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ORLICZ CENTROID BODIES

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Abstract

The sharp affine isoperimetric inequality that bounds the volume of the centroid body of a star body (from below) by the volume of the star body itself is the Busemann-Petty centroid inequality. A decade ago, the L_p analogue of the classical Busemann-Petty centroid inequality was proved. Here, the definition of the centroid body is extended to an Orlicz centroid body of a star body, and the corresponding analogue of the Busemann-Petty centroid inequality is established for convex bodies.

The centroids of the intersections of an origin-symmetric body with half-spaces form the surface of a convex body. This "centroid body" is a concept that dates back at least to Dupin.

The classical affine isoperimetric inequality that relates the volume of a convex body with that of its centroid body was conjectured by Blaschke and established in a landmark work by Petty [53]. Petty's inequality became known as the Busemann-Petty centroid inequality because, in establishing his inequality, Petty not only made critical use of Busemann's random simplex inequality, but as Petty stated, he "reinterpreted" it. (See, e.g., the books by Gardner [12], Leichtweiss [25], Schneider [55], and Thompson [58] for reference.)

The concept of a centroid body had a natural extension in what became known as the L_p Brunn-Minkowski theory and its dual. This theory had its origins in the early 1960s when Firey (see, e.g., Schneider [55]) introduced his concept of L_p compositions of convex bodies. Three decades later, in [34] and [35] these *Firey-Minkowski L_p combinations* were shown to lead to an embryonic L_p Brunn-Minkowski theory. This theory (and its dual) has witnessed a rapid growth. (See, e.g., [1–9, 17–23, 26–32, 34–44, 46–48, 54, 56, 57, 59, 62].)

The L_p analogues of centroid bodies became a central focus within the L_p Brunn-Minkowski theory and its dual and establishing the L_p-analogue of the Busemann-Petty centroid inequality became a major goal. This was accomplished by the authors of the present paper in [37] with an independent approach presented by Campi and Gronchi [3].

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The L_p centroid bodies quickly became objects of interest in asymptotic geometric analysis (see, e.g., [10], [11], [24], [49], [50], [51], [52]) and even the theory of stable distributions (see, e.g., [48]).

Using concepts introduced by Ludwig [29], Haberl and Schuster [21] were led to establish "asymmetric" versions of the L_p Busemann-Petty centroid inequality that, for bodies that are not origin-symmetric, are stronger than the L_p Busemann-Petty centroid inequality presented in [37] and [3]. The "asymmetric" inequalities obtained by Haberl and Schuster are ideally suited for non-symmetric bodies. This can be seen by looking at the Haberl-Schuster version of the L_p analogue of the classical Blaschke-Santaló inequality that was presented in [46]. While for origin symmetric bodies, the L_p extension of [46] does recover the original Blaschke-Santaló inequality as $p \to \infty$, for arbitrary bodies only the Haberl-Schuster version does so.

The works of Haberl and Schuster [21] (see also [22]), Ludwig and Reitzner [32], and Ludwig [31] have demonstrated the clear need to move beyond the L_p Brunn-Minkowski theory to what we are calling an Orlicz Brunn-Minkowski theory. This need is not only motivated by compelling geometric considerations (such as those presented in Ludwig and Reitzner [32]), but also by the desire to obtain Sobolev bounds (see [22]) of a far more general nature.

This paper is the second in a series intended to develop a few of the elements of an Orlicz Brunn-Minkowski theory and its dual. Here we define the Orlicz centroid body, establish some of its basic properties, and most importantly establish (what we call) the Orlicz Busemann-Petty centroid inequality (for Orlicz centroid bodies).

We consider convex $\phi : \mathbb{R} \to [0, \infty)$ such that $\phi(0) = 0$. This means that ϕ must be decreasing on $(-\infty, 0]$ and increasing on $[0, \infty)$. We require that one of these is happening strictly so; i.e., ϕ is either strictly decreasing on $(-\infty, 0]$ or strictly increasing on $[0, \infty)$. The class of such ϕ will be denoted by $\mathcal{C}$.

