}_{j',-n',-s'} \tag{4.56}$$

$g_{1,1}$ corresponds to the generator $a = \mathrm{e}^{-\pi i J_0}$ of the axial rotations in the gauged WZW model; $g_{0,2}$ is the element $(-1)^{\widehat{F}}$ that distinguishes the RR and NSNS sectors.

Orbifold

We consider the orbifold by the subgroup $\mathbb{Z}_{k+2} \times \mathbb{Z}_2$ generated by the currents $g_{2,0}$ and $g_{0,2}$, whose space of states is

$$\mathcal{H}^M = \bigoplus_{(j,n,s)\in \mathrm{M}_k} \mathscr{H}_{j,n,s} \otimes \mathscr{H}_{j,n,s}. \tag{4.57}$$

This can be regarded as the mirror of the original model. The mirror map $\Psi : \mathcal{H} \to \mathcal{H}^M$ acts on states as $\Psi = V_M \otimes 1 : |j,n,s\rangle \otimes |j,-n,-s\rangle \to |j,-n,-s\rangle \otimes |j,-n,-s\rangle$.

4.4.2 A-type parities

We now turn to the construction of the standard PSS parities and crosscaps, which we shall call A-type crosscaps. For each simple current $(0,n,s)$ there is a crosscap state given by

$$|\mathscr{C}_{n,s}\rangle = \sum_{(j',n',s')\in \mathrm{M}_k} \frac{P_{(0,n,s)(j',n',s')}}{\sqrt{S_{(0,0,0)(j',n',s')}}}\, |\mathscr{C}, j', n', s'\rangle\!\rangle. \tag{4.58}$$

Explicit expressions for the A-type crosscap states can be found in Appendix D. Note that the crosscap states with n, s even contain only Ishibashi states in the NSNS-sector, whereas those with n, s odd contain only Ishibashi states in the RR-sector. One can compute the Klein bottle amplitudes using the Y-tensor. The result is

$$\begin{aligned}
&\langle \mathscr{C}_{\bar n,\bar s}|\, \mathrm{e}^{-\frac{\pi i}{2\tau}H} |\mathscr{C}_{n,s}\rangle = \\
&\quad \sigma_{0,\bar n,\bar s}\sigma_{0,n,s}\delta^{(2)}_{\bar n+n}\delta^{(2)}_{\bar s+s}\Bigg\{ \sum_{2j+\frac{\bar n-n}{2}+\frac{\bar s-s}{2}\ \text{even}} (-1)^{2j}\big(\chi_{j,\frac{\bar n-n}{2},\frac{\bar s-s}{2}} + (-1)^s \chi_{j,\frac{\bar n-n}{2},\frac{\bar s-s}{2}+2}\big)(2\tau) \\
&\qquad\qquad + \delta^{(2)}_k\, \mathrm{e}^{\frac{\pi i}{2}(-\bar n+\bar s)}\big(\chi_{\frac{k}{4},\frac{\bar n-n}{2}+\frac{k+2}{2},\frac{\bar s-s}{2}+1} + (-1)^s \chi_{\frac{k}{4},\frac{\bar n-n}{2}+\frac{k+2}{2},\frac{\bar s-s}{2}-1}\big)(2\tau)\Bigg\}.
\end{aligned} \tag{4.59}$$

For $n=\bar n$ and $s=\bar s$, this expression simplifies to

$$\begin{aligned}
&\langle \mathscr{C}_{n,s}|\, \mathrm{e}^{-\frac{\pi i}{2\tau}H} |\mathscr{C}_{n,s}\rangle \\
&\quad = \sum_{j\in\mathbb{Z}} (\chi_{j,0,0} + (-1)^s \chi_{j,0,2})(2\tau) + \delta^{(2)}_k\, \mathrm{e}^{\frac{\pi i}{2}(-n+s)} (\chi_{\frac{k}{4},\frac{k+2}{2},1} + (-1)^s \chi_{\frac{k}{4},\frac{k+2}{2},-1})(2\tau).
\end{aligned}$$

The crosscaps correspond to involutive parity symmetries $P_{n,s}$ which are related among themselves as

$$P_{n,s} = g_{n,s} P_{0,0}.$$

The above expression for the Klein bottle function $\mathrm{Tr} P_{n,s} q^H = \langle \mathscr{C}_{n,s} | q_t^H | \mathscr{C}_{n,s} \rangle$ is consistent with this relation.

4.4.3 B-type parities

Another class of crosscap states can be constructed from A-type crosscap states in the $G = \mathbb{Z}_{k+2} \times \mathbb{Z}_2$ orbifold model, with an application of the mirror map. We shall call them B-type crosscaps. We refer to [30] for notation and conventions. For an RCFT $\mathcal{C}$ with the charge conjugation modular invariant partition function there are $|G|$ A-type crosscap states in the orbifold theory for each G-orbit of simple currents. These crosscap states are given by

$$\left| \mathscr{C}_{P^{\theta}_{g'}} \right\rangle^{\mathcal{C}/G} = \frac{e^{i\omega_{g'}}}{\sqrt{|G|}} \sum_{g \in G} e^{-\pi i (\theta(g) - \hat{Q}_{g'}(g))} \left| \mathscr{C}_{P_{gg'}} \right\rangle^{\mathcal{C}}, \tag{4.60}$$

where $\hat{Q}_g(i) := h_g + h_i - h_{g(i)}$ and g' is a fixed representative of a simple current orbit. θ is a solution to the constraint equation

$$\theta(g_1 g_2) = \theta(g_1) + \theta(g_2) - \hat{Q}_{g_2}(g_1) + 2q(g_1, g_2) \qquad \mathrm{mod}\, 2 \tag{4.61}$$

where q is a symmetric bilinear form of G that determines the orbifold theory. (q is a form obeying $q(g,g) = -h_g$ and $Q_{g_1}(g_2) = 2q(g_1, g_2)$ mod 1.) The crosscap state corresponds to a parity symmetry $P^{\theta}_{g'}$, which squares to

$$(P^{\theta}_{g'})^2 = \mathrm{e}^{2\pi i (\theta(g) - Q_{g'}(g))} \qquad \text{on the } g\text{-twisted Hilbert space } \mathcal{H}_g. \tag{4.62}$$

We apply this construction to the orbifold of $\mathcal{C} = SU(2)_k \times U(1)_2 / U(1)_{k+2}$ by $G = \mathbb{Z}_{k+2} \times \mathbb{Z}_2$ with the Hilbert space (4.57). This is the orbifold with respect to the bilinear form given by

$$q(g_{n,s}, g_{n',s'}) = \frac{nn'}{4(k+2)} - \frac{ss'}{8}. \tag{4.63}$$

($g_{n,s}$ is in G if n and s are both even.) We first need expressions modulo 2 for $\hat{Q}_g(h)$. The conformal weight of a simple current $(0, n, s)$ with $(0, n, s) \neq (0, \pm 1, \mp 1)$ is given by

$$h_{(0,n,s)} = -\frac{n^2}{4(k+2)} + \frac{s^2}{8} + \frac{|n| - |s|}{2} \quad \text{for} \quad (0, n, s) \neq (0, \pm 1, \mp 1). \tag{4.64}$$

Using this we find

$$\begin{aligned}\hat{Q}_{g_{n,s}}(g_{n',s'}) &= h_{(0,n,s)} + h_{(0,n',s')} - h_{(0,n\hat{+}n',s\hat{+}s')} \\ &= -\frac{n^2}{4(k+2)} + \frac{s^2}{8} - \frac{(n')^2}{4(k+2)} + \frac{(s')^2}{8} + \frac{(n\hat{+}n')^2}{4(k+2)} - \frac{(s\hat{+}s')^2}{8} \\ &\quad + \frac{1}{2}(|n| - |s| + |n'| - |s'| - |n\hat{+}n'| + |s\hat{+}s'|) \\ &= \frac{nn'}{2(k+2)} - \frac{ss'}{4} - \frac{n\hat{+}n'}{2} + \frac{n+n'}{2} + \frac{s\hat{+}s'}{2} - \frac{s+s'}{2} \\ &\quad + \frac{1}{2}(|n| - |s| + |n'| - |s'| - |n\hat{+}n'| + |s\hat{+}s'|)\end{aligned} \tag{4.65}$$

In the last step, we have used n, n', s, s' even. We thus conclude that

$$\hat{Q}_{g_{n,s}}(g_{n',s'}) = \frac{nn'}{2(k+2)} - \frac{ss'}{4} \mod 2, \tag{4.66}$$

in particular $\hat{Q} = 2q \bmod 2$. Therefore, we obtain a homogeneous equation for θ, $\theta(gh) = \theta(g) + \theta(h)$, whose solutions are given by

$$\theta_{rq}(g_{n,s}) = -\frac{rn}{k+2} + \frac{qs}{2}\ . \tag{4.67}$$

The set of simple currents $(0, n, s)$ splits up into two orbits under the orbifold group $\mathbb{Z}_{k+2} \times \mathbb{Z}_2$. The first orbit is the one of $(0, 0, 0)$, which contains only currents $(0, n, s)$ with n, s even; the other orbit is the one of $(0, 1, 1)$, which contains only currents with n, s odd. Accordingly, there are two types of crosscap states. Following the general procedure, one first constructs A-type crosscaps in the orbifold and then applies the mirror map. These steps are performed in the appendix, and here we merely list the results. B-type crosscap states are labelled by an element $(r, q) \in \mathbb{Z}_{k+2} \times \mathbb{Z}_2$ and an orbit label p which can take the values 0 and 1. They are given by

$$|\mathscr{C}_{rqp}\rangle = (2(k+2))^{\frac{1}{4}} \sum_j \sigma_{j,-2r-p,-2q-p} \frac{P_{j\frac{k}{2}}}{\sqrt{S_{0j}}} (-1)^{\frac{\widehat{2r+p}-p}{2}+q} |\mathscr{C}, j, 2r+p, 2q+p\rangle\rangle_B, \tag{4.68}$$

where $P_{j\frac{k}{2}}$ is the P-matrix of the $SU(2)_k$-theory. These states are elements of the sector twisted by $g_{4r+p,2p}$. Hence, the square of the parity action P_{rqp} is given as

$$P_{rqp}^2 = g_{4r+2p,2p}. \tag{4.69}$$

One can compute the Klein bottle amplitudes using the average formula (4.60) and the results for A-type Klein bottles (4.59). The result is

$$\langle\mathscr{C}_{rqp}| e^{-\frac{\pi i}{2\tau}H} |\mathscr{C}_{rqp}\rangle = \sum_{(j,n,s)\in \mathrm{M}_k} e^{\pi i \frac{\widehat{(2r+p)}n}{k+2}}\ \mathrm{e}^{-\pi i \frac{\widehat{(2q+p)}s}{2}} \chi_{j,n,s}(2\tau) \tag{4.70}$$

We also note that

$$\langle \mathscr{C}_{r,q,p}|e^{-\frac{\pi i}{2\tau}H}|\mathscr{C}_{r,q+1,p}\rangle = 0,$$

because $|\mathscr{C}_{r,q,p}\rangle$ and $|\mathscr{C}_{r,q+1,p}\rangle$ belong to orthogonal subspaces. The expression (4.70) for the Klein bottle function $\mathrm{Tr}P_{rqp}q^H = \langle \mathscr{C}_{rqp}|q_t^H|\mathscr{C}_{rqp}\rangle$ implies the following relations among the parities,

$$P_{rqp} = g_{2r+p,2q+p}P_{000}, \tag{4.71}$$

at least in the action on closed string states. This is also consistent with the square formula (4.69).

4.5 Crosscaps in the theory before GSO projection

In the previous subsection, we obtained the crosscaps for the theory in which the non-chiral GSO projection is imposed. In this subsection, we use this result to reconstruct the crosscaps in the theory before the GSO projection. This enables us to compute the overlaps with the supersymmetric ground states, as well as to reproduce the Witten index with a twist by supersymmetric parities.

GSO projection is in a sense an *orbifold* by the symmetry $(-1)^F$ where NSNS and RR sectors are regarded as the untwisted and twisted sectors respectively. Thus, one can find the relation of the crosscaps before and after the GSO projection by following the argument used in finding the relation of crosscaps before and after orbifolding [30]. In this subsection, we shall refer to the theory before GSO projection simply as 'the theory' or 'the $\mathcal{N}=2$ theory' and the theory after GSO projection as 'the (GSO) projected theory' or 'the RCFT'. We shall also put a superscript 'GSO' to the space of states, the crosscaps, etc, of the projected theory. Let P be a parity of the theory, and consider the parity of the projected theory induced from P. The twisted partition function of the latter is

$$\mathop{\mathrm{Tr}}_{\mathcal{H}^{\mathrm{GSO}}} Pq^H = \frac{1}{2}\mathop{\mathrm{Tr}}_{\mathcal{H}_{\mathrm{NSNS}}}\big((1+(-1)^F)Pq^H\big) + \frac{1}{2}\mathop{\mathrm{Tr}}_{\mathcal{H}_{\mathrm{RR}}}\big((1+(-1)^F)Pq^H\big)\,.$$

Using (2.24) and (2.25), the four terms of the right hand side can be expressed as

$$\frac{1}{2}\langle \mathscr{C}_{(-1)^FP}|q_t^H|\mathscr{C}_{(-1)^FP}\rangle + \frac{1}{2}\langle \mathscr{C}_P|q_t^H|\mathscr{C}_P\rangle + \frac{1}{2}\langle \mathscr{C}_P|q_t^H|\mathscr{C}_{(-1)^FP}\rangle + \frac{1}{2}\langle \mathscr{C}_{(-1)^FP}|q_t^H|\mathscr{C}_P\rangle.$$

This shows that the crosscap of the projected theory is given by

$$|\mathscr{C}_P\rangle^{\mathrm{GSO}} = \mathrm{e}^{i\theta}\left[\frac{1}{\sqrt{2}}|\mathscr{C}_{(-1)^FP}\rangle + \frac{1}{\sqrt{2}}|\mathscr{C}_P\rangle\right].$$

One may also consider the parity $(-1)^{\widehat{F}}P$ of the projected theory, where $(-1)^{\widehat{F}}$ is 1 on NSNS sector and -1 on RR sector. Repeating the same procedure, we find

$$|\mathscr{C}_{(-1)^{\widehat{F}}P}\rangle^{\rm GSO} = {\rm e}^{i\theta'}\left[\frac{1}{\sqrt{2}}|\mathscr{C}_{(-1)^{F}P}\rangle - \frac{1}{\sqrt{2}}|\mathscr{C}_{P}\rangle\right].$$

We would like to invert these equations to express $|\mathscr{C}_P\rangle$ and $|\mathscr{C}_{(-1)^F P}\rangle$ as linear combinations of $|\mathscr{C}_P\rangle^{\rm GSO}$ and $|\mathscr{C}_{(-1)^{\widehat{F}}P}\rangle^{\rm GSO}$, which we know from the RCFT computation of the previous section. However, in order to find the right combination it is important to know the phases ${\rm e}^{i\theta}$ and ${\rm e}^{i\theta'}$, but there is no canonical way to fix them. Moreover, the phases of the RCFT crosscaps $|\mathscr{C}_P\rangle^{\rm GSO}$ and $|\mathscr{C}_{(-1)^{\widehat{F}}P}\rangle^{\rm GSO}$ are highly non-canonical.

4.5.1 The right combination

In fact, one can overcome this difficulty by making use of one independent constraint — the supercurrent condition. If P is an $A_{\alpha,\beta}$-parity or a $B_{\alpha,\beta}$-parity of the $\mathcal{N}=2$ theory, the crosscap $|\mathscr{C}_P\rangle$ obeys a certain supercurrent condition, which is satisfied only for a particular linear combination of $|\mathscr{C}_P\rangle^{\rm GSO}$ and $|\mathscr{C}_{(-1)^{\widehat{F}}P}\rangle^{\rm GSO}$. The other crosscap, $|\mathscr{C}_{(-1)^F P}\rangle$, obeys a different condition which is satisfied by another linear combination. In this way one can find the right linear combinations, up to an overall phase.

In what follows we carry out this program. It turns out that A-type (resp. B-type) parities in the RCFT correspond to $A_{\alpha,\beta}$-parities (resp. $B_{\alpha,\beta}$-parities) of the $\mathcal{N}=2$ theory. We thus separate the discussions into the two types.

A-type

The A-type crosscaps in RCFT are the PSS crosscaps $|\mathscr{C}_{n,s}\rangle$ labelled by simple currents, (n,s) with $n+s$ even. The symmetry $(-1)^{\widehat{F}}$ is nothing but the global symmetry labelled by $(n,s)=(0,2)$. Thus, $|\mathscr{C}_{n,s}\rangle$ and $|\mathscr{C}_{n,s+2}\rangle$ corresponds to $|\mathscr{C}_{P_{n,s}}\rangle^{\rm GSO}$ and $|\mathscr{C}_{(-1)^{\widehat{F}}P_{n,s}}\rangle^{\rm GSO}$ for a suitable $P_{n,s}$. The task is to identify $P_{n,s}$ and find the right combination to express $|\mathscr{C}_{P_{n,s}}\rangle$ and $|\mathscr{C}_{(-1)^F P_{n,s}}\rangle$. Since $|\mathscr{C}_{n,s}\rangle$ belongs to NSNS-sector (resp. RR-sector) for even s (resp. odd s), $P_{n,s}$ squares to $(-1)^F$ if s is even and it is involutive if s is odd.

The crosscap state of an $A_{\alpha,\beta}$-parity must obey the supercurrent condition (2.20). In terms of the Fourier modes, it is

$$G_r + i\,{\rm e}^{-i\alpha}(-1)^{r+\frac{\beta}{\pi}}\overline{\widetilde{G}}_{-r} = \widetilde{G}_r - i\,{\rm e}^{-i\alpha}(-1)^{r+\frac{\beta}{\pi}}\overline{G}_{-r} = 0, \quad r\in\mathbb{Z}-\tfrac{\beta}{\pi}. \tag{4.72}$$

Let us put

$$|\mathscr{C}_{n,s}(\pm)\rangle := \frac{1}{\sqrt{2}}|\mathscr{C}_{n,s}\rangle \mp \frac{1}{\sqrt{2}}\frac{\sqrt{T_{0,n,s}}}{\sqrt{T_{0,n,s+2}}}|\mathscr{C}_{n,s+2}\rangle. \tag{4.73}$$

One can show that they obey the conditions of the types described in the following table (see Appendix E for the proof):

	s odd	s even
$\|\mathscr{C}_{n,s}(+)\rangle$	$A_{0,0}$	$A_{\frac{\pi}{2},\frac{\pi}{2}}$
$\|\mathscr{C}_{n,s}(-)\rangle$	$A_{\pi,0}$	$A_{\frac{\pi}{2},-\frac{\pi}{2}}$

We know that the $A_{0,0}$, $A_{\pi,0}$, $A_{\frac{\pi}{2},-\frac{\pi}{2}}$, $A_{\frac{\pi}{2},\frac{\pi}{2}}$-parities of the theory are $a^{even}P_A$, $(-1)^F a^{even}P_A$, $a^{odd}P_A$, $(-1)^F a^{odd}P_A$ respectively. We also know that the axial symmetry a induces the global symmetry of the projected theory labelled by $(n,s)=(1,\pm 1)$. These are enough to show that

$$|\mathscr{C}_{a^{2m}P_A}\rangle = |\mathscr{C}_{2m-1,2m-1}(+)\rangle, \tag{4.74}$$

$$|\mathscr{C}_{(-1)^F a^{2m}P_A}\rangle = |\mathscr{C}_{2m-1,2m-1}(-)\rangle, \tag{4.75}$$

$$|\mathscr{C}_{a^{2m+1}P_A}\rangle = |\mathscr{C}_{2m,2m}(-)\rangle, \tag{4.76}$$

$$|\mathscr{C}_{(-1)^F a^{2m+1}P_A}\rangle = |\mathscr{C}_{2m,2m}(+)\rangle. \tag{4.77}$$

The overall phases are not fixed by this argument. Here we have chosen the ones that will be justified by later computations.

B-type

B-type crosscaps in RCFT are $|\mathscr{C}^B_{r,q,p}\rangle$ given in (4.68) labelled by $(r,q)\in\mathbb{Z}_{k+2}\times\mathbb{Z}_2$ and $p\in\{0,1\}$. Combining the parity with $(-1)^{\widehat{F}}$ corresponds to the shift $q\to q+1$. Thus, we need to find the right combinations of $|\mathscr{C}^B_{r,q,p}\rangle$ and $|\mathscr{C}^B_{r,q+1,p}\rangle$. Since $|\mathscr{C}^B_{r,q,p}\rangle$ belongs to the sector twisted by the symmetry labelled by $(n,s)=(4r+2p,2p)$, it corresponds to the parity that squares to $a^{4r}(-1)^F$ if $p=0$ and a^{4r+2} if $p=1$. This shows that $P_B^{r,q,p}$ is induced from $a^{2r+p}P_B$ or $(-1)^F a^{2r+p}P_B$ (or those combined with axial R-symmetry).

The supercurrent condition on the crosscap state of a $B_{\alpha,\beta}$-parity is (2.21), or in terms of the Fourier modes

$$G_r + i\,\mathrm{e}^{i\beta}(-1)^{r-\frac{\alpha}{\pi}}\widetilde{G}_{-r} = \overline{\widetilde{G}}_r - i\,\mathrm{e}^{i\beta}(-1)^{r-\frac{\alpha}{\pi}}\overline{G}_{-r} = 0, \quad r\in\mathbb{Z}+\tfrac{\alpha}{\pi}. \tag{4.78}$$

Let us put

$$|\mathscr{C}^B_{r,p}(\pm)\rangle := \frac{1}{\sqrt{2}}|\mathscr{C}^B_{r,0,p}\rangle \pm \frac{1}{\sqrt{2}}\frac{\sqrt{T^{(2)}_{2+p}}}{\sqrt{T^{(2)}_p}}|\mathscr{C}^B_{r,1,p}\rangle, \tag{4.79}$$

where $\sqrt{T_n^{(2)}} = \mathrm{e}^{\pi i \hat{n}^2/8}$ is the square-root of the T-matrix of rational $U(1)$ at level 2. One can show that $|\mathscr{C}^B_{r,p}(\pm)\rangle$ obey the conditions of the types described in the following table (see Appendix E for the proof):

	$p=1$	$p=0$
$\lvert\mathscr{C}^B_{r,p}(+)\rangle$	$B_{0,0}$	$B_{\frac{\pi}{2},\frac{\pi}{2}}$
$\lvert\mathscr{C}^B_{r,p}(-)\rangle$	$B_{0,\pi}$	$B_{\frac{\pi}{2},-\frac{\pi}{2}}$

Since $B_{0,0}$, $B_{0,\pi}$, $B_{\frac{\pi}{2},-\frac{\pi}{2}}$, $B_{\frac{\pi}{2},\frac{\pi}{2}}$-parities are $a^{odd}P_B$, $(-1)^F a^{odd} P_B$, $a^{even}P_B$, $(-1)^F a^{even} P_B$, we can conclude that

$$|\mathscr{C}_{a^{2m+1}P_B}\rangle = (-1)^m |\mathscr{C}^B_{m,1}(+)\rangle, \tag{4.80}$$

$$|\mathscr{C}_{(-1)^F a^{2m+1}P_B}\rangle = |\mathscr{C}^B_{m,1}(-)\rangle, \tag{4.81}$$

$$|\mathscr{C}_{a^{2m}P_B}\rangle = (-1)^m |\mathscr{C}^B_{m,0}(-)\rangle, \tag{4.82}$$

$$|\mathscr{C}_{(-1)^F a^{2m}P_B}\rangle = |\mathscr{C}^B_{m,0}(+)\rangle. \tag{4.83}$$

Again, the overall phases cannot be fixed by this argument. The choice we made will be justified by later computations.

4.5.2 Overlaps with supersymmetric ground states

Using the above results, one can compute the overlaps of the supersymmetric crosscaps and supersymmetric ground states.

A-parities

To compute the overlaps of the crosscap for the basic A-parity P_A and the RR ground states $|j\rangle_{\rm RR}$ one has to read off the coefficient of $|\mathscr{C}, j, -2j-1, -1\rangle\rangle$ in the expansion of $|\mathscr{C}_{-1,-1}(+)\rangle$. Since $\sqrt{T_{0,-1,-1}/T_{0,-1,1}} = -1$ by supersymmetry, the latter is $(|\mathscr{C}_{-1,-1}\rangle + |\mathscr{C}_{-1,1}\rangle)/\sqrt{2}$. The result is

$${}_{\rm RR}\langle j|\mathscr{C}_{P_A}\rangle = \sqrt{\frac{2}{(k+2)\sin(\frac{\pi(2j+1)}{k+2})}}\,\mathrm{e}^{\frac{\pi i(2j+1)}{2(k+2)}}\left\{-i\delta^{(2)}_{2j+k}\sin\left(\tfrac{\pi(2j+1)}{2(k+2)}\right) + \delta^{(2)}_{2j}\cos\left(\tfrac{\pi(2j+1)}{2(k+2)}\right)\right\}. \tag{4.84}$$

Also, we find

$$\langle\mathscr{C}_{(-1)^F P_A}|j\rangle_{\rm RR} = \sqrt{\frac{2}{(k+2)\sin(\frac{\pi(2j+1)}{k+2})}}\,\mathrm{e}^{\frac{-\pi i(2j+1)}{2(k+2)}}\left\{\delta^{(2)}_{2j+k}\sin\left(\tfrac{\pi(2j+1)}{2(k+2)}\right) + i\delta^{(2)}_{2j}\cos\left(\tfrac{\pi(2j+1)}{2(k+2)}\right)\right\}. \tag{4.85}$$

For other A-parities the overlaps can be easily obtained by using $|\mathscr{C}_{a^{2m}P_A}\rangle = \pm a^m |\mathscr{C}_{P_A}\rangle$,

$$
\begin{aligned}
{}_{\rm RR}\langle j|\mathscr{C}_{a^{2m}P_A}\rangle &= \pm\, \mathrm{e}^{-2\pi i m(\frac{2j+1}{k+2}-\frac{1}{2})}\, {}_{\rm RR}\langle j|\mathscr{C}_{P_A}\rangle, \\
\langle \mathscr{C}_{(-1)^F a^{2m}P_A}|j\rangle_{\rm RR} &= \pm\, \mathrm{e}^{2\pi i m(\frac{2j+1}{k+2}-\frac{1}{2})} \langle \mathscr{C}_{(-1)^F P_A}|j\rangle_{\rm RR}.
\end{aligned}
$$

B-parities

Since the B-parity $a^{2m+1}P_B$ squares to $a^{2(2m+1)}$, the crosscap state belongs to the sector with the twist parameter $(a,\widetilde{a})=(2m+1,0)$. Such a sector has a $(2,2)$ supersymmetry and has a unique supersymmetric ground state $|\mathrm{G}\rangle_{2m+1,0}$ which is an element of $B_{j_*,-2j_*-1,-1}\otimes B_{j_*,-2j_*-1,-1}$ where $j_*\in \mathrm{P}_k$ is defined by $2j_*+1\equiv -(a+\widetilde{a}) = -(2m+1) \bmod (k+2)$. Namely,

$$
j_* = \begin{cases} \frac{k}{2}-m & \text{if } m=0,1,...,[\frac{k}{2}], \\ k+1-m & \text{if } m=[\frac{k}{2}]+1,...,k+1. \end{cases}
$$

In the two cases, $(j_*,-2j_*-1,-1)$ are equivalent to $(m,2m+1,1)$ and $(k+1-m,2m+1,-1)$ respectively. We are interested in the overlaps of this ground state $|\mathrm{G}\rangle_{a,\widetilde{a}}$ and the crosscap states

$$
\left.\begin{array}{l} |\mathscr{C}_{a^{2m+1}P_B}\rangle(-1)^m \\ |\mathscr{C}_{(-1)^F a^{2m+1}P_B}\rangle \end{array}\right\} = \frac{1}{\sqrt{2}}|\mathscr{C}^B_{m,0,1}\rangle \pm \frac{1}{\sqrt{2}}|\mathscr{C}^B_{m,1,1}\rangle.
$$

The overlaps are obtained by reading the coefficient of $|\mathscr{C},m,2m+1,1\rangle\!\rangle$ or $|\mathscr{C},k+1-m,2m+1,-1\rangle\!\rangle$ depending on $m\equiv 0,1,...,[\frac{k}{2}]$ or $m\equiv[\frac{k}{2}]+1,...,k+1$ (mod $(k+2)$). The result is

$$
{}_{2m+1,0}\langle \mathrm{G}|\mathscr{C}_{a^{2m+1}P_B}\rangle = \begin{cases} (-1)^m\sqrt{\cot\left(\frac{\pi(2m+1)}{2(k+2)}\right)} & m\equiv 0,1,...,[\frac{k}{2}], \\ (-1)^{m+k+1}\sqrt{-\cot\left(\frac{\pi(2m+1)}{2(k+2)}\right)} & m\equiv [\frac{k}{2}]+1,...,k+1, \end{cases} \tag{4.86}
$$

and

$$
{}_{2m+1,0}\langle \mathrm{G}|\mathscr{C}_{(-1)^F a^{2m+1}P_B}\rangle = \begin{cases} \sqrt{\cot\left(\frac{\pi(2m+1)}{2(k+2)}\right)} & m\equiv 0,1,...,[\frac{k}{2}], \\ (-1)^{k}\sqrt{-\cot\left(\frac{\pi(2m+1)}{2(k+2)}\right)} & m\equiv [\frac{k}{2}]+1,...,k+1. \end{cases} \tag{4.87}
$$

This is sufficient to show that the twisted Witten index is

$$
I_{a^{2m+1}P_B} = \langle \mathscr{C}_{(-1)^F a^{2m+1}P_B}|q_t^H|\mathscr{C}_{a^{2m+1}P_B}\rangle = (-1)^m \cot\left(\frac{\pi(2m+1)}{2(k+2)}\right), \tag{4.88}
$$

which reproduces the loop channel result (4.48), provided $\varepsilon_{\rm RR}=1$.

4.5.3 Partition function and Witten index

The expressions for the crosscap states obtained above can now be used to compute the parity-twisted partition functions, or equivalently Klein bottle amplitudes, $\mathrm{Tr}_{\mathrm{NSNS}}(Pq^H) = \langle \mathscr{C}_{(-1)^F P}|q_t^H|\mathscr{C}_{(-1)^F P}\rangle$, $\mathrm{Tr}_{\mathrm{RR}}(Pq^H) = \langle \mathscr{C}_{(-1)^F P}|q_t^H|\mathscr{C}_P\rangle$.

A-type

We first evaluate those amplitudes for A and $\widetilde{A}$-parities $P = a^\ell P_A, (-1)^F a^\ell P_A$. Since each crosscap is a sum of two RCFT crosscaps, the partition function is a sum of four terms. The summands are

$$\begin{aligned}
\langle \mathscr{C}_{\ell-1,\ell-1}|q_t^H|\mathscr{C}_{\ell-1,\ell-1}\rangle &= \sum_{j\in\mathbb{Z}}(\chi_{j00} - (-1)^\ell \chi_{j02}) + \delta_k^{(2)}(\chi_{\frac{k}{4},\frac{k+2}{2},1} - (-1)^\ell \chi_{\frac{k}{4},\frac{k+2}{2},-1})\\
\langle \mathscr{C}_{2m-1,2m-1}|q_t^H|\mathscr{C}_{2m-1,2m+1}\rangle &= 0,\\
\langle \mathscr{C}_{2m,2m}|q_t^H|\mathscr{C}_{2m,2m+2}\rangle &= -\frac{\sigma_{0,2m,2m}}{\sigma_{0,2m,2m+2}}\sum_{j\in\mathbb{Z}+\frac{1}{2}}(\chi_{j,0,1} + \chi_{j,0,-1}),
\end{aligned}$$

where the argument of the characters are all 2τ. Using these formulae and also the relation $\sqrt{\frac{T_{0,2m,2m}}{T_{0,2m,2m+2}}}\frac{\sigma_{0,2m,2m}}{\sigma_{0,2m,2m+2}} = \sqrt{\frac{T^{(2)}_{2m}}{T^{(2)}_{2m+2}}} = -i(-1)^m$, we find

$$\mathrm{Tr}_{\mathrm{NSNS}}(-1)^{\nu F} a^\ell P_A q^H = \sum_{j\in\mathbb{Z}}(\chi_{j00} - (-1)^\ell \chi_{j02}), \tag{4.89}$$

$$\begin{aligned}
\mathrm{Tr}_{\mathrm{RR}}(-1)^{\nu F} a^\ell P_A q^H &= -\,\mathrm{e}^{\frac{\pi i \ell}{2}}(-1)^\nu \sum_{j\in\mathbb{Z}+\frac{1}{2}}(\chi_{j,0,1} - (-1)^\ell \chi_{j,0,-1})\\
&\quad + \delta_k^{(2)}(\chi_{\frac{k}{4},\frac{k+2}{2},1} - (-1)^\ell \chi_{\frac{k}{4},\frac{k+2}{2},-1}).
\end{aligned} \tag{4.90}$$

Note that this reproduces the results (4.39) and (4.40) obtained in the gauged WZW model, where the constants undetermined there are now fixed as

$$\begin{aligned}
\epsilon_{\mathrm{NSNS}} &= 1,\\
\epsilon_{\mathrm{RR}} &= 1,\\
\pm(-1)^{\frac{k}{2}} &= 1.
\end{aligned}$$

In particular, the Witten index is fixed as

$$I_{a^{2m}P_A} = \begin{cases} 1 & k \text{ even} \\ 0 & k \text{ odd.} \end{cases} \tag{4.91}$$

B-type

We now consider the partition functions for B- and $\widetilde{B}$-parities. The computation is simpler here since the pairings of crosscaps of different q (for the same r,p) vanish, $\langle \mathscr{C}^B_{r,q,p}|\mathscr{C}^B_{r,q+1,p}\rangle = 0$. Using the formula (4.70), we find

$$\begin{aligned}
\mathrm{Tr}_{\mathrm{NSNS}}(-1)^{\nu F} a^{2m+p} P_B q^H &= \frac{1}{2}\langle \mathscr{C}_{m,0,p}|q_t^H|\mathscr{C}_{m,0,p}\rangle + \frac{1}{2}\langle \mathscr{C}_{m,1,p}|q_t^H|\mathscr{C}_{m,1,p}\rangle \\
&= \sum \mathrm{e}^{\pi i \frac{(2m+p)n}{k+2}} \frac{\mathrm{e}^{-\pi i \frac{ps}{2}} + \mathrm{e}^{\pi i \frac{(p+2)s}{2}}}{2} \chi_{jns}(2\tau) \\
&= \sum_{s\,\mathrm{even}} \mathrm{e}^{\pi i (2m+p)(\frac{n}{k+2} - \frac{s}{2})} \chi_{jns}(2\tau), \qquad (4.92)
\end{aligned}$$

$$\begin{aligned}
\mathrm{Tr}_{\mathrm{RR}}(-1)^{\nu F} a^{2m+p} P_B q^H &= \frac{(-1)^m}{2}\langle \mathscr{C}_{m,0,p}|q_t^H|\mathscr{C}_{m,0,p}\rangle - \frac{(-1)^m}{2}\langle \mathscr{C}_{m,1,p}|q_t^H|\mathscr{C}_{m,1,p}\rangle \\
&= (-1)^m \sum \mathrm{e}^{\pi i \frac{(2m+p)n}{k+2}} \frac{\mathrm{e}^{-\pi i \frac{ps}{2}} - \mathrm{e}^{\pi i \frac{(p+2)s}{2}}}{2} \chi_{jns}(2\tau) \\
&= \sum_{s\,\mathrm{odd}} \mathrm{e}^{\pi i (2m+p)(\frac{n}{k+2} - \frac{s}{2})} \chi_{jns}(2\tau). \qquad (4.93)
\end{aligned}$$

This reproduces the results (4.46) and (4.47) we obtained in the gauged WZW model. In particular, the undetermined coefficients there are determined now as

$$\varepsilon_{\mathrm{NSNS}} = 1,$$
$$\varepsilon_{\mathrm{RR}} = 1.$$

The Witten index $I_{a^{2m+1}P_B}$ is given by (4.88).

5 Orientifold of $\mathcal{N}=2$ Minimal Models II — Open Strings

This is a continuation of the previous section. We include D-branes into the discussion.

5.1 Facts on D-branes in $\mathcal{N}=2$ minimal models

We start with reviewing some known facts about D-branes in the minimal model [21].

5.1.1 A geometrical picture

It is convenient to provide a basic geometrical picture of the branes. As we have seen, the $SU(2)$ mod $U(1)$ supersymmetric gauged WZW model can be interpreted as the sigma model on the disk $|z| \leq 1$ with the metric $\mathrm{d}s^2 = k|\mathrm{d}z|^2/(1-|z|^2)$, with non-trivial dilaton. As in ordinary sigma models, A-branes are wrapped on Lagrangian submanifolds (1-dimension) and B-branes are at complex submanifolds (0 or 2-dimensions).

A-branes

A-branes are denoted as $\mathscr{B}_{j,n,s}$ where $(j,n,s) \in \mathrm{M}_k$. There are $2k+4$ special points at the boundary of the disk. They fall into two classes, the even and the odd points. The A-branes are D1 branes which stretch between these special points. The branes with s even extend between the even points, while those with s odd extend between the odd points. More precisely, the brane $\mathscr{B}_{jns}$ stretches between boundary points

$$z_i = \mathrm{e}^{\frac{\pi i}{k+2}(n-2j-1)} \quad \text{and} \quad z_f = \mathrm{e}^{\frac{\pi i}{k+2}(n+2j+1)} \tag{5.1}$$

For $s = 0, -1$ the orientation of the brane is from z_i to z_f, and for $s = 1, 2$ the other way around meaning that $s \to s+2$ is an orientation flip. The corresponding boundary condition on the right boundary of the string preserves the combination $\overline{Q}_+ - (-1)^s Q_-$ of the supersymmetry. The boundary condition on the left boundary preserves the opposite combination $\overline{Q}_+ + (-1)^s Q_-$ for the standard reason. The axial rotation a is a rotation of the disc by angle $\pi/(k+2)$ and thus rotates the brane as $n \to n+1$.

B-branes

There are unoriented B-branes denoted as $\mathscr{B}^B_{[j,s]}$, where $j \in \mathrm{P}_k$, $j < [\frac{k}{4}]$ and $s \in \mathbb{Z}_2$. They are located at concentric smaller disks, whose radius depends on the label j. The $j = 0$ states represent D0 branes at the center of the disk whereas the higher j states correspond to D2-branes. For even k, there are also oriented B-branes $\mathscr{B}^B_{\frac{k}{4},s}$ where $s \in \mathbb{Z}_4$. They are D2 branes wrapping the whole disk. $s \to s+2$ corresponds to an orientation flip. As before, the corresponding boundary condition on the right boundary preserves $Q_+ - (-1)^s Q_-$, while on the left boundary the opposite combination $Q_+ + (-1)^s Q_-$ is preserved. The axial rotation a does the exchanges $\mathscr{B}^B_{[j,0]} \leftrightarrow \mathscr{B}^B_{[j,1]}$ and also $\mathscr{B}^B_{\frac{k}{4},s} \leftrightarrow \mathscr{B}^B_{\frac{k}{4},s\pm 1}$.[1]

[1]Note that 'orientation of branes' is defined in the GSO projected theory, while the 'axial rotation a' is defined before the GSO. This and the above paragraphs contain a certain abuse of language. This is the reason for the arbitrariness in, say, the "action of a on the branes." There is of course a well-defined description in both before and after GSO, separately (as given in more detail in the following discussions).

5.1.2 Boundary states and one-loop amplitudes

We next provide the boundary states and cylinder amplitudes, both in the GSO projected theory.

A-branes

In the RCFT, the A-branes $\mathscr{B}_{j,n,s}$ are described by the Cardy states [73]

$$|\mathscr{B}_{j,n,s}\rangle = \sum_{(j'n's')\in \mathrm{M}_k} \frac{S_{(j,n,s)(j',n',s')}}{\sqrt{S_{(0,0,0)(j',n',s')}}} |j', n', s'\rangle\rangle .$$

Under the global symmetry $g_{n,s}$, the branes are transformed as

$$g_{n,s} : \mathscr{B}_{j,n',s'} \rightarrow \mathscr{B}_{j,n'+n,s'+s} .$$

The cylinder amplitudes are

$$\langle \mathscr{B}_{j,n,s}|\, \mathrm{e}^{-\frac{\pi i}{\tau}H}\, |\mathscr{B}_{j',n',s'}\rangle = \sum_{j''\in \mathrm{P}_k} N_{jj'}^{j''}\, \chi_{j'',n-n',s-s'}(\tau) \tag{5.2}$$

This shows that the open string Hilbert space is the sum of $\mathscr{H}_{j'',n-n',s-s'}$ where j'' runs over P_k such that $N_{jj'}^{j''} = 1$.

B-branes

B-type boundary states are obtained by taking the $\mathbb{Z}_{k+2} \times \mathbb{Z}_2$ orbit of A-type boundary states, followed by an application of the mirror map. They are labelled by (j, n, s) modulo the action of the group, $(j, n, s) \sim (j, n + 2, s) \sim (j, n, s + 2)$. For each j there are only two orbits distinguished by s even or s odd. n is then even or odd as required by the selection rule. We also note that $(j, n, s) \sim (\frac{k}{2} - j, n + k, s)$. Thus, the states are labelled by $[j, s]$ where $j \leq [\frac{k}{2}]$ and $s \in \mathbb{Z}_2$. The boundary states are

$$\begin{aligned} |\mathscr{B}^B_{[j,s]}\rangle &= \frac{1}{\sqrt{2k+4}} \sum_{n',r} (V_M \otimes 1)\, |\mathscr{B}_{j,n+2n',s+2r}\rangle \\ &= (k+2)^{\frac{1}{4}} \sum_{j'} \delta^{(2)}_{2j'} \frac{S_{jj'}}{\sqrt{S_{0j'}}} \left(|j', 0, 0\rangle\rangle_B + (-1)^s |j', 0, 2\rangle\rangle_B\right) \end{aligned} \tag{5.3}$$

Since there is no RR-part, the brane is unoriented. If k is even, the orbifold action has fixed points at $j = \frac{k}{4}$. The above boundary states for $j = \frac{k}{4}$ should be further resolved

[21] (see [74] for a general discussion and [?, ?] for the original discussion in the context of WZW models with non-diagonal modular invariant) as

$$\left|\mathscr{B}_{\frac{k}{4},S}\right\rangle = \frac{1}{2}\left(\left|\mathscr{B}^B_{[\frac{k}{4},S]}\right\rangle + \sqrt{k+2}\,\mathrm{e}^{-i\frac{\pi S^2}{2}} \sum_{s=\pm1} \mathrm{e}^{-i\frac{\pi S s}{2}} |\frac{k}{4}, \frac{k+2}{2}, s\rangle\!\rangle_B\right) \tag{5.4}$$

Here, S can take the values $-1, 0, 1, 2$. $S \to S+2$ flips the sign of the RR part and thus corresponds to the orientation flip.

The action of the global symmetry $g_{n,s}$ on the branes can be read off from the boundary states.

$$\begin{array}{ll} g_{1,1} : \mathscr{B}^B_{[j,s]} \to \mathscr{B}^B_{[j,s+1]}, & g_{0,2} : \mathscr{B}^B_{[j,s]} \to \mathscr{B}^B_{[j,s]}, \\ g_{1,1} : \mathscr{B}^B_{\frac{k}{4},S} \to \mathscr{B}^B_{\frac{k}{4},S-(-1)^S}, & g_{0,2} : \mathscr{B}^B_{\frac{k}{4},S} \to \mathscr{B}^B_{\frac{k}{4},S+2}. \end{array} \tag{5.5}$$

In particular, the unoriented brane $\mathscr{B}^B_{[j,s]}$ is invariant under the subgroup $\mathbb{Z}_{k+2}\times\mathbb{Z}_2$ generated by $g_{2,0}$ and $g_{0,2}$, while the oriented brane $\mathscr{B}^B_{\frac{k}{4},S}$ is invariant under the subgroup $\mathbb{Z}_{k+2}$ generated by $g_{2,2}$.

We will later compute parity-twisted partition functions. Since the B-parities are not always involutive (4.69), we need to have the boundary state on the circles twisted by $P^2_{rqp} = g_{4r+2p,2p}$. All B-branes are invariant under this, and one can indeed think about the boundary states on the circle twisted by this symmetry. For the construction, we use the fact [30] that boundary states of the orbifold $\mathcal{C}/G$ on the circle twisted by a quantum symmetry g_ρ associated with the character $g \to \mathrm{e}^{2\pi i\rho(g)}$ are given by

$$\left|\mathscr{B}_{[i]}\right\rangle^{\mathcal{C}/G}_{g_\rho} = \frac{e^{i\lambda}}{\sqrt{|G|}} \sum_{g\in G} \mathrm{e}^{-2\pi i\rho(g)} \left|\mathscr{B}_{g(i)}\right\rangle^{\mathcal{C}}. \tag{5.6}$$

Since the boundary states associated to long orbits remain invariant under the action of the symmetry group $\mathbb{Z}_{k+2}\times\mathbb{Z}_2$, one can consider boundary states on circles with $\mathbb{Z}_{k+2}\times\mathbb{Z}_2$ twisted boundary conditions. To construct those states, note that the group element $g^r_{2,0}$ is mapped to a quantum symmetry of the orbifold model associated to the character $g_{2n,2q} \to \mathrm{e}^{-2\pi i\frac{rn}{k+2}}$. Similarly, $g^r_{0,2}$ is mapped to the character $g_{2n,2q} \to e^{\pi i rq}$. Hence, the boundary states on the twisted circles are

$$\begin{aligned} \left|\mathscr{B}^B_{[j,s]}\right\rangle_{g_{2n',2s'}} &= \frac{e^{i\lambda}}{\sqrt{2(k+2)}} \sum_{\hat{n},\hat{s}:\,\text{even}} \mathrm{e}^{\pi i\frac{\hat{n}n'}{k+2}}\, e^{-\pi i\frac{\hat{s}s'}{2}}\, (V_M\otimes 1)\, \left|\mathscr{B}_{j,n\hat{+}\hat{n},s\hat{+}\hat{s}}\right\rangle \\ &= (2k+4)^{\frac{1}{4}}\,\mathrm{e}^{i\lambda}\,\mathrm{e}^{-\frac{\pi inn'}{k+2}}\,\mathrm{e}^{\frac{\pi iss'}{2}} \sum_{j'} \frac{S_{jj'}}{\sqrt{S_{0j'}}}\left(|j',n',s'\rangle\!\rangle_B + (-1)^s|j',n',s'+2\rangle\!\rangle_B\right) \end{aligned} \tag{5.7}$$

The boundary state is an element of the twisted Hilbert space with twist $g_{2n',2s'}$ as indicated by the subscript. Note that $|\mathscr{B}^B_{[j,s]}\rangle_{g_{2n',2s'}} = |\mathscr{B}^B_{[j,s]}\rangle_{g_{2(n'+k+2),2(s'+2)}}$. We now choose the n', s' dependent phase $e^{i\lambda} = \mathrm{e}^{\pi i \frac{nn'}{k+2}} \mathrm{e}^{-\frac{\pi i s s'}{2}}$, which makes the boundary state real. This choice breaks the explicit invariance of the expression under shifts of n' by $k+2$ and of s' by 2, effectively reducing the range of n', s' to $n' \in \{-\frac{k}{2}, \dots, \frac{k+2}{2}\}$ and $s' \in \{0, 1\}$. With this choice, the equations $\frac{2n'}{2} = n'$ and $\frac{2s'}{2} = s'$ hold exactly, not only modulo $k+2$ or 2. The short orbit state (5.4) is only invariant under the elements of the subgroup generated by $g_{2,2}$, therefore, we can only construct twisted boundary states with the corresponding twists. They are

$$\left|\mathscr{B}_{\frac{k}{4},S}\right\rangle_{g_{2n',2n'}} = \frac{1}{2}\left(\left|\mathscr{B}^B_{[k/4,S]}\right\rangle_{g_{2n',2n'}} + \sqrt{k+2}\,\mathrm{e}^{-i\frac{\pi S^2}{2}} \sum_{s=\pm 1} \mathrm{e}^{-i\frac{\pi S s}{2}} |\frac{k}{4}, \frac{k+2}{2} + n', s + n'\rangle\!\rangle_B\right) \tag{5.8}$$

The one-loop amplitudes

The cylinder amplitude makes sense for any two boundary states on the same twisted circle. Hence, we can consider the following amplitudes between long orbit branes

$$\langle \mathscr{B}^B_{[j,s]}|e^{-\frac{\pi i}{\tau}H}|\mathscr{B}^B_{[j's']}\rangle_{g_{2\bar{n},2\bar{s}}} = \sum_{2j''+n''+s''\ \mathrm{even}} N^{j''}_{jj'}\,\delta^{(2)}_{s'-s+s''}\,\mathrm{e}^{\frac{\pi i \bar{n} n''}{k+2} - \frac{\pi i \bar{s} s''}{2}} \chi_{j'',n'',s''}(\tau). \tag{5.9}$$

For short orbit branes, we obtain

$$\langle \mathscr{B}^B_{\frac{k}{4},S}|e^{-\frac{\pi i}{\tau}H}|\mathscr{B}^B_{\frac{k}{4},S'}\rangle_{g_{2\bar{n},2\bar{n}}}$$
$$= \sum_{(j,n,s)\in \mathrm{M}_k} \delta^{(2)}_{2j}\,\delta^{(2)}_{S-S'-s} \frac{1+(-1)^{\frac{2j+n-s}{2}}\,\mathrm{e}^{\frac{i\pi}{2}(S^2+S-S'^2-S')}}{2}\,\mathrm{e}^{\frac{\pi i \bar{n} n}{k+2} - \frac{\pi i \bar{n} s}{2}} \chi_{j,n,s}(\tau). \tag{5.10}$$

Since the long orbit branes only have non-vanishing overlap with the orbit part of the short orbit boundary state, one can easily obtain the cylinder involving a long and a short orbit brane from (5.9) by setting $j' = k/4$ and dividing by two.

5.2 Parity Actions on D-branes and open strings

Let us now compute the Möbius strip amplitudes in the GSO projected theory in order to find how the parity acts on the D-branes and on the open string states.

5.2.1 Geometrical picture

Before doing the CFT computation, consider the actions in the geometrical picture. In Section 4.2.5, we have seen that the A-parities act as the complex conjugation (4.26), folding the disk along the diameters, while B-parities act as the rotations (4.27), including the identity $z \to z$ as well as the inversion $z \to -z$. This can already tell, roughly, how these parities act on the D-branes.

Let us first look how A-branes are transformed. The A-type parity $a^\ell P_A$ acts on the disk as the reflection $z \to e^{\frac{\pi i}{k+2}\ell}\overline{z}$, mapping the initial/final points $z_i(j,n)$ and $z_f(j,n)$ of the brane $\mathscr{B}_{j,n,s}$ (5.1) as $z_i(j,n) \to z_f(j,\ell-n)$ and $z_f(j,n) \to z_i(j,\ell-n)$. Thus, the label (j,n) is mapped to $(j,\ell-n)$ and the orientation is flipped. If we denote the oriented segment from $z_i(j,n)$ to $z_f(j,n)$ by $\overrightarrow{L}_{j,n}$ and the one with reversed orientation by $\overleftarrow{L}_{j,n}$, the parity maps them as

$$a^\ell P_A : \overrightarrow{L}_{j,n} \to \overleftarrow{L}_{j,\ell-n}. \tag{5.11}$$

The B-type parity $a^\ell P_B$ acts on the disk as the rotation $z \to e^{\frac{\pi i}{k+2}\ell}z$, mapping the initial/final points as $z_{i,f}(j,n) \to z_{i,f}(j,n+\ell)$. Thus, the transformation rule is

$$a^\ell P_B : \overrightarrow{L}_{j,n} \to \overrightarrow{L}_{j,n+\ell}. \tag{5.12}$$

B-branes are D0-brane at the center or D2-branes located at the concentric disks whose radii are determined by j. Since concentric disks are invariant under both reflections $z \to e^{\frac{\pi i}{k+2}\ell}\overline{z}$ and rotations $z \to e^{\frac{\pi i}{k+2}\ell}z$, both A-type and B-type parities preserve the j-label of the B-branes. To find the action on the s (or S) label, we need a geometrical interpretation of s (or S), which is currently missing. This is found, however, by computing and reading the Möbius strip amplitudes, which we now do.

5.2.2 Möbius strips in RCFT

A-type

A-type parities corresponds to PSS crosscaps and A-branes correspond to Cardy states. Thus, the actions of A-type parities on A-branes follow the general rule in RCFT:

$$P_{\bar{n},\bar{s}} : \mathscr{B}_{j,n,s} \to \mathscr{B}_{j,\bar{n}-n,\bar{s}-s}. \tag{5.13}$$

To compare this with the geometric action (5.11), we recall that the parity $a^\ell P_A$ becomes the PSS-parity $P_{\ell-1,\ell\mp1}$ after GSO projection (4.74)-(4.77), where the ambiguity in the s-index is due to the symmetry $(-1)^{\widehat{F}}$ absent before GSO projection. Thus, (5.11) appears to suggest the action $P_{\bar{n},\bullet} : \mathscr{B}_{j,n,\bullet} \to \mathscr{B}_{j,\bar{n}+1-n,\bullet}$, where the s-indices are hidden by $\bullet$ due to the ambiguity mentioned. The j-index is invariant as in (5.13) but the transformation of the n-index is different from the RCFT rule (5.13). A possible resolution of this problem is that the correspondence between the geometry and the boundary state depends on whether the boundary is on the left or on the right of the string. If the boundary state for the brane located at $L_{j,n}$ is $|\mathscr{B}_{j,n,\bullet}\rangle$ on the right-boundary but is $\langle\mathscr{B}_{j,n-1,\bullet}|$ on the right-boundary, then the two transformation rules (5.11) and (5.13) are consistent with each other. This is exactly the case in the Landau–Ginzburg description of the model where there is a similar geometric picture of the branes [22, 23], as we will see in the next section.

To find the parity action on the states, we compute the Möbius strip amplitudes, which is represented in the tree-channel by the overlap of the crosscap states and the boundary states. A computation as shown in Appendix D leads to the following result

$$\langle\mathscr{C}_{\bar{n},\bar{s}}|q_t^H|\mathscr{B}_{j,n,s}\rangle = \sum_{j'\in \mathrm{P}_k} N_{jj}^{j'}\delta^{(2k+4)}_{2n+n'-\bar{n}}\delta^{(4)}_{2s+s'-\bar{s}}\epsilon^{j,n,s}_{\bar{n},\bar{s}}(j',n',s')\widehat{\chi}_{j',n',s'}(\tau), \tag{5.14}$$

where $\epsilon^{j,n,s}_{\bar{n},\bar{s}}$ is the sign factor

$$\epsilon^{j,n,s}_{\bar{n},\bar{s}}(j',n',s') = (-1)^{2j+j'+\bar{s}\cdot\frac{s-\frac{\widehat{\bar{s}}-\widehat{s'}}{2}}{2}+(\bar{n}+k)\frac{n-\frac{\widehat{\bar{n}}-\widehat{n'}}{2}}{k+2}}\sigma_{0,\bar{n},\bar{s}}\sigma_{j'n's'}.$$

The factor $N_{jj}^{j'}\delta^{(2k+4)}_{2n+n'-\bar{n}}\delta^{(4)}_{2s+s'-\bar{s}}$ indeed selects the character $\chi_{j',n',s'}$ that appears in the $\mathscr{B}_{j,\bar{n}-n,\bar{s}-s}$-$\mathscr{B}_{j,n,s}$ open string. From the above result one can read that the parity $P_{\bar{n},\bar{s}}$ acts on the open string Hilbert space as

$$P_{\bar{n},\bar{s}} = \epsilon^{j,n,s}_{\bar{n},\bar{s}}(j',n',s')\,\mathrm{e}^{\pi i(L_0-h_{j,n,s})} \qquad \text{on the subspace } \mathscr{H}_{j',n',s'}. \tag{5.15}$$

B-type

For B-type parity, there is no general rule on the action on B-branes, but one can read it from Möbius strip amplitudes.

The Möbius strip with an unoriented B-brane at the boundary is given by

$$\begin{aligned}
&\langle\mathscr{C}_{rqp}|q_t^H|\mathscr{B}^B_{[j,s]}\rangle_{g_{4r+2p,2p}} \\
&= (-1)^{sq+p}\sum_{(j',n',s'):\,\text{even}} \delta^{(2)}_{s',p}N_{jj}^{j'}\,\mathrm{e}^{\frac{\pi i\widehat{(2r+p)}n'}{2(k+2)}-\frac{\pi i\widehat{(2q+p)}s'}{4}}(-1)^{\frac{2j'+n'+s'}{2}}\sigma_{j',n',s'}\widehat{\chi}_{j',n',s'}(\tau)
\end{aligned} \tag{5.16}$$

A comparison of the Möbius and cylinder amplitude shows that the boundary label j is mapped to itself by all B-type parities, as expected from the geometrical consideration. The selection factor $\delta^{(2)}_{s',p}$ shows that the $p=0$ parities leave also the label s invariant, whereas those with $p = 1$ exchange even and odd boundary labels. Furthermore, the structure of the Möbius strip implies that the B-parity P_{rqp} can be obtained from P_{000} by combination with the symmetry elements $g_{2r+p,2q+p}$. This extends the claim (4.71) from the closed string sector to the open string sector.

The Möbius strip with an oriented B-brane at the boundary is given by

$$\begin{aligned}&\langle \mathscr{C}_{rqp}|q_t^H|\mathscr{B}^B_{\frac{k}{4},S}\rangle_{g_{4r+2p,2p}} \\ &\quad = (-1)^{Sq+r+q+p} \sum_{(j,n,s)\in \mathrm{M}_k} \delta^{(2)}_{2j}\delta^{(2)}_{s,p} \frac{1+(-1)^{r+q}(-1)^{\frac{2j+n-s}{2}}}{2} e^{\frac{\pi i(2r+p)n}{2(k+2)} - \frac{\pi i(2q+p)s}{4}} \sigma_{j,n,s}\hat{\chi}_{j,n,s}(\tau)\end{aligned} \tag{5.17}$$

Again, we see that the Cardy label $j = k/4$ remains invariant as expected from geometry. Also, parities with $p = 0$ map S-even branes to S-even branes and odd branes to odd branes, while $p = 1$ parities exchange them, as before. In the present case, however, the label S is defined mod 4. To obtain the refined information, note that the $\mathscr{B}_{\frac{k}{4},S'}$-$\mathscr{B}_{\frac{k}{4},S}$ open string partition function (5.10) has the selection factor $(1+(-1)^{\frac{2j+n-s}{2}}\,\mathrm{e}^{\frac{\pi i}{2}((S')^2+S'-S^2-S)})/2$. Comparing with this, we find that the label S' of the image brane is identified as

$$(-1)^{r+q} = \mathrm{e}^{\frac{\pi i}{2}(S'^2+S'-S^2-S)}. \tag{5.18}$$

This determines the parity action on S mod 4:

$$\begin{aligned} p=0,\ r+q \text{ even} &: \ \mathscr{B}^B_{\frac{k}{4},S} \to \mathscr{B}^B_{\frac{k}{4},S} \\ p=0,\ r+q \text{ odd} &: \ \mathscr{B}^B_{\frac{k}{4},S} \to \mathscr{B}^B_{\frac{k}{4},S+2} \\ p=1,\ r+q \text{ even} &: \ \mathscr{B}^B_{\frac{k}{4},0} \leftrightarrow \mathscr{B}^B_{\frac{k}{4},-1}, \quad \mathscr{B}^B_{\frac{k}{4},2} \leftrightarrow \mathscr{B}^B_{\frac{k}{4},1} \\ p=1,\ r+q \text{ odd} &: \ \mathscr{B}^B_{\frac{k}{4},0} \leftrightarrow \mathscr{B}^B_{\frac{k}{4},1}, \quad \mathscr{B}^B_{\frac{k}{4},2} \leftrightarrow \mathscr{B}^B_{\frac{k}{4},-1} \end{aligned}$$

In comparison with the action of $g_{n,s}$ on the branes (5.5), we see that this action is consistent with the claim $P_{rqp} = g_{2r+p,2q+p}P_{000}$.

5.3 Resolving GSO

Let us now entangle the GSO projection and derive the Möbius strip amplitudes in the original $\mathcal{N} = 2$ minimal model. This in particular enables us to compute the open string Witten indices.

5.3.1 Oriented branes vs unoriented branes

The first step is to find the relation of the boundary states of the model before and after the GSO projection. We use the following prescription. Let us consider a theory involving fermions with a mod-2 fermion number $(-1)^F$ that can be used to define the non-chiral GSO projection. We are interested in how the open string amplitudes of the GSO projected theory can be defined in terms of the underlying theory. Let $\{O_a\}$ be oriented D-branes and $\{U_j\}$ be unoriented branes of the GSO projected theory. For each oriented brane O_a there is another brane $O_{r(a)}$ which corresponds to the same boundary condition $[a]$ but has an opposite sign in the GSO projection on the open string sector:

$$\operatorname*{Tr}_{\mathcal{H}^{\rm GSO}_{a,b}} q^H = \operatorname*{Tr}_{\mathcal{H}^{\rm GSO}_{r(a),r(b)}} q^H = \operatorname*{Tr}_{\mathcal{H}_{[a],[b]}} \frac{1+(-1)^F}{2} q^H, \tag{5.19}$$

$$\operatorname*{Tr}_{\mathcal{H}^{\rm GSO}_{a,r(b)}} q^H = \operatorname*{Tr}_{\mathcal{H}^{\rm GSO}_{r(a),b}} q^H = \operatorname*{Tr}_{\mathcal{H}_{[a],[b]}} \frac{1-(-1)^F}{2} q^H. \tag{5.20}$$

$O_{r(a)}$ is the orientation reversal of O_a in this sense. The partition functions involving unoriented branes are

$$\operatorname*{Tr}_{\mathcal{H}^{\rm GSO}_{i,j}} q^H = \operatorname*{Tr}_{\mathcal{H}_{i,j}} q^H, \tag{5.21}$$

$$\operatorname*{Tr}_{\mathcal{H}^{\rm GSO}_{a,i}} q^H = \frac{1}{\sqrt{2}} \operatorname*{Tr}_{\mathcal{H}_{[a],i}} q^H. \tag{5.22}$$

The factor of $1/\sqrt{2}$ may appear odd. However, in the spectrum between oriented and unoriented branes, there is always an odd number of real (or Majorana) fermion zero modes whose partition functions are odd powers of $\sqrt{2}$. Thus, it is only with the factor of $1/\sqrt{2}$ that the open string partition functions of the GSO projected theory have integer coefficients.

The above definition leads to the following expressions for the boundary states of the GSO projected theory

$$|\mathscr{B}_{r^s(a)}\rangle^{\rm GSO} = \frac{1}{\sqrt{2}} |\mathscr{B}_{[a]}\rangle_{\rm NSNS} + \frac{(-1)^s}{\sqrt{2}} |\mathscr{B}_{[a]}\rangle_{\rm RR}, \tag{5.23}$$

$$|\mathscr{B}_i\rangle^{\rm GSO} = |\mathscr{B}_i\rangle_{\rm NSNS}. \tag{5.24}$$

Example: free Dirac fermion

For illustration, we consider the free Dirac fermion $\psi_\pm, \overline{\psi}_\pm$, and the following two boundary conditions

$$A : \psi_- = \overline{\psi}_+, \;\; \overline{\psi}_- = \psi_+;$$

$$B:\psi_-=\psi_+,\ \ \overline{\psi}_-=\overline{\psi}_+.$$

For both AA and BB open strings, the space of states is the Fock space of the complex Clifford algebra generated by $\psi_n, \overline{\psi}_n=\psi_{-n}^\dagger$ $(n\in\mathbb{Z})$ obeying the relation $\{\psi_n,\overline{\psi}_m\}=\delta_{n+m,0}$, $\{\psi_n,\psi_m\}=0$. In particular, the partition functions $\mathrm{Tr}_{AA}\,\mathrm{e}^{i\alpha F_A}q^H$ and $\mathrm{Tr}_{BB}\,\mathrm{e}^{i\alpha F_V}q^H$ have integer coefficients in the expansion by $\mathrm{e}^{i\alpha Q_N}q^{E_N}$, where $F_A=\overline{\psi}_-\psi_-+\overline{\psi}_+\psi_+$ and $F_V=-\overline{\psi}_-\psi_-+\overline{\psi}_+\psi_+$ are the fermion numbers conserved in the respective open string system. Let us now consider the AB-string. There is one real fermion zero mode, the zero mode of $\mathrm{Re}(\psi_-)$ or equivalently of $\mathrm{Re}(\psi_+)$, whose partition function is [75]

$$\sqrt{2}.$$

The non-zero modes are positive-integer as well as positive-half-integer moded complex fermions, with the zero point energy $-\frac{1}{24}+\frac{1}{16}=\frac{1}{48}$. Thus the total partition function is

$$\mathrm{Tr}_{AB}q^H=\sqrt{2}q^{\frac{1}{48}}\prod_{n=1}^{\infty}(1+q^n)(1+q^{n-\frac{1}{2}}).$$

Let us now consider GSO projection by the operator

$$(-1)^F=\mathrm{e}^{\pi i F_A}.$$

The projected theory is the sigma model on the circle of radius $R=\sqrt{2}$, $X\equiv X+2\pi R$, where the correspondence is $\overline{\psi}_\pm\psi_\pm=(\partial_t X\pm\partial_\sigma X)/\sqrt{2}$. The boundary condition A implies $\overline{\psi}_-\psi_-=-\overline{\psi}_+\psi_+$ or equivalently the Dirichlet boundary condition $\partial_t X=0$ (for a D0-brane), while the boundary condition B implies $\overline{\psi}_-\psi_-=\overline{\psi}_+\psi_+$ or equivalently the Neumann boundary condition $\partial_\sigma X=0$ (for a D1-brane). A more careful inspection of the boundary states shows that the location of the D0-brane is at $X=0$ or $X=\pi R$, and the Wilson line for the D1-brane is zero. Namely,

$$|D_{\pi sR}\rangle=\frac{1}{\sqrt{2}}|\mathscr{B}_A\rangle_{\rm NSNS}+\frac{(-1)^s}{\sqrt{2}}|\mathscr{B}_A\rangle_{\rm RR},\tag{5.25}$$

$$|N_0\rangle=|\mathscr{B}_B\rangle_{\rm NSNS}.\tag{5.26}$$

In fact, the DN-string has odd-integer moded bosonic field and the partition function is

$$\begin{aligned}\mathrm{Tr}_{DN}q^H&=q^{\frac{1}{48}}\prod_{n=1}^{\infty}(1-q^{n-\frac{1}{2}})^{-1}\\&=q^{\frac{1}{48}}\prod_{n=1}^{\infty}(1+q^n)(1+q^{n-\frac{1}{2}})=\frac{1}{\sqrt{2}}\mathrm{Tr}_{AB}q^H.\end{aligned}$$

This is indeed in agreement with the relation $\langle D|q_t^H|N\rangle=\frac{1}{\sqrt{2}}{}_{\rm NSNS}\langle\mathscr{B}_A|q_t^H|\mathscr{B}_B\rangle_{\rm NSNS}$ that follows from (5.25) and (5.26).

5.3.2 A-type

All the A-branes $\mathscr{B}_{j,n,s}$ are oriented where the orientation is flipped by the shift $s \to s+2$. Thus, the boundary states of the projected theory is composed of the NSNS and RR sector states as

$$|\mathscr{B}_{j,n,s}\rangle = \frac{1}{\sqrt{2}}|\mathscr{B}_{j,n}\rangle_{\rm NSNS} + \frac{1}{\sqrt{2}}|\mathscr{B}_{j,n,(s)}\rangle_{\rm RR}. \tag{5.27}$$

In other words, the boundary states before the GSO projection are given by

$$\begin{aligned}|\mathscr{B}_{j,n}\rangle_{\rm NSNS} &= \frac{1}{\sqrt{2}}|\mathscr{B}_{j,n,s}\rangle + \frac{1}{\sqrt{2}}|\mathscr{B}_{j,n,s+2}\rangle,\\ |\mathscr{B}_{j,n,(s)}\rangle_{\rm RR} &= \frac{1}{\sqrt{2}}|\mathscr{B}_{j,n,s}\rangle - \frac{1}{\sqrt{2}}|\mathscr{B}_{j,n,s+2}\rangle.\end{aligned}$$

We have retained the orientation dependence for the RR-boundary states, so that the orientation flip corresponds to the sign flip $|\mathscr{B}_{j,n,(s+2)}\rangle_{\rm RR} = -|\mathscr{B}_{j,n,(s)}\rangle_{\rm RR}$.

The spectrum of the open string before the GSO projection is found by computing the ordinary partition function ${\rm Tr}_{(j,n),(j',n')}q^H = {}_{\rm NSNS}\langle\mathscr{B}_{j,n}|q_t^H|\mathscr{B}_{j',n'}\rangle_{\rm NSNS}$. Using the above identification of $|\mathscr{B}_{j,n}\rangle_{\rm NSNS}$ and the formula (5.2), we find

$$\mathcal{H}_{(j,n),(j',n')} = \bigoplus_{2j''+n-n'+s''\ \rm even} N_{jj'}^{j''}\,\mathscr{H}^{\mathcal{N}=2}_{j'',n-n',s''} \tag{5.28}$$

where $\mathscr{H}^{\mathcal{N}=2}_{j,n,0}$ and $\mathscr{H}^{\mathcal{N}=2}_{j,n,1}$ are the representations of the Neveu-Schwarz and Ramond $\mathcal{N}=2$ super-Virasoro algebra which is defined as

$$\mathscr{H}^{\mathcal{N}=2}_{j,n,[s]} = \mathscr{H}_{j,n,s} \oplus \mathscr{H}_{j,n,s+2}. \tag{5.29}$$

where $j \in {\rm P}_k, n \in \mathbb{Z}_{k+2}$, and $[s] \in \mathbb{Z}_2$ is the mod 2 reduction of $s \in \mathbb{Z}_4$.

Let us now compute the open string Witten index. $\mathscr{B}_{j,n,(s)}$ at the left boundary and $\mathscr{B}_{j',n',(s')}$ at the right boundary preserve the same supersymmetry if and only if $s-s'$ is odd. Then, the index is given by

$$\begin{aligned}I(\mathscr{B}_{j,n,(s)},\mathscr{B}_{j',n',(s')}) &= {}_{\rm RR}\langle\mathscr{B}_{j,n,(s)}|q_t^H|\mathscr{B}_{j',n',(s')}\rangle_{\rm RR}\\ &= \langle\mathscr{B}_{j,n,s}|q_t^H|\mathscr{B}_{j',n',s'}\rangle - \langle\mathscr{B}_{j,n,s}|q_t^H|\mathscr{B}_{j',n',s'+2}\rangle\\ &= \sum_{j'':ev} N_{jj'}^{j''}\delta^{(2)}_{s-s'+1}\left(\chi_{j'',n-n',s-s'} - \chi_{j'',n-n',s-s'+2}\right)(\tau)\\ &= \sum_{j''} N_{jj'}^{j''}(\delta^{(4)}_{s-s',1} - \delta^{(4)}_{s-s',-1})(\delta^{(2k+4)}_{n-n',2j''+1} - \delta^{(2k+4)}_{n-n',-2j''-1})\\ &= (-1)^{\frac{s-s'+1}{2}} N_{jj'}^{\frac{n'-n-1}{2}}\end{aligned} \tag{5.30}$$

Here, N is the periodically continued fusion rule coefficient [20], $N_{jj'}^{j''} = N_{jj'}^{j''+(k+2)} = -N_{jj'}^{-j''-1}$, $N_{jj'}^{\frac{1}{2}} = N_{jj'}^{\frac{k+1}{2}} = 0$. This continuation is the same as the analytic continuation using the Verlinde formula, which expresses the fusion rule coefficient N in terms of elements of the modular S-matrix.