If K is a star body (see Section 1 for precise definitions) with respect to the origin in $\mathbb{R}^n$ with volume $|K|$, and $\phi \in \mathcal{C}$ then we define the *Orlicz centroid body* $\Gamma_\phi K$ of K as the convex body whose support function at $x \in \mathbb{R}^n$ is given by

$$h(\Gamma_\phi K; x) = \inf \left\{ \lambda > 0 : \frac{1}{|K|} \int_K \phi\left(\frac{x \cdot y}{\lambda}\right) \, dy \ \leq 1 \right\},$$

where $x \cdot y$ denotes the standard inner product of x and y in $\mathbb{R}^n$ and the integration is with respect to Lebesgue measure in $\mathbb{R}^n$.

When $\phi_p(t) = |t|^p$, with $p \geq 1$, then

$$\Gamma_{\phi_p} K = \Gamma_p \, K,$$

where $\Gamma_p K$ is the L_p centroid body of K, whose support function is given by

$$h(\Gamma_p K; x)^p = \frac{1}{|K|} \int_K |x \cdot y|^p \, dy.$$

For $p = 1$ the body $\Gamma_p K$ is the classical centroid body, ΓK, of K.

We will establish the following affine isoperimetric inequality for Orlicz centroid bodies.

Theorem. *If $\phi \in \mathcal{C}$ and K is a convex body in $\mathbb{R}^n$ that contains the origin in its interior, then the volume ratio*

$$|\Gamma_\phi K|/|K|$$

is minimized if and only if K is an ellipsoid centered at the origin.

The theorem contains as a special case the classical Busemann-Petty centroid inequality for convex bodies [53], as well as the L_p Busemann-Petty centroid inequality for convex bodies that was established in [37] and [3] and the "asymmetric" version of the L_p Busemann-Petty centroid inequality for convex bodies that was established by Haberl and Schuster [21].

For quick later reference, we list in Section 1 some basic, and for the most part well-known, facts regarding convex bodies. The basic properties of the Orlicz centroid operator are developed in Section 2. In Section 3 the Theorem is established. Section 4 concludes with some open problems.

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1. Basics regarding convex and star bodies

The setting will be Euclidean n-space $\mathbb{R}^n$. We write $e_1, \ldots, e_n$ for the standard orthonormal basis of $\mathbb{R}^n$ and when we write $\mathbb{R}^n = \mathbb{R}^{n-1} \times \mathbb{R}$ we always assume that e_n is associated with the last factor.

We will attempt to use x, y for vectors in $\mathbb{R}^n$ and x', y' for vectors in $\mathbb{R}^{n-1}$. We will also attempt to use a, b, s, t for numbers in $\mathbb{R}$ and c, λ for strictly positive reals. If Q is a Borel subset of $\mathbb{R}^n$ and Q is contained in an i-dimensional affine subspace of $\mathbb{R}^n$ but in no affine subspace of lower dimension, then $|Q|$ will denote the i-dimensional Lebesgue measure of Q. If $x \in \mathbb{R}^n$ then by abuse of notation we will write $|x|$ for the norm of x.

For $A \in \mathrm{GL}(n)$ write A^t for the transpose of A and A^{-t} for the inverse of the transpose (contragradient) of A. Write $|A|$ for the absolute value of the determinant of A.

We say that a sequence $\{\phi_i\}$, of $\phi_i \in \mathcal{C}$, is such that $\phi_i \to \phi_o \in \mathcal{C}$ provided

$$|\phi_i - \phi_o|_I := \max_{t \in I} |\phi_i(t) - \phi_o(t)| \longrightarrow 0,$$

for every compact interval $I \subset \mathbb{R}$. For $\phi \in \mathcal{C}$ define $\phi^\star \in \mathcal{C}$ by

$$\phi^\star(t) = \int_0^1 \phi(ts)\, ds^n, \tag{1.1}$$

where $ds^n = ns^{n-1}ds$. Obviously, $\phi_i \to \phi_o \in \mathcal{C}$ implies $\phi_i^\star \to \phi_o^\star$.