Let us next compute the open string index twisted by the parity symmetry P_A. We recall that P_A commutes with the supercharge $\overline{Q}_+ + Q_-$ which is the combination preserved by the $\mathscr{B}_{j,n,(s)}$-$\mathscr{B}_{j',n',(s')}$-string with s even and s' odd. Recall also that P_A corresponds to $P_{-1,-1}$ in the GSO projected theory under which the $\mathscr{B}_{j,n,s}$-$\mathscr{B}_{j,-n-1,-s-1}$-string is invariant. Thus, we consider the twisted Witten index for the $\mathscr{B}_{j,n,(s)}$-$\mathscr{B}_{j,-n-1,(-s-1)}$-string in the original minimal model:

$$\begin{aligned} I_{P_A}(\mathscr{B}_{j,n,(s)}, \mathscr{B}_{j,-n-1,(-s-1)}) &= {}_{\rm RR}\langle \mathscr{B}_{j,n,(s)} | q_t^H | \mathscr{C}_{P_A} \rangle \\ &= \langle \mathscr{B}_{j,n,s} | q_t^H | \mathscr{C}_{-1,-1} \rangle + \langle \mathscr{B}_{j,n,s} | q_t^H | \mathscr{C}_{-1,1} \rangle . \end{aligned}$$

Using the formula (D.3), we find that the summands are

$$\begin{aligned} &\langle \mathscr{B}_{j,n,s} | q_t^H | \mathscr{C}_{-1,\mp 1} \rangle \\ &\quad = (-1)^{\frac{s}{2}} \sum_{\substack{2j' \text{ even} \\ n' \text{ odd}}} \left(\delta^{(2k+4)}_{n,\frac{-1-\widehat{n'}}{2}} - (-1)^k \delta^{(2k+4)}_{n,\frac{-1-\widehat{n'}}{2}+k+2} \right) (-1)^{2j+j'} N_{jj}^{j'} \frac{\sigma_{0,-1,\mp 1}}{\sigma_{j',n',\mp 1}} \widehat{\chi}_{j',-n',\pm 1} . \end{aligned}$$

Note that $\sigma_{0,-1,\mp 1} = \pm 1$ and

$$\frac{\widehat{\chi}_{j',-n',1}}{\sigma_{j',n',-1}} - \frac{\widehat{\chi}_{j',-n',-1}}{\sigma_{j',n',1}} = \delta_{n',-2j'-1} - \delta_{n',2j'+1} .$$

This shows that

$$\begin{aligned} &I_{P_A}(\mathscr{B}_{j,n,(s)}, \mathscr{B}_{j,-n-1,(-s-1)}) \\ &\quad = (-1)^{\frac{s}{2}} \sum_{j' \in \mathrm{P}_k} (-1)^{2j+j'} N_{jj}^{j'} \left(\delta^{(2k+4)}_{n,j'} - (-1)^k \delta^{(2k+4)}_{n,j'+k+2} - \delta^{(2k+4)}_{n,-j'-1} + (-1)^k \delta^{(2k+4)}_{n,-j'-1+k+2} \right) \\ &\quad = (-1)^{\frac{s}{2}} \sum_{j' \in \mathrm{P}_k} N_{jj}^{j'} \left(\delta^{(2k+4)}_{j',n} - \delta^{(2k+4)}_{j',n+(k+2)} + \delta^{(2k+4)}_{j',-n-1} - \delta^{(2k+4)}_{j',(k+2)-n-1} \right) . \end{aligned} \tag{5.31}$$

Similar to the untwisted open string Witten index, this index can be expressed as a periodically continued fusion rule coefficient:

$$I_{P_A}(\mathscr{B}_{j,n,(s)}, \mathscr{B}_{j,-n-1,(-s-1)}) = (-1)^{\frac{s}{2}} \widetilde{N}_{jj}^{n}, \tag{5.32}$$

where $\widetilde{N}_{jj}^{j'}$ is the $SU(2)$ fusion rule coefficient with the periodic continuation $\widetilde{N}_{jj}^{j'} = -\widetilde{N}_{jj}^{j'+k+2} = \widetilde{N}_{jj}^{-j'-1}$. Note that this periodic continuation is different than the one that appears in the untwisted open string Witten index (5.30). While the fusion rule coefficients appear in the open string channel of the cylinder diagram, the quantity that governs the Möbius strip amplitude is the Y-tensor, and the periodicity is in fact the one inherited from the Y-tensor. To see this, recall that

$$N_{jj}^{j'} = (-1)^{j'+2j} Y_{j0}^{j'} = (-1)^{j'+2j} \sum_{j''} \frac{4}{k+2} \frac{\sin\pi\frac{(2j'+1)(2j''+1)}{2(k+2)} \sin\pi\frac{(2j+1)(2j''+1)}{k+2} \sin\pi\frac{(2j''+1)}{2(k+2)}}{\sin\pi\frac{2j''+1}{k+2}},$$

whose analytic continuation indeed agrees with the one derived above.

We will compute the same index in the Landau–Ginzburg description of the model in the next section.

5.3.3 B-type

We now consider B-branes and B-parities. We shall omit the superscript "B" for the branes in this subsubsection. The boundary states of the unoriented branes $\mathscr{B}_{[j,s]}$ and the oriented branes $\mathscr{B}_{\frac{k}{4},S}$ are written in terms of those before GSO projection as

$$|\mathscr{B}_{[j,s]}\rangle = |\mathscr{B}_{[j,s]}\rangle_{\rm NSNS}, \tag{5.33}$$

$$|\mathscr{B}_{\frac{k}{4},S}\rangle = \frac{1}{\sqrt{2}}|\mathscr{B}_{\frac{k}{4},[S]}\rangle_{\rm NSNS} + \frac{{\rm e}^{-\frac{\pi i}{2}(S^2+S)}}{\sqrt{2}}|\mathscr{B}_{\frac{k}{4},[S]}\rangle_{\rm RR}. \tag{5.34}$$

$[S]$ is the mod 2 reduction of the mod 4 integer S, and the phase factor ${\rm e}^{-\frac{\pi i}{2}(S^2+S)}$ represents the sign flip of the RR-part under the orientation reversal $S \to S+2$.

Let us find the open string spectrum before the GSO projection. As in the A-type case, this can be read off from the ordinary partition function ${\rm Tr}_{ab} q^H = {}_{\rm NSNS}\langle a|q_t^H|b\rangle_{\rm NSNS}$, which is computable using the relation to the boundary states of the GSO projected theory, (5.33)-(5.34), and the formulae (5.9)-(5.10). We find that the spectrum of $\mathscr{B}_{[j,s]}$-$\mathscr{B}_{[j',s']}$-strings and $\mathscr{B}_{\frac{k}{4},[S]}$-$\mathscr{B}_{\frac{k}{4},[S']}$-strings is

$$\mathcal{H}_{[j,s],[j',s']} = \bigoplus_{2j''+n''+s-s'\ \rm even} N_{jj'}^{j''} \mathscr{H}^{\mathcal{N}=2}_{j'',n'',s-s'}, \tag{5.35}$$

$$\mathcal{H}_{(\frac{k}{4},[S]),(\frac{k}{4},[S'])} = \bigoplus_{\substack{2j,n+S-S'\ \rm even \\ (j,n)\equiv(\frac{k}{2}-j,n+k+2)}} \mathscr{H}^{\mathcal{N}=2}_{j,n,[S-S']}, \tag{5.36}$$

where $\mathscr{H}^{\mathcal{N}=2}_{j,n,0/1}$ are the $\mathcal{N}=2$ NS/R modules (5.29). We would like to identify the action of the symmetries a^{2m} and $(-1)^F a^{2m}$ on these open string Hilbert spaces. We shall find it using the following guideline:

• a^{2m} must commute with the supersymmetry,
• $(-1)^F a^{2m}$ must anticommute with the supersymmetry,
• They must induce the symmetry $g_{2m,2m}$ or $g_{2m,2m+2}$ after GSO projection.

Let us first look at the $\mathscr{B}_{[j,s]}$-$\mathscr{B}_{[j',s']}$-string. Looking at the Cylinder amplitudes (5.9) one observes that $g_{2\bar{n},2\bar{s}}$ acts as the phase $\mathrm{e}^{\pi i(\frac{\bar{n}n''}{k+2}-\frac{\bar{s}s''}{2})}$ on the subspace $\mathscr{H}_{j''n''s''}$. This shows that $g_{2m,0}$ commutes with the supersymmetry and $g_{2m,2}$ anti-commutes with the supersymmetry. This fixes the identification as

$$\begin{aligned} a^{2m} &\equiv g_{2m,0}, \\ (-1)^F a^{2m} &\equiv g_{2m,2}. \end{aligned} \tag{5.37}$$

The boundary state identification $|\mathscr{B}_{[j,s]}\rangle = |\mathscr{B}_{[j,s]}\rangle_{\mathrm{NSNS}} := |\mathscr{B}_{[j,s]}\rangle_{(-1)^F}$ is now generalized by the twists as

$$|\mathscr{B}_{[j,s]}\rangle_{g_{2m,0}} = |\mathscr{B}_{[j,s]}\rangle_{(-1)^F a^{2m}}, \tag{5.38}$$

$$|\mathscr{B}_{[j,s]}\rangle_{g_{2m,2}} = |\mathscr{B}_{[j,s]}\rangle_{a^{2m}}. \tag{5.39}$$

Next, consider the $\mathscr{B}_{\frac{k}{4},[S]}$-$\mathscr{B}_{\frac{k}{4},[S']}$-string. Looking at the formula (5.10), we find that $g_{2\bar{n},2\bar{n}}$ acts as $\mathrm{e}^{-\pi i\bar{n}J_0}$ on the subspace of $\mathcal{H}_{(\frac{k}{4},[S]),(\frac{k}{4},[S'])}$ that survives the GSO projection. The operator $\mathrm{e}^{-\pi i\bar{n}J_0}$ commutes (anti-commutes) with the supersymmetry for even $\bar{n}$ (odd $\bar{n}$). Thus, we identify

$$g_{2m,2m} = \mathrm{e}^{-\pi i m J_0} = (-1)^{mF} a^{2m}. \tag{5.40}$$

In particular, the twisted version of the boundary state identification (5.34) is

$$|\mathscr{B}_{\frac{k}{4},S}\rangle_{g_{2m,2m}} = \frac{1}{\sqrt{2}}|\mathscr{B}_{\frac{k}{4},[S]}\rangle_{(-1)^{(m+1)F}a^{2m}} + \frac{\mathrm{e}^{-\frac{\pi i}{2}(S^2+S)}}{\sqrt{2}}|\mathscr{B}_{\frac{k}{4},[S]}\rangle_{(-1)^{mF}a^{2m}}. \tag{5.41}$$

In particular, we find

$$\begin{aligned} |\mathscr{B}_{\frac{k}{4},[S]}\rangle_{a^{2m}} &= \frac{\mathrm{e}^{-i\theta_S}}{\sqrt{2}}\left[|\mathscr{B}_{\frac{k}{4},S}\rangle_{g_{2m,2m}} + (-1)^{m-1}|\mathscr{B}_{\frac{k}{4},S+2}\rangle_{g_{2m,2m}}\right] \\ &= \begin{cases} \frac{1}{\sqrt{2}}|\mathscr{B}_{[\frac{k}{4},S]}\rangle_{g_{2m,2m}} & m \text{ odd} \\ \mathrm{e}^{-i\theta_S}\sqrt{\frac{k+2}{2}}\,\mathrm{e}^{\frac{-\pi i S^2}{2}}\sum_{s=\pm1}\mathrm{e}^{\frac{-\pi i S s}{2}}|\frac{k}{4},\frac{k+2}{2}+m,s+m\rangle\!\rangle_B & m \text{ even} \end{cases} \end{aligned} \tag{5.42}$$

where $\mathrm{e}^{i\theta_S}=1$ for odd m and $\mathrm{e}^{i\theta_S} = \mathrm{e}^{-\frac{\pi i}{2}(S^2+S)}$ for even m.

Open string Witten index

As an application of (5.38), (5.39) and (5.41), let us compute the open string Witten index from the tree-channel.

We start with the open string stretched between unoriented B-branes. $\mathscr{B}_{[j,s]}$ on the left boundary and $\mathscr{B}_{[j',s']}$ on the right boundary preserve the same supersymmetry if $s-s'$ is odd. Thus we consider the Witten index for odd $s-s'$, twisted by an axial rotation symmetry:

$$\begin{aligned}
\mathrm{Tr}_{[j,s],[j',s']} a^{2m}(-1)^F q^H &= {}_{a^{2m}}\langle \mathscr{B}_{[j,s]}| q_t^H | \mathscr{B}_{[j',s']}\rangle_{a^{2m}} \\
&= {}_{g_{2m,2}}\langle \mathscr{B}_{[j,s]}| q_t^H | \mathscr{B}_{[j',s']}\rangle_{g_{2m,2}} \\
&= \sum_{j''\in \mathrm{P}_k} N_{jj'}^{j''}\Big(\mathrm{e}^{\pi i(\frac{m(2j''+1)}{k+2}-\frac{1}{2})} - \mathrm{e}^{\pi i(-\frac{m(2j''+1)}{k+2}-\frac{1}{2})}\Big)
\end{aligned} \tag{5.43}$$

The ordinary index (the one with $m=0$) vanishes. This means that there are equal number of bosonic and fermionic supersymmetric ground states — for each j'' with $N_{jj'}^{j''}\neq 0$ there is one ground state from $\mathscr{H}^{\mathcal{N}=2}_{j'',2j''+1,1}$ and another from $\mathscr{H}^{\mathcal{N}=2}_{j'',-2j''-1,1}$ which contribute to the index with opposite signs. The index escapes from vanishing if twisted by an operator a^{2m} that acts differently on those ground states.

Let us next consider the index for the string stretched between oriented B-branes. The boundary conditions $\mathscr{B}_{\frac{k}{4},[S]}$ on the left and $\mathscr{B}_{\frac{k}{4},[S']}$ on the right preserve the same supersymmetry. Thus, the open string Witten index is defined for odd $S-S'$ and is given by

$$\begin{aligned}
\mathrm{Tr}_{(\frac{k}{4},[S]),(\frac{k}{4},[S'])} a^{2m}(-1)^F q^H &= {}_{a^{2m}}\langle \mathscr{B}_{\frac{k}{4},[S]}| q_t^H | \mathscr{B}_{\frac{k}{4},[S']}\rangle_{a^{2m}} \\
&= \begin{cases} \frac{1}{2}\langle \mathscr{B}_{[\frac{k}{4},S]}|\mathscr{B}_{[\frac{k}{4},S']}\rangle_{g_{2m,2m}} & (m \text{ odd}) \\ \mathrm{e}^{i(\theta_S-\theta_{S'})}\Big[\langle \mathscr{B}_{\frac{k}{4},S}|\mathscr{B}_{\frac{k}{4},S'}\rangle - \langle \mathscr{B}_{\frac{k}{4},S}|\mathscr{B}_{\frac{k}{4},S'+2}\rangle\Big]_{g_{2m,2m}} & (m \text{ even}) \end{cases} \\
&= \sum_{j\in \mathrm{P}_k\cap \mathbb{Z}} \mathrm{e}^{\pi i m(\frac{2j+1}{k+2}-\frac{1}{2})}
\end{aligned} \tag{5.44}$$

This is consistent with the fact that there are $\frac{k+2}{2}$ ground states from $\mathscr{H}^{\mathcal{N}=2}_{j,2j+1,1}$, $j\in \mathrm{P}_k\cap\mathbb{Z}$. They are all regarded bosonic and a^{2m} acts on them by phase $\mathrm{e}^{\pi i m(\frac{2j+1}{k+2}-\frac{1}{2})}$. For even m, the result can also be evaluated as

$$\begin{cases} 0 & (m \text{ even}, m\neq 0) \\ \frac{k+2}{2} & (m=0) \end{cases}$$

The vanishing for twists with m even, $m\neq 0$, is most easily understood in terms of the closed string channel: by (5.42) we see that the propagating closed string states are in the representation $(\frac{k}{4},\frac{k+2}{2}+m, s+m)$. There are no supersymmetric ground state in this sector, unless $m=0$.

In both cases, the index behaves in the way it should. This may be regarded as a strong consistency test of the twisted versions of the boundary state identification, (5.39) and (5.41).[1] Now let us use them to compute the Möbius strip amplitudes of the theory before GSO projection.

Parity actions on boundary conditions

From the actions of parities P_{rqp} on the B-branes in the GSO projected theory, we see that the parities in the original model transforms the boundary conditions as

$$(-1)^{\nu F} a^{2m+1} P_B \quad : \quad \begin{cases} \mathscr{B}_{[j,s]} \longrightarrow \mathscr{B}_{[j,s+1]} \\ \mathscr{B}_{\frac{k}{4},[S]} \longrightarrow \mathscr{B}_{\frac{k}{4},[S+1]} \end{cases}$$

$$(-1)^{\nu F} a^{2m} P_B \quad : \quad \begin{cases} \mathscr{B}_{[j,s]} \longrightarrow \mathscr{B}_{[j,s]} \\ \mathscr{B}_{\frac{k}{4},[S]} \longrightarrow \mathscr{B}_{\frac{k}{4},[S]} \end{cases}$$

In particular, the B-parities $a^{2m+1}P_B$ transform the $\mathscr{B}_{[j,s]}$-$\mathscr{B}_{[j,s+1]}$-string to itself and $\mathscr{B}_{\frac{k}{4},[S]}$-$\mathscr{B}_{\frac{k}{4},[S+1]}$-string to itself.

parity-twisted open string Witten index

The B-parities $a^{2m+1}P_B$ commute with the supercharge $\overline{Q}_+ + \overline{Q}_-$ which is the supersymmetry preserved by the $\mathscr{B}_{[j,0]}$-$\mathscr{B}_{[j,1]}$-string as well as by the $\mathscr{B}_{\frac{k}{4},[0]}$-$\mathscr{B}_{\frac{k}{2},[1]}$-string. Thus, we consider parity-twisted Witten index in such open string sectors.

We first consider the $\mathscr{B}_{[j,0]}$-$\mathscr{B}_{[j,1]}$-string. The index is expressed as

$$I_{a^{2m+1}P_B}(\mathscr{B}_{[j,0]}, \mathscr{B}_{[j,1]}) := \mathrm{Tr}_{[j,0],[j,1]}(-1)^F a^{2m+1} P_B q^H = \langle \mathscr{B}_{[j,0]} | q_t^H | \mathscr{C}_{a^{2m+1}P_B} \rangle,$$

where the boundary state should be the one on the circle twisted by the square of the parity $(a^{2m+1}P_B)^2 = a^{4m+2}$, which is ${}_{g_{4m+2,2}}\langle \mathscr{B}_{[j,0]}|$ by the identification (5.39). The crosscap state is the one given in Eq. (4.80), and hence

$$I_{a^{2m+1}P_B}(\mathscr{B}_{[j,0]}, \mathscr{B}_{[j,1]}) = {}_{g_{4m+2,2}}\langle \mathscr{B}_{[j,0]} | q_t^H | \mathscr{C}_{m,1}(+) \rangle \times (-1)^m.$$

Since $\sqrt{\frac{T_{-1}^{(2)}}{T_1^{(2)}}} = 1$, we find $|\mathscr{C}_{m,1}(+)\rangle = [|\mathscr{C}_{m01}\rangle + |\mathscr{C}_{m11}\rangle]/\sqrt{2}$. The sum of the two pairings is the complex conjugate of the following

$$\langle \mathscr{C}_{m01} | \mathscr{B}_{[j,0]} \rangle_{g_{4m+2,2}} + \langle \mathscr{C}_{m11} | \mathscr{B}_{[j,0]} \rangle_{g_{4m+2,2}}$$

[1] It would also be interesting to study the open string stretched between oriented brane and unoriented brane. Just as in the case of free fermion, we obtain $\sqrt{2}$ in the partition function. This signals the presence of an odd number of real fermion zero modes in the open string system. We do not, however, study it further in this paper.

$$= \sum_{\substack{j'\in \mathrm{P}_k\cap\mathbb{Z}\\ n'\,\mathrm{odd},\, s'=\pm 1}} N_{jj}^{j'}\,\mathrm{e}^{\frac{\pi i(\widehat{2m+1})n'}{2(k+2)}}(\,\mathrm{e}^{-\frac{\pi i s'}{4}}+\,\mathrm{e}^{\frac{\pi i s'}{4}})(-1)^{\frac{2j'+n'+s'}{2}+1}\sigma_{j'n's'}\widehat{\chi}_{j'n's'}$$

$$= \sqrt{2}\sum_{\substack{j'\in \mathrm{P}_k\cap\mathbb{Z}\\ n'\,\mathrm{odd}}} N_{jj}^{j'}\,\mathrm{e}^{\frac{\pi i(\widehat{2m+1})n'}{2(k+2)}}(-1)^{\frac{2j'+n'+s'}{2}+1}(\sigma_{j'n'1}\widehat{\chi}_{j'n'1}-\sigma_{j'n',-1}\widehat{\chi}_{j'n',-1})$$

$$= \sqrt{2}\sum_{j'\in \mathrm{P}_k\cap\mathbb{Z}} N_{jj}^{j'}\Big(\,\mathrm{e}^{\frac{\pi i(\widehat{2m+1})(2j'+1)}{2(k+2)}}+\,\mathrm{e}^{-\frac{\pi i(\widehat{2m+1})(2j'+1)}{2(k+2)}}\Big).$$

Thus, we find

$$I_{a^{2m+1}P_B}(\mathscr{B}_{[j,0]},\mathscr{B}_{[j,1]}) = (-1)^m\sum_{j'\in \mathrm{P}_k\cap\mathbb{Z}} N_{jj}^{j'}\left(\,\mathrm{e}^{\pi i\frac{(\widehat{2m+1})(2j'+1)}{2(k+2)}}+\,\mathrm{e}^{-\pi i\frac{(\widehat{2m+1})(2j'+1)}{2(k+2)}}\right) \qquad (5.45)$$

This shows that the B-parity $a^{2m+1}P_B$ acts on the supersymmetric ground states in $\mathscr{H}^{\mathcal{N}=2}_{j',\pm(2j'+1),1}$ by the phase $\pm(-1)^m\,\mathrm{e}^{\pm\pi i\frac{(\widehat{2m+1})(2j'+1)}{2(k+2)}}$.

We next consider the $\mathscr{B}_{\frac{k}{4},[0]}$-$\mathscr{B}_{\frac{k}{2},[1]}$-string. The index is represented in the tree-channel as the pairing $\langle \mathscr{B}_{\frac{k}{4},[0]}|q_t^H|\mathscr{C}_{a^{2m}P_B}\rangle$. The boundary state is the one on the circle twisted by $a^{4m+2}=a^{2(2m+1)}$, which is ${}_{g_{4m+2,4m+2}}\langle\mathscr{B}_{[\frac{k}{4},0]}|\times\frac{1}{\sqrt{2}}$ by the identification (5.42):

$$I_{a^{2m+1}P_B}(\mathscr{B}_{\frac{k}{4},[0]},\mathscr{B}_{\frac{k}{4},[1]}) = {}_{g_{4m+2,2}}\langle\mathscr{B}_{[\frac{k}{4},0]}|\mathscr{C}_{m,1}(+)\rangle\times\frac{(-1)^m}{\sqrt{2}}.$$

The computation thus reduces to the special case $j=\frac{k}{4}$ of the unoriented branes, the difference being a division by $\sqrt{2}$. Since $N_{\frac{k}{4}\frac{k}{4}}^{j'}=1$ for any $j'\in\mathrm{P}_k\cap\mathbb{Z}$ we find

$$\begin{aligned} I_{a^{2m+1}P_B}(\mathscr{B}_{\frac{k}{4},[0]},\mathscr{B}_{\frac{k}{4},[1]}) &= \frac{(-1)^m}{\sqrt{2}}\sum_{j'\in \mathrm{P}_k\cap\mathbb{Z}}\Big(\,\mathrm{e}^{\frac{\pi i(\widehat{2m+1})(2j'+1)}{2(k+2)}}+\,\mathrm{e}^{-\frac{\pi i(\widehat{2m+1})(2j'+1)}{2(k+2)}}\Big)\\ &= (-1)^m\sum_{j'\in \mathrm{P}_k\cap\mathbb{Z}}\frac{1+\,\mathrm{e}^{-\pi i\frac{\widehat{2m+1}}{2}}}{\sqrt{2}}\,\mathrm{e}^{\frac{\pi i(\widehat{2m+1})(2j'+1)}{2(k+2)}}, \end{aligned}$$

where, in the second step, we made the change of variable $j'\to\frac{k}{2}-j'$ for the second term of the summand. Note at this point that $(1+\,\mathrm{e}^{-\pi i\frac{\widehat{2m+1}}{2}})/\sqrt{2}=\,\mathrm{e}^{\pm\pi i/4}$. This shows that the index is given by

$$I_{a^{2m+1}P_B}(\mathscr{B}_{\frac{k}{4},[0]},\mathscr{B}_{\frac{k}{4},[1]}) = \alpha_m\sum_{j\in \mathrm{P}_k\cap\mathbb{Z}}\,\mathrm{e}^{\pi i\frac{\widehat{2m+1}}{2}(\frac{2j'+1}{k+2}-\frac{1}{2})}, \qquad (5.46)$$

where α_m is a phase $\pm1,\pm i$ depending only on m. This shows that the B-parity $a^{2m+1}P_B$ acts on the supersymmetric ground state in $\mathscr{H}^{\mathcal{N}=2}_{j,2j+1,1}$ as the phase multiplication by $\alpha_m\,\mathrm{e}^{\pi i\frac{\widehat{2m+1}}{2}(\frac{2j'+1}{k+2}-\frac{1}{2})}$.

6 Landau–Ginzburg Orientifolds

The $\mathcal{N} = 2$ minimal model is realized as the infra-red fixed point of the $\mathcal{N} = 2$ Landau–Ginzburg model of a single chiral superfield Φ with superpotential

$$W = \Phi^{k+2}. \tag{6.1}$$

In this section, we study parity invariance and orientifolds of Landau–Ginzburg models and apply the result to the particular example (6.1). We start by determining the condition for a parity to preserve A-type or B-type supersymmetry. We next find the integral expression of the overlap of the crosscap states and the supersymmetric ground states. Also, the twisted Witten indices are interpreted as the 'intersection numbers' of O-planes and O-planes or O-planes and D-branes. We then specialize to the model with the superpotential (6.1). The results on the overlaps of crosscaps and supersymmetric ground states and Witten indices agree with the ones of the $\mathcal{N} = 2$ minimal model obtained in Section 4.5.

6.1 A-parity and B-parity

Let us consider a Landau–Ginzburg model of chiral superfields $\Phi = (\Phi^i)$ with the Lagrangian

$$\mathcal{L} = \int \mathrm{d}^4\theta\, K(\Phi, \overline{\Phi}) + \int \mathrm{d}\theta^- \mathrm{d}\theta^+\, W(\Phi)|_{\overline{\theta}^\pm=0} + \int \mathrm{d}\overline{\theta}^+ \mathrm{d}\overline{\theta}^-\, \overline{W(\Phi)}|_{\theta^\pm=0}. \tag{6.2}$$

$K(\phi, \overline{\phi})$ is the Kähler potential for a non-degenerate Kähler metric $g_{i\overline{j}} = \partial^2 K/\partial\phi^i\partial\overline{\phi}^{\overline{j}}$, and the superpotential $W(\phi)$ is a holomorphic function of $(\phi^1, ..., \phi^n)$. In terms of the component fields $(\phi^i, \psi^i_\pm)$ of Φ^i, the Lagrangian has the kinetic and four-Fermi terms for the non-linear sigma model and also a potential term $-g^{i\overline{j}}\partial_i W \partial_{\overline{j}}\overline{W}$ as well as the fermion mass term (or 'Yukawa coupling') $-(D_i\partial_j W \psi^i_+\psi^j_- + c.c.)$. We may regard ϕ^i as local complex coordinates of some Kähler manifold X, but we assume that the first Chern class of X is zero, so that the B-twist is possible. For the existence of a non-trivial holomorphic function W, X has to be non-compact.

The supercharges are expressed as

$$Q_+ = \int \mathrm{d}x^1 \left(g_{i\overline{j}}(\partial_0 + \partial_1)\overline{\phi}^{\overline{j}}\psi^i_+ - i\overline{\psi}^{\overline{i}}_-\partial_{\overline{i}}\overline{W} \right)$$
$$Q_- = \int \mathrm{d}x^1 \left(g_{i\overline{j}}(\partial_0 - \partial_1)\overline{\phi}^{\overline{j}}\psi^i_- + i\overline{\psi}^{\overline{i}}_+\partial_{\overline{i}}\overline{W} \right)$$

$$\overline{Q}_+ = \int \mathrm{d}x^1 \left(g_{i\bar{j}} \overline{\psi}_+^{\bar{j}} (\partial_0 + \partial_1)\phi^i + i\psi_-^i \partial_i W \right)$$
$$\overline{Q}_- = \int \mathrm{d}x^1 \left(g_{i\bar{j}} \overline{\psi}_-^{\bar{j}} (\partial_0 - \partial_1)\phi^i - i\psi_+^i \partial_i W \right).$$

We would like to find a parity symmetry that transforms the supercharges as (2.1) or (2.2) — A-parity or B-parity.

Let us first consider the A-parity that exchanges $Q_+ \leftrightarrow \overline{Q}_-$. By looking at the expression for the supercharges, it is clear that an A-parity should map the holomorphic coordinates to antiholomorphic coordinates. Furthermore, the superpotential should be conjugated. Namely, an A-parity is given by $\tau_A \Omega$, where τ_A *is an antiholomorphic involution of the Kähler manifold* X *such that*

$$W(\tau_A \phi) = \overline{W(\phi)} + \text{constant}. \tag{6.3}$$

This is anomaly-free since we assume that the first Chern class of X is zero. Suppose that the Kähler potential is the flat one $K = \sum_i |\Phi^i|^2$ and the superpotential $W = \sum a_{i_1 \ldots i_r} \Phi^{i_1} \cdots \Phi^{i_r}$ has all real coefficients, $a_{i_1 \ldots i_r} \in \mathbb{R}$. Then, for the complex conjugation $\tau : \phi^i \to \overline{\phi^i}$, $\tau\Omega$ is an A-parity. In terms of the superfields, this is given by

$$\Phi^i \longrightarrow \overline{\Omega_A^* \Phi^i},$$

and the Lagrangian (6.2) is manifestly invariant since $\Omega_A : \theta^{\pm} \to -\overline{\theta}^{\mp}$ maps the measure $\mathrm{d}\theta^- \mathrm{d}\theta^+$ to $\mathrm{d}\overline{\theta}^+ \mathrm{d}\overline{\theta}^-$.

Next we consider the B-parity $Q_+ \leftrightarrow Q_-$. Again by looking at the expression of the supercharges, we find that a B-parity should map the holomorphic coordinates to holomorphic coordinates. Since the coefficients of $\overline{\partial W}$ terms are opposite between Q_+ and Q_-, we also find that W should be mapped to minus itself, $-W$, up to a constant addition. Thus a B-parity is given by $\tau_B \Omega$ where τ_B *is a holomorphic involution of the Kähler manifold* X *such that*

$$W(\tau_B \phi) = -W(\phi) + \text{constant}. \tag{6.4}$$

The minus sign can also be understood by looking at the fermion mass term $W''(\phi)\psi_+\psi_-$: the worldsheet chirality flip Ω exchanges ψ_+ and ψ_- and thus maps $\psi_+\psi_- \to \psi_-\psi_+ = -\psi_+\psi_-$. To compensate this minus sign, the superpotential itself has to flip its sign. In terms of the superfields, the parity action is given by

$$\Phi^i \longrightarrow \tau_B^i(\Omega_B^* \Phi).$$

The Lagrangian expressed in the superspace (6.2) is manifestly invariant under this, since $W(\tau_B\Phi) = -W(\Phi) + \text{constant}$ and $\Omega_B : \theta^\pm \to \theta^\mp$ flips the sign of the measure $\mathrm{d}\theta^-\mathrm{d}\theta^+$.

One may also consider the variants of A- and B-parities. For an antiholomorphic involution τ_A such that $W(\tau_A\phi) = \mathrm{e}^{2i\alpha}\overline{W(\phi)}$+const, one can define an $A_{\alpha,0}$-parity (that does $Q_+ \leftrightarrow \mathrm{e}^{-i\alpha}\overline{Q}_-$) by combining $\tau_A\Omega$ and a vector R-rotation. Starting with a B-parity, the $B_{0,\beta}$-parity (that transforms $Q_+ \leftrightarrow \mathrm{e}^{-i\beta}Q_-$) is obtained by an additional action of an axial R-rotation (which is a symmetry of the model under the assumption of $c_1(X) = 0$). For an antiholomorphic involution τ_A such that $W(\tau_A\phi) = -\overline{W(\phi)}$+const, the operator $(-1)^{F_R}\tau_A\Omega$ is an $\widetilde{A}$-parity (that does $Q_+ \to -\overline{Q}_-, \overline{Q}_- \to Q_+$). If τ_B is a holomorphic involution such that $W(\tau_B\phi) = W(\phi)$+const rather than (6.4), then $(-1)^{F_R}\tau_B\Omega$ is a $\widetilde{B}$-parity (transforming $Q_+ \to -Q_-, Q_- \to Q_+$).

6.2 Overlap of crosscap and RR ground states

Let us consider the A-parity associated with an antiholomorphic involution τ of X such that $W(\tau\phi) = \overline{W(\phi)}$+const. We will compute the overlaps of the crosscap states and the RR ground states. If we use the ground state corresponding (via a B-twist) to the *cc* ring elements, the overlaps do not depend on the twisted chiral parameters. In particular, one can take the large-volume limit of X where the zero-mode approximation is exact. This is precisely as in the case of overlaps of the boundary state for A-branes and the same set of ground states [22, 23].

We recall that an A-brane in a massive LG model is wrapped on a Lagrangian submanifold γ consisting of the collection of gradient flow lines of $\mathrm{Re}(W)$ starting from a critical point. Its image in the W-plane is a straight line emanating from the critical value and extending in the real-positive direction. The overlaps with RR ground states are expressed as the integrals over γ of the ground state wavefunctions of the zero mode quantum mechanics. The ground states are middle-dimensional forms on X annihilated by the supercharges

$$\overline{Q}_+ = \overline{\partial} - i\partial W\wedge, \quad Q_+ = *(-\partial + i\overline{\partial W}\wedge)*,$$
$$Q_- = \partial + i\overline{\partial W}\wedge, \quad \overline{Q}_- = *(-\overline{\partial} - i\partial W\wedge) * .$$

We choose two sets of such ground states $\{\omega_i\}$ and $\{\omega_{\tilde{\imath}}\}$ to be used in the overlaps $\Pi_i^\gamma = \langle \mathscr{B}_\gamma | i\rangle$ and $\widetilde{\Pi}_{\tilde{\imath}}^\gamma = \langle \tilde{\imath} | \mathscr{B}_\gamma\rangle$. The overlaps are expressed as

$$\Pi_i^\gamma = \int_\gamma \mathrm{e}^{-i(W-\overline{W})}\omega_i, \quad \widetilde{\Pi}_{\tilde{\imath}}^\gamma = \int_\gamma \mathrm{e}^{i(W-\overline{W})} * \omega_{\tilde{\imath}}. \tag{6.5}$$

The factors $e^{\mp i(W-\overline{W})}$ are simply some constants on γ, but they turn the integrands into closed forms on X.[1] Since the integrands are closed, we can deform the cycle γ without changing the integrals. We deform it to γ^- for Π_i^γ and to γ^+ for $\widetilde{\Pi}_{\bar{\imath}}^\gamma$, where $\gamma^\mp$ are such that the W-image is rotated by the small phase $e^{\mp i\epsilon}$ around the critical value. Even though the boundaries are moved, this rotation does not change the integral since $e^{\mp i(W-\overline{W})}$ works as a convergence factor. At this point, one can replace ω_i (resp. $\omega_{\bar{\imath}}$) by another representative of the $(\overline{Q}_+ + Q_-)$ cohomology class (resp. $(Q_+ + \overline{Q}_-)$ cohomology class). Convenient representatives are of the forms $e^{-i\overline{W}}\phi_i\Omega$ for ω_i and $e^{-i\overline{W}}\overline{\phi_i\Omega}$ for $\omega_{\bar{\imath}}$, where ϕ_i are chiral ring elements and Ω is the nowhere vanishing holomorphic n-form of X. Thus, we find alternative expressions

$$\Pi_i^\gamma = \int_{\gamma^-} e^{-iW}\phi_i\Omega, \quad \widetilde{\Pi}_{\bar{\imath}}^\gamma = \int_{\gamma^+} e^{-i\overline{W}}\overline{\phi_i} * \overline{\Omega}. \tag{6.6}$$

It is useful to think of this in terms of cohomology theory. Note that $\gamma^\pm$ defines an element of the relative homology group $H_n(X, B_\pm; \mathbb{Z})$, where $B_\pm$ is a region in X such that $\pm\mathrm{Im}(W) > R$ for a large positive R. On the other hand, $e^{-iW}\phi_i\Omega$ and $e^{-i\overline{W}}\overline{\phi_i} * \overline{\Omega}$ form a basis of the relative cohomology groups $H^n(X, B_-; \mathbb{C})$ and $H^n(X, B_+; \mathbb{C})$. The overlaps are simply the natural pairings.

In the above argument, we have assumed that the critical points are all non-degenerate and the corresponding vacua are massive. Suppose now that W has degenerate critical points and the theory flows to a non-trivial superconformal field theory. Let us assume that the theory can be deformed to a massive theory by deforming the superpotential ΔW, which does not affect the asymptotic behaviour at infinity. This is the case, for example, in the minimal model $W = \Phi^{k+2}$, where the addition of a generic lower-order term splits the degenerate critical point to $k+1$ non-degenerate critical points, without changing the asymptotic behaviour. In such a case, we can apply the above to the deformed theory, obtaining the integral formula for the overlap of the boundary states and RR ground states. Since the expression obtained in this way is analytic in the perturbation ΔW, the overlaps for the theory before deformation are obtained by just setting $\Delta W \to 0$. In this way, we see that for such a class of SCFT, we also have the same integral formula for the overlaps of the boundary states and RR ground states. In what follows, this line of argument will be frequently used or assumed. In the computation involving crosscap states, the only thing to be careful of is that the deformation ΔW must preserve the parity under consideration. As for the A-parity $\Phi \to \overline{\Omega_A^* \Phi}$ for the minimal model, this is

[1] The factors $e^{\mp i(W-\overline{W})}$ are also naturally induced from the modification of the boundary term so that the open string ground state energy is zero. They also make the overlaps obey the parallelism $\nabla\Pi = \nabla\widetilde{\Pi} = 0$.

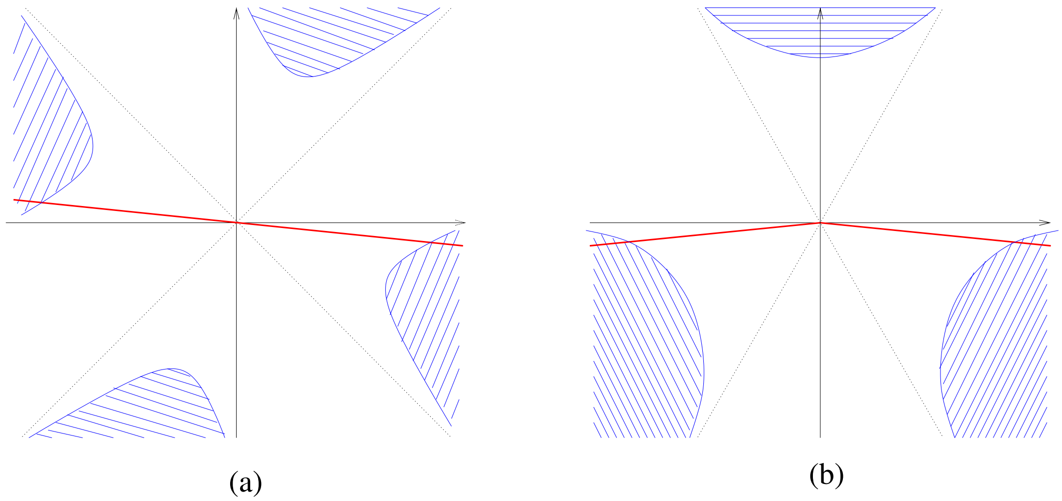

Figure 6: Deformed orientifold planes for $\tau : \phi \to \overline{\phi}$ in the models with $X = \mathbb{C}$ and superpotential (a) $W = \Phi^4$ and (b) $W = \Phi^3$. The shaded regions are B_- and the bold (broken) lines are the deformed orientifold planes $(X^\tau)^-$. The other ones B_+ and $(X^\tau)^+$ are obtained by reflection with respect to the real line.

ensured by using a real polynomial ΔW of lower order.

As in the case of D-branes, the overlaps $\Pi_i^{\tau\Omega} = \langle \mathscr{C}_{(-1)^F\tau\Omega} | i \rangle$ and $\widetilde{\Pi}_{\bar{\imath}}^{\tau\Omega} = \langle \bar{\imath} | \mathscr{C}_{\tau\Omega} \rangle$ of the crosscap states and the ground states are expressed as the integration of ω_i or $*\omega_{\bar{\imath}}$ over the τ-fixed locus $X^\tau \subset X$ — the *orientifold plane*. As in the case of branes, one can put the constant factor $\mathrm{e}^{\mp i(W - \overline{W})}$ in the integrand and deform the integration submanifold X^τ, in order to express the integral as the holomorphic (or antiholomorphic) integrals over deformed orientifold planes. We note that the W-image of X^τ is parallel to the real axis but not necessarily extending in the real-positive direction; it could also be extending in the real-negative direction or in both the real-positive and negative directions. Thus, in order for $\mathrm{e}^{\mp i(W - \overline{W})}$ to work as the convergence factor, we deform the plane X^τ so that the W-image is rotated by the phase $\mathrm{e}^{\mp i\epsilon}$ in the real-positive direction and by $\mathrm{e}^{\pm i\epsilon}$ in the real-negative direction. We call the resulting cycle $X^{\tau\mp}$. (See Fig. 6.) Thus the overlaps are expressed as

$$\Pi_i^{\tau\Omega} = \int_{X^\tau} \mathrm{e}^{-i(W - \overline{W})} \omega_i = \int_{X^{\tau-}} \mathrm{e}^{-iW} \phi_i \mathit{\Omega}, \tag{6.7}$$

$$\widetilde{\Pi}_{\bar{\imath}}^{\tau\Omega} = \int_{X^\tau} \mathrm{e}^{i(W - \overline{W})} * \omega_{\bar{\imath}} = \int_{X^{\tau+}} \mathrm{e}^{-i\overline{W}} \overline{\phi_i} * \overline{\mathit{\Omega}}. \tag{6.8}$$

The deformed orientifold planes define relative homology classes, $[X^{\tau\mp}] \in H_n(X, B_\mp; \mathbb{Z})$. Then, one can consider the expressions (6.7) and (6.8) as the pairing of $[X^{\tau\mp}]$ and the basis of the group $H^n(X, B_\pm; \mathbb{C})$ defined by the chiral ring elements ϕ_i.

6.3 The twisted Witten index

Using the expressions for the overlaps with the RR ground states, one can find a homological formula for the Witten index. Inserting the expressions (6.5) into the bilinear identity (2.34), we find

$$I(a,b) = \sum_{i,\bar{\jmath}} \int_{\gamma_a^-} \mathrm{e}^{-\beta(W-\overline{W})}\omega_i \, g^{i\bar{\jmath}} \int_{\gamma_b^+} \mathrm{e}^{i\beta(W-\overline{W})} * \omega_{\bar{\jmath}},$$

where $g^{i\bar{\jmath}}$ is the inverse of the matrix $g_{i\bar{\jmath}} = \int_X \omega_i \wedge *\omega_{\bar{\jmath}}$. We have similar expressions for the twisted Witten indices $I_{\tau\Omega}$ and $I_{\tau\Omega}(a)$.

By Riemann's bilinear identity, such expressions can be identified as certain intersection numbers (see [23]). Here, an essential role is played by Poincaré duality. For each homology class $[C^+]$ in $H_n(X,B_+)$ we find a cohomology class $\mathrm{Pd}[C^+]$ in $H^n(X,B_-)$ such that $\int_X \mathrm{Pd}[C^+] \wedge \eta_+ = \int_{C^+} \eta_+$ for any $\eta_+ \in H^n(X,B_+)$ and $\int_{D^-} \mathrm{Pd}[C^+] = \#(D^- \cap C^+)$ for any $D^+ \in H_n(X,B_+)$. It is roughly a delta function supported on C^+. Since the group $H^n(X,B_-)$ is spanned by $\mathrm{e}^{-i\beta(W-\overline{W})}\omega_i$, one can express the cohomology class $\mathrm{Pd}[C^+]$ as the linear combination $\sum_i c^i \, \mathrm{e}^{-i\beta(W-\overline{W})}\omega_i$. The coefficients c^i can be found by taking the wedge product with $\mathrm{e}^{i\beta(W-\overline{W})} * \omega_{\bar{\jmath}} \in H^n(X,B_+)$ and integrating over X. This shows $c^i = \sum_{\bar{\jmath}} g^{i\bar{\jmath}} \int_{C^+} \mathrm{e}^{i\beta(W-\overline{W})} * \omega_{\bar{\jmath}}$. Thus, we find the expression for $\mathrm{Pd}[C^+]$, or

$$\sum_{i,\bar{\jmath}} \int_{D^-} \mathrm{e}^{-i\beta(W-\overline{W})}\omega_i \, g^{i\bar{\jmath}} \int_{C^+} \mathrm{e}^{i\beta(W-\overline{W})} * \omega_{\bar{\jmath}} = \int_{D^-} \mathrm{Pd}[C^+] = \#(D^- \cap C^+)$$

for $D^- \in H_n(X,B_-)$.

Applying this to $D^- = \gamma_a^-$ and $C^+ = \gamma_b^+$, we find

$$I(\gamma_a,\gamma_b) = \#(\gamma_a^- \cap \gamma_b^+). \tag{6.9}$$

For the parity-twisted Witten index of the closed string, we find

$$I_{\tau\Omega} = \#(X^{\tau-} \cap X^{\tau+}). \tag{6.10}$$

This is the Landau–Ginzburg version of the index formula (3.5) in non-linear sigma models. For the parity-twisted Witten index for the γ-$\tau\gamma$ open string, we find

$$I_{\tau\Omega}(\gamma) = \#(\gamma^- \cap X^{\tau+}). \tag{6.11}$$

This is the LG counterpart of the index formula (3.6). We also find $I_{\tau\Omega}(\tau\gamma) = \#(X^{\tau-} \cap \gamma^+)$, which is consistent with the above since $\#(\gamma^- \cap X^{\tau+}) = \#(X^{\tau-} \cap (\tau\gamma)^+)$.

6.4 The case of $W = \Phi^{k+2}$

The LG model with superpotential (6.1) flows to the level k $\mathcal{N}=2$ minimal model [16, 17] (see also [18]). The system has both vector and axial R-symmetries

$$U(1)_V : \Phi(\theta^\pm) \to \mathrm{e}^{\frac{2i\alpha}{k+2}} \Phi(\mathrm{e}^{-i\alpha}\theta^\pm), \quad U(1)_A : \Phi(\theta^\pm) \to \Phi(\mathrm{e}^{\mp i\beta}\theta^\pm),$$

which correspond to two combinations of the $U(1)_R$ and $U(1)_L$ R-symmetries (4.3). There is also a discrete symmetry generated by

$$\Phi(\theta^\pm) \to \mathrm{e}^{\frac{\pi i}{k+2}} \Phi(\pm\theta^\pm),$$

which corresponds to the $\mathbb{Z}_{2(k+2)}$ symmetry (4.11). This was found by a subtle analysis of the anomaly in the gauged WZW model, but it is evident at the level of the classical Lagrangian in the LG model.

The chiral ring of the model is $\mathbb{C}[\phi]/W'(\phi)$, namely, generated by ϕ and subject to the relation $\phi^{k+1} = 0$. The ring elements correspond to the states

$$|j\rangle_{cc} \leftrightarrow \phi^{2j}. \tag{6.12}$$

The antichiral fields correspond to the states

$$|j\rangle_{aa} \leftrightarrow \overline{\phi}^{2j}. \tag{6.13}$$

They can also be related to the RR ground states $|j\rangle_{\mathrm{RR}}$ by spectral flow.

In what follows, we study orientifolds (and D-branes) of this LG model, and compare with the results obtained in the previous section.

6.4.1 A-orientifolds

Let us first find the A-parities of the system. As discussed above, an A-parity is of the form $\tau_A\Omega$, where τ_A is an antiholomorphic map such that $(\tau_A\phi)^{k+2} = \overline{\phi}^{k+2}$ up to a possible addition of a constant. We find $k+2$ of them given by

$$\tau_A^{2m} : \phi \to \mathrm{e}^{\frac{2\pi i m}{k+2}} \overline{\phi}, \qquad m = 0, 1, \ldots, k+1. \tag{6.14}$$

We also find the same number of $\widetilde{A}$-parities $(-1)^{F_R}\tau_A^{2m+1}\Omega$, where

$$\tau_A^{2m+1} : \phi \to \mathrm{e}^{\frac{\pi i(2m+1)}{k+2}} \overline{\phi}, \qquad m = 0, 1, \ldots, k+1. \tag{6.15}$$

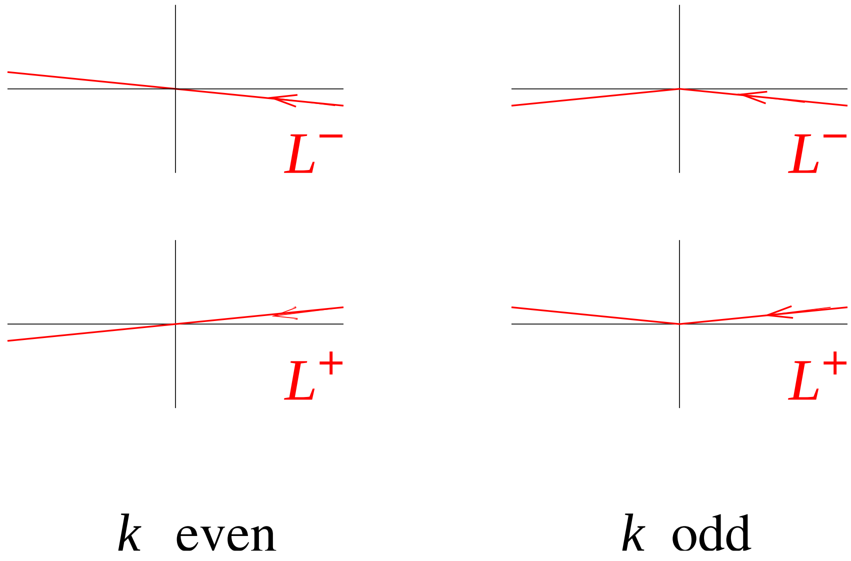

Figure 7: The deformed orientifold planes

Note that τ_A^ℓ maps the chiral primary fields ϕ^{2j} to antichiral primary fields $e^{\frac{2\pi i\ell j}{k+2}}\overline{\phi}^{2j}$. Comparing with the action of P_A^ℓ, which maps $|j\rangle_{cc}$ to $e^{\frac{2\pi i\ell j}{k+2}}|j\rangle_{aa}$ up to an ℓ-independent phase multiplication, we find that these are related as

$$a^{2m}P_A = \tau_A^{2m}\Omega, \quad a^{2m+1}P_A = (-1)^{F_R}\tau_A^{2m+1}\Omega, \tag{6.16}$$

possibly with a uniform shift of m.

In what follows we study some properties of the parity for the involution $\tau = \tau_A^0$. Other A-parities are simply combinations with the discrete axial symmetries.

Closed string Witten index

The orientifold plane L for $\tau\Omega$ is the real line $\phi \in \mathbb{R}$. Its W-image is the semi-infinite line $\mathbb{R}_{\geq 0}$ for even k and the real line itself for odd k. Thus, the deformed orientifold planes $L^\pm$ are straight lines $e^{\pm i\epsilon}\mathbb{R}$ for even k while they are infinitesimally bent lines $e^{\pm i\epsilon}\mathbb{R}_{\geq 0} \cup e^{\mp i\epsilon}\mathbb{R}_{\leq 0}$ for odd k. This is shown in Fig. 7, where we have chosen an orientation of each of the planes. We find from this that the parity-twisted Witten index for the closed string is given by

$$I_{\tau\Omega} = \#(L^- \cap L^+) = \begin{cases} 1 & k \text{ even} \\ 0 & k \text{ odd.} \end{cases} \tag{6.17}$$

This is in agreement with the result (4.91).

Overlap of the crosscap and the RR ground states

Let us compute the overlap of the crosscap states and the RR ground states using the LG model. The differential form $\phi_j \Omega$ corresponding to the normalized ground state $|j\rangle_{_{\rm RR}}$ is

$$\phi_j \Omega = c_j \phi^{2j} {\rm d}\phi, \tag{6.18}$$

where

$$c_j = -i\, {\rm e}^{-\frac{\pi i(2j+1)}{2(k+2)}} \frac{\sqrt{k+2}}{\Gamma(\frac{2j+1}{k+2})\sqrt{2\sin(\frac{\pi(2j+1)}{k+2})}}. \tag{6.19}$$

We have chosen this normalization so that the associated ground states have unit norm $\langle \overline{\jmath}|j'\rangle = \delta_{j,j'}$. (See Appendix F for an explanation of this point.) It is straightforward to compute the integrals (6.7) and (6.8)

$$\Pi_j^{\tau\Omega} = c_j \int_{L^-} {\rm e}^{-i\phi^{k+2}} \phi^{2j} {\rm d}\phi,$$
$$\widetilde{\Pi}_{\overline{\jmath}}^{\tau\Omega} = \overline{c_j} \int_{L^+} {\rm e}^{-i\overline{\phi}^{k+2}} \overline{\phi}^{2j} * {\rm d}\overline{\phi}.$$

They are

$$\Pi_j^{\tau\Omega} = \begin{cases} i\, {\rm e}^{-\frac{\pi i(2j+1)}{(k+2)}} \dfrac{1+(-1)^{2j}}{\sqrt{2(k+2)\sin(\frac{\pi(2j+1)}{k+2})}} & k \text{ even}, \\ i\, {\rm e}^{-\frac{\pi i(2j+1)}{2(k+2)}} \dfrac{{\rm e}^{-\frac{\pi i(2j+1)}{2(k+2)}} + (-1)^{2j}\, {\rm e}^{\frac{\pi i(2j+1)}{2(k+2)}}}{\sqrt{2(k+2)\sin(\frac{\pi(2j+1)}{k+2})}} & k \text{ odd}, \end{cases} \tag{6.20}$$

$$\widetilde{\Pi}_{\overline{\jmath}}^{\tau\Omega} = \begin{cases} \dfrac{1+(-1)^{2j}}{\sqrt{2(k+2)\sin(\frac{\pi(2j+1)}{k+2})}} & k \text{ even}, \\ {\rm e}^{\frac{\pi i(2j+1)}{2(k+2)}} \dfrac{{\rm e}^{-\frac{\pi i(2j+1)}{2(k+2)}} + (-1)^{2j}\, {\rm e}^{\frac{\pi i(2j+1)}{2(k+2)}}}{\sqrt{2(k+2)\sin(\frac{\pi(2j+1)}{k+2})}} & k \text{ odd}. \end{cases} \tag{6.21}$$

Let us compare these with the results (4.84) and (4.85) from the RCFT analysis. It is straightforward to check that they completely agree:

$$\Pi_j^{\tau\Omega} = \langle \mathscr{C}_{(-1)^F P_A} | j\rangle_{_{\rm RR}},$$
$$\widetilde{\Pi}_{\overline{\jmath}}^{\tau\Omega} = {}_{_{\rm RR}}\langle j|\mathscr{C}_{P_A}\rangle.$$

The A-branes of the model are studied in [22], using the deformation to a massive model, and found to be D1-branes at the wedge-shaped broken lines A_{a_f,a_i}. The broken line A_{a_f,a_i} is the inward half-line $\{\phi \in \mathrm{e}^{ia_i}\mathbb{R}_{\geq 0}\}$ joined at the origin to the outward half-line $\{\phi \in \mathrm{e}^{ia_f}\mathbb{R}_{\geq 0}\}$ where the angles are distinct, $a_i \neq a_f$, and are quantized as $a_i, a_f \in \frac{2\pi}{k+2}\mathbb{Z}_{k+2}$. See the left figure in Fig. 8. The Cardy brane $\mathscr{B}_{j,n,s}$ with odd s corresponds to the broken line $A_{j,n,s}$ where $A_{j,n,s=\pm 1} = A_{a_\pm,a_\mp} = \pm A_{a_+,a_-}$ where $a_\pm = \frac{\pi(n\pm 2j\pm 1)}{k+2}$. The (untwisted) open string index is computed to be [22]

$$I(A_{jns}, A_{j'n's'}) = \#(A^-_{jns} \cap A^+_{j'n's'}) = (-1)^{\frac{s-s'}{2}} N_{j\,j'}^{\frac{n'-n}{2}}, \tag{6.22}$$

where $N_{j_1\,j_2}^{j_3}$ is the level k $SU(2)$ fusion coefficients, which are extended outside the standard region $0 \leq j_3 \leq \frac{k}{2}$ by $N_{j_1\,j_2}^{j_3} = -N_{j_1\,j_2}^{-j_3-1} = N_{j_1\,j_2}^{j_3+(k+2)}$ and $N_{j_1\,j_2}^{-\frac{1}{2}} = 0$. Let us now compare it with the results obtained in the RCFT analysis. We see that the above result agrees with $I(\mathscr{B}_{jns}, \mathscr{B}_{j'n's'})$ given in (5.30), if we make a shift $(n, s) \to (n-1, s-1)$ in the latter. This implies the following identification of the brane and the boundary states

$$\begin{array}{ccc} & A_{jns} & \\ \swarrow & & \searrow \\ \langle \mathscr{B}_{j,n-1,s-1}| & & |\mathscr{B}_{j,n,s}\rangle. \end{array} \tag{6.23}$$

Furthermore, under this correspondence, the overlaps of the boundary states and the RR ground states are exactly identical in the two descriptions [22] (see also Appendix F):

$$\Pi_j^{A_{jns}} = c_j \int_{A^-_{jns}} \mathrm{e}^{-i\phi^{k+2}} \phi^{2j} \mathrm{d}\phi = {}_{\mathrm{RR}}\langle \mathscr{B}_{j,n-1,(s-1)}|j\rangle_{\mathrm{RR}},$$
$$\widetilde{\Pi}_{\bar{j}}^{A_{jns}} = \overline{c_j} \int_{A^+_{jns}} \mathrm{e}^{-i\overline{\phi}^{k+2}} \overline{\phi}^{2j} * \mathrm{d}\overline{\phi} = {}_{\mathrm{RR}}\langle j|\mathscr{B}_{j,n,(s)}\rangle_{\mathrm{RR}}.$$

Let us now look at the parity action on the branes. Since the parity acts on the ϕ-plane by complex conjugation which inverts the phases, the A-branes are mapped under τ as $A_{a_f,a_i} \to A_{-a_f,-a_i}$. (See Fig. 8.) Since $-a_\pm(j,n) = \frac{\pi(-n\mp 2j\mp 1)}{k+2} = a_\mp(j,-n)$, this is equivalent to

$$\tau : A_{j,n,s=1} \to A_{j,-n,s=-1}. \tag{6.24}$$

This is in agreement with the RCFT result (5.13) under the correspondence (6.23).

Let us next compute the parity-twisted Witten index for the open string stretched from A_{a_f,a_i} to $\tau A_{a_f,a_i}$. The index is given by the intersection number of the deformed

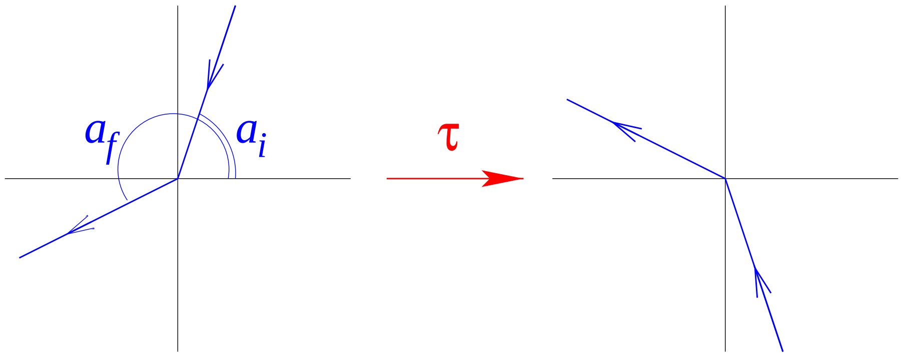

Figure 8: D-brane transform $\tau : A_{a_f,a_i} \to A_{-a_f,-a_i}$

D-brane $A^-_{a_f,a_i}$ and the deformed orientifold plane L^+. It is easy to see that

$$I_{\tau\Omega}(A_{a_f,a_i}, A_{-a_f,-a_i}) = \#(A^-_{a_f,a_i} \cap L^+) = \begin{cases} -1 & 0 < a_i \le \pi < a_f \le 2\pi \\ 1 & 0 < a_f \le \pi < a_i \le 2\pi \\ 0 & \text{otherwise.} \end{cases} \tag{6.25}$$

One could reduce the computation to the computation of the brane intersection number by using the relation $\#(A^- \cap L^+) = \pm\#(A^- \cap \tau A^+)$. For $A_{jns=1}$, the sign here is $-$ for $0 < n \le k+2$ and $+$ for $k+2 < n \le 2(k+2)$. On the other hand, we know the brane intersection number by formula (6.22), $\#(A^-_{jn1} \cap \tau A^+_{jn1}) = \#(A^-_{jn1} \cap A^+_{j,-n,-1}) = N_{j\,j}^{n-1}$ in which the extension of $N_{j_1 j_2}^{j_3}$ outside $0 \le j_3 \le \frac{k}{2}$ is understood. Thus, we find

$$I_{\tau\Omega}(A_{j,n,1}, A_{j,-n,-1}) = \begin{cases} N_{j\,j}^{n-1} & \frac{1}{2} \le n \le \frac{k+2}{2} + \frac{1}{2} \\ -N_{j\,j}^{(k+2)-n} & \frac{k+2}{2} + \frac{1}{2} \le n \le (k+2) \\ -N_{j\,j}^{n-(k+2)-1} & (k+2) + \frac{1}{2} \le n \le \frac{3(k+2)}{2} + \frac{1}{2} \\ N_{j\,j}^{2(k+2)-n} & \frac{3(k+2)}{2} + \frac{1}{2} \le n \le 2(k+2). \end{cases} \tag{6.26}$$

This is in complete agreement with (5.31) under the correspondence (6.23). As in (5.32), this can be concisely written as

$$I_{\tau\Omega}(A_{j,n,s}, A_{j,-n,s+2}) = (-1)^{\frac{s-1}{2}} \widetilde{N}_{j,j}^{n-1}, \tag{6.27}$$

where $\widetilde{N}_{j_1,j_2}^{j_3} = N_{j_1,j_2}^{j_3}$ in $-\frac{1}{2} \le j_3 \le \frac{k+1}{2}$ and is extended by $\widetilde{N}_{j_1,j_2}^{j_3} = \widetilde{N}_{j_1,j_2}^{-j_3-1} = -\widetilde{N}_{j_1,j_2}^{j_3+(k+2)}$ outside that region. Yet another expression is obtained by noting that the orientifold plane L is equal or close to one of the branes. This leads to

$$I_{\tau\Omega}(A_{jns}) = (-1)^{\frac{s-1}{2}} N_{j,\frac{1}{2}[\frac{k+1}{2}]}^{\frac{n+[\frac{k-1}{2}]}{2}}. \tag{6.28}$$

6.4.2 B-orientifolds

We next consider B-parities. A B-parity should be of the form $\tau_B\Omega$, where τ_B is a holomorphic map such that $(\tau_B\phi)^{k+2} = -\phi^{k+2}$ plus a possible constant. We find $k+2$ of them given by

$$\tau_B^{2m+1} : \phi \to \mathrm{e}^{\frac{\pi i(2m+1)}{k+2}}\phi, \qquad m = 0, 1, \ldots, k+1. \tag{6.29}$$

We also find $\widetilde{B}$-parities $(-1)^{F_R}\tau_B^{2m}\Omega$ with

$$\tau_B^{2m} : \phi \to \mathrm{e}^{\frac{2\pi i m}{k+2}}\phi, \qquad m = 0, 1, \ldots, k+1. \tag{6.30}$$

τ_B^ℓ maps the chiral primary fields ϕ^{2j} to $\mathrm{e}^{\frac{2\pi i \ell j}{k+2}}\phi^{2j}$. Comparing with the action of P_B^ℓ which maps $|j\rangle_{cc}$ to $\mathrm{e}^{\frac{2\pi i\ell j}{k+2}}|j\rangle_{cc}$, up to an ℓ-independent phase multiplication, we find that these are related as

$$a^{2m+1}P_B = \tau_B^{2m+1}\Omega, \quad a^{2m}P_B = (-1)^{F_R}\tau_B^{2m}\Omega, \tag{6.31}$$

possibly with a uniform shift of m.

For even k, there is no involutive B-parity, and therefore one cannot think about the crosscap states. For odd k there is a unique involutive B-parity, $P_B^{k+2} = \tau_B^{k+2}\Omega$. The crosscap state $|P_B^{k+2}\rangle$ of course consists of RR-sector states. However, it has no overlap with RR ground states, since $|P_B^{k+2}\rangle$ has to have zero R-charge but, for odd k, there is simply no RR ground states with vanishing R-charge. Thus, no non-trivial computation can be done in the LG realization. [1]

In the gauged WZW model, it was a non-trivial task to find anomaly-free B-parity symmetries. For instance, the transform $\mathcal{I}_B\Omega$ was anomalous since it flips the sign of the fermion path-integral measure in an odd instanton background. In the LG realization, it can readily be found at the level of the classical Lagrangian: The analogue of $\mathcal{I}_B\Omega$ would be $\tau_B^0\Omega$ and it flips the sign of the fermion mass term $W''(\phi)\psi_+\psi_-$. This reminds us of mirror symmetry where the effect of worldsheet instantons in one theory is classical in the mirror. In fact, in a closely related $SL(2,\mathbb{R})/U(1)$ gauged WZW model, one can find a mirror transform to a LG model whose superpotential reflects the instanton effects [76]. The present observation suggests that this story may extend also to the $SU(2)/U(1)$ model. In the next section, we will see among other things that parity anomalies in non-linear sigma models are reflected as the explicit breaking by the superpotential in the mirror LG models.

[1] This is unlike the result of the study of B-branes, where for even k there is one B-brane with a non-trivial overlap with the RR ground state $|\frac{k}{4}\rangle$ of zero R-charge, which is computable in the LG realization [23].

7 Orientifolds of Linear Sigma Models and Mirror Symmetry

Linear sigma models provide a way to realize and study non-linear sigma models on a class of target spaces from a global view point, and allow us to find a picture of the moduli space of theories. In this and the next section we study parity symmetries of linear sigma models. We determine the conditions on the parameters for theory to be invariant under A- and B-type parities. The condition agrees in the large volume limit with the one derived from the non-linear sigma model and at the LG orbifold point to the one from the LG model. We also determine the corresponding parity in the mirror Landau–Ginzburg model. In particular, we find that the information on the parity actions on the line bundle $\mathcal{L}_{\tau^* B+B}$ (which we found to be required in Section 3.3.4) has a natural counterpart in the mirror LG model as the type of the orientifold planes. The general results are applied in specific examples.

7.1 Parity symmetry of linear sigma models

Let us consider a $(2,2)$ supersymmetric $U(1)^k$ gauge theory with N matter fields $\Phi_1, \ldots, \Phi_N$ of charge Q_i^a, where $i = 1, \ldots, N$ labels the matter fields and $a = 1, \ldots, k$ labels the gauge group. The basic Lagrangian of the model is given by

$$L_1 = \int \mathrm{d}^4\theta \left[\sum_{i=1}^{N} \overline{\Phi}_i \, \mathrm{e}^{Q_i \cdot V} \Phi_i - \sum_{a=1}^{k} \frac{1}{e_a^2} |\Sigma_a|^2 \right] + \mathrm{Re} \int \mathrm{d}^2\widetilde{\theta} \sum_{a=1}^{k} \Big(-t^a \Sigma_a \Big). \tag{7.1}$$

V_a in $Q_i \cdot V := \sum_{a=1}^{k} Q_i^a V_a$ are the vector superfields, $\Sigma_a := \overline{D}_+ D_- V_a$ are the fieldstrengths (twisted chiral superfields), and e_a are the gauge coupling constants (with dimension of mass). The measure is $\mathrm{d}^2\widetilde{\theta} = \mathrm{d}\overline{\theta}^- \mathrm{d}\theta^+$ for the twisted chiral superfields and t^a are the complex combinations of the Fayet–Iliopoulos parameters and theta angles:

$$t^a = r^a - i\theta^a. \tag{7.2}$$

If one can find a gauge-invariant holomorphic polynomial $W(\Phi_1, \ldots, \Phi_N)$, one may also consider adding an F-term

$$L_2 = \mathrm{Re} \int \mathrm{d}^2\theta \, W(\Phi_1, \ldots, \Phi_N). \tag{7.3}$$

With or without this additional term, the FI parameter is renormalized as $r^a(\mu') = r^a(\mu) + \sum_{i=1}^{N} b_1^a \log(\mu'/\mu)$, where

$$b_1^a := \sum_{i=1}^{N} Q_i^a.$$

At certain energies lower than the gauge couplings, the system reduces to the non-linear sigma model on the vacuum manifold if r^a is in a suitable region. If the Lagrangian is given only by L_1, the vacuum manifold is a toric manifold X determined by the symplectic reduction $\sum_{i=1}^N Q_i^a|\phi_i|^2 = r^a$ mod $U(1)^k$. With the additional term L_2 it is a subspace of X determined by the equation $\partial_i W = 0$. In either case, it has a second cohomology group of rank k. With respect to an integral basis $\{\omega_a\}$, the Kähler class is roughly written as $\sum_{a=1}^k r^a\omega_a$ and the first Chern class is given by $\sum_{a=1}^k b_1^a\omega_a$. There are, however, regions of r^a, where the system reduces to Landau–Ginzburg orbifolds or some hybrid of them, rather than a non-linear sigma model.

We are interested in the parity symmetries of this class of gauge systems.

7.1.1 A-parity

Let us consider the A-parity transformation

$$\begin{aligned} \Phi_i &\longrightarrow \overline{\Omega_A^*\Phi_i}, \\ V_a &\longrightarrow \Omega_A^* V_a. \end{aligned} \tag{7.4}$$

This is compatible with the gauge symmetry, as can be seen from the action on the components $(\phi(x), v_\mu(x)) \to (\overline{\phi}(\widetilde{x}), -(-1)^\mu v_\mu(\widetilde{x}))$, or by looking how the superfield gauge transformation $\Phi \to \mathrm{e}^{i\Lambda}\Phi, V \to V + i(\bar{\Lambda} - \Lambda)$ is affected. One can also check that the Lagrangian L_1 is invariant: The term $\int \mathrm{d}^4\theta\, \overline{\Phi}\, \mathrm{e}^V \Phi$ is obviously invariant. To see the rest, we note that the field strength transforms as

$$\Sigma_a \longrightarrow -\Omega_A^*\Sigma_a.$$

Then, the kinetic term $\int \mathrm{d}^4\theta|\Sigma|^2$ is also invariant. The twisted superpotential $-t\Sigma$ flips sign, but the sign is cancelled since the measure $\mathrm{d}^2\widetilde{\theta}$ is odd under Ω_A. This shows the invariance of L_1. For the potential term L_2 to be invariant, however, the superpotential W has to obey the condition $W(\overline{\Phi}_i) = \overline{W(\Phi_i)}$, that is, the coefficients of the polynomial W all have to be real. If the system describes a non-linear sigma model at certain energies, the map $\phi_i \to \overline{\phi}_i$ reduces to an antiholomorphic involution of the target space, and the parity corresponds to the A-parity of the non-linear sigma model associated with it.

In the quantum theory, however, we have to see if the path-integral measure is also invariant. Here we encounter a possible anomaly of the type considered in the minimal model (and the non-linear sigma model), since the topology of any gauge field background is preserved under the parity we are considering, $v_a \to -\Omega^* v_a$. Following the argument of Section 4.2.1, we see that the fermion measure changes as follows

$$\mathcal{D}_v\Psi \longrightarrow (-1)^{b_1\cdot c_1}\mathcal{D}_v\Psi, \tag{7.5}$$

where $b_1 \cdot c_1 := \sum_{a=1}^{k} b_1^a c_1(V_a)$ in which V_a is the $U(1)$ bundle of the a-th gauge field. Thus, if some of the b_1^a are odd, the parity symmetry is anomalous. This corresponds to the anomaly of an A-parity in the non-linear sigma model in the case where the target space is not spin, since b_1^a provides the first Chern class of the manifold.

One may consider combining (7.4) with an internal action on the fields. Here we describe the case where this action involves a permutation of the Φ_i's. A permutation $\Phi_i \to \Phi_{\sigma(i)}$ is compatible with the gauge symmetry if there is a linear transformation $V \to V'$ such that $Q_i \cdot V' = Q_{\sigma(i)} \cdot V$. Namely, it should be accompanied by an action on the gauge field $V_a \to \sigma_a^{\ b} V_b$ where $\sigma_a^{\ b}$ is a matrix such that

$$\sum_{b=1}^{k} Q_i^b \sigma_b^{\ a} = Q_{\sigma(i)}^a. \tag{7.6}$$

Thus, we consider the combined transformation

$$\begin{aligned} \Phi_i &\longrightarrow \overline{\Omega_A^* \Phi_{\sigma(i)}}, \\ V_a &\longrightarrow \sigma_a^{\ b}\Omega_A^* V_b. \end{aligned} \tag{7.7}$$

Under this, the Lagrangian L_1 is invariant if the FI-theta parameters obey $t^b\sigma_b^{\ a} = t^a$ (mod $2\pi i\mathbb{Z}$). In the quantum theory, because of the transformation of the path-integral measure (7.5), the condition for this to be a symmetry is

$$t^b\sigma_b^{\ a} = t^a + \pi i b_1^a \qquad \text{mod } 2\pi i\mathbb{Z}. \tag{7.8}$$

If this is not satisfied, or if there is no solution to this equation, the parity symmetry is anomalous. If the model has the potential term L_2, there is a further condition that

$$W(\overline{\Phi}_{\sigma(i)}) = \overline{W(\Phi_i)}. \tag{7.9}$$

In the picture of the non-linear sigma model, $\phi_i \to \overline{\phi_{\sigma(i)}}$ corresponds to an antiholomorphic diffeomorphism f of the target space. The symmetry condition (7.8) is nothing but the geometric condition (3.1) in the sigma model, since one can identify $c_1(X)^a = b_1^a$ and $f_a^{\ b} = -\sigma_a^{\ b}$. The latter holds because the $\mathrm{Re}\Sigma_a$ correspond to the integral basis ω_a of $H^2(X)$, and they are transformed under the parity as

$$\Sigma_a \longrightarrow -\sigma_a^{\ b}\Omega_A^*\Sigma_b,$$

while $f_a^{\ b}$ is defined as $\omega_a \to f^*\omega_a = f_a^{\ b}\omega_b$.