Associated with each $\phi \in \mathcal{C}$ is $c_\phi \in (0,\infty)$ defined by

$$c_\phi = \min\{c > 0 : \max\{\phi(c), \phi(-c)\} \geq 1\}.$$

Let $\rho(K;\cdot) = \rho_K : \mathbb{R}^n \setminus \{0\} \to [0,\infty)$ denote the radial function of the set $K \subset \mathbb{R}^n$, star-shaped about the origin; i.e. $\rho(K,x) = \rho_K(x) = \max\{\lambda > 0 : \lambda x \in K\}$. If ρ_K is strictly positive and continuous, then we call K a star body and we denote the class of star bodies in $\mathbb{R}^n$ by $\mathcal{S}_o^n$. If $c > 0$, then obviously for the dilate $cK = \{cx : x \in K\}$ we have

$$\rho_{cK} = c\rho_K. \tag{1.2}$$

The *radial distance* between $K, L \in \mathcal{S}_o^n$ is

$$|\rho_K - \rho_L|_\infty = \max_{u \in S^{n-1}} |\rho_K(u) - \rho_L(u)|.$$

Let $h(K;\cdot) = h_K : \mathbb{R}^n \to \mathbb{R}$ denote the *support function* of the convex body (compact convex) $K \subset \mathbb{R}^n$; i.e., $h(K;x) = \max\{x \cdot y : y \in K\}$. The *Hausdorff distance* between the convex bodies K and L is

$$|h_K - h_L|_\infty = \max_{u \in S^{n-1}} |h_K(u) - h_L(u)|.$$

We write $\mathcal{K}^n$ for the space of convex bodies of $\mathbb{R}^n$. We write $\mathcal{K}_o^n$ for the set of convex bodies that contain the origin in their interiors. On $\mathcal{K}_o^n$ the Hausdorff metric and the radial metric induce the same topology.

We shall require the obvious facts that for $K, L \in \mathcal{K}^n$, we have

$$K \subset L \qquad \text{if and only if} \qquad h_K \leq h_L, \tag{1.3}$$

and that for $c > 0$ and $x \in \mathbb{R}^n$,

$$h_{cK}(x) = ch_K(x) \qquad \text{and} \qquad h_K(cx) = ch_K(x). \tag{1.4}$$

More generally, from the definition of the support function it follows immediately that for $A \in \mathrm{GL}(n)$ the support function of the image $AK = \{Ay : y \in K\}$ of K is given by

$$h_{AK}(x) = h_K(A^t x). \tag{1.5}$$

For $K \in \mathcal{S}_o^n$, define the real numbers R_K and r_K by

$$R_K = \max_{u \in S^{n-1}} \rho_K(u) \qquad \text{and} \qquad r_K = \min_{u \in S^{n-1}} \rho_K(u). \tag{1.6}$$

Note that the definition of $\mathcal{S}_o^n$ is such that $0 < r_K \leq R_K < \infty$, for all $K \in \mathcal{S}_o^n$.

Throughout, $B = \{x \in \mathbb{R}^n : |x| \le 1\}$ will denote the unit ball centered at the origin, and $\omega_n = |B|$ will denote its n-dimensional volume. We shall make use of the trivial fact that for $u_o \in S^{n-1}$,

$$\omega_{n-1} = \int_{S^{n-1}} (u_o \cdot u)_+ \, dS(u) = \frac{1}{2} \int_{S^{n-1}} |u_o \cdot u| \, dS(u), \tag{1.7}$$

where $(t)_+ = \max\{t, 0\}$ for $t \in \mathbb{R}$, and where S denotes Lebesgue measure on S^{n-1}; i.e., S is $(n-1)$-dimensional Hausdorff measure.