One may consider yet another modification of the parity transformation. It is to combine (7.4) or (7.7) with one of the $U(1)^{N-k}$ torus actions, $\Phi_i \to \mathrm{e}^{i\theta_i}\Phi_i$. This is a symmetry under the same condition on t^a as before but, if there is an L_2 term, under the modified condition on the superpotential $W(\mathrm{e}^{i\theta_i}\overline{\Phi_{\sigma(i)}}) = \overline{W(\Phi_i)}$.

7.1.2 B-parity

Let us next consider the B-parity transformation

$$\begin{aligned} \Phi_i &\longrightarrow \mathrm{e}^{i\theta_i}\Omega_B^*\Phi_i, \\ V_a &\longrightarrow \Omega_B^* V_a. \end{aligned} \tag{7.10}$$

It acts on the component fields as $(\phi(x), v_\mu(x)) \to (\phi(\widetilde{x}), (-1)^\mu v_\mu(\widetilde{x}))$ and is compatible with the gauge symmetry. The field strength transforms as

$$\Sigma_a \longrightarrow \overline{\Omega_B^*\Sigma_a},$$

and therefore the action L_1 is invariant under the condition that the t^a are all real. The potential term L_2, if it is present, is always invariant. This parity symmetry is anomaly-free. In the sigma model picture, this corresponds to the B-parity associated with the identity action on the target space.

As before, one may also consider combining this with an internal action on the fields. We discuss the combination with a permutation of the fields

$$\begin{aligned} \Phi_i &\longrightarrow \mathrm{e}^{i\theta_i}\Omega_B^*\Phi_{\sigma(i)}, \\ V_a &\longrightarrow \sigma_a^{\ b}\Omega_B^* V_b, \end{aligned} \tag{7.11}$$

where we need the relation (7.6) for compatibility with the gauge symmetry. The condition of invariance of L_1 is

$$t^b\sigma_b^{\ a} = \overline{t^a} \qquad \mathrm{mod}\ 2\pi i\mathbb{Z}, \tag{7.12}$$

and the condition for invariance of L_2 is

$$W(\mathrm{e}^{i\theta_i}\Phi_{\sigma(i)}) = -W(\Phi_i). \tag{7.13}$$

The symmetry (7.11) is an exact symmetry of the quantum theory. In the sigma model picture, it corresponds to the B-parity symmetry associated with a holomorphic automorphism f of the target space. The condition (7.12) is nothing but the geometric condition (3.2) since $\sigma_a^{\ b} = f_a^{\ b}$. The latter holds because the real part of the field strength is transformed as $\mathrm{Re}\Sigma_a \to \sigma_a^{\ b}\Omega_B^*\mathrm{Re}\Sigma_b$ while $f_a^{\ b}$ is defined as the action on the integral basis of H^2, $\omega_a \to f^*\omega_a = f_a^{\ b}\omega_b$.

7.2 Description in the mirror LG model

The model with Lagrangian L_1 has a mirror description [32]. It is obtained by dualization of the phase of the charged matter fields, taking into account the effect of the vortex instantons. The dualization of $\arg(\Phi_i)$ yields a twisted chiral superfield Y_i with periodic identification $Y_i \equiv Y_i + 2\pi i$. The twisted superpotential of the dual theory is

$$\widetilde{W} = \sum_{a=1}^{k} \left(\sum_{i=1}^{N} Q_i^a Y_i - t^a \right) \Sigma_a + \sum_{i=1}^{N} \mathrm{e}^{-Y_i},$$

where the Y-linear terms originate from the dualization and the exponential terms come from instanton effects. In the non-linear sigma model limit where the gauge coupling is taken to be large, it is appropriate to integrate out the heavy fields Σ_a, and we are left with the theory of fields Y_i obeying the constraints

$$\sum_{i=1}^{N} Q_i^a Y_i = t^a \qquad \bmod 2\pi i\mathbb{Z}, \tag{7.14}$$

having the twisted superpotential

$$\widetilde{W} = \sum_{i=1}^{N} \mathrm{e}^{-Y_i}. \tag{7.15}$$

Namely, the mirror is the LG model on the algebraic torus $Y \cong (\mathbb{C}^\times)^{N-k}$ defined by (7.14) with the above superpotential. We study how the A- and B-parities we considered above are described in this mirror theory.

7.2.1 A-parity (B-parity in LG)

Let us consider the A-parity (7.7) of the original linear sigma model. Looking at the action on the charged matter fields $\Phi_i \to \overline{\Omega_A^* \Phi_{\sigma(i)}}$, we expect that their dual fields are transformed similarly, $Y_i \to \Omega_A^* Y_{\sigma(i)}$. This would transform the superpotential as $\widetilde{W} \to \Omega_A^* \widetilde{W}$. However, we recall an important condition for unbroken symmetry: since the twisted F-term measure $\mathrm{d}^2\widetilde{\theta}$ flips sign under Ω_A, the twisted superpotential must also flip sign under the transformation. The only way to make the sign of $\sum_{i=1}^{N} \mathrm{e}^{-Y_i}$ flip is to add πi to each field Y_i. Thus, it can be concluded that the dual transformation of the fields, required by the condition that it be a symmetry, is

$$Y_i \longrightarrow \Omega_A^* Y_{\sigma(i)} + \pi i. \tag{7.16}$$

In addition, we should make sure that it is compatible with the constraint (7.14). Let us examine this. The left hand side of the constraint $\sum_{i=1}^{N} Q_i^a Y_i = t^a \pmod{2\pi i\mathbb{Z}}$ is transformed to

$$\begin{aligned}\sum_{i=1}^{N} Q_i^a(\Omega_A^* Y_{\sigma(i)} + \pi i) &= \sum_{i=1}^{N} Q_{\sigma^{-1}(i)}^a(\Omega_A^* Y_i + \pi i) \\ &= \sum_{i,b} Q_i^b(\sigma^{-1})_b^a(\Omega_A^* Y_i + \pi i) = \sum_b (\sigma^{-1})_b^a(t^b + \pi i b_1^b).\end{aligned}$$

In the last step, we have used the fact that the original fields obey the constraint. Now, inserting the anomaly-free condition (7.8), we infer that $(\sigma^{-1})_b^a(t^b + \pi i b_1^b) = t^a \bmod 2\pi i\mathbb{Z}$. Thus the transformed fields indeed respect the constraint. If the condition (7.8) were broken, the transformation (7.16) would not be consistent with the constraint (7.14), or we would not be able to find a transformation such that $\widetilde{W} \to -\Omega_A^* \widetilde{W}$. One of the conclusions is that the parity anomaly, which is a non-trivial quantum effect in the original (linear) sigma model, is reflected in the dual theory as the explicit breaking by the potential term.

We recall that one could consider the modification of A-parity using the $U(1)^{N-k}$ torus actions. How does it affect the parity symmetry in the mirror side? As we have discussed, there is no freedom to change the action (7.16) on the dual fields. In fact, the change appears simply in the way it acts on the winding sector. Recall that the LG field Y_i takes values in the algebraic torus $Y = (\mathbb{C}^\times)^{N-k}$ with non-trivial topology $\pi_1(Y) = \mathbb{Z}^{N-k}$, and the momentum in the sigma model on X associated with $U(1)^{N-k}$ symmetry is dual to the winding number of Y in the mirror LG model.

A transformation of this type has been used in [77] in the application of mirror symmetry to compute the space-time superpotential in Type II orientifolds on non-compact Calabi–Yau threefolds. We now consider some examples where the parity anomaly and its cancellation play important roles.

7.2.2 Examples and applications

Example 1.

Let us consider the case $X = \mathbb{CP}^n$ whose mirror is the LG model of n periodic variables $Y_1, ..., Y_n$ with superpotential

$$\widetilde{W} = \mathrm{e}^{-Y_1} + \cdots + \mathrm{e}^{-Y_n} + \mathrm{e}^{-t+Y_1+\cdots+Y_n},$$

where t corresponds to the complexified Kähler class of $\mathbb{CP}^n$. Consider the involution

$$\widetilde{\tau} : Y_i \to Y_i + \pi i.$$

If n is odd, it flips the sign of the superpotential $\widetilde{W} \to -\widetilde{W}$ and hence is a symmetry of the system. If n is even, it fails to flip the superpotential — the first n terms of $\widetilde{W}$ do flip but the last term does not. Thus, $\widetilde{\tau}\Omega$ is not a symmetry of the system. Note that the above holomorphic involution $\widetilde{\tau}$ corresponds to the antiholomorphic involution $\tau : \Phi_i \to \overline{\Phi}_i$ in the original $\mathbb{CP}^n$ sigma model. The above trouble for even n corresponds to the anomaly of this A-parity. Moreover one can show that there is no way of transforming the fields Y_i so that $\widetilde{W} \to -\widetilde{W}$. This corresponds to the A-parity anomaly on non-spin manifolds with $b^2(X) = 1$.

Example 2.

Let us next consider the case $X = \mathbb{CP}^n \times \mathbb{CP}^n$ whose mirror is the direct sum

$$\widetilde{W} = \sum_{i=1}^{n} \mathrm{e}^{-Y_i^{(1)}} + \mathrm{e}^{-t_1 + Y_1^{(1)} + \cdots + Y_n^{(1)}} + \sum_{i=1}^{n} \mathrm{e}^{-Y_i^{(2)}} + \mathrm{e}^{-t_2 + Y_1^{(2)} + \cdots + Y_n^{(2)}}.$$

Here t_1 and t_2 correspond to the complexified Kähler class of the first and the second $\mathbb{CP}^n$. Consider the involution

$$\widetilde{\tau} : (Y_i^{(1)}, Y_i^{(2)}) \mapsto (Y_i^{(2)} + \pi i, Y_i^{(1)} + \pi i).$$

Suppose $t_1 = t_2$. Then, as in Example 1, $\widetilde{\tau}$ flips the sign of the superpotential if and only if n is odd. The failure in the cases where n is odd is again caused by the anomaly of the A-parity associated with $\tau : (z_1, z_2) \mapsto (\overline{z}_2, \overline{z}_1)$ in $\mathbb{CP}^n \times \mathbb{CP}^n$. However, in this case, one can enforce a sign flip of the superpotential under the parity action by taking $t_1 = t_2 + \pi i$. This corresponds to the cancellation of the anomaly by a B-field, which we have seen in Section 3.2.

Application: Mirror pair of D-branes

In Section 3.5, we have seen that one can associate a D-brane to a parity symmetry, as long as the parity has a fixed point. One can use this to find the mirror pairs of D-branes. Let us consider the parity symmetry in Example 2. The fixed-point set is

$$Y^{\widetilde{\tau}} = \left\{ Y_i^{(1)} = Y_i^{(2)} + \pi i \right\} \cong (\mathbb{C}^\times)^n.$$

In the original sigma model on $X = \mathbb{CP}^n \times \mathbb{CP}^n$, the fixed-point set is the 'skew-diagonal'

$$X^\tau = \Big\{ z_1 = \overline{z}_2 \Big\} \cong \mathbb{CP}^n.$$

Thus, we see that the A-brane wrapped on X^τ is mirror to the B-brane wrapped on $Y^{\widetilde{\tau}}$. [1] To be precise, this is true only when the parity $\tau\Omega$ (or equivalently $\widetilde{\tau}\Omega$) is a symmetry of the theory. For example let us consider the case with $t_1 = t_2$ and n even. We know that $\tau\Omega$ is anomalous and the D-brane wrapped on X^τ is expected to suffer from some pathology. This is in fact evident on the mirror side: The superpotential $\widetilde{W}$ is not constant on the fixed-point set $Y^{\widetilde{\tau}}$ (that is, $Y^{\widetilde{\tau}}$ does not lie in one of the level sets of $\widetilde{W}$), which means that $Y^{\widetilde{\tau}}$ violates the condition of $\mathcal{N} = 2_B$ supersymmetry. On the other hand, for odd n (with $t_1 = t_2$), $Y^{\widetilde{\tau}}$ lies in the level set

$$\widetilde{W}|_{Y_i^{(1)} = Y_i^{(2)} + \pi i} = 0$$

and is indeed a good B-brane. For n even, $Y^{\widetilde{\tau}}$ can be placed in the level set $\widetilde{W} = 0$ by taking $t_1 = t_2 + \pi i$. Thus, X^τ becomes a consistent A-brane by adding a B-field of period π in one of the $\mathbb{CP}^n$'s of $X = \mathbb{CP}^n \times \mathbb{CP}^n$. This cannot be seen directly in the sigma model without a detailed analysis of the subtleties of the moduli space of holomorphic discs [59]. In the mirror description, this is evident at the classical level.

We note that D-branes of the above type are oriented, namely, they can have non-trivial overlaps with the RR ground states. This is because the brane $Y^{\widetilde{\tau}}$ in the LG description is middle-dimensional [23].

7.2.3 B-parity (A-parity in LG)

Let us next consider the B-parity (7.11) of the linear sigma model. The action on the charged matter fields $\Phi_i \to \Omega_B^* \Phi_{\sigma(i)}$ indicates that it transforms the dual fields as

$$Y_i \longrightarrow \overline{\Omega_B^* Y_{\sigma(i)}}. \tag{7.17}$$

This is consistent with the constraint (7.14) if the t^a obey the reality condition (7.12), $t^b \sigma_b^{\ a} = \overline{t^a} \bmod 2\pi i\mathbb{Z}$. This is indeed a symmetry of the dual system since the superpotential is transformed as $\widetilde{W} \to \overline{\Omega_B^* \widetilde{W}}$.

We recall that one could modify the B-parity using the $U(1)^{N-k}$ torus actions. As in the case of A-parities, the modification will change the way the parity acts on the

[1] Mirror symmetry between skew diagonal $\mathbb{CP}^n$ and $\{Y_i^{(1)} = Y_i^{(2)} + \pi i\}$ was first reported to KH by C. Vafa. At that time (October, 2000) the reason for the necessity of a B-field difference for the even n case (discussed below) was not clear to them.

winding sector in the mirror theory and induce the mixture of the (SO vs. Sp) types of orientifold planes. This is a generalization of the phenomenon that the orientifold of S^1 by a half-period shift is T-dual to the orientifold of the dual circle $\widetilde{S^1}$ by an inversion, with two opposite types of orientifold points [78, 79, 30].

Let us examine this in more detail, in the case where the parity is associated with the simple complex conjugation $Y_i \to \overline{Y_i}$, which is a symmetry of the system if the t^a are all real, $\theta^a = 0\ \forall a$. The fixed points of this action are $Y_i = \overline{Y_i}$ mod $2\pi i\mathbb{Z}$, namely

$$Y_i \in \mathbb{R} + \pi i p_i,$$

where p_i are integers obeying $\sum_{i=1}^N Q_i^a p_i = 0 \bmod 2\mathbb{Z}$. The set of points with $Y_i \in \mathbb{R}+\pi i p_i$ is a middle-dimensional plane $\mathbb{R}^{N-k}$ and will be denoted as $L_{p_1 \dots p_N}$. There are 2^{N-k} such planes. The fixed-point set $Y^{\widetilde{\tau}}$ is the union of these 2^{N-k} L_p's. The type of the orientifold plane L_p (whether it is SO or Sp) depends on the original parity symmetry on the sigma model side. Let us first consider the basic parity of SO-type associated with the identity $\Phi_i \to \Phi_i$ of X. Its mirror should be such that all 2^{N-k} orientifold planes are of SO type. Then, they have the same orientations so that the total homology class is

$$Y^{\widetilde{\mathrm{id}}} = \sum_p L_{p_1 \dots p_N}. \tag{7.18}$$

Here the orientation of all L_p's are related by translations by $\pi i p_i$. Let us next consider the parity of SO-type associated with an involution of X of the form $\tau_r : \Phi_i \to (-1)^{r_i}\Phi_i$. This action corresponds on the mirror side to the multiplication by

$$(-1)^{w_1 r_1 + \dots + w_N r_N}$$

in the sector with winding number $(w_1, ..., w_N)$ (where w_i are integers obeying $\sum_{i=1}^N Q_i^a w_i = 0 \bmod 2\mathbb{Z}$). For such a parity action, the type of the orientifold plane $L_{p_1 \dots p_N}$ is still SO-type if $(-1)^{p_1 r_1 + \dots + p_N r_N} = 1$, but is flipped to Sp-type if $(-1)^{p_1 r_1 + \dots + p_N r_N} = -1$. Thus the total homology class of the orientifold plane is

$$Y^{\widetilde{\tau_r}} = \sum_p (-1)^{p_1 r_1 + \dots + p_N r_N} L_{p_1 \dots p_N}. \tag{7.19}$$

Note that the involution τ_r is gauge-equivalent to the involution $\tau_{r+Q\cdot\lambda}$, with $\lambda_a \in \mathbb{Z}$. However, thanks to the condition $\sum_{i=1}^N Q_i^a p_i = 0 \bmod 2$, the gauge equivalent replacement $r \to r + Q \cdot \lambda$ does not alter the type $(-1)^{p_1 r_1 + \dots + p_N r_N}$ of the orientifold plane $L_{p_1 \dots p_N}$.

Actually, the parity associated with $Y_i \to \overline{Y_i}$ is a symmetry as long as $t^a = \overline{t^a}$ mod $2\pi i\mathbb{Z}$. The theta angle (or B-field) θ^a only has to be in $\pi\mathbb{Z}$ and does not have to vanish.

The above story can be applied also to the case with non-zero θ^a. An important difference appears however in the constraint on p_i — the p_i must still be integers but obey a different constraint

$$\sum_{i=1}^{N} Q_i^a p_i = -\frac{\theta^a}{\pi} \mod 2\mathbb{Z}.$$

This causes an apparent trouble. Under the gauge equivalent replacement $r_i \to r_i + Q_i^a \lambda_a$ ($\lambda_a \in \mathbb{Z}$), the type $(-1)^{p_1 r_1 + \cdots + p_N r_N}$ of the orientifold plane $L_{p_1 \ldots p_N}$ changes by a factor $\mathrm{e}^{i\theta^a \lambda_a}$. However, rather than being a trouble, it is consistent with the observation we made in the study of parity symmetry in non-linear sigma models with non-zero B-field. The observation was: *In addition to the geometric action $\tau : X \to X$, we have to specify the parity action on the line bundle $\mathcal{L}_{\tau^* B + B}$ with first Chern class $(\tau^* B + B)/2\pi$.* In the present case, $\tau^* B = B$ and the bundle is $\mathcal{L}_{2B}$ with first Chern class B/π. We propose that different choices of r_i within a fixed gauge equivalence class correspond to different choices of the parity action on $\mathcal{L}_{2B}$ if they are different in the numbers $(-1)^{p_1 r_1 + \cdots + p_N r_N}$. In fact, the field Φ_i may be regarded as a section of a line bundle $\mathcal{L}_i = \otimes_a \mathcal{L}_a^{Q_i^a}$ where $\omega_a = c_1(\mathcal{L}_a)$ are basis elements of $H^2(X, \mathbb{Z}) \cong \mathbb{Z}^{\oplus k}$. The bundle $\mathcal{L}_{2B}$ has first Chern class $c_1(\mathcal{L}_{2B}) = \sum_{a=1}^{k} \omega_a \theta_a / \pi$. Let us choose p_i (from the same mod 2 integer class) so that $\sum_{i=1}^{N} Q_i^a p_i = -\theta^a/\pi$ holds exactly (i.e. not just mod 2). Then we find

$$\bigotimes_{i,a} \mathcal{L}_a^{-Q_i^a p_i} = \bigotimes_{a} \mathcal{L}_a^{\theta^a/\pi} \xrightarrow{c_1} \sum_{a=1}^{k} \omega_a \frac{\theta^a}{\pi} = \frac{B}{\pi}.$$

Namely, $\otimes_i \mathcal{L}_i^{-p_i} \cong \mathcal{L}_{2B}$. Thus, $\prod_i \Phi_i^{-p_i}$ can be regarded as the section of the bundle $\mathcal{L}_{2B}$. The (r_i), which determine the numbers $(-1)^{p_1 r_1 + \cdots + p_N r_N}$, thus specify the parity action of the bundle $\mathcal{L}_{2B}$. In other words, the information of the pair $(\tau_r : X \to X, \tau_r : \mathcal{L}_{2B} \to \mathcal{L}_{2B})$ is mapped under mirror symmetry to the information of the type of the orientifold planes $L_{p_1 \ldots p_N}$.

We illustrate these rules in the example of $X = \mathbb{CP}^1$.

7.2.4 Example: B-parities of $\mathbb{CP}^1$ and their mirrors

For $X = \mathbb{CP}^1$ there are two possible geometric actions of the above type. One is the identity, and the other is the rotation by π along the $U(1)$-fibre, $R : \Phi_1/\Phi_2 \to -\Phi_1/\Phi_2$. Also, there are two possible values of the theta angle $\theta = 0$ and $\theta = \pi$. The mirror is the sine-Gordon model with superpotential

$$W = \mathrm{e}^{-Y} + \mathrm{e}^{-t+Y}.$$

The parity action on the dual field is the complex conjugation $Y \to \overline{Y}$ which is a symmetry if $\theta = 0, \pi$. The fixed-point set consists of $L_0 = \{Y \in \mathbb{R}\}$ and $L_1 = \{Y \in \mathbb{R} + \pi i\}$. They are respectively L_{00} and L_{11} if $\theta = 0$ or L_{01} and L_{10} if $\theta = \pi$, in the notation of the above general discussion. We compute the parity-twisted Witten index in both sigma model and LG model and compare the results.

We recall that the cohomology of $X = \mathbb{CP}^1$ is generated by $1 \in H^0(\mathbb{CP}^1)$ and $H \in H^2(\mathbb{CP}^1)$ with $\int_{\mathbb{CP}^1} H = 1$. We discuss the cases of $\theta = 0$ and $\theta = \pi$ separately.

$\theta = 0$

Let us first consider the basic parity Ω ($\tau = \mathrm{id}$). This maps the bundle $\mathcal{O}(n)$ to $\mathcal{O}(-n)$. The fixed-point set is $\mathbb{CP}^1$ itself and hence the normal bundle is zero. The characteristic classes relevant to the computation of the index are $L(\mathbb{CP}^1) = 0$ and $\mathrm{td}(\mathbb{CP}^1) = 1 + H = \mathrm{e}^H$. We then find

$$I_\Omega = 0, \tag{7.20}$$

$$I_\Omega(\mathcal{O}(n), \mathcal{O}(-n)) = \int_{\mathbb{CP}^1} \mathrm{e}^{-2nH}\, \mathrm{e}^H = 1 - 2n. \tag{7.21}$$

Let us next consider the parity $R\Omega$ associated with the π-rotation. This also maps $\mathcal{O}(n)$ to $\mathcal{O}(-n)$. The fixed-point set consists of two points, say, the north pole $\mathbf{N}$ and south pole $\mathbf{S}$. The normal bundle is real two dimensional (complex one-dimensional) and hence $e(N(\mathbf{N})) = e(N(\mathbf{S})) = 0$ and $\mathrm{ch}(\wedge \overline{N}_{\mathbf{N}}) = \mathrm{ch}(\wedge \overline{N}_{\mathbf{S}}) = 2$. This yields

$$I_{R\Omega} = 0, \tag{7.22}$$

$$I_{R\Omega}(\mathcal{O}(n), \mathcal{O}(-n)) = \sum_{p=\mathbf{N}}^{\mathbf{S}} \int_p \mathrm{e}^{-2nH} \frac{1}{2} = \frac{1}{2} + \frac{1}{2} = 1. \tag{7.23}$$

According to the above discussion, the orientifold planes on the mirror side are

$$Y^{\widetilde{\mathrm{id}}} = L_0 + L_1, \tag{7.24}$$

$$Y^{\widetilde{R}} = L_0 - L_1. \tag{7.25}$$

We also know that the mirror of $\mathcal{O}(0)$ is the same as L_0 and the mirror of $\mathcal{O}(n)$ is topologically

$$\gamma_{\mathcal{O}(n)} = L_0 + n\gamma_0, \tag{7.26}$$

where γ_0 (the mirror of the 0-brane) is a circle that winds once around the Y-cylinder (See Fig. 9). Under the parity action $Y \to \overline{Y}$, it is mapped to $L_0 - n\gamma_0$ which is indeed the

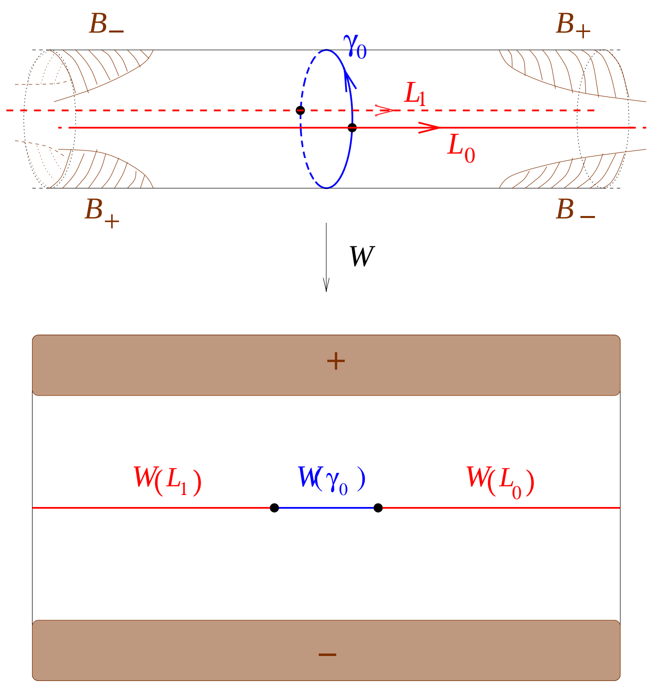

Figure 9: Orientifold planes (L_0 and L_1) and the mirror of the D0-brane (γ_0). Their images under the superpotential W are also depicted.

mirror of $\mathcal{O}(-n)$. The Witten index is obtained by taking the intersection numbers. We recall that the intersection of C_1 and C_2 is defined as $\#(C_1^- \cap C_2^+)$ where $C_i^\pm$ is obtained from C_i by moving the asymptotes toward the region $B_\pm$. It is visible in Fig. 9 that

$$\#(L_0^- \cap L_0^+) = 1; \quad \#(L_1^- \cap L_1^+) = -1;$$
$$\#(L_i^- \cap L_j^+) = 0 \text{ if } i \neq j; \quad \#(\gamma_0^- \cap L_i) = -1.$$

Using these, we find

$$I_{\widetilde{\Omega}} = \#((Y^{\widetilde{\text{id}}})^- \cap (Y^{\widetilde{\text{id}}})^+) = 1 - 1 = 0, \tag{7.27}$$
$$I_{\widetilde{\Omega}}(\gamma_{\mathcal{O}(n)}, \gamma_{\mathcal{O}(-n)}) = \#(\gamma_{\mathcal{O}(n)}^- \cap (Y^{\widetilde{\text{id}}})^+) = 1 + n(-1) + n(-1) = 1 - 2n, \tag{7.28}$$

which reproduces (7.20), (7.21), and

$$I_{\widetilde{R\Omega}} = \#((Y^{\widetilde{R}})^- \cap (Y^{\widetilde{R}})^+) = 1 - 1 = 0, \tag{7.29}$$
$$I_{\widetilde{R\Omega}}(\gamma_{\mathcal{O}(n)}, \gamma_{\mathcal{O}(-n)}) = \#(\gamma_{\mathcal{O}(n)}^- \cap (Y^{\widetilde{R}})^+) = 1 + n(-1) - n(-1) = 1. \tag{7.30}$$

which reproduces (7.22), (7.23).

$\theta = \pi$

We now consider the case with a non-zero theta angle $\theta = \pi$. This corresponds to turning on the B-field $B = \pi H$. For all holomorphic involutions τ, the twist bundle $\mathcal{L}_{\tau^* B+B}$ has first Chern class $B/\pi = H$ and hence it is $\mathcal{O}(1)$. Thus, the parity transforms the bundles as

$$\tau\Omega : \mathcal{O}(n) \longrightarrow \overline{\mathcal{O}(n)} \otimes \mathcal{L}_{\tau^* B+B} = \mathcal{O}(1-n). \tag{7.31}$$

The parities to be considered are τ_{r_1,r_2} that acts on the sections Φ_1 and Φ_2 of $\mathcal{L}_{\tau^* B+B} = \mathcal{O}(1)$ as $(\Phi_1, \Phi_2) \mapsto ((-1)^{r_1}\Phi_1, (-1)^{r_2}\Phi_2)$. τ_{00} and τ_{11} project to the identity map of $\mathbb{CP}^1$ while τ_{10} and τ_{01} project to the π-rotation $R : \mathbb{CP}^1 \to \mathbb{CP}^1$. The sign function at the fixed-point set is

$$\varepsilon_B^{\tau_{00}} = 1, \quad \varepsilon_B^{\tau_{11}} = -1, \quad \text{on } \mathbb{CP}^1$$
$$\varepsilon_B^{\tau_{10}} = \begin{cases} 1 & \text{at } \mathbf{N} \\ -1 & \text{at } \mathbf{S} \end{cases} \qquad \varepsilon_B^{\tau_{01}} = \begin{cases} -1 & \text{at } \mathbf{N} \\ 1 & \text{at } \mathbf{S} \end{cases}$$

where we assumed that $\mathbf{N}$ and $\mathbf{S}$ are loci of $\Phi_1 = 0$ and $\Phi_2 = 0$ respectively. It is straightforward to compute the index.

$$I_{\tau_{00}\Omega} = 0, \tag{7.32}$$
$$I_{\tau_{00}\Omega}(\mathcal{O}(n), \mathcal{O}(1-n)) = \int_{\mathbb{CP}^1} \mathrm{e}^{-2nH} \varepsilon_B^{\tau_{00}}\, \mathrm{e}^{H}\, \mathrm{e}^{H} = 2 - 2n. \tag{7.33}$$

For $\tau_{11}\Omega$ the opposite sign occurs. Also,

$$I_{\tau_{10}\Omega} = 0, \tag{7.34}$$
$$I_{\tau_{10}\Omega}(\mathcal{O}(n), \mathcal{O}(1-n)) = \sum_{p=\mathbf{N}}^{\mathbf{S}} \frac{\varepsilon_B^{\tau_{10}}(p)}{2} = \frac{1}{2} - \frac{1}{2} = 0. \tag{7.35}$$

For $\tau_{01}\Omega$ the sign is opposite and hence the same. By the general discussion, the orientifold planes in the mirror are

$$Y^{\widetilde{\tau}_{00}} = L_0 + L_1, \tag{7.36}$$
$$Y^{\widetilde{\tau}_{10}} = L_0 - L_1, \tag{7.37}$$

and $Y^{\widetilde{\tau}_{11}} = -L_0 - L_1$ and $Y^{\widetilde{\tau}_{01}} = -L_0 + L_1$. The mirror of $\mathcal{O}(0)$ and $\mathcal{O}(1)$ are the wavefront trajectories originating in the two critical points $Y = r/2 - \pi i/2$ and $Y = r/2 + \pi i/2$ respectively. They are depicted as γ_a and γ_b in Fig. 10 They are homologically related as $\gamma_b = \gamma_a + \gamma_0$. The mirror of the other bundles are homologically

$$\gamma_{\mathcal{O}(n)} = \gamma_a + n\gamma_0. \tag{7.38}$$

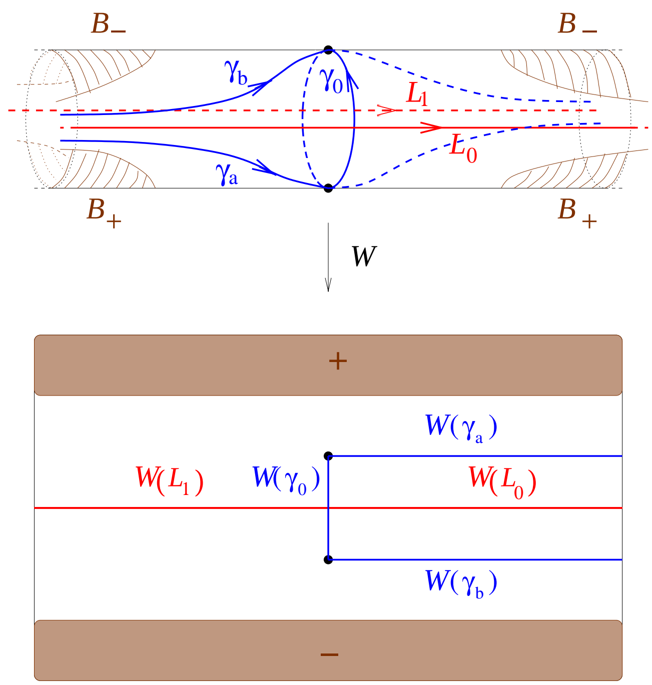

Figure 10: Orientifold planes (L_0 and L_1) and the mirror of $\mathcal{O}(0)$ (γ_a) and or $\mathcal{O}(1)$ (γ_b). Notice that the location of B_+ and B_- have changed from the previous case $\theta = 0$.

Under the parity action $Y \to \overline{Y}$, γ_a and $\gamma_b = \gamma_a + \gamma_0$ are exchanged while γ_0 is reversed. Thus $\gamma_{\mathcal{O}(n)} = \gamma_a + n\gamma_0$ is mapped to $\gamma_b - n\gamma_0 = \gamma_a + (1-n)\gamma_0$ which is indeed the mirror of $\mathcal{O}(1-n)$. To compute the intersection numbers, we note the following

$$\#(L_i^- \cap L_j^+) = 0 \quad \forall i,\ \forall j;$$
$$\#(\gamma_a^- \cap L_i^+) = 1 \quad i = 0, 1; \qquad \#(\gamma_0^- \cap L_i) = -1.$$

Using these, we find

$$I_{\widetilde{\tau_{00}\Omega}} = \#((Y^{\widetilde{\tau}_{00}})^- \cap (Y^{\widetilde{\tau}_{00}})^+) = 0, \tag{7.39}$$
$$I_{\widetilde{\tau_{00}\Omega}}(\gamma_{\mathcal{O}(n)}, \gamma_{\mathcal{O}(1-n)}) = \#(\gamma_{\mathcal{O}(n)}^- \cap (Y^{\widetilde{\tau}_{00}})^+) = 1 + 1 + n(-1) + n(-1) = 2 - 2n, \tag{7.40}$$

which reproduces (7.32), (7.33), and

$$I_{\widetilde{\tau_{10}\Omega}} = \#((Y^{\widetilde{\tau}_{10}})^- \cap (Y^{\widetilde{\tau}_{10}})^+) = 0, \tag{7.41}$$
$$I_{\widetilde{\tau_{10}\Omega}}(\gamma_{\mathcal{O}(n)}, \gamma_{\mathcal{O}(1-n)}) = \#(\gamma_{\mathcal{O}(n)}^- \cap (Y^{\widetilde{\tau}_{10}})^+) = 1 - 1 + n(-1) - n(-1) = 0, \tag{7.42}$$

which reproduces (7.34), (7.35).

8 Orientifolds of Compact Calabi–Yau: A First Step

In this section, we initiate the discussion of orientifolds of compact Calabi–Yau manifolds, which are of vital phenomenological relevance. This is a first step and full detail must be clarified in future works. However, the considerations using linear sigma models and mirror symmetry already provide a basic global picture and directions to proceed.

8.1 LSM for compact CY and parity symmetry

Let us consider the $U(1)$ gauge theory with matter fields $P, \Phi_1, ..., \Phi_N$ of charge $-N, 1, ..., 1$ having the term L_2 with gauge invariant superpotential

$$W = PG(\Phi_1, ..., \Phi_N),$$

where $G(\Phi_i)$ is a polynomial of degree N. In this system, the FI parameter r does not run and is a parameter of the theory. At large positive r, the model corresponds to the non-linear sigma model on the degree N hypersurface of $\mathbb{CP}^{N-1}$ defined by the equation $G(z_1, ..., z_N) = 0$ for the homogeneous coordinates z_i, which is a Calabi–Yau manifold of dimension $(N-2)$. At large negative r, p acquires a fixed value $\langle p \rangle$ and the model corresponds to a LG orbifold — the LG model with superpotential $W = \langle p \rangle G(\Phi_1, ..., \Phi_N)$ modded out by the $\mathbb{Z}_N$ action generated by $\Phi_i \to \mathrm{e}^{\frac{2\pi i}{N}} \Phi_i$. The worldsheet theory is singular exactly at one point

$$t = t_* := N \log(-N) = N \log N + \pi i N. \tag{8.1}$$

The moduli space of the complexified Kähler class t has three notable points — the large-volume limit ($r = +\infty$), the conifold point ($t = t_*$), and the LG orbifold point or equivalently the Gepner point ($r = -\infty$). The moduli space is connected and the theory is singular only at the conifold point. The moduli space of complex structure is a complex space of complex dimension $\binom{2N-1}{N-1} - N^2 + \delta_{N,4}$.