For a convex body K and a direction $u \in S^{n-1}$, let K_u denote the image of the orthogonal projection of K onto $u^\perp$, the subspace of $\mathbb{R}^n$ orthogonal to u. We write $\underline{\ell}_u(K;\cdot) : K_u \to \mathbb{R}$ and $\overline{\ell}_u(K;\cdot) : K_u \to \mathbb{R}$ for the *undergraph* and *overgraph functions* of K in the direction u; i.e.

$$K = \left\{ y' + tu : \ -\underline{\ell}_u(K;y') \le t \le \overline{\ell}_u(K;y') \text{ for } y' \in K_u \right\}.$$

Thus the *Steiner symmetral* $\mathrm{S}_u K$ of $K \in \mathcal{K}_o^n$ in direction u can be defined as the body whose orthogonal projection onto $u^\perp$ is identical to that of K and whose undergraph and overgraph functions are given by

$$\underline{\ell}_u(\mathrm{S}_u K; y') = \frac{1}{2}[\underline{\ell}_u(K;y') + \overline{\ell}_u(K;y')] \tag{1.8a}$$

and

$$\overline{\ell}_u(\mathrm{S}_u K; y') = \frac{1}{2}[\underline{\ell}_u(K;y') + \overline{\ell}_u(K;y')]. \tag{1.8b}$$

For $y' \in K_u$, define $m_{y'} = m_{y'}(u)$ by

$$m_{y'}(u) = \frac{1}{2}[\overline{\ell}_u(K;y') - \underline{\ell}_u(K;y')]$$

so that the midpoint of the chord $K \cap (y' + \mathbb{R}u)$ is $y' + m_{y'}(u)u$. The length $|K \cap (y' + \mathbb{R}u)|$ of this chord will be denoted by $\sigma_{y'} = \sigma_{y'}(u)$. Note that the midpoints of the chords of K in the direction u lie in a subspace if and only if there exists an $x'_o \in K_u$ such that

$$x'_o \cdot y' = m_{y'}, \qquad \text{for all } y' \in K_u.$$

In this case $\{y' - \underline{\ell}_u(K;y')u : y' \in \operatorname{relint} K_u\}$, the undergraph of K with respect to u, is mapped into the overgraph by the linear transformation

$$y' + tu \ \longmapsto \ y' + [2(x'_o \cdot y') - t]u.$$

A classical characterization of the ellipsoid is the following: A convex body $K \in \mathcal{K}_o^n$ is an origin centered ellipsoid if and only if for each direction $u \in S^{n-1}$ all of the midpoints of the chords of K parallel to u lie in a subspace of $\mathbb{R}^n$. Gruber [15] showed how the following Lemma is a consequence of the Gruber-Ludwig theorem [16]

Lemma 1.1. *A convex body $K \in \mathcal{K}_o^n$ is an origin centered ellipsoid if and only if there exists an $\varepsilon_K > 0$ such that for each direction $u \in S^{n-1}$ all of the chords of K that come within a distance of ε_K of the origin and are parallel to u, have midpoints that lie in a subspace of $\mathbb{R}^n$.*

When considering the convex body $K \subset \mathbb{R}^{n-1} \times \mathbb{R}$, for $(x', t) \in \mathbb{R}^{n-1} \times \mathbb{R}$ we will usually write $h(K; x', t)$ rather than $h(K; (x', t))$.

The following is well known:

Lemma 1.2. *Suppose $K \in \mathcal{K}_o^n$ and $u \in S^{n-1}$. For $y' \in \operatorname{relint} K_u$, the overgraph and undergraph functions of K in direction u are given by*

$$\overline{\ell}_u(K; y') = \min_{x' \in u^\perp} \left\{ h(K; x', 1) - x' \cdot y' \right\}, \tag{1.9a}$$

and

$$\underline{\ell}_u(K; y') = \min_{x' \in u^\perp} \left\{ h(K; x', -1) - x' \cdot y' \right\}. \tag{1.9b}$$

See [3] for an application of (1.9a) and (1.9b) to the proof of the L_p Busemann-Petty centroid inequality.

Proof. Suppose $x' \in u^\perp$. From the definition of the overgraph it follows immediately that $y' + \overline{\ell}_u(K; y')u \in K$, and thus the definition of the support function shows that

$$(y' + \overline{\ell}_u(K; y')u) \cdot (x' + u) \leq h_K(x' + u).$$

Thus,

$$x' \cdot y' + \overline{\ell}_u(K; y') \leq h_K(x' + u) = h(K; x', 1),$$

for all $x' \in u^\perp$.