The model has a mirror description, which is the non-linear sigma model on the degree N-hypersurface in $\mathbb{CP}^{N-1}$

$$\widetilde{G}(\widetilde{z}_i) := \widetilde{z}_1^N + \cdots + \widetilde{z}_N^N + \mathrm{e}^{t/N} \widetilde{z}_1 \cdots \widetilde{z}_N = 0,$$

modded out by the orbifold group $(\mathbb{Z}_N)^{N-2}$ acting as $\widetilde{z}_i \to \omega_i \widetilde{z}_i$, where $\omega_i^N = \omega_1 \cdots \omega_N = 1$. At the singular point $t = t_*$, the mirror manifold has a conifold singularity at $\widetilde{z}_i = 1$. It has one complex structure modulus e^t and $\binom{2N-1}{N-1} - N^2 + \delta_{N,4}$ Kähler moduli coming from the Kähler class of $\mathbb{CP}^{N-1}$ and the resolutions of the orbifold singularities. The mirror may also be described as the LG orbifold with superpotential $\widetilde{W} = \widetilde{G}(\widetilde{\Phi}_i)$ and orbifold group $(\mathbb{Z}_N)^{N-1}$: $\widetilde{\Phi}_i \to \omega_i \widetilde{\Phi}_i$. This mirror can be found by using the dualization $\arg(\Phi_i) \to Y_i$ followed by the change of variables $\mathrm{e}^{-Y_i} \sim \widetilde{\Phi}_i^N$ [32].

Let us study the parity symmetries of this system. We will consider A-parity and B-parity separately.

8.1.1 A-parity (B-parity of mirror)

The transformation

$$P \to \overline{\Omega_A^* P}, \quad \Phi_i \to \overline{\Omega_A^* \Phi_{\sigma(i)}}, \quad V \to \Omega_A^* V \tag{8.2}$$

yields an A-parity that is an exact symmetry of the system if $G(z_i)$ is a polynomial obeying $G(\overline{z_{\sigma(i)}}) = \overline{G(z_i)}$. Note that there is no anomaly since the sum of the charges is zero $-N+1+\cdots+1=0$ (and hence even), or equivalently, since M is Calabi–Yau (and hence spin). There is no condition on the complexified Kähler modulus, but the complex structure moduli are reduced by the constraint $G(\overline{z_{\sigma(i)}}) = \overline{G(z_i)}$. The reduction is by one-half if σ is an involution.

Following the duality transformation and change of variables, we find that the corresponding parity in the mirror side (in the LG description) is of the form

$$\widetilde{\Phi}_i \longrightarrow \mathrm{e}^{\pi i/N} \Omega_A^* \widetilde{\Phi}_{\sigma(i)}, \tag{8.3}$$

where the phase is chosen uniquely by the requirement that the superpotential be flipped $\widetilde{W} \to -\Omega_A^* \widetilde{W}$. In the geometric description of the mirror, the parity is simply the one associated with the holomorphic map $\widetilde{z}_i \to \widetilde{z}_{\sigma(i)}$ of the mirror CY manifold.

If the defining polynomial obeys $G(\mathrm{e}^{i\theta_i}\overline{z_{\sigma(i)}}) = \mathrm{e}^{-i\theta_P}\overline{G(z_i)}$, then we have a parity symmetry given by (8.2), combined with the torus action $P \to \mathrm{e}^{i\theta_P} P$, $\Phi_i \to \mathrm{e}^{i\theta_i}\Phi_i$. In this case, the parity action on the mirror LG fields is still given by (8.3), but a difference appears in the action on the twisted sector states. In the geometric picture, this will change the type of orientifold planes, just as in [77].

8.1.2 B-parity (A-parity of the mirror)

The transformation

$$P \to -\Omega_B^* P, \quad \Phi_i \to \Omega_B^* \Phi_{\sigma(i)}, \quad V \to \Omega_B^* V \tag{8.4}$$

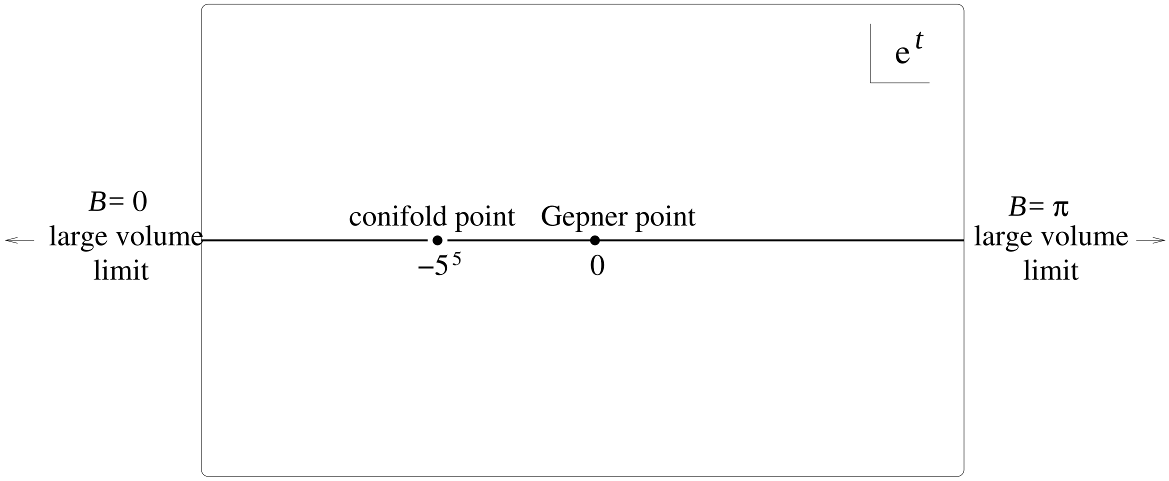

Figure 11: Kähler moduli space for a B-orientifold of the quintic

yields a B-parity. It is an exact symmetry of the system if the defining polynomial is invariant under the permutation $G(z_{\sigma(i)}) = G(z_i)$ and also if the FI-theta parameter t obeys $t = \bar{t} \bmod 2\pi i\mathbb{Z}$, namely

$$\mathrm{Im}\, t = 0 \quad \text{or} \quad \pi i. \tag{8.5}$$

Thus, there is a holomorphic constraint on the complex structure moduli, and the complexified Kähler modulus is reduced by a half — one real dimension. The reduced moduli space $\mathrm{Im} t \in \pi i\mathbb{Z}$ passes directly through the singular point $t_* = N \log N + \pi i N$. Thus, the real moduli space of the worldsheet theory is separated at $t = t_*$. It has two parts — one includes the large-volume limit with trivial B-field and the other includes the large volume limit with a B-field of period π. The LG orbifold point (Gepner point) belongs to one of them and is separated from the other. Note that the analytic continuation in the t-space to go around the singular point $t = t_*$ [31, 80] is not applicable here since the orientifold forbids us to continuously change the θ angle. Figure 11 depicts the restriction of the Kähler moduli by the B-parity in the case of $N = 5$, the quintic hypersurface in $\mathbb{CP}^4$ which will be discussed in some detail below (together with the A-parities).[1]

[1] We thank M. Douglas for pointing out an error in the identification of the value of the B-field at the two large volume regions. The relation of the B-field and the theta angle is shifted by $N\pi$ when the P-field is integrated [81].

8.2 The case of the quintic

Let us consider the case $N = 5$, quintic hypersurfaces M in $\mathbb{CP}^4$. We classify the involutive parity symmetries that are present in the Fermat-type hypersurfaces:

$$z_1^5 + z_2^5 + z_3^5 + z_4^5 + z_5^5 = 0.$$

Without orientifold, there is one complexified Kähler modulus parametrized by a function of t, and 101 complex structure moduli. The latter is the number of monomials $\binom{9}{4} = 126$ minus the dimension $5^5 = 25$ of the group $GL(5, \mathbb{C})$ of coordinate transformations. The mirror is the resolution of the orbifold of

$$\widetilde{z}_1^5 + \widetilde{z}_2^5 + \widetilde{z}_3^5 + \widetilde{z}_4^5 + \widetilde{z}_5^5 + \mathrm{e}^{t/5}\widetilde{z}_1\widetilde{z}_2\widetilde{z}_3\widetilde{z}_4\widetilde{z}_5 = 0.$$

by the $\mathbb{Z}_5^3$ action $\widetilde{z}_i \to \omega_i \widetilde{z}_i$, $\omega_i^5 = \omega_1\omega_2\omega_3\omega_4\omega_5 = 1$, $\omega_i \equiv \omega\omega_i$ [82]. Without orientifold, there are 101 complexified Kähler moduli and one complex structure modulus e^t. 101 comes from the $1+100$ harmonic two-forms where the extra 100 is counted at the orbifold point as follows. Note that there are ten $\mathbb{Z}_5 \times \mathbb{Z}_5$ invariant points (points of the form $(0, 0, 0, 1, -1)$). At each point there are 6 localized blow-up modes of a $\mathbb{C}^3/\mathbb{Z}_5 \times \mathbb{Z}_5$ singularity. For some pairs of $\mathbb{Z}_5 \times \mathbb{Z}_5$ invariant points, there is a curve of $\mathbb{C}^2/\mathbb{Z}_5$ singularity, which carry 4 blow-up modes. There are ten such pairs. Thus, in total, there are $10 \times 6 + 10 \times 4 = 100$ blow-up modes.

8.2.1 A-parities

We start with the A-parities. In all cases, the 101 complex structure moduli are reduced to one half (101 real ones), but the Kähler moduli space remains the same.

The simplest A-parity is associated with the complex conjugation $\tau_A : z_i \mapsto \overline{z_i}$. (This example was also studied in [83].) The fixed-point set is the real quintic which is a (3-dimensional) submanifold of the quintic defined by $z_i \in \mathbb{R}$. To find its topology, let us consider the map $M \to \mathbb{CP}^3$ obtained by forgetting the last coordinate z_5. It maps the real quintic M^{τ_A} homeomorphically onto the real locus of $\mathbb{CP}^3$, namely $\mathbb{RP}^3$. This is because any real number has a unique real 5-th root, $x_5 = \sqrt[5]{-x_1^2 - x_2^2 - x_3^2 - x_4^2}$, and 'taking the 5-th root' defines a continuous map $\mathbb{R} \to \mathbb{R}$. Thus, the real quintic has the topology of a real projective space $\mathbb{RP}^3$. On the mirror side, this maps to the B-parity associated with the identity $\widetilde{\tau}_B : \widetilde{z}_i \mapsto \widetilde{z}_i$, whose fixed-point set is the mirror quintic itself (6-dimensional). One may consider a modification of τ_A of the form $z_i \mapsto \mathrm{e}^{i\theta_i}\overline{z_i}$, which is a symmetry if $\mathrm{e}^{5i\theta_i}$ is independent of i. But this is simply the complex conjugation with respect to the coordinate $z_i' = \mathrm{e}^{-i\theta_i/2} z_i$. Thus, there is no essential difference. This will be the same for the rest of the cases as well, and hence will not be mentioned.

Next, we consider the A-parity associated with $\tau_A^{12} : (z_1, z_2, z_3, z_4, z_5) \mapsto (\overline{z_2}, \overline{z_1}, \overline{z_3}, \overline{z_4}, \overline{z_5})$. The τ_A^{12}-fixed-point set is the (3-dimensional) submanifold defined by $z_2 = \overline{z_1}$, $z_i \in \mathbb{R}$ $(i = 3, 4, 5)$. Using the z_5-forgetting map $M \to \mathbb{CP}^3$, we find that it has the topology of $\mathbb{RP}^3$. This corresponds to the B-parity of the mirror associated with the holomorphic involution $\widetilde{\tau}_B^{12} : (\widetilde{z}_1, \widetilde{z}_2, \widetilde{z}_3, \widetilde{z}_4, \widetilde{z}_5) \mapsto (\widetilde{z}_2, \widetilde{z}_1, \widetilde{z}_3, \widetilde{z}_4, \widetilde{z}_5)$. The fixed-point set is a point $\{(1, -1, 0, 0, 0)\}$ (0-dimensional) and a complex hypersurface $\{(z, z, z_3, z_4, z_5))\}$ (4-dimensional).

The final A-parity is the one associated with $\tau_A^{12,34} : (z_1, z_2, z_3, z_4, z_5) \mapsto (\overline{z_2}, \overline{z_1}, \overline{z_4}, \overline{z_3}, \overline{z_5})$. The fixed-point set is the (3-dimensional) submanifold defined by $z_2 = \overline{z_1}, z_4 = \overline{z_3}$, $z_5 \in \mathbb{R}$. Using the z_5-forgetting map $M \to \mathbb{CP}^3$, we again find that it has a topology of $\mathbb{RP}^3$. This corresponds to the B-parity of the mirror associated with $\widetilde{\tau}_B^{12,34}$: $(\widetilde{z}_1, \widetilde{z}_2, \widetilde{z}_3, \widetilde{z}_4, \widetilde{z}_5) \mapsto (\widetilde{z}_2, \widetilde{z}_1, \widetilde{z}_4, \widetilde{z}_3, \widetilde{z}_5)$. The fixed-point set is a line $\{(\widetilde{z}, -\widetilde{z}, \widetilde{w}, -\widetilde{w}, 0)\}$ and a curve $\{(\widetilde{z}, \widetilde{z}, \widetilde{w}, \widetilde{w}, \widetilde{u})\}$ (both 2-dimensional).

8.2.2 B-parities

We now consider B-parities. In all the cases, the complexified Kähler modulus (complex) is reduced to a real one, $\mathrm{Im}\, t \in \pi i \mathbb{Z}$. The reduction of the complex structure moduli is holomorphic and depends on the particular case.

The basic B-parity is associated with the identity $\tau_B : z_i \mapsto z_i$. The fixed-point set is the quintic itself (6-dimensional). All the complex 101 complex structure moduli survive. In the mirror description, this corresponds to the A-parity associated with the complex conjugation $\widetilde{\tau}_A : \widetilde{z}_i \mapsto \overline{\widetilde{z}_i}$. The fixed-point set is the real locus $\widetilde{z}_i \in \mathbb{R}$ (3-dimensional). One may consider modifying τ_B by a phase multiplication. For $z_i \mapsto \mathrm{e}^{i\theta_i} z_i$ to be a symmetry, $\mathrm{e}^{5i\theta_i}$ must be i independent, and for this to be an involution, $\mathrm{e}^{2i\theta_i}$ must be i-independent. This requires $\mathrm{e}^{i\theta_i} = \mathrm{e}^{5i\theta}/(\mathrm{e}^{2i\theta_i})^2$ to be i-independent. Namely, $\mathrm{e}^{i\theta_i}$ are all the same and the transformation is a part of the $U(1)$ gauge symmetry. The 'modification' had no effect. (It turns out that this is the same for the rest of the cases, and hence will not be mentioned.)

Next, we consider the B-parity associated with $\tau_B^{12} : (z_1, z_2, z_3, z_4, z_5) \mapsto (z_2, z_1, z_3, z_4, z_5)$. Let us count the number of complex structure moduli. The polynomial $G(z_i)$ must be invariant under the exchange of z_1 and z_2. Monomials that are invariant under $z_1 \leftrightarrow z_2$ are of the form $(z_1 z_2)^a z_3^b z_4^c z_5^d$ and there are 34 of them. From the rest, $126 - 34 = 92$, a half (46 of them) is invariant. Thus, invariant polynomials can have $46 + 34 = 80$ terms. Coordinate transformations commuting with the exchange $z_1 \leftrightarrow z_2$ form a subgroup of $GL(5,\mathbb{C})$ of dimension 17. Thus, the number of complex structure moduli is $80 - 17 = 63$. The fixed-point set is the point $\{(1,-1,0,0,0)\}$ (0-dimensional) and the hypersurface $\{(z,z,z_3,z_4,z_5)\}$ (4-dimensional). The mirror description of this parity is associated with $\widetilde{\tau}_A^{12} : (\widetilde{z}_1,\widetilde{z}_2,\widetilde{z}_3,\widetilde{z}_4,\widetilde{z}_5) \mapsto (\overline{\widetilde{z}_2},\overline{\widetilde{z}_1},\overline{\widetilde{z}_3},\overline{\widetilde{z}_4},\overline{\widetilde{z}_5})$. The complexified Kähler moduli correspond to $\widetilde{\tau}_A^{12}$-anti-invariant harmonic $(1,1)$ forms, by the condition (3.1). Let us count the number at the orbifold point. We recall that the 101 consists of 1 'bulk' modulus, 10×4 moduli of the resolution of $\mathbb{C}^2/\mathbb{Z}_5$ sigularity along ten curves, and 10×6 moduli of the isolated resolution of ten $\mathbb{C}^3/\mathbb{Z}_5 \times \mathbb{Z}_5$ singularity. Since $\widetilde{\tau} = \widetilde{\tau}_A^{12}$ is antiholomorphic, 1 from the bulk remains (subtotal **1**). Out of the ten $\mathbb{Z}_5$ curves, six of them (three pairs) are exchanged by $\widetilde{\tau}$ while four of them are $\widetilde{\tau}$-invariant. Out of 6×4 from the three pairs, a half of them are $\widetilde{\tau}$-anti-invariant (**12**). $\widetilde{\tau}$ acts non-trivially on three of the four invariant curves, and all of the 3×4 remain (**12**). $\widetilde{\tau}$ acts identically on the last invariant curve $\{(0,0,*,*,*)\}$ exchanging the four exceptional curves (two pairs) of the $\mathbb{C}^2/\mathbb{Z}_5$ resolution, leaving half of 4 (**2**). Among the ten $\mathbb{Z}_5 \times \mathbb{Z}_5$ points, six of them (three pairs) are exchanged by $\widetilde{\tau}$ while four of them are τ-invariant. Out of the 6×6 from the three pairs, a half of them are τ-anti-invariant (**18**). $\widetilde{\tau}$ acts near three of the four invariant points as $(z_1,z_2,z_3) \to (\overline{z}_2,\overline{z}_1,\overline{z}_3)$ in $\mathbb{C}^3/\mathbb{Z}_5 \times \mathbb{Z}_5$. One can show by an orbifold analysis that there are 4 anti-invariant modulus at each such singularity, thus total of 3×4 (**12**). Near the last invariant point $(1,-1,0,0,0)$, $\widetilde{\tau}$ acts as $(z_1,z_2,z_3) \to (\overline{z}_1,\overline{z}_2,\overline{z}_3)$ in $\mathbb{C}^3/\mathbb{Z}_5 \times \mathbb{Z}_5$, and all moduli remain (**6**). The total number of moduli is $1+12+12+2+18+12+6 = 63$, in agreement with the number of complex moduli of the original side. The fixed-point set is the (3-dimensional) submanifold defined by $\widetilde{z}_2 = \overline{\widetilde{z}_1}$ and $\widetilde{z}_i \in \mathbb{R}$ $(i = 3,4,5)$.

The final B-parity is associated with $\tau_B^{12,34} : (z_1,z_2,z_3,z_4,z_5) \mapsto (z_2,z_1,z_4,z_3,z_5)$. Let us count the number of complex structure moduli. Monomials that are invariant under $z_1 \leftrightarrow z_2$, $z_3 \leftrightarrow z_4$ are of the form $(z_1z_2)^a(z_3z_4)^b z_5^c$ and there are 6 of them. Thus, the invariant polynomials can have $\frac{126-6}{2} + 6 = 66$ terms. The group of coordinate transformation has dimension 13. Thus, the number of complex structure moduli is $66 - 13 = 53$. The fixed-point set is the line $\{(z,-z,w,-w,0)\}$ and the genus 6 curve $\{(z,z,w,w,u)\}$ (both 2-dimensional). On the mirror side, the parity is the one associated with $\widetilde{\tau}_A^{12,34} : (\widetilde{z}_1,\widetilde{z}_2,\widetilde{z}_3,\widetilde{z}_4,\widetilde{z}_5) \mapsto (\overline{\widetilde{z}_2},\overline{\widetilde{z}_1},\overline{\widetilde{z}_4},\overline{\widetilde{z}_3},\overline{\widetilde{z}_5})$. The complexified Kähler moduli correspond to $\widetilde{\tau}_A^{12,34}$-anti-invariant harmonic $(1,1)$ forms. Counting as in the case of $\widetilde{\tau}_A^{12}$-anti-invariants, the number of moduli is $1 + 16 + 4 + 24 + 8 = 53$, again in agreement with the original side. The fixed-point set is the (3-dimensional) submanifold defined by $\widetilde{z}_2 = \overline{\widetilde{z}_1}$, $\widetilde{z}_4 = \overline{\widetilde{z}_3}$, and $\widetilde{z}_i \in \mathbb{R}$ $(i = 3,4,5)$.

Table 1: Orientifolds of Quintic and their Mirrors

quintic M $(1_{\mathbb{C}}, 101_{\mathbb{C}})$	mirror quintic $\widetilde{M}$ $(101_{\mathbb{C}}, 1_{\mathbb{C}})$
$\tau_A : z_i \longrightarrow \overline{z_i}$ $(1_{\mathbb{C}}, 101_{\mathbb{R}})$ O6 at the real quintic ($\mathbb{RP}^3$)	$\widetilde{\tau}_B : \widetilde{z}_i \longrightarrow \widetilde{z}_i$ $(101_{\mathbb{R}}, 1_{\mathbb{C}})$ O9 at $\widetilde{M}$
$\tau_A^{12} : (z_1, z_2, z_3, z_4, z_5) \longrightarrow (\overline{z_2}, \overline{z_1}, \overline{z_3}, \overline{z_4}, \overline{z_5})$ $(1_{\mathbb{C}}, 101_{\mathbb{R}})$ O6 at $\mathbb{RP}^3 = \{(z, \overline{z}, x_3, x_4, x_5) \| x_i \in \mathbb{R}\}$	$\widetilde{\tau}_B^{12} : (\widetilde{z}_1, \widetilde{z}_2, \widetilde{z}_3, \widetilde{z}_4, \widetilde{z}_5) \longrightarrow (\widetilde{z}_2, \widetilde{z}_1, \widetilde{z}_3, \widetilde{z}_4, \widetilde{z}_5)$ $(101_{\mathbb{R}}, 1_{\mathbb{C}})$ O3 at $\{(1, -1, 0, 0, 0)\}$ O7 at $\{(\widetilde{z}, \widetilde{z}, \widetilde{z}_3, \widetilde{z}_4, \widetilde{z}_5)\}$
$\tau_A^{12,34} : (z_1, z_2, z_3, z_4, z_5) \longrightarrow (\overline{z_2}, \overline{z_1}, \overline{z_4}, \overline{z_3}, \overline{z_5})$ $(1_{\mathbb{C}}, 101_{\mathbb{R}})$ O6 at $\mathbb{RP}^3 = \{(z, \overline{z}, w, \overline{w}, x) \| x \in \mathbb{R}\}$	$\widetilde{\tau}_B^{12,34} : (\widetilde{z}_1, \widetilde{z}_2, \widetilde{z}_3, \widetilde{z}_4, \widetilde{z}_5) \longrightarrow (\widetilde{z}_2, \widetilde{z}_1, \widetilde{z}_4, \widetilde{z}_3, \widetilde{z}_5)$ $(101_{\mathbb{R}}, 1_{\mathbb{C}})$ O5 at $\{(\widetilde{z}, -\widetilde{z}, \widetilde{w}, -\widetilde{w}, 0)\}$ O5 at $\{(\widetilde{z}, \widetilde{z}, \widetilde{w}, \widetilde{w}, \widetilde{u})\}$
$\tau_B : z_i \longrightarrow z_i$ $(1_{\mathbb{R}}, 101_{\mathbb{C}})$ O9 at M	$\widetilde{\tau}_A : \widetilde{z}_i \longrightarrow \overline{\widetilde{z}_i}$ $(101_{\mathbb{C}}, 1_{\mathbb{R}})$ O6 at real quintic
$\tau_B^{12} : (z_1, z_2, z_3, z_4, z_5) \longrightarrow (z_2, z_1, z_3, z_4, z_5)$ $(1_{\mathbb{R}}, 63_{\mathbb{C}})$ O3 at $\{(1, -1, 0, 0, 0)\}$ O7 at $\{(z, z, z_3, z_4, z_5)\}$	$\widetilde{\tau}_A^{12} : (\widetilde{z}_1, \widetilde{z}_2, \widetilde{z}_3, \widetilde{z}_4, \widetilde{z}_5) \longrightarrow (\overline{\widetilde{z}_2}, \overline{\widetilde{z}_1}, \overline{\widetilde{z}_3}, \overline{\widetilde{z}_4}, \overline{\widetilde{z}_5})$ $(63_{\mathbb{C}}, 1_{\mathbb{R}})$ O6 at $\{(\widetilde{z}, \overline{\widetilde{z}}, \widetilde{x}_3, \widetilde{x}_4, \widetilde{x}_5) \| \widetilde{x}_i \in \mathbb{R}\}$
$\tau_B^{12,34} : (z_1, z_2, z_3, z_4, z_5) \longrightarrow (z_2, z_1, z_4, z_3, z_5)$ $(1_{\mathbb{R}}, 53_{\mathbb{C}})$ O5 at $\{(z, -z, w, -w, 0)\}$ (line) O5 at $\{(z, z, w, w, z_5)\}$ (genus 6)	$\widetilde{\tau}_A^{12,34} : (\widetilde{z}_1, \widetilde{z}_2, \widetilde{z}_3, \widetilde{z}_4, \widetilde{z}_5) \longrightarrow (\overline{\widetilde{z}_2}, \overline{\widetilde{z}_1}, \overline{\widetilde{z}_4}, \overline{\widetilde{z}_3}, \overline{\widetilde{z}_5})$ $(53_{\mathbb{C}}, 1_{\mathbb{R}})$ O6 at $\{(\widetilde{z}, \overline{\widetilde{z}}, \widetilde{w}, \overline{\widetilde{w}}, \widetilde{x}) \| \widetilde{x} \in \mathbb{R}\}$

8.2.3 Summary and remarks

The results are summarized in Table 1. The left column and the right column are mirror of each other. The number (n, m) in each column means that there are n complexified Kähler moduli and m complex structure moduli. The subscript $_\mathbb{R}$ or $_\mathbb{C}$ shows that the moduli is real or complex. We have in mind to embed these orientifolds in Type II superstring theory on $\mathrm{CY}^3 \times \mathbb{R}^{3+1}$ in which the parity acts trivially on the 3+1 dimensional Minkowski-coordinates. Thus, the fixed-point sets of dimension p are called orientifold $(p+3)$-planes, or simply $\mathrm{O}(p+3)$.

We have not determined the types of the orientifold planes. In the large volume limit, each orientifold plane can be of several types — roughly two types, *SO* or *Sp*. We have not determined which combinations of the types are possible and how they are related by mirror symmetry. This will be determined, for example, by going to the Gepner point at which an exact construction of the crosscap states is now available as an application of the construction in Section 4 (see [45, 46] for existing results on Type I strings on Calabi–Yaus). Also, for a consistent compactification in string theory, we need to cancel the tadpole generated by the O-planes by, say, including D-branes. This too can be done once we know the types of the orientifold planes. This will be carried out in a future publication.

8.3 Spacetime picture

So far, we have been studying the theory on the string worldsheet with a focus on the parity symmetries that commute with halves of the $(2, 2)$ worldsheet supersymmetry. In the full string theory, more emphasis is put on the spacetime physics, especially when we consider Type II orientifolds of the form $M^6 \times \mathbb{R}^{3+1}/\tau$, where τ is an involution of M^6 and acts trivially on the Minkowski coordinates. There are several important issues, one of which is tadpole cancellation mentioned above. Here we comment on three others — supersymmetry, massless fields, and superpotential (all in the sense of the 3+1 dimensional spacetime). In what follows, M is a general simply connected Calabi–Yau 3-fold with an involution τ. We note that the number of moduli of complexified Kähler class and complex structures of M (before orientifolding) are

$$\begin{aligned} &\dim_\mathbb{C} H^{1,1}(M) =: h^{1,1}, \\ &\dim_\mathbb{C} H^1(M, T_M) = \dim_\mathbb{C} H^{2,1}(M) =: h^{2,1}. \end{aligned}$$

It is useful to bear in mind the example of the quintic and the six involutions in Table 1.

8.3.1 Spacetime supersymmetry

We have seen that $\tau\Omega$ preserves an $\mathcal{N}=2$ worldsheet supersymmetry if $\tau: M \to M$ is an antiholomorphic involution (A-parity) a holomorphic involution (B-parity). However, not all of them preserve a spacetime supersymmetry, that is, $\mathcal{N}=1$ supersymmetry in $3+1$ dimensions. There is no extra condition on B-parities, but the orientifold by an A-parity preserves a spacetime supersymmetry only if τ maps the holomorphic three-form $\mathit{\Omega}$ to its complex conjugate with a possible constant phase, $\tau^*\mathit{\Omega} = \mathrm{e}^{i\theta}\overline{\mathit{\Omega}}$. For the quintic, all the three A-parities in fact satisfy this constraint. Let us consider the Fermat type quintic. Using the inhomogeneous coordinates at the $z_5 \neq 0$ patch, $\zeta_i = z_i/z_5$ obeying $\sum_{i=1}^4 (\zeta_i)^5 + 1 = 0$, the holomorphic three-form is expressed as $\mathit{\Omega} = \mathrm{d}\zeta_1 \wedge \mathrm{d}\zeta_2 \wedge \mathrm{d}\zeta_3/(\zeta_4)^4 = -\mathrm{d}\zeta_1 \wedge \mathrm{d}\zeta_2 \wedge \mathrm{d}\zeta_4/(\zeta_3)^4$. Then it is easy to see that

$$\tau_A^*\mathit{\Omega} = \overline{\mathit{\Omega}}, \quad \overset{12}{\tau_A}{}^*\mathit{\Omega} = -\overline{\mathit{\Omega}}, \quad \overset{12,34}{\tau_A}{}^*\mathit{\Omega} = \overline{\mathit{\Omega}}.$$

For the corresponding mirror involutions, we find

$$\widetilde{\tau}_B^*\widetilde{\mathit{\Omega}} = \widetilde{\mathit{\Omega}}, \quad \overset{12}{\widetilde{\tau}_B}{}^*\widetilde{\mathit{\Omega}} = -\widetilde{\mathit{\Omega}}, \quad \overset{12,34}{\widetilde{\tau}_B}{}^*\widetilde{\mathit{\Omega}} = \widetilde{\mathit{\Omega}}.$$

Note that the signs on the right hand sides have an invariant meaning for B-parities. It is $+1$ for orientifolds with O9/O5-planes and it is -1 for those with O7/O3-planes [77].

8.3.2 Light fields

Moduli of the worldsheet theory give rise to massless scalar fields in the $3+1$ dimensional spacetime. We have seen that the moduli compatible with parity symmetry can be real — orientifolds by a B-parity (resp. A-parity) reduce the complexified Kähler moduli (resp. complex structure moduli) by a real constraint. For example, see Fig. 11, which depicts the Kähler moduli constrained by a real condition. Thus, they give rise to real massless scalar fields in $3+1$ dimensions. However, in the full string theory, these real fields are paired with real fields from RR gauge potentials, and they form complex fields, or chiral superfields together with their fermionic superpartners.

To see the detail, let us recall how the NSNS and RR fields transform under the parity symmetry. The NSNS fields, dilaton-gravity-B-field (ϕ, g, B), are always transformed as

$$\phi \to \tau^*\phi, \quad g \to \tau^* g, \quad B \to -\tau^* B.$$

Transformation of the RR fields depends on the type of the involution. Let $\tau = \mathcal{I}_p$ be the inversion of $(9-p)$-coordinates. [1] Type IIB RR fields are transformed by the parity as

$$p=9,5,1:\begin{cases}A_0\to-\tau^*A_0\\A_2\to\tau^*A_2\\A_4^+\to-\tau^*A_4^+,\end{cases}\qquad p=7,3:\begin{cases}A_0\to\tau^*A_0\\A_2\to-\tau^*A_2\\A_4^+\to\tau^*A_4^+,\end{cases}$$

and for Type IIA,

$$p=8,4,0:\begin{cases}A_1\to\tau^*A_1\\A_3\to-\tau^*A_3,\end{cases}\qquad p=6,2:\begin{cases}A_1\to-\tau^*A_1\\A_3\to\tau^*A_3.\end{cases}$$

It is also useful to recall the massless spectrum before orientifolding where the spacetime theory has $\mathcal{N}=2$ supersymmetry, given in Table 2 (see for example [84]). In Type

Table 2: Massless Fields in $\mathcal{N}=2$ Compactifications

	hypermultiplets	vector multiplets
IIA	$h^{2,1}+1$	$h^{1,1}$
IIB	$h^{1,1}+1$	$h^{2,1}$

IIA, there are $h^{1,1}$ vector multiplets from the complexified Kähler moduli and A_3 reduced on $H^{1,1}(M)$, $h^{2,1}$ hypermultiplets from the complex structure moduli and A_3 reduced on $H^{2,1}(M)\oplus H^{1,2}(M)$, and one hypermultiplet from (ϕ,B) and A_3 reduced on $H^{3,0}(M)\oplus H^{0,3}(M)$, and a gravity multiplet from (g,A_1). In Type IIB, there are $h^{2,1}$ vector multiplets from the complex structure moduli and A_4^+ reduced on $H^{2,1}(M)\oplus H^{1,2}(M)$, $h^{1,1}$ hypermultiplets from the complexified Kähler moduli and (A_2,A_4^+) reduced on $(H^{1,1}(M),H^{2,2}(M))$, one hypermultiplet from (ϕ,B) and (A_0,A_2), and a gravity multiplet from g and A_4^+ reduced on $H^{3,0}(M)\oplus H^{0,3}(M)$. In terms of the $\mathcal{N}=1$ supersymmetry, an $\mathcal{N}=2$ vector multiplet splits into a vector and a chiral multiplets while a hypermultiplet splits into two chiral multiplets. We would like to see which of them survive and which of them are projected out by the orientifolds.

The involution τ induces an involution on the space of Harmonic forms by $\eta\to\tau^*\eta$. We denote by $H^\bullet_+$ and $H^\bullet_-$ the subspaces of $H^\bullet(M)$ consisting of τ^*-invariant and τ^*-anti-invariant forms. We also use $H^{p,q}_\pm$ when applicable, and denote its dimension by $h^{p,q}_\pm$.

[1] As is well-known, in Type II superstring theory, the involutive parity symmetry giving rise to the BPS Op-plane is of the form $\mathcal{I}_p\Omega$ for $p=9,8,5,4,1,0$ and $(-1)^{\widehat{F}_L}\mathcal{I}_p\Omega$ for $p=8,7,6,3,2$ where $(-1)^{\widehat{F}_L}$ is -1 on the left-moving Ramond sector. p is even for Type IIA and odd for Type IIB.

Type IIA orientifolds

Let us first consider Type IIA orientifold associated with an antiholomorphic involution τ (such that $\tau^*\Omega = \text{const} \times \overline{\Omega}$). Since τ flips 3 real coordinates of M, it is of the type $p = 6$, and therefore the RR fields are transformed as $A_1 \to -\tau^* A_1$ and $A_3 \to \tau^* A_3$. The moduli space of complex structure is reduced to a half by the orientifold and thus has real dimension $h^{2,1}$. The moduli space of complexified the Kähler class is reduced by the constraint (3.1). The reduced moduli space has complex dimension $h_-^{1,1}$, the dimension of the space $H_-^{1,1}$ of τ-anti-invariant harmonic $(1,1)$-forms.

The surviving fields out of the $h^{1,1}$ ($\mathcal{N} = 2$) vector multiplets are $h_-^{1,1}$ ($\mathcal{N} = 1$) chiral multiplets from the complexified Kähler moduli and $h_+^{1,1}$ ($\mathcal{N} = 1$) vector multiplets from A_3 reduced on τ-invariant harmonic 2-forms. Let us next see which of the $h^{2,1}$ hypermultiplet fields survives. We find $h^{2,1}$ real scalars from the complex structure moduli. We also find $h^{2,1}$ real scalars from A_3 reduced on the $\tau^* = 1$ subspace of $H^{2,1}(M) \oplus H^{1,2}(M)$. Note that τ, being antiholomorphic, exchanges $H^{p,q}(M)$ and $H^{q,p}(M)$ and hence $\tau^* = 1$ (or $\tau^* = -1$) on half of $H^{p,q}(M) \oplus H^{q,p}(M)$ if $p \neq q$. They combine into complex scalars in $h^{2,1}$ chiral multiplets. From the hypermultiplet including the dilaton, we find one chiral multiplet whose lowest component consists of the dilaton and A_3 reduced on the $\tau^* = 1$ subspace of $H^{3,0}(M) \oplus H^{0,3}(M)$. The $\mathcal{N} = 2$ gravity multiplet is projected to a $\mathcal{N} = 1$ gravity multiplet.