Since K has a support hyperplane at $y' + \overline{\ell}_u(K; y')u \in \partial K$, and since $y' \in \operatorname{relint} K_u$, there exists a vector of the form $x'_o + u$, with $x'_o \in u^\perp$, so that

$$(y' + \overline{\ell}_u(K; y')u) \cdot (x'_o + u) = h_K(x'_o + u) = h(K; x'_o, 1).$$

Therefore,

$$\overline{\ell}_u(K; y') = \min_{x' \in u^\perp} \left\{ h(K; x', 1) - x' \cdot y' \right\}.$$

Formula (1.9b) can be shown in the same way. q.e.d.

We shall require the following crude estimate.

Lemma 1.3. *Suppose $K \in \mathcal{K}_o^n$ and $u \in S^{n-1}$. If $y' \in (r_K/2)B \cap u^\perp$, and $x'_1, x'_2 \in u^\perp$ are such that*

$$\overline{\ell}_u(K; y') = h(K; x'_1, 1) - x'_1 \cdot y'$$

and

$$\underline{\ell}_u(K; y') = h(K; x'_2, -1) - x'_2 \cdot y',$$

then both

$$|x'_1|, |x'_2| \leq \frac{2R_K}{r_K}.$$

Proof. Note that since $y' \in (r_K/2)B \cap u^\perp$, it follows that both $\overline{\ell}_u(K; y') > 0$ and $\underline{\ell}_u(K; y') > 0$.

Observe that since $r_K B \subset K$, we have from (1.3),

$$h_K\left(\frac{(x_1', 1)}{\sqrt{1+|x_1'|^2}}\right) \geq r_K. \tag{1.10}$$

From the fact that K contains the origin in its interior, the definition of R_K, the hypothesis, (1.10) and (1.4), we have

$$\begin{aligned} R_K &\geq \overline{\ell}_u(K; y') \\ &= h(K; x_1', 1) - x_1' \cdot y' \\ &\geq r_K(1 + |x_1'|^2)^{1/2} - x_1' \cdot y' \\ &\geq r_K|x_1'| - x_1' \cdot y' \\ &\geq r_K|x_1'| - \frac{r_K}{2}|x_1'|, \end{aligned}$$

where the last step comes from the hypothesis that $|y'| \leq r_K/2$.

The estimate for $|x_2'|$ can be established in the identical manner.

q.e.d.

2. Definition and basic properties of Orlicz centroid bodies

If $\phi \in \mathcal{C}$, then the Orlicz centroid body $\Gamma_\phi K$ of $K \in \mathcal{S}_o^n$ is defined as the body whose support function is given by

$$h_{\Gamma_\phi K}(x) = \inf\left\{\lambda > 0 : \frac{1}{|K|}\int_K \phi\left(\frac{x \cdot y}{\lambda}\right) dy \leq 1\right\}, \tag{2.1}$$

where the integration is with respect to Lebesgue measure on $\mathbb{R}^n$. Observe that since $\lim_{s\to\infty}\phi(s) = \infty$ or $\lim_{s\to-\infty}\phi(s) = \infty$ we have $h_{\Gamma_\phi K}(x) > 0$ whenever $x \neq 0$.

It will be helpful to also use the alternate definition:

$$h_{\Gamma_\phi K}(x) = \inf\left\{\lambda > 0 : \int_{S^{n-1}} \phi^\star\left(\tfrac{1}{\lambda}(x \cdot u)\rho_K(u)\right) dV_K^*(u) \ \leq 1\right\}, \tag{2.2}$$

where $\phi^\star$ is defined by (1.1) and dV_K^* is the volume-normalized dual conical measure of K, defined by

$$|K| dV_K^* = \frac{1}{n}\rho_K^n \, dS,$$

where S is Lebesgue measure on S^{n-1} (i.e., $(n-1)$-dimensional Hausdorff measure). We shall make use of the fact that the volume-normalized dual conical measure

$$V_K^* \text{ is a probability measure on } S^{n-1}. \tag{2.3}$$

The equivalence of the two definitions is a consequence of the fact that

$$\int_K \phi\left(\frac{1}{\lambda}x \cdot y\right)\, dy = \frac{1}{n}\int_{S^{n-1}} \phi^\star\left(\frac{1}{\lambda}(x\cdot v)\rho_K(v)\right)\rho_K(v)^n\, dS(v). \tag{2.4}$$