Type IIB orientifolds with O9 or O5-planes

Let us next consider Type IIB orientifold associated with a holomorphic involution τ of the type $p = 9$ or $p = 5$ (such as τ_B and $\tau_B^{12,34}$ in the quintic case). The RR fields are transformed as $A_0 \to -\tau^* A_0$, $A_2 \to \tau^* A_2$, $A_4^+ \to -\tau^* A_4^+$. The holomorphic 3-form Ω is preserved by τ, $\tau^*\Omega = \Omega$. The moduli space of the complexified Kähler class is reduced to a half and has real dimension $h^{1,1}$. The complex structure deformations are generated by the $\tau = 1$ subspace of $H^1(M, T_M)$. Since the isomorphism $H^1(M, T_M) \cong H^{2,1}(M)$ is given by contraction with Ω and since Ω is τ^*-invariant, we see that the deformation space is isomorphic to $H_+^{2,1}$. Thus, the complex structure moduli are reduced to $h_+^{2,1}$.

Out of the $h^{2,1}$ $\mathcal{N} = 2$ vector multiplet, we find $h_+^{2,1}$ chiral multiplets from the complex structure moduli and $h_-^{2,1}$ vector multiplets from A_4^+ reduced on $H_-^{2,1} \oplus H_-^{1,2}$. Let us next see which of the $h^{1,1}$ hypermultiplet fields are unprojected. First, we find $h^{1,1}$ real scalars from the reduced Kähler moduli. We also find real scalars from A_2 reduced on τ-invariant harmonic 2-forms and from A_4^+ reduced on τ-anti-invariant harmonic 4-forms. Since τ

preserves the orientation and the metric of M, τ^* commutes with the Hodge $*$-operator which in particular sends $H^{1,1}(M)$ to $H^{2,2}(M)$. We thus find $H_+^{1,1}$ is isomorphic to $H_+^{2,2}$ by the $*$-operation. This means that

$$\dim(H_+^{1,1} \oplus H_-^{2,2}) = \dim(H_-^{1,1} \oplus H_+^{2,2}) = h^{1,1}.$$

Thus, the RR potentials A_2 and A_4^+ reduce to $h^{1,1}$ real scalars. They combine with the real massless scalars from the Kähler moduli and form $h^{1,1}$ chiral multiplet fields. Next, the single hypermultiplet including the dilaton is projected to a single chiral multiplet consisting of the dilaton and the dual of A_2, plus fermions. Finally, the $\mathcal{N}=2$ gravity multiplet projects to an $\mathcal{N}=1$ gravity multiplet (A_4^+ reduced on $H^{3,0}(M) \oplus H^{0,3}(M)$ is projected out since $A_4^+ \to -\tau^* A_4^+$ and $\tau^*\Omega = \Omega$).

Type IIB orientifolds with O7 and/or O3-planes

Finally, let us consider Type IIB orientifolds associated with a holomorphic involution τ of the type $p = 7$ and/or $p = 3$ (such as τ_B^{12} in the quintic case). The RR fields are transformed as $A_0 \to \tau^* A_0$, $A_2 \to -\tau^* A_2$, $A_4^+ \to \tau^* A_4^+$, and τ^* flips the sign of the holomorphic 3-form Ω. The moduli space of Kähler class has real dimension $h^{1,1}$. The complex structure deformation are generated by the $\tau = 1$ subspace of $H^1(M, T_M)$, as in the previous case but that is isomorphic to $H_-^{2,1}$ in this case since $\tau^*\Omega = -\Omega$. Thus, there are $h_-^{2,1}$ complex structure moduli.

The counting of unprojected fields proceeds as before. We find $h_-^{2,1}$ chiral multiplets (complex structure moduli), $h_+^{2,1}$ vector multiplets (A_4^+ reduced on $H_+^{2,1} \oplus H_+^{1,2}$), $h^{1,1}$ chiral multiplets (real Kähler moduli plus real fields from (A_2, A_4^+) reduced on $(H_-^{1,1}, H_+^{2,2})$), one chiral multiplet (ϕ, A_0), and a gravity multiplet (A_4^+ reduced on $H^{3,0}(M) \oplus H^{0,3}(M)$ is projected out since $A_4^+ \to \tau^* A_4^+$ and $\tau^*\Omega = -\Omega$).

To summarize, we list in Table 3 the number of light $\mathcal{N}=1$ supermultiplets. Table 4 shows the numbers for the six orientifolds of the quintic. In general, D-branes should also be included in the setup for tadpole cancellation. Thus, there are other fields than in Table 3, associated with open string modes. Some of them correspond to the location of the D-branes and others are Yang-Mills gauge fields and charged matter fields on the branes or brane intersections.

Table 3: Light Fields from Closed Strings

	chiral multiplets	vector multiplets
IIAO(6)	$h^{1,1}_- + h^{2,1} + 1$	$h^{1,1}_+$
IIBO(9,5)	$h^{2,1}_+ + h^{1,1} + 1$	$h^{2,1}_-$
IIBO(7,3)	$h^{2,1}_- + h^{1,1} + 1$	$h^{2,1}_+$

Table 4: The number for the six orientifolds of the quintic

	chiral multiplets	vector multiplets
IIAO by $\tau_A, \tau_A^{12}, \tau_A^{12,34}$	103	0
IIBO by τ_B	103	0
IIBO by τ_B^{12}	65	38
IIBO by $\tau_B^{12,34}$	55	48

8.3.3 Spacetime superpotential

An important part of the low energy effective theory is the superpotential. It is a holomorphic function of the fields from the above 'light' chiral multiplets as well as other chiral multiplet fields such as those associated with D-branes. We must first compute the superpotential, find the minima of the potential, and expand around a chosen vacuum. Only after that one can discuss the actual mass spectrum.

Certain origins of the superpotential terms are known (see e.g. [77]). For Type IIA orientifolds, superpotential can be generated by holomorphic disks ending on A-branes [85–88] as well as "holomorphic $\mathbb{RP}^2$'s" — holomorphic maps of $\mathbb{CP}^1$ to X which are equivariant with respect to the anti-podal map of $\mathbb{CP}^1$ and the involution τ of X. For Type IIB orientifolds, we have the so called flux superpotential $W = \int_X \Omega \wedge G$ where G is a linear combination of RR and NSNS 3-form field strengths with dilaton-axion as the coefficients [89]. (The flux superpotential is computed or applied in the context of (mostly toroidal) orientifolds in [90, 91]. The geometry underlying $\mathcal{N} = 1$ superpotential is discussed in [92].) For D5-branes wrapped on 2-cycles, the superpotential for the 3-form flux generated by D5 is equivalent [77, 87] to the holomorphic Chern-Simons action for the open topological B-model [93, 94], which in turn is equivalent [85] to the superpotential associated with the obstruction to the deformation of holomorphic curves [20, 42].

One may wonder whether the deformation theory of the orientifold itself is obstructed as well. Namely, whether the deformation of a Calabi–Yau with holomorphic involution is obstructed. We claim that it is not. This can be shown as follows.[2] Let us fix the underlying differentiable manifold M. A 'Calabi–Yau with holomorphic involution' is just a pair (J, τ) where J is a complex structure of M whose canonical bundle is holomorphically trivial and τ is an involutive diffeomorphism of M that commutes with J. We are interested in the deformation space of the pair (J, τ), especially whether it is smooth or not. (We identify pairs that are related by diffeomorphisms.) First, we note that there is no deformation of τ itself — any involution close to τ is diffeomorphic to τ itself. Thus one can fix τ and consider deformations only of J commuting with τ. Let us for now forget about τ and consider the full space of deformations of the Calabi–Yau. This deformation theory is not obstructed. Namely, the Teichmüller space (the space of Calabi–Yau J divided by the group of diffeomorphisms homotopic to the identity) is smooth. On the Teichmüller space there is a $\mathbb{Z}_2$ action generated by $[J] \mapsto [\tau_* J \tau_*^{-1}]$. The τ-invariant J's we are interested in are nothing but the fixed points of this action. Since the fixed-point locus of a $\mathbb{Z}_2$ action on a smooth manifold is always smooth, our deformation space is smooth. This proves that the deformation theory is not obstructed.

This does not mean, however, that there is no superpotential of purely orientifold origin. For example if there are two O5-planes C_+ and C_- of opposite RR charges, we do have a flux superpotential which is $\int_Y \Omega$ where Y is a three dimensional submanifold of M bounded by $C_+ - C_-$, as occured in the example studied in [77].

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Appendix

A Index $\mathrm{Tr}(-1)^F P$ in Non-linear Sigma Models

Let (X, g) be a Riemannian manifold of real dimension n. We consider the supersymmetric non-linear sigma model in $1+1$ dimensions whose classical action on Minkowski space reads

$$\begin{aligned}\mathscr{L} \;=\;& \frac{1}{2}g_{IJ}(\partial_0\phi^I\partial_0\phi^J - \partial_1\phi^I\partial_1\phi^J) + \frac{i}{2}g_{IJ}\psi_-^I(D_0+D_1)\psi_-^J + \frac{i}{2}g_{IJ}\psi_+^I(D_0-D_1)\psi_+^J \\ & -\frac{1}{4}R_{IJKL}\psi_+^I\psi_-^J\psi_+^K\psi_-^L.\end{aligned}$$

Supersymmetry is generated by $\delta\phi^I = i\epsilon_+\psi_-^I + i\epsilon_-\psi_+^I$, $\delta\psi_\pm^I = -\epsilon_\mp(\partial_0 \pm \partial_1)\phi^I$. Let $\tau : X \to X$ be an isometric involution. The classical action is invariant under the parity symmetry $\tau\Omega$

$$\begin{aligned}&\phi^I(x) \to \tau^I(\phi(\widetilde{x})),\\ &\psi_\pm^I(x) \to \tau_{*J}^I\psi_\mp^J(\widetilde{x}),\end{aligned}$$

where $\widetilde{x} = (x^0, -x^1)$ for $x = (x^0, x^1)$, and we assume that it is anomaly-free. We shall compute the twisted supersymmetric index $\mathrm{Tr}(-1)^F\tau\Omega$ for both closed and open strings.

The basic strategy is to represent the index as the partition function on a flat surface and localize the path-integral on the fixed point of the supersymmetry. For example, the ordinary Witten index $\mathrm{Tr}(-1)^F$ on the RR sector is path-integral on the 2-torus with periodic boundary condition in both directions. The boundary condition is fully supersymmetric and the fixed points are the constant maps. In a constant background, the action consists only of the four-fermi terms. The 1-loop integral on non-constant modes is 1 as a consequence of boson-fermion cancellation, and we are left with the zero mode integral, with the exp(four-fermi terms) as the weight. This leads to integration over X of the Pfaffian of the Riemannian curvature. The latter is the Euler class of the tangent bundle $T(X)$ of X, and hence

$$\mathop{\mathrm{Tr}}_{\mathcal{H}_{\mathrm{RR}}}(-1)^F = \int_X e(T(X)) = \chi(X). \tag{A.1}$$

This method was introduced in [52, 53].

Now we compute the twisted index $\mathrm{Tr}(-1)^F\tau\Omega$ on the RR-sector (closed string). It is represented as the path-integral on the Klein bottle $(x_1, x_2) \equiv (x_1 + L_1, x_2) \equiv$

$(-x_1, x_2+L_2)$ with the periodic boundary condition along x_1, but with the twisted boundary condition along x_2:

$$\phi^I(x_1, x_2) = \tau^I(\phi(-x_1, x_2 + L_2)), \tag{A.2}$$

$$\psi^I_\pm(x_1, x_2) = \tau^I_{*J}\psi^J_\mp(-x_1, x_2 + L_2). \tag{A.3}$$

This periodicity preserves the diagonal part of the $(1,1)$ supersymmetry and one can still use the localization method. The fixed points of the supersymmetry are constant maps into the set X^τ of τ-fixed points. We first note that the integral of modes that are not constant along x_1 yields 1 due to boson-fermion cancellation. Thus, we can focus on modes that depend only on x_2. This computation has been done beautifully in [51]. Here is the outline. We first separate the coordinates into tangent directions ϕ^μ to X^τ and normal directions ϕ^i. Let n_1 be the real dimension of X^τ, so that $\mu = 1, ..., n_1$ and $i = 1, ..., n - n_1$. By (A.2), ϕ^μ is periodic and ϕ^i is antiperiodic along x_2. By (A.3), the periodicity of the fermions are

$$\begin{aligned}
\psi^\mu_+ + \psi^\mu_- &: \quad \text{periodic} \\
\psi^\mu_+ - \psi^\mu_- &: \quad \text{anti-periodic} \\
\psi^i_+ + \psi^i_- &: \quad \text{anti-periodic} \\
\psi^i_+ - \psi^i_- &: \quad \text{periodic}
\end{aligned}$$

We also note that $R_{IJKL}\psi^I_+\psi^J_-\psi^K_+\psi^L_-$ is proportional to $R_{IJKL}(\psi_+ + \psi_-)^I(\psi_+ + \psi_-)^J(\psi_+ - \psi_-)^K(\psi_+ - \psi_-)^L$. It follows that the zero mode action is the curvature of the normal bundle to X^τ. The one-loop integral of the non-zero modes yields

$$\begin{aligned}
\phi^\mu &: \quad \det{}_P^{-\frac{1}{2}}(\partial^2 + iR_T\partial) = (2\pi\beta)^{-\frac{n_1}{2}} \prod_{\lambda_T} \frac{\lambda_T\beta/2}{\sinh(\lambda_T\beta/2)} \\
\psi^\mu_+ + \psi^\mu_- &: \quad \mathrm{Pf}_P(i\partial) = 1 \\
\psi^\mu_+ - \psi^\mu_- &: \quad \mathrm{Pf}_A(i\partial + R_T) = \prod_{\lambda_T} 2\cosh(\lambda_T\beta/2) \\
\phi^i &: \quad \det{}_A^{-\frac{1}{2}}(\partial^2 + iR_N\partial) = \prod_{\lambda_N} \frac{1}{4\cosh(\lambda_N\beta/2)} \\
\psi^i_+ + \psi^i_- &: \quad \mathrm{Pf}_A(i\partial) = \prod_{\lambda_N} 2 \\
\psi^i_+ - \psi^i_- &: \quad \mathrm{Pf}_P(i\partial + R_N) = \prod_{\lambda_N} \frac{\sinh(\lambda_N\beta/2)}{\lambda_N\beta/2}
\end{aligned}$$

Here ∂ is the derivative with respect to x_2. R_T is the curvature of the tangent bundle of X^τ and $(i\lambda_T, -i\lambda_T)$ are its eigenvalues, and similarly for the curvature of the normal bundle R_N. (Here we assume that both n and n_1 are even, for simplicity.) β is the circumference L_2 in the x_2 direction, on which the final expression should not depend. In this notation, the zero mode action contributes by a factor

$$\prod_{\lambda_N} \lambda_N \beta.$$

Collecting all together, the zero mode integral is

$$\int_{X^\tau} (\pi\beta)^{-\frac{n_1}{2}} \prod_{\lambda_T} \frac{\lambda_T \beta/2}{\tanh(\lambda_T \beta/2)} \prod_{\lambda_N} \frac{\tanh(\lambda_N \beta/2)}{\lambda_N \beta/2} \prod_{\lambda_N} \lambda_N \beta/2.$$

This indeed does not depend on β because $n_1 = \dim X^\tau$ and λ_T, λ_N are 2-forms. Setting $\beta = 1/\pi$, we find the following index formula

$$\operatorname*{Tr}_{\mathcal{H}_{\rm RR}} (-1)^F \tau\Omega = \int_{X^\tau} \frac{L(T(X^\tau))}{L(N(X^\tau))} e(N(X^\tau)), \tag{A.4}$$

$L(V)$ is the Hirzebruch L-genus defined by $\prod_{\lambda_V} (\lambda_V/2\pi)/\tanh(\lambda_V/2\pi)$.

We next consider the twisted index for an open string. The string has one end on a D-brane wrapped on $W \subset X$ and supporting a complex vector bundle E, and the other end on the image brane $(\tau W, \tau E)$. The boundary condition preserves the same supersymmetry as $\tau\Omega$ preserves, and one can consider twisted Witten index $\mathrm{Tr}(-1)^F \tau\Omega$. This is represented as the path-integral on the Möbius strip $(x_1, x_2) = (L_1 - x_1, x_2 + L_2)$, $0 \leq x_1 \leq L_1$, with the boundary condition at one end

$$\partial_2 \phi^I(0, x_2),\ (\psi_+^I + \psi_-^I)(0, x_2): \quad \text{tangent to } W, \tag{A.5}$$

$$\partial_1 \phi^I(0, x_2),\ (\psi_+^I - \psi_-^I)(0, x_2): \quad \text{normal to } W, \tag{A.6}$$

and similar condition at the other end (L_1, x_2). The coupling to the gauge field A for the bundle E is through the Chan-Paton factor

$$\mathrm{tr}_{\overline{E}} P \exp \left\{ -i \int_{x_1=0} \Big(-{}^t\! A_M \partial_2 \phi^M - i\, {}^t\! F_{MN} (\psi_+ + \psi_-)^M (\psi_+ + \psi_-)^N \Big) \mathrm{d}x_2 \right\}, \tag{A.7}$$

(similarly at $x_1 = L_1$) but not through the change in the boundary condition. In the above formula, M, N are coordinate indices on the brane W. Also, we have $\overline{E}$ with the dual gauge field $-{}^tA$ because the left boundary of the string worldsheet is oriented toward negative time direction. Periodicity along x_2 is the same as (A.2)-(A.3) except that $-x_1$ is replaced by $L_1 - x_1$. Because of supersymmetry, path-integral localizes on the fixed points, which are constant maps to the intersection of the brane and the orientifold plane, $W \cap X^\tau$. By the deformation invariance of the index, we deform W continuously so that

W and X^τ intersects normally. This means that τ sends W to itself in a neighborhood of X^τ. One can then choose the coordinates $(x^a, x^\mu, x^\alpha, x^i)$ where (x^a, x^μ) are coordinates on W and the involution acts as

$$\tau : (x^a, x^\mu, x^\alpha, x^i) \mapsto (-x^a, x^\mu, x^\alpha, -x^i)$$

so that (x^μ, x^α) are coordinates on X^τ. We name the dimensions of the respective directions as

$$a = 1, ..., n_1; \quad \mu = 1, ..., n_2; \quad \alpha = 1, ..., n_3; \quad i = 1, ..., n_4.$$

For simplicity we assume that all four dimensions are even. Since τW is the same as W in a neighborhood of X^τ, the boundary conditions are the same on the two end points of the string. In the intersection $W \cap X^\tau$, τE can be topologically identified with the complex conjugate of E. Thus, the Chan-Paton factor at $x_1 = L_1$ is the same as (A.7). We can thus ignore the x_1 dependence of the fields — the integral of x_1-dependent modes will give 1 by boson-fermion cancellation. The boundary condition (A.5)-(A.6) then forces the constraints $\psi^a_+ = \psi^a_- =: \psi^a$, $\psi^\mu_+ = \psi^\mu_- =: \psi^\mu$, $\phi^\alpha = 0$, $\psi^\alpha_+ = -\psi^\alpha_- =: \psi^\alpha$, $\phi^i = 0$, $\psi^i_+ = -\psi^i_- =: \psi^i$. By (A.2) and (A.3), the remaining fields obey the following periodicity

$$\begin{array}{ll} \phi^a : & \text{anti-periodic} \\ \psi^a : & \text{anti-periodic} \\ \phi^\mu : & \text{periodic} \\ \psi^\mu : & \text{periodic} \\ \psi^\alpha : & \text{anti-periodic} \\ \psi^i : & \text{periodic} \end{array}$$

The one-loop integral of the non-zero modes yields

$$\begin{array}{ll} \phi^a : & \det_A^{-\frac{1}{2}}(\partial^2 + iR_1\partial) = \prod_{\lambda_1} \dfrac{1}{4\cosh(\lambda_1\beta/2)} \\ \psi^a : & \mathrm{Pf}_A(i\partial) = \prod_{\lambda_1} 2 \\ \phi^\mu : & \det_P^{-\frac{1}{2}}(\partial^2 + iR_2\partial) = (2\pi\beta)^{-\frac{n_2}{2}} \prod_{\lambda_2} \dfrac{\lambda_2\beta/2}{\sinh(\lambda_2\beta/2)} \\ \psi^\mu : & \mathrm{Pf}_P(i\partial) = 1 \\ \psi^\alpha : & \mathrm{Pf}_A(i\partial + R_3) = \prod_{\lambda_3} 2\cosh(\lambda_3\beta/2) \\ \psi^i : & \mathrm{Pf}_P(i\partial + R_4) = \prod_{\lambda_4} \dfrac{\sinh(\lambda_4\beta/2)}{\lambda_4\beta/2} \end{array}$$

The zero mode bulk action yields the factor

$$\prod_{\lambda_4} \lambda_4 \beta,$$

and the boundary action provides

$$\mathrm{tr}_{\overline{E}}\, \mathrm{e}^{2\beta\, {}^t F},$$

where the exponent has the factor 2β rather than β because the circumference of the closed boundary circle is twice as large as $\beta = L_2$. Collecting all together, we find the following expression for the zero mode integral

$$\int_{W \cap X^\tau} \mathrm{tr}_{\overline{E}}\, \mathrm{e}^{2\beta\, {}^t F} \times$$
$$2^{-\frac{n_1}{2}+\frac{n_3}{2}} (2\pi\beta)^{-\frac{n_2}{2}} \prod_{\lambda_1} \frac{1}{\cosh(\frac{\lambda_1 \beta}{2})} \prod_{\lambda_2} \frac{\frac{\lambda_2 \beta}{2}}{\sinh(\frac{\lambda_2 \beta}{2})} \prod_{\lambda_3} \cosh(\frac{\lambda_3 \beta}{2}) \prod_{\lambda_4} \frac{\sinh(\frac{\lambda_4 \beta}{2})}{\frac{\lambda_4 \beta}{2}} \prod_{\lambda_4} \lambda_4 \beta.$$

This is indeed independent of the value of β on dimensional ground, $n_2 = \dim W \cap X^\tau$. At this stage, we use the elementary relations

$$\frac{\sinh(\frac{x}{2})}{\frac{x}{2}} = \sqrt{\frac{\sinh(x)}{x} \frac{\tanh(\frac{x}{2})}{\frac{x}{2}}}, \qquad \cosh\left(\frac{x}{2}\right) = \sqrt{\frac{\sinh(x)}{x} \frac{\frac{x}{2}}{\tanh(\frac{x}{2})}}.$$

to re-express the integral as

$$2^{-\frac{n_1}{2}+\frac{n_3}{2}} (2\pi\beta)^{-\frac{n_2}{2}} \int_{W \cap X^\tau} \mathrm{tr}_{\overline{E}}\, \mathrm{e}^{2\beta\, {}^t F} \sqrt{\prod_{\lambda_1} \frac{\lambda_1 \beta}{\sinh(\lambda_1 \beta)} \frac{\tanh(\frac{\lambda_1 \beta}{2})}{\frac{\lambda_1 \beta}{2}} \prod_{\lambda_2} \frac{\lambda_2 \beta}{\sinh(\lambda_2 \beta)} \frac{\frac{\lambda_2 \beta}{2}}{\tanh(\frac{\lambda_2 \beta}{2})}} \times$$
$$\times \sqrt{\prod_{\lambda_3} \frac{\sinh(\lambda_3 \beta)}{\lambda_3 \beta} \frac{\frac{\lambda_3 \beta}{2}}{\tanh(\frac{\lambda_3 \beta}{2})} \prod_{\lambda_4} \frac{\sinh(\lambda_4 \beta)}{\lambda_4 \beta} \frac{\tanh(\frac{\lambda_4 \beta}{2})}{\frac{\lambda_4 \beta}{2}}} \prod_{\lambda_4} \lambda_4 \beta$$

Setting $\beta = 1/4\pi$ we find that the index is expressed as

$$\begin{aligned} &\mathop{\mathrm{Tr}}_{\mathcal{H}_{(W,E),(\tau W, \tau E)}} (-1)^F \tau \Omega \\ &\quad = 2^{-\frac{n_1}{2}+\frac{n_3}{2}+\frac{n_2}{2}-\frac{n_4}{2}} \int_{W \cap X^\tau} \mathrm{ch}(\overline{E}) \sqrt{\frac{\widehat{A}(T(W))}{\widehat{A}(N(W))}} \sqrt{\frac{L(\frac{1}{4} T(X^\tau))}{L(\frac{1}{4} N(X^\tau))}} e(N(W) \cap N(X^\tau)). \end{aligned} \tag{A.8}$$

Here $\widehat{A}(V)$ is the A-roof genus $\prod_{\lambda_V} (\lambda_V / 4\pi) / \sinh(\lambda_V / 4\pi)$ and $L(\frac{1}{4} V)$ is the modified L-genus $\prod_{\lambda_V} (\lambda_V / 8\pi) / \tanh(\lambda_V / 8\pi)$. We note here that the power of 2 in the formula is

$$-\frac{n_1}{2} + \frac{n_3}{2} + \frac{n_2}{2} - \frac{n_4}{2} = n_2 + n_3 - \frac{n_1 + n_2 + n_3 + n_4}{2} = \dim_{\mathbb{R}} X^\tau - \frac{1}{2} \dim_{\mathbb{R}} X. \tag{A.9}$$

It way happen that τ has fixed-point submanifolds of various dimensions. It is clear form the above derivation that, in such a case, the index is the sum over components of intersection where the summand is the above in each component of intersection.

One may consider the application of the result to superstring theory. In particular, one can read off from the index the WZ coupling of the D-branes and O-planes to the bulk RR fields [95–102]. The index in a special case has been computed in such a context [103].

The case $W = X$

In the case $W = X$, the formula (A.8) with a modification $\hat{A}(X) \to \mathrm{td}(X)$ simplifies considerably. It is straightforward to see that

$$\int_{X^\tau} 2^{\dim_r X^\tau - \frac{1}{2}\dim_r X} \mathrm{ch}(\overline{E}) \sqrt{\mathrm{td}(X)\frac{L(\frac{1}{4}T(X^\tau))}{L(\frac{1}{4}N(X^\tau))}} = \int_{X^\tau} \mathrm{ch}(\overline{E}) \prod_{\lambda_t} \frac{\frac{\lambda_t}{2\pi}}{1-\mathrm{e}^{-\frac{\lambda_t}{4\pi}}} \prod_{\lambda_n} \frac{1}{1+\mathrm{e}^{-\frac{\lambda_n}{4\pi}}}.$$

In the integrand, only the term of the same degree as $\dim X^\tau$ contribute. This allows us to replace it by

$$\int_{X^\tau} \mathrm{ch}(2\overline{E}) \prod_{\lambda_t} \frac{\frac{\lambda_t}{2\pi}}{1-\mathrm{e}^{-\frac{\lambda_t}{2\pi}}} \prod_{\lambda_n} \frac{1}{1+\mathrm{e}^{-\frac{\lambda_n}{2\pi}}}.$$

which is nothing but

$$\int_{X^\tau} \mathrm{ch}(2\overline{E})\, \mathrm{td}(X^\tau) \frac{1}{\mathrm{ch}(\wedge \overline{N}_{X^\tau})}.$$

B Computation of the weights in the coset construction

In this appendix, we compute the weights of all fields of the minimal model $SU(2)_k \times U(1)_2/U(1)_{k+2}$ exactly, in particular not only modulo integers. From this we can then infer exact expressions for $\sqrt{T}$.

As is standard in the coset construction, one starts by embedding the denominator theory $U(1)_{k+2}$ into the numerator $SU(2)_k \times U(1)_2$. This is achieved by the identification

$$J^H(z) = J^3(z) - J^{(2)}(z), \tag{B.1}$$

which is the same as in (4.10). As before, $J^3(z)$ is the current associated with the Cartan-subalgebra of $su(2)$ and J^H and $J^{(2)} = \frac{1}{2}J^f$ are the level $k+2$ and 2 $U(1)$ currents. Note however that here we are talking about the model *after* GSO projection.

The Hilbert space of the numerator theory can then be decomposed in the following way

$$\mathcal{H}_j \otimes \mathcal{H}_s = \bigoplus_{n\in\mathbb{Z}_{2k+4}, 2j+n+s\, even} \mathcal{H}_{(j,n,s)} \otimes \mathcal{H}_{-n}, \tag{B.2}$$

where $\mathcal{H}_j$, $\mathcal{H}_s$, $\mathcal{H}_{j,n,s}$ and $\mathcal{H}_{-n}$ denote representation spaces of the $SU(2)_k$, $U(1)_2$, the GSO projected minimal model and $U(1)_{k+2}$ respectively. Compared to the discussion of the unprojected theory, $\hat{V}_j^{G,k} = \mathcal{H}_j$, but the Hilbert spaces for the two $U(1)$'s are different.

To determine the weights of the primary fields of the coset theory, one has to understand in detail how the ground states $|j,n,s\rangle \otimes |-n\rangle$ of $\mathcal{H}_{j,n,s} \otimes \mathcal{H}_{-n}$ are realized within the representation space $\mathcal{H}_j \otimes \mathcal{H}_s$.

To find these ground states we fix j, s and choose n such that $2j+n+s$ is even. Within the subspace

$$\mathcal{H}_{js}^{(n)} \ = \ (\mathcal{H}_j \otimes \mathcal{H}_s)^{(n)} \ := \ \{\psi \in \mathcal{H}_j \otimes \mathcal{H}_s \ | \ \mathrm{e}^{\frac{2\pi i}{2k+4} J_0^H}\, \psi \ = \ \mathrm{e}^{\frac{2\pi i n}{2k+4}}\, \psi \ \} \tag{B.3}$$

we then search for eigenstates $\psi_{js}^{(n)}$ of $L_0^{SU(2)_k} + L_0^{U(1)_2}$ with minimal eigenvalue.

Some of these states are easily identified. These are the states which are realized in terms of ground states of the numerator theory,

$$\psi_{j\,s}^{(n)} \ = \ |j,n,s\rangle \otimes |-n\rangle \ = \ |j, \nu = -n+s\rangle \otimes |s\rangle \tag{B.4}$$

where n is restricted by $|n-s| \leq 2j$. In this way we have realized all fields from the so-called standard range of $\mathcal{N}=2$ minimal models,

$$2j \ \leq \ k \ , \qquad |n-s| \ \leq \ 2j, \quad 2j+n+s \text{ even} \tag{B.5}$$

$$n \in \{-k-1, \ldots, k+2\}, \quad s \in \{-1,0,1,2\} \tag{B.6}$$

For these fields, the following formula for their conformal weights holds exactly (not just up to an integer),

$$h_{(j,n,s)} \ = \ \frac{j(j+1)}{k+2} - \frac{n^2}{4(k+2)} + \frac{s^2}{8} \ . \tag{B.7}$$

But the $2j+1$ states we have found do not exhaust the ground states of the denominator theory. Additional states can be constructed with the help of

$$(J^+)_{-1}^{\mu} |j, 2j\rangle \ , \ (J^-)_{-1}^{\mu} |j,-2j\rangle \ \in \mathcal{H}_j \quad \text{for} \quad \mu = 1,2,\ldots \ ,$$

where $J^+(z)$ and $J^-(z)$ are the raising and lowering operators of the $su(2)$ algebra. These states carry the charge $\nu = 2j+2\mu$ or $\nu = -2j-2\mu$, respectively. When combined with appropriate states from $\mathcal{H}_s$ (not necessarily the ground state) they furnish all the ground states for the denominator theory of the coset.

There is a second class of fields whose weight is easy to write down, namely those which can be reflected to the standard range by field identification. More precisely, these are primaries labelled (j, n, s), for which $(k/2-j, n\hat{+}(k+2), s\hat{+}2)$ is in the standard range. The notation $\hat{+}$ denotes the addition modulo $2k+4$ for the label n and modulo 4 for the label s, in such a way that the result of the addition is in the range $-k-1, \ldots, k+2$ for the labels n and $-1, 0, 1, 2$ for s. The hatted sum can be rewritten as $n\hat{+}(k+2) = n - \frac{n}{|n|}(k+2)$, where we made use of the choice that n itself is in the range $-k-1, \ldots, k+2$. We can now apply the standard range formula for the weights to the reflected field, with the result that

$$h_{(j,n,s)} = \frac{j(j+1)}{k+2} - \frac{n^2}{4(k+2)} + \frac{s^2}{8} + \frac{|n| - |s| - 2j}{2} \quad \text{for} \quad (\frac{k}{2} - j, n\hat{+}(k+2), s\hat{+}2) \in S.R. \quad . \tag{B.8}$$

There is a list of primaries which cannot be reflected to the standard range, namely $(j, -2j, 2), (j, 2j+2, 0), (j, 2j+1, -1)$ and $(j, -2j-1, 1)$. The primaries $(j, -2j, 2)$ with $j \geq 1$ have a coset realization with minimal weight as $|j, 2j-2\rangle \otimes |-2\rangle \in \mathcal{H}_j \otimes \mathcal{H}_s$. Note that the $U(1)_2$-label is not in the standard domain for s. We can then use the expression (B.7) for the weight and obtain

$$h_{(j,-2j,2)} = \frac{j}{(k+2)} + \frac{1}{2} \qquad j \geq 1. \tag{B.9}$$

In the special case $j = 0$ corresponding to the primary $(0, 0, 2)$ one picks the realization $J^+_{-1}\,|0, 0\rangle \otimes |2\rangle$, resulting in the weight

$$h_{(0,0,2)} = \frac{3}{2}. \tag{B.10}$$

Note that the primaries with higher j's could have similarly been represented as $J^+_{-1}\,|j, 2j\rangle \otimes |2\rangle$, but these states do not have minimal weight and hence are not ground states.

The primaries $(j, 2j+2, 0)$, $j \neq k/2$, have a coset representation as $J^-_{-1}\,|j, -2j\rangle \otimes |0\rangle$, which is of minimal weight. This results in a weight given by (B.7) shifted by one, or explicitly

$$h_{(j,2j+2,0)} = -\frac{j+1}{(k+2)} + 1 \tag{B.11}$$

For the special case $j = k/2$ we consider the state $|k/2, -k+2\rangle \otimes |4\rangle$, leading to the result that the standard range result has to be shifted by 2, or

$$h_{(\frac{k}{2},k+2,0)} = \frac{3}{2} \tag{B.12}$$

The set of states $(j,-2j,2)$ is mapped to $(j,2j+2,0)$ under field identification and our computation of the weights is compatible with that. For later use, we emphasize that the standard range formula (B.7) still holds for $(j,-2j,2)$, whereas the reflected formula (B.8) applies to the field identified primaries $(j,2j+2,0)$. Therefore, these primaries can for our purposes be effectively included in the discussion of the standard range fields and no further case distinction is required.

In the Ramond sector, we realize the exceptional cases $(j,2j+1,-1)$ as $J^-_{-1}\,|j,-2j\rangle\otimes|-1\rangle$ leading to a shift of the standard range weight by 1

$$h_{(j,2j+1,-1)}=\frac{c}{24}+1. \tag{B.13}$$

Similarly, we have $|j,-2j-1,1\rangle=J^+_{-1}\,|j,2j\rangle\otimes|1\rangle$ and a weight of

$$h_{(j,-2j-1,1)}=\frac{c}{24}+1. \tag{B.14}$$

In the same way as in the NS sector, $(j,2j+1,-1)$ get mapped to $(j,-2j-1,1)$ under field identification and our computation of the weights is compatible with that. However, neither (B.7) nor (B.8) holds in the R-sector, so that we need to consider the fields that cannot be reflected to the standard range separately.

In terms of the T matrices, we can conclude from the above discussion that

$$T^{\frac{1}{2}}_{(j,n,s)}=\sigma_{j,n,s}\;T_j^{\frac{1}{2}}\;T_n^{-\frac{1}{2}}\;T_s^{\frac{1}{2}} \tag{B.15}$$

where

$$\sigma_{j,n,s}:=\begin{cases}1 & (j,n,s)\in S.R.\\ 1 & (j,-2j,2),\;\; l\geq 2\\ 1 & (\frac{k}{2},k+2,0)\\ (-1)^{\frac{|n|-|s|-2j}{2}} & (\frac{k}{2}-j,n\hat{+}(k+2),s\hat{+}2)\in S.R.\\ (-1)^{\frac{|n|-|s|-2j}{2}} & (j,2j+2,0),\;\; j\neq\frac{k}{2}\\ -1 & (j,\pm(2j+1),\mp 1)\\ -1 & (0,0,2)\end{cases}$$

One can derive the following general relation between the σ

$$\sigma_{\frac{k}{2}-j,n\hat{+}(k+2),s\hat{+}2}=(-1)^{\frac{|n|-|s|-2j}{2}}\sigma_{j,n,s} \tag{B.16}$$

This holds independent of whether (j,n,s) is or can be brought to the standard range. It is however important that the label n is chosen in the range $-k-1,\ldots,k+2$ and s in $-1,0,1,2$. Note also that the sign factor $(-1)^{\frac{|n|-|s|-2j}{2}}$ is invariant under field identification $j\to\frac{k}{2}-j, n\to n\hat{+}(k+2), s\to s\hat{+}2$.