To see (2.4) observe that:

$$\begin{aligned}
&\int_K \phi\left(\frac{1}{\lambda}x \cdot y\right)\, dy\\
&= \int_{S^{n-1}}\int_0^{\rho_K(v)} \phi\left(\frac{1}{\lambda}(x\cdot v)r\right) r^{n-1}\, dr\, dS(v)\\
&= \int_{S^{n-1}}\left(\int_0^1 \phi\left(\frac{1}{\lambda}(x\cdot v)\rho_K(v)t\right) t^{n-1}\, dt\right)\rho_K(v)^n dS(v)\\
&= \frac{1}{n}\int_{S^{n-1}} \phi^\star\left(\frac{1}{\lambda}(x\cdot v)\rho_K(v)\right)\rho_K(v)^n\, dS(v).
\end{aligned}$$

Since $\phi^\star$ is strictly increasing on $[0,\infty)$ or strictly decreasing on $(-\infty, 0]$ it follows that the function

$$\lambda \longmapsto \int_{S^{n-1}} \phi^\star(\tfrac{1}{\lambda}(x\cdot v)\rho_K(v))\, dV_K^*(v)$$

is strictly decreasing in $(0,\infty)$. It is also continuous. Thus, we have:

Lemma 2.1. *Suppose $K \in \mathcal{S}_o^n$ and $u_o \in S^{n-1}$. Then*

$$\int_{S^{n-1}} \phi^\star\left(\tfrac{1}{\lambda_o}(u_o\cdot v)\rho_K(v)\right) dV_K^*(v) \; = 1$$

if and only if

$$h_{\Gamma_\phi K}(u_o) \;=\; \lambda_o.$$

Observe that (1.4) now shows that Lemma 2.1 holds for all $u_o \in \mathbb{R}^n \setminus \{0\}$.

We now demonstrate that (2.1) defines a convex body that contains the origin in its interior.

Lemma 2.2. *If $K \in \mathcal{S}_o^n$ then $h_{\Gamma_\phi K}$ is the support function of a body in $\mathcal{K}_o^n$.*

Proof. Observe that it follows immediately from definition (2.1) that for $x \in \mathbb{R}^n$ and $c > 0$,

$$h_{\Gamma_\phi K}(cx) = c\, h_{\Gamma_\phi K}(x).$$

We show that indeed for nonzero $x_1, x_2 \in \mathbb{R}^n$,

$$h_{\Gamma_\phi K}(x_1 + x_2) \;\le\; h_{\Gamma_\phi K}(x_1) + h_{\Gamma_\phi K}(x_2).$$

To that end let $h_{\Gamma_\phi K}(x_i) = \lambda_i$; i.e.,

$$\int_{S^{n-1}} \phi^\star\left(\frac{x_i\cdot u}{\lambda_i}\rho_K(u)\right) dV_K^*(u) = 1. \tag{2.5}$$

The convexity of the function $s \mapsto \phi^\star(s\,\rho_K(u))$ shows that

$$\phi^\star\left(\frac{x_1\cdot u + x_2\cdot u}{\lambda_1+\lambda_2}\,\rho_K(u)\right) \le \frac{\lambda_1}{\lambda_1+\lambda_2}\,\phi^\star\left(\frac{x_1\cdot u}{\lambda_1}\,\rho_K(u)\right) \;+\; \frac{\lambda_2}{\lambda_1+\lambda_2}\,\phi^\star\left(\frac{x_2\cdot u}{\lambda_2}\,\rho_K(u)\right).$$

Integrating both sides of this inequality with respect to the measure V_K^* and using (2.5) gives

$$\int_{S^{n-1}} \phi^\star\left(\frac{(x_1+x_2)\cdot u}{\lambda_1+\lambda_2}\,\rho_K(u)\right) dV_K^*(u) \;\le\; 1,$$

which, using (2.2), gives the desired result that

$$h_{\,\Gamma_\phi K}(x_1+x_2) \;\le\; \lambda_1+\lambda_2.$$

Thus $h_{\,\Gamma_\phi K}$ is indeed the support function of a compact convex set, and since $h_{\,\Gamma_\phi K} > 0$ on $\mathbb{R}^n\backslash\{0\}$, we see that $\Gamma_\phi K \in \mathcal{K}_o^n$. q.e.d.