C P-matrix for the minimal model

In this appendix, we write down an explicit expressions of the P-matrix of the $\mathcal{N}=2$ minimal model. We start with the expressions for the P-matrix and Y-tensor of the constituent theories.

$SU(2)_k$

The P-matrix of the level k $SU(2)$ WZW model is

$$P_{jj'} = \frac{2}{\sqrt{k+2}} \sin\left[\frac{\pi(2j+1)(2j'+1)}{2(k+2)}\right] \delta^{(2)}_{2j+2j'+k}.$$

Some part of Y-tensor is given by

$$Y^{j'}_{j0} = (-1)^{2j+j'} N^{j'}_{j\,j}, \quad Y^{j'}_{j\frac{k}{2}} = N^{j'}_{j\,(\frac{k}{2}-j)}.$$

In particular, $Y^{0}_{j\,0} = (-1)^{2j}$ and $Y^{\frac{k}{2}}_{j\,\frac{k}{2}} = 1$ for any $j \in \mathrm{P}_k$.

$U(1)_k$

The P-matrix and Y-tensor of the level k $U(1)$ is

$$P_{nn'} = \tfrac{1}{\sqrt{k}}\, \mathrm{e}^{-\frac{\pi i \widehat{n}\widehat{n'}}{2k}} \delta^{(2)}_{n+n'+k},$$

$$Y^{n''}_{nn'} = \delta^{(2)}_{n'+n''}\left(\delta^{(2k)}_{n+\frac{\widehat{n'-n''}}{2}} + (-1)^{n'+k}\delta^{(2k)}_{n+\frac{\widehat{n'-n''}}{2}+k}\right),$$

where $\widehat{n}$ is the unique member of $n+2k\mathbb{Z}$ in the standard range $\{-k+1,...,k-1,k\}$. In the following, we will omit the $\hat{}$, but it is understood that all labels are chosen in this way.

Minimal model

We first note that the Q-matrix of the minimal model can be expressed in terms of the Q-matrices of the constituent theories in the following way

$$Q_{(j,n,s)(j',n',s')} = Q_{jj'} Q^{*}_{nn'} Q_{ss'} + Q_{j(\frac{k}{2}-j')} Q^{*}_{n(n'\hat{+}(k+2))} Q_{s(s'\hat{+}2)} \tag{C.1}$$

The P-matrix is then obtained as

$$
\begin{aligned}
P_{(j,n,s)(j'n's')} &= T^{\frac{1}{2}}_{(j,n,s)} Q_{(j,n,s)(j',n',s')} T^{\frac{1}{2}}_{(j',n',s')} \\
&= \sigma_{j,n,s}\sigma_{j'n's'} P_{jj'} P^*_{nn'} P_{ss'} \\
&\quad + \sigma_{j,n,s}\sigma_{\frac{k}{2}-j',n'\hat{+}(k+2),s'\hat{+}2} P_{j,\frac{k}{2}-j'} P^*_{n,n'\hat{+}(k+2)} P_{s,s'\hat{+}2} \\
&= \sigma_{j,n,s}\sigma_{j'n's'} \left(P_{jj'} P^*_{nn'} P_{ss'} + (-1)^{\frac{|n'|-|s'|-2j'}{2}} P_{j,\frac{k}{2}-j'} P^*_{n,n'\hat{+}(k+2)} P_{s,s'\hat{+}2} \right)
\end{aligned}
\tag{C.2}
$$

In the last step, we have used (B.16). One can further evaluate this formula as

$$
\begin{aligned}
P^*_{n,n'\hat{+}(k+2)} &= \frac{1}{\sqrt{k+2}} \delta^{(2)}_{n+n'}\, \mathrm{e}^{\frac{\pi i n(n'\hat{+}(k+2))}{2(k+2)}} = \frac{1}{\sqrt{k+2}} \delta^{(2)}_{n+n'}\, \mathrm{e}^{\frac{\pi i n n'}{k+2}}\, \mathrm{e}^{\frac{\pi i n}{2}} (-1)^{\frac{|n'|+n'}{2}} \\
P_{s,s'\hat{+}2} &= \frac{1}{\sqrt{2}} \delta^{(2)}_{s+s'}\, \mathrm{e}^{-\frac{\pi i s(s'\hat{+}2)}{4}} = \frac{1}{\sqrt{2}} \delta^{(2)}_{s+s'}\, \mathrm{e}^{-\frac{\pi i s s'}{4}}\, \mathrm{e}^{\frac{\pi i s}{2}} (-1)^{\frac{|s'|-s'}{2}}.
\end{aligned}
$$

Note that these expressions are only valid if n, n', s, s' are chosen in the range $-k-1, \ldots, k+2$ and $-1, 0, 1, 2$.

The explicit expression for the P-matrix is

$$
\begin{aligned}
P_{(j,n,s)(j'n's')} = \sigma_{j,n,s}\sigma_{j'n's'} \frac{\sqrt{2}}{k+2} \delta^{(2)}_{s+s'}\ \mathrm{e}^{\frac{\pi i n n'}{2(k+2)}}\ \mathrm{e}^{-\frac{\pi i s s'}{4}} \Big(&\sin\left[\pi \tfrac{(2j+1)(2j'+1)}{2(k+2)}\right] \delta^{(2)}_{2j+2j'+k}\ \delta^{(2)}_{n+n'+k} \\
&+ (-1)^{\frac{2j'+n'+s'}{2}}\ \mathrm{e}^{\frac{\pi i (n+s)}{2}} \sin\left[\pi \tfrac{(2j+1)(k-2j'+1)}{2(k+2)}\right] \delta^{(2)}_{2j+2j'}\ \delta^{(2)}_{n+n'} \Big)
\end{aligned}
\tag{C.3}
$$

In the calculations involving B-type parities, it is often convenient to use the following form of the P-matrix, where one inserts the $U(1)$ data explicitly, and then uses general formulas for the $SU(2)$ part:

$$
\begin{aligned}
P_{(j,n,s)(j'n's')} &= \sigma_{j,n,s}\sigma_{j'n's'} \frac{1}{\sqrt{2(k+2)}}\, \mathrm{e}^{\frac{\pi i n n'}{2(k+2)}}\, \mathrm{e}^{-\frac{\pi i s s'}{4}} \delta^{(2)}_{s+s'} \\
&\quad \left(P_{jj'} \delta^{(2)}_{n+n'+k} + (-1)^{\frac{2j'+n'+s'}{2}}\, \mathrm{e}^{\frac{\pi i (n+s)}{2}} P_{j,\frac{k}{2}-j'} \delta^{(2)}_{n+n'} \right)
\end{aligned}
\tag{C.4}
$$

D Formulae for the crosscap states

D.1 A-type

We compute the explicit coefficients of the A-type crosscap states

$$
|\mathscr{C}_{\bar{n},\bar{s}}\rangle = \sum_{(j,n,s)\in \mathrm{M}_k} \frac{P_{(0,\bar{n},\bar{s})(j,n,s)}}{\sqrt{S_{(0,0,0)(j,n,s)}}}\ |\mathscr{C}, (j,n,s)\rangle\rangle \tag{D.1}
$$

using the expressions for the P-matrix given in the previous section:

$$|\mathscr{C}_{\bar{n},\bar{s}}\rangle = \sqrt{\frac{2}{k+2}} \sum_{(j,n,s)\in \mathrm{M}_k} \frac{\sigma_{j,n,s}\sigma_{0,\bar{n},\bar{s}}}{\sqrt{\sin\pi\frac{2j+1}{k+2}}}\ \delta^{(2)}_{\bar{s}+s}\ \ \mathrm{e}^{\pi i \frac{\bar{n}n}{2(k+2)}}\ \ \mathrm{e}^{-\pi i\frac{\bar{s}s}{4}}$$
$$\left(\sin\left[\pi\tfrac{2j+1}{2(k+2)}\right]\delta^{(2)}_{2j+k}\delta^{(2)}_{\bar{n}+n+k} + (-1)^{\frac{2j+n+s}{2}+\frac{\bar{n}+\bar{s}}{2}}\cos\left[\pi\tfrac{2j+1}{2(k+2)}\right]\delta^{(2)}_{2j}\delta^{(2)}_{\bar{n}+n}\right)|\mathscr{C},(j,n,s)\rangle\!\rangle \tag{D.2}$$

Möbius strip

The Möbius strips are expressed using the Y-tensor,

$$\begin{aligned}
\langle\mathscr{B}_{j,n,s}|q_t^H|\mathscr{C}_{\bar{n},\bar{s}}\rangle &= \sum_{(j',n',s')\in\mathrm{M}_k} Y_{(j,-n,-s)(0,\bar{n},\bar{s})}^{\ \ (j',n',s')}\widehat{\chi}_{j',-n',-s'}(\tau) \\
&= \frac{1}{2}\sum_{(j',n',s')\colon\mathrm{even}}\sqrt{\frac{T_{0\bar{n}\bar{s}}}{T_{j'n's'}}}\widetilde{Y}_{(j,-n,-s)(0,\bar{n},\bar{s})}^{\ \ (j',n',s')}\widehat{\chi}_{j',-n',-s'}(\tau) \\
&= \sum_{(j',n',s')\colon\mathrm{even}}\sqrt{\frac{T_{0\bar{n}\bar{s}}}{T_{j'n's'}}}\widetilde{Y}_{j0}^{j'}\widetilde{\overline{Y}}{}^{n'}_{-n,\bar{n}}\widetilde{Y}^{s'}_{-s,\bar{s}}\widehat{\chi}_{j',-n',-s'}(\tau) \\
&= \sum_{(j',n',s')\colon\mathrm{even}}\frac{\sigma_{0\bar{n}\bar{s}}}{\sigma_{j'n's'}}Y_{j0}^{j'}\overline{Y}{}^{n'}_{-n,\bar{n}}Y^{s'}_{-s,\bar{s}}\widehat{\chi}_{j',-n',-s'}(\tau) \\
&= \sum_{(j',n',s')\colon\mathrm{even}}\frac{\sigma_{0\bar{n}\bar{s}}}{\sigma_{j'n's'}}\delta^{(2)}_{n'+\bar{n}}\delta^{(2)}_{s'+\bar{s}}\left(\delta^{(2k+4)}_{n,\frac{\widehat{\bar{n}}-\widehat{n'}}{2}} + (-1)^{\bar{n}+k}\delta^{(2k+4)}_{n,\frac{\widehat{\bar{n}}-\widehat{n'}}{2}+k+2}\right) \\
&\qquad\times\left(\delta^{(4)}_{s,\frac{\widehat{\bar{s}}-\widehat{s'}}{2}} + (-1)^{\bar{s}}\delta^{(4)}_{s,\frac{\widehat{\bar{s}}-\widehat{s'}}{2}+2}\right)(-1)^{2j+j'}N_{jj}^{j'}\widehat{\chi}_{j',-n',-s'}(\tau)
\end{aligned}$$

Simplifying the delta functions for the s-indices by

$$\delta^{(4)}_{s,\frac{\widehat{\bar{s}}-\widehat{s'}}{2}} + (-1)^{\bar{s}}\delta^{(4)}_{s,\frac{\widehat{\bar{s}}-\widehat{s'}}{2}+2} = (-1)^{\bar{s}\cdot\frac{s-\frac{\widehat{\bar{s}}-\widehat{s'}}{2}}{2}}\delta^{(4)}_{s',\bar{s}-2s},$$

we obtain the formula

$$\begin{aligned}
\langle\mathscr{B}_{j,n,s}|q_t^H|\mathscr{C}_{\bar{n},\bar{s}}\rangle &= \sum_{(j',n',s')\colon\mathrm{even}}\delta^{(2)}_{n'+\bar{n}}\left(\delta^{(2k+4)}_{n,\frac{\widehat{\bar{n}}-\widehat{n'}}{2}} + (-1)^{\bar{n}+k}\delta^{(2k+4)}_{n,\frac{\widehat{\bar{n}}-\widehat{n'}}{2}+k+2}\right)\delta^{(4)}_{s',\bar{s}-2s} \\
&\qquad\times(-1)^{2j+j'+\bar{s}\cdot\frac{s-\frac{\widehat{\bar{s}}-\widehat{s'}}{2}}{2}}N_{jj}^{j'}\frac{\sigma_{0\bar{n}\bar{s}}}{\sigma_{j'n's'}}\widehat{\chi}_{j',-n',-s'}(\tau).
\end{aligned} \tag{D.3}$$

Taking the complex conjugation, we also find

$$\langle\mathscr{C}_{\bar{n},\bar{s}}|q_t^H|\mathscr{B}_{j,n,s}\rangle = \sum_{(j',n',s')\colon\mathrm{even}}\delta^{(2)}_{n'+\bar{n}}\left(\delta^{(2k+4)}_{n,\frac{\widehat{\bar{n}}-\widehat{n'}}{2}} + (-1)^{\bar{n}+k}\delta^{(2k+4)}_{n,\frac{\widehat{\bar{n}}-\widehat{n'}}{2}+k+2}\right)\delta^{(4)}_{s',\bar{s}-2s}$$

$$\times(-1)^{2j+j'+\bar{s}\cdot\frac{s-\frac{\widehat{\bar{s}}-\widehat{s'}}{2}}{2}} N_{jj}^{j'} \frac{\sigma_{0\bar{n}\bar{s}}}{\sigma_{j'n's'}} \widehat{\chi}_{j',n',s'}(\tau). \tag{D.4}$$

One may further simplify the δ-function for n-indices. This leads to the following expression

$$\langle \mathscr{C}_{\bar{n},\bar{s}} | q_t^H | \mathscr{B}_{j,n,s} \rangle = \sum_{j' \in \mathrm{P}_k} N_{jj}^{j'} \delta^{(2k+4)}_{2n+n'-\bar{n}} \delta^{(4)}_{2s+s'-\bar{s}} \epsilon^{j,n,s}_{\bar{n},\bar{s}}(j',n',s') \widehat{\chi}_{j',n',s'}(\tau),$$

where

$$\epsilon^{j,n,s}_{\bar{n},\bar{s}}(j',n',s') := (-1)^{2j+j'+\bar{s}\cdot\frac{s-\frac{\widehat{\bar{s}}-\widehat{s'}}{2}}{2} + (\bar{n}+k)\frac{n-\frac{\widehat{\bar{n}}-\widehat{n'}}{2}}{k+2}} \frac{\sigma_{0\bar{n}\bar{s}}}{\sigma_{j'n's'}}.$$

D.2 B-type

The set of simple currents $(0,n,s)$ splits up into two orbits under the orbifold group $\mathbb{Z}_{k+2} \times \mathbb{Z}_2$. The first orbit is the one of $(0,0,0)$, which contains only currents $(0,n,s)$ with n,s even, the other orbit is the one of $(0,1,1)$, which contains only currents with n,s odd. Accordingly, there are two types of crosscap states. We first construct A-type crosscaps in the orbifold and then apply the mirror map. The crosscaps of the orbifold corresponding to the even orbit are

$$\left| \mathscr{C}_{P^{\theta_{rq}}_{(0,0,0)}} \right\rangle = \frac{1}{\sqrt{2(k+2)}} \sum_{n,s} \delta^{(2)}_n \; \delta^{(2)}_s \; \mathrm{e}^{-\pi i \theta_{rq}(n,s)} \, |\mathscr{C}_{n,s}\rangle \,, \tag{D.5}$$

where $\theta_{rq}(n,s) = -rn/(k+2) + qs/2$, as explained in the main text. The resulting crosscap states on the orbifold are

$$\left| \mathscr{C}_{P^{\theta_{rq}}_{(0,0,0)}} \right\rangle = 2 \sum_j \delta^{(2)}_{2j} \; (-1)^j \; (-1)^{\frac{\widehat{2r}}{2}+q} \; \frac{\cos\pi\frac{2j+1}{2(k+2)}}{\sqrt{\sin\pi\frac{2j+1}{k+2}}} \; \sigma_{j,-2r,-2q} \; |\mathscr{C}, j, -2r, -2q\rangle\!\rangle \tag{D.6}$$

Applying the mirror map, one obtains the B-type states

$$\left| \mathscr{C}^B_{P^{\theta_{rq}}_{(0,0,0)}} \right\rangle = 2 \sum_j \delta^{(2)}_{2j} \; (-1)^j \; (-1)^{\frac{\widehat{2r}}{2}+q} \; \frac{\cos\pi\frac{2j+1}{2(k+2)}}{\sqrt{\sin\pi\frac{2j+1}{k+2}}} \; \sigma_{j,-2r,-2q} \; |\mathscr{C}, j, 2r, 2q\rangle\!\rangle_B \tag{D.7}$$

The crosscaps of the orbifold corresponding to the odd currents are

$$\left| \mathscr{C}_{P^{\theta_{rq}}_{(0,1,1)}} \right\rangle = \frac{1}{\sqrt{2(k+2)}} \sum_{n,s} \delta^{(2)}_n \; \delta^{(2)}_s \; \mathrm{e}^{-\pi i (\theta_{rq}(n,s) - \hat{Q}_{(0,1,1)}(0,n,s))} \; |\mathscr{C}_{n+1,s+1}\rangle \tag{D.8}$$

We first compute

$$\mathrm{e}^{\pi i \hat{Q}_{(0,1,1)}(0,n,s)} = \mathrm{e}^{\pi i (\frac{n}{2(k+2)} - \frac{s}{4})} \; \mathrm{e}^{\pi i \frac{|n|}{2}} \; (-1)^{(k+1)\frac{n+1-(\widehat{n+1})}{2k+4}} \; \sigma_{0,n+1,s+1}$$

In the calculation, one has to replace various hatted sums by ordinary sums. One uses

$$\begin{aligned}
e^{\pi i \frac{(n\hat{+}1)n'}{2(k+2)}}\delta^{(2)}_{1+k+n+n'} &= \mathrm{e}^{\pi i \frac{(n+1)n'}{2(k+2)}}(-1)^{(k+1)\frac{n+1-(n\hat{+}1)}{2k+4}}\delta^{(2)}_{1+k+n+n'} \\
e^{\pi i \frac{(n\hat{+}1)n'}{2(k+2)}}(-1)^{\frac{n\hat{+}1}{2}}\delta^{(2)}_{1+n+n'} &= \mathrm{e}^{\pi i \frac{(n+1)n'}{2(k+2)}}(-1)^{\frac{n+1}{2}}(-1)^{(k+1)\frac{n+1-(n\hat{+}1)}{2k+4}}\delta^{(2)}_{1+n+n'} \\
e^{-\frac{(s\hat{+}1)s'}{4}}\delta^{(2)}_{s}\delta^{(2)}_{s+s'} &= \mathrm{e}^{-\frac{(s+1)s'}{4}}(-1)^{\frac{s}{2}}\ \delta^{(2)}_{s}\delta^{(2)}_{s+s'}
\end{aligned} \tag{D.9}$$

Inserting this, one obtains the following crosscap states for the orbifold theory

$$\begin{aligned}
\left|\mathscr{C}_{P^{\theta_{rq}}_{(0,1,1)}}\right\rangle &= 2\,\mathrm{e}^{i\omega_{(0,1,1)}}\sum_j \frac{\sigma_{j,-2r-1,-2q-1}}{\sqrt{\sin\pi\frac{2j+1}{k+2}}}\,\mathrm{e}^{-\pi i\frac{\widehat{2r+1}}{k+2}}\ \ \mathrm{e}^{\pi i\frac{\widehat{2q+1}}{4}}\ \ \delta^{(2)}_{2j} \\
&\qquad (-1)^{j+1}(-1)^{\frac{\widehat{2r+1}+2q+1}{2}}\cos\pi\frac{2j+1}{2(k+2)}|\mathscr{C},j,-2r-1,-2q-1\rangle\!\rangle
\end{aligned} \tag{D.10}$$

An application of the mirror map leads to the following odd B-type crosscap states

$$\begin{aligned}
\left|\mathscr{C}^B_{P^{\theta_{rq}}_{(0,1,1)}}\right\rangle &= 2\,\mathrm{e}^{i\omega_{(0,1,1)}}\sum_j \frac{\sigma_{j,-2r-1,-2q-1}}{\sqrt{\sin\pi\frac{2j+1}{k+2}}}\,\mathrm{e}^{-\pi i\frac{\widehat{2r+1}}{2(k+2)}}\ \ \mathrm{e}^{\pi i\frac{\widehat{2q+1}}{4}}\ \ \delta^{(2)}_{2j} \\
&\qquad (-1)^{j+1}(-1)^{\frac{\widehat{2r+1}+2q+1}{2}}\cos\pi\frac{2j+1}{2(k+2)}|\mathscr{C},j,2r+1,2q+1\rangle\!\rangle_B
\end{aligned} \tag{D.11}$$

One can now choose $\omega_{(0,1,1)}$ in such a way that the crosscap becomes real

$$e^{i\omega_{(0,1,1)}} = \mathrm{e}^{\pi i\frac{\widehat{2r+1}}{2(k+2)}}\ \ \mathrm{e}^{-\pi i\frac{\widehat{2q+1}}{4}}$$

At this point, it is convenient to relabel the states: We keep the label $(r,q)\in\mathbb{Z}_{k+2}\times\mathbb{Z}_2$ and introduce a new orbit-label $p\in 0,1$. $p=0$ refers to the orbit of $(0,0,0)$ and $p=1$ to the orbit of $(0,1,1)$. We can then give a closed formula for the crosscap states

$$|\mathscr{C}_{rqp}\rangle = (2(k+2))^{\frac{1}{4}}\sum_j \sigma_{j,-2r-p,-2q-p}\ \frac{P_{\frac{k}{2}j}}{\sqrt{S_{0j}}}(-1)^{\frac{\widehat{2r+p}-p}{2}+q}|\mathscr{C},j,2r+p,2q+p\rangle\!\rangle_B \tag{D.12}$$

Here, S and P are the modular matrices of $SU(2)$.

Möbius strip

We present the computation of the Möbius strip.

$$\begin{aligned}
&\langle\mathscr{C}_{rqp}|q_t^H|\mathscr{B}^B_{[j,s]}\rangle_{g_{4r+2p,2p}} \\
&= (-1)^{sq}\sqrt{2k+4}\sum_j \frac{S_{jj''}P_{\frac{k}{2}j''}}{S_{0j''}}(-1)^{\frac{\widehat{2r+p}-p}{2}+q}\sigma_{j'',-2r-p,-2q-p}\hat{\chi}_{j'',2r+p,2q+p}(\tau).
\end{aligned} \tag{D.13}$$

A modular transformation using the P-matrix as given in (C.4) yields

$$\begin{aligned}
&\langle \mathscr{C}_{rqp}| q_t^H |\mathscr{B}^B_{[j,s]}\rangle_{g_{4r+2p,2p}} \\
&= (-1)^{sq} \sum_{(j'n's')\in \mathrm{M}_k} \delta^{(2)}_{s'+p} \Big(Y^{\frac{k}{2}}_{jj'} (-1)^{\frac{\widehat{2r+p}-p}{2}+q} \delta^{(2)}_{n'+k+p} + (-1)^{\frac{2j'+n'+s'}{2}+p} Y^{\frac{k}{2}}_{j\frac{k}{2}-j'} \delta^{(2)}_{n'+p} \Big) \\
&\qquad\qquad \times \mathrm{e}^{\frac{\pi i(\widehat{2r+p})n'}{2(k+2)} - \frac{\pi i(\widehat{2q+p})s'}{4}} \sigma_{j',n',s'} \hat{\chi}_{j'n's'}(\tau) \\
&= (-1)^{sq} \sum_{(j'n's')\in \mathrm{M}_k} \delta^{(2)}_{s'+p} \Big(N^{\frac{k}{2}-j'}_{jj} (-1)^{\frac{\widehat{2r+p}-p}{2}+q} \delta^{(2)}_{n'+p+k} + (-1)^{\frac{2j'+n'+s'}{2}+p} N^{j'}_{jj} \delta^{(2)}_{n'+p} \Big) \\
&\qquad\qquad \times \mathrm{e}^{\frac{\pi i(\widehat{2r+p})n'}{2(k+2)} - \frac{\pi i(\widehat{2q+p})s'}{4}} \sigma_{j',n',s'} \hat{\chi}_{j'n's'}(\tau) \\
&= (-1)^{sq} \sum_{(j'n's'):\,\mathrm{even}} \delta^{(2)}_{s'+p} \delta^{(2)}_{n'+p} (-1)^{\frac{2j'+n'+s'}{2}+p} N^{j'}_{jj} e^{\frac{\pi i(\widehat{2r+p})n'}{2(k+2)} - \frac{\pi i(\widehat{2q+p})s'}{4}} \sigma_{j',n',s'} \hat{\chi}_{j'n's'}(\tau)
\end{aligned} \tag{D.14}$$

Here we have used some relations between $SU(2)$ the Y-matrix and the fusion rules: $Y^{\frac{k}{2}}_{jj'} = N^{j'}_{j,\frac{k}{2}-j} = N^{\frac{k}{2}-j'}_{jj}, Y^{j'}_{j,\frac{k}{2}-j'} = N^{j'}_{jj}$. To determine the Möbius strip involving the short orbit boundary states note that there is no overlap of the crosscap state with the extra RR-part of the boundary state. The formula presented in the main text is obtained from the second line of the above equation by inserting $N^{j'}_{\frac{k}{4}\frac{k}{4}} = N^{\frac{k}{2}-j'}_{\frac{k}{4}\frac{k}{4}} = 1$ for all $j' = 0, \ldots, \frac{k}{2}$.

E Supercurrent Conditions

In this appendix, we derive the supercurrent conditions obeyed by the A-type and B-type crosscaps of the superparafermion RCFT. The starting point is the conditions on the boundary states. The Cardy states $|\mathscr{B}_{j,n,s=\pm 1}\rangle$ obey the A-type supercurrent condition

$$\overline{\widetilde{G}}_{-r} - iG_r = \widetilde{G}_r - i\overline{G}_{-r} = 0, \tag{E.1}$$

where $r \in \mathbb{Z}$ (resp. $r \in \mathbb{Z}+\frac{1}{2}$) when they act on the RR-part (resp. NSNS-part) of the boundary state. This shows that the combinations of the Ishibashi states

$$|\mathscr{B}, j, n, s\rangle\!\rangle - |\mathscr{B}, j, n, s+2\rangle\!\rangle$$

obey the same condition (E.1) with $r \in \mathbb{Z}+\frac{1}{2}$ if $s = 0$ and $r \in \mathbb{Z}$ if $s = -1$. Then, the following combination of the crosscap Ishibashi states

$$\sqrt{T_{j,n,s}}|\mathscr{C}, j, n, s\rangle\!\rangle - \sqrt{T_{j,n,s+2}}|\mathscr{C}, j, n, s+2\rangle\!\rangle = \mathrm{e}^{\pi i L_0}\Big(|\mathscr{B}, j, n, s\rangle\!\rangle - |\mathscr{B}, j, n, s+2\rangle\!\rangle\Big)$$

obey the condition $\mathrm{e}^{\pi i L_0}\left(\widetilde{\overline{G}}_{-r} - iG_r\right)\mathrm{e}^{-\pi i L_0} = \mathrm{e}^{\pi i L_0}\left(\widetilde{G}_r - i\overline{G}_{-r}\right)\mathrm{e}^{-\pi i L_0} = 0$. This is nothing but the supercurrent condition on the $A_{0,0}$-parity for $s=-1$ and $A_{\frac{\pi}{2},\frac{\pi}{2}}$-parity for $s=0$. It is also easy to see that the other combination

$$\sqrt{T_{j,n,s}}|\mathscr{C},j,n,s\rangle\!\rangle + \sqrt{T_{j,n,s+2}}|\mathscr{C},j,n,s+2\rangle\!\rangle$$

obey the $A_{\pi,0}$-condition for $s=-1$ and the $A_{\frac{\pi}{2},-\frac{\pi}{2}}$-condition for $s=0$.

We now consider the PSS crosscaps

$$|\mathscr{C}_{j,n,s}\rangle = \sqrt{T_{j,n,s}}\sum_{j,n,s}\frac{Q_{j,n,s,}^{\;j',n',s'}}{\sqrt{S_{0,0,0}^{\;j',n',s'}}}\sqrt{T_{j',n',s'}}|\mathscr{C},j',n',s'\rangle\!\rangle.$$

Using the "factorized form" (C.1) of Q-matrix and the following property of $Q_{s,s'}$

$$\begin{pmatrix} Q_{0,0} & Q_{0,2} \\ Q_{2,0} & Q_{2,2}\end{pmatrix} \propto \begin{pmatrix} 1 & -i \\ -i & 1\end{pmatrix}, \qquad \begin{pmatrix} Q_{-1,-1} & Q_{-1,1} \\ Q_{1,-1} & Q_{1,1}\end{pmatrix} \propto \begin{pmatrix} -i & 1 \\ 1 & -i\end{pmatrix},$$

it is easy to see that the s' dependence of $Q_{j,n,s}^{\;j',n',s'} \mp Q_{j,n,s+2}^{\;j',n',s'}$ is proportional to 1 for s' and ∓ 1 for $s'+2$. This shows that

$$\frac{1}{\sqrt{T_{j,n,s}}}|\mathscr{C}_{j,n,s}\rangle \mp \frac{1}{\sqrt{T_{j,n,s+2}}}|\mathscr{C}_{j,n,s+2}\rangle$$

is spanned by $\sqrt{T_{j',n',s}}|\mathscr{C},j',n',s\rangle\!\rangle \mp \sqrt{T_{j',n',s+2}}|\mathscr{C},j',n',s+2\rangle\!\rangle$, which obey the supercurrent conditions of the types given above. This shows the first table in Section 4.5.1.

Next, let us apply the mirror map to the combinations of the crosscap Ishibashi states considered above. This shows that

$$\sqrt{T_{j,n,s}}|\mathscr{C},j,n,s\rangle\!\rangle_B \mp \sqrt{T_{j,n,s+2}}|\mathscr{C},j,n,s+2\rangle\!\rangle_B$$

obey for the $(-)$-sign the supercurrent conditions of the type $B_{0,0}$ (s odd) and $B_{\frac{\pi}{2},\frac{\pi}{2}}$ (s even), while they obey for the $(+)$-sign the conditions of the type $B_{0,\pi}$ (s odd) and $B_{\frac{\pi}{2},-\frac{\pi}{2}}$ (s even). Note that the B-type crosscaps that appear in (4.68) ((D.7) and (D.11) or their combined form (D.12)) has the following structure

$$|\mathscr{C}^B_{r,q,p}\rangle = \sum_{j\,\text{integer}} c_{j,r,p}\frac{(-1)^q}{\sqrt{T^{(2)}_{2q+p}}}\sqrt{T_{j,2r+p,2q+p}}|\mathscr{C},j,2r+p,2q+p\rangle\!\rangle_B,$$

where $c_{j,r,p}$ is a number depending on (j,r,p) only. This motivates us to find the combinations (4.79) which obey the supercurrent conditions of the types in the second table of Section 4.5.1.

F Normalization of RR Ground States

In this appendix, we review the Landau–Ginzburg computation of the overlaps of RR ground states and A-branes in $\mathcal{N}=2$ minimal model [22, 23]. This explains the choice of the normalization constant (6.19) for the ground state wavefunctions.

For the ground state wave functions, we use those of the form

$$\omega_{\frac{l}{2}} = c_{\frac{l}{2}}\,\mathrm{e}^{-i\overline{W}}\phi^l\mathrm{d}\phi + (\overline{Q}_+ + Q_-)(\cdots), \quad l=0,1,\ldots,k, \tag{F.1}$$

for some constant $c_{\frac{l}{2}}$. Their inner-products are computed as

$$\begin{aligned} g_{\frac{l}{2},\overline{\frac{l'}{2}}} &= \int \omega_{\frac{l}{2}}\wedge *\overline{\omega_{\frac{l'}{2}}} \\ &= c_{\frac{l}{2}}\overline{c_{\frac{l'}{2}}}\int \mathrm{e}^{-i(\phi^{k+2}+\overline{\phi}^{k+2})}\phi^l\overline{\phi}^{l'}\,\mathrm{d}\phi\wedge *\mathrm{d}\overline{\phi} \\ &= 2c_{\frac{l}{2}}\overline{c_{\frac{l'}{2}}}\int \mathrm{e}^{-2ir^{k+2}\cos((k+2)\theta)}\,\mathrm{e}^{i(l-l')\theta}r^{l+l'+1}\mathrm{d}r\mathrm{d}\theta, \end{aligned}$$

where we have used the polar coordinates $\phi = r\,\mathrm{e}^{i\theta}$. We see that they are orthogonal, $g_{\frac{l}{2},\overline{\frac{l'}{2}}}=0$ if $l\neq l'$. Expanding the exponential and performing the θ integral, we find

$$\begin{aligned} g_{\frac{l}{2},\overline{\frac{l}{2}}} &= 4\pi|c_{\frac{l}{2}}|^2\int_0^\infty \sum_{m=0}^\infty \frac{(-1)^m}{(m!)^2}(r^{k+2})^{2m}r^{2l+1}\mathrm{d}r \\ &= 4\pi|c_{\frac{l}{2}}|^2\int_0^\infty J_0(2r^{k+2})r^{2l+1}\mathrm{d}r \\ &= |c_{\frac{l}{2}}|^2\frac{2}{k+2}\left[\Gamma\left(\frac{l+1}{k+2}\right)\right]^2\sin\left(\frac{\pi(l+1)}{k+2}\right), \end{aligned}$$

where $J_0(x)=\sum_{m=0}^\infty(-1)^m(x/2)^{2m}/(m!)^2$ is the Bessel function, and we have used the integral formula $\int_0^\infty x^{\mu-1}J_0(ax)\mathrm{d}x = 2^{\mu-1}[\Gamma(\mu/2)]^2\sin(\pi\mu/2)/(\pi a^\mu)$. Thus, they form an orthonormal basis if $c_{\frac{l}{2}}$ are chosen as

$$c_{\frac{l}{2}} = \mathrm{e}^{i\gamma_l}\frac{\sqrt{k+2}}{\Gamma(\frac{l+1}{k+2})\sqrt{2\sin(\frac{\pi(l+1)}{k+2})}}, \tag{F.2}$$

where $\mathrm{e}^{i\gamma_l}$ is some phase.

Now, let us perform the overlap integrals. We consider the brane $A_{jns=1} = A_{a_+,a_-}$ where $a_\pm = \frac{\pi(n\pm 2j\pm 1)}{k+2}$. It is a broken line coming from the infinity in the direction $z_i = \mathrm{e}^{\frac{\pi i(n-2j-1)}{k+2}}$, cornering at $\phi=0$, and then going out to infinity in the direction

$z_f = \mathrm{e}^{\frac{\pi i(n+2j+1)}{k+2}}$. The overlap is

$$\begin{aligned}
\Pi^{A_{jn1}}_{\frac{l}{2}} &= c_{\frac{l}{2}} \int_{A^-_{jn1}} \mathrm{e}^{-i\phi^{k+2}} \phi^l \mathrm{d}\phi \\
&= c_{\frac{l}{2}} (z_f^{l+1} - z_i^{l+1}) \int_0^{+\infty - i\cdot 0} \mathrm{e}^{-it^{k+2}} t^l \mathrm{d}t \\
&= c_{\frac{l}{2}} \frac{1}{k+2} \mathrm{e}^{\frac{\pi i(n-1/2)(l+1)}{k+2}} 2i \sin\left[\frac{\pi(2j+1)(l+1)}{k+2}\right] \Gamma\left(\frac{l+1}{k+2}\right)
\end{aligned}$$

and

$$\begin{aligned}
\widetilde{\Pi}^{A_{jn1}}_{\frac{l}{2}} &= \overline{c_{\frac{l}{2}}} \int_{A^+_{jn1}} \mathrm{e}^{-i\overline{\phi}^{k+2}} \overline{\phi}^l * \mathrm{d}\overline{\phi} \\
&= i\,\overline{c_{\frac{l}{2}}} \overline{\int_{A^+_{jn1}} \mathrm{e}^{i\phi^{k+2}} \phi^l \mathrm{d}\phi} \\
&= i\,\overline{c_{\frac{l}{2}}} \frac{1}{k+2} \mathrm{e}^{\frac{-\pi i(n+1/2)(l+1)}{k+2}} (-2i) \sin\left[\frac{\pi(2j+1)(l+1)}{k+2}\right] \Gamma\left(\frac{l+1}{k+2}\right)
\end{aligned}$$

Using the normalization formula (F.2) with the phase choice $\mathrm{e}^{i\gamma_l} = -i\,\mathrm{e}^{\frac{-\pi i(l+1)}{2(k+2)}}$, we find that the above reproduce the known formula for the overlaps:

$$\Pi^{A_{jn1}}_{\frac{l}{2}} = {}_{\mathrm{RR}}\langle \mathscr{B}_{j,n-1,(0)} | \tfrac{l}{2} \rangle_{\mathrm{RR}}, \quad \widetilde{\Pi}^{A_{jn1}}_{\frac{l}{2}} = {}_{\mathrm{RR}}\langle \tfrac{l}{2} | \mathscr{B}_{j,n,(1)} \rangle_{\mathrm{RR}}. \tag{F.3}$$

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