We shall require more than $h_{\,\Gamma_\phi K} > 0$. Specifically,

Lemma 2.3. *If $K \in \mathcal{S}_o^n$, then*

$$\frac{\omega_{n-1} r_K^{n+1}}{n c_{\phi^\star}\,|K|} \;\le\; h_{\,\Gamma_\phi K}(u) \;\le\; \frac{R_K}{c_{\phi^\star}},$$

for all $u \in S^{n-1}$.

Proof. Suppose $u_o \in S^{n-1}$ and let $h_{\,\Gamma_\phi K}(u_o) = \lambda_o$; i.e.

$$\int_{S^{n-1}} \phi^\star\left(\frac{u_o\cdot u}{\lambda_o}\,\rho_K(u)\right)\frac{\rho_K(u)^n\,dS(u)}{n\,|K|} \;=\; 1\;. \tag{2.6}$$

To obtain the lower estimate we proceed as follows. From the definition of $c_{\phi^\star}$, either $\phi^\star(c_{\phi^\star}) = 1$ or $\phi^\star(-c_{\phi^\star}) = 1$. Suppose $\phi^\star(c_{\phi^\star}) = 1$. Then from the fact that $\phi^\star$ is non-negative and $\phi^\star(0) = 0$, Jensen's inequality, and definition (1.6) together with the fact that $\phi^\star$ is monotone increasing on $[0,\infty)$ and (1.7), we have

$$\begin{aligned}
\phi^\star(c_{\phi^\star}) &= 1\\
&= \int_{S^{n-1}} \phi^\star\left(\frac{u_o\cdot u}{\lambda_o}\,\rho_K(u)\right)\frac{\rho_K(u)^n\,dS(u)}{n|K|}\\
&\ge \int_{S^{n-1}} \phi^\star\left(\frac{(u_o\cdot u)_+}{\lambda_o}\,\rho_K(u)\right)\frac{\rho_K(u)^n\,dS(u)}{n|K|}\\
&\ge \phi^\star\left(\frac{1}{n}\int_{S^{n-1}}\frac{(u_o\cdot u)_+}{\lambda_o}\,\frac{\rho_K(u)^{n+1}\,dS(u)}{|K|}\right)\\
&\ge \phi^\star\left(\frac{\omega_{n-1} r_K^{n+1}}{n\lambda_o\,|K|}\right).
\end{aligned}$$

Since $\phi^\star$ is monotone increasing on $[0,\infty)$, from this we obtain the lower bound for $h_{\Gamma_\phi K}$:

$$\frac{\omega_{n-1} r_K^{n+1}}{n c_{\phi^\star} |K|} \ \le\ \lambda_o.$$

The case where $\phi^\star(-c_{\phi^\star}) = 1$ is handled the same way and gives the same result.

To obtain the upper estimate, observe that from the definition of $c_{\phi^\star}$, the fact that $\phi^\star$ is monotone decreasing on $(-\infty, 0]$ and monotone increasing on $[0,\infty)$, together with the fact that the function $t \mapsto \max\{\phi^\star(t), \phi^\star(-t)\}$ is monotone increasing on $[0,\infty)$, definition (1.6), and finally (2.3) it follows that

$$\begin{aligned}
&\max\{\phi^\star(c_{\phi^\star}), \phi^\star(-c_{\phi^\star})\}\\
&\qquad = 1\\
&\qquad = \int_{S^{n-1}} \phi^\star\left(\frac{u_o \cdot u}{\lambda_o}\rho_K(u)\right) dV_K^*(u)\\
&\qquad \le \int_{S^{n-1}} \max\{\phi^\star\left(\frac{|u_o \cdot u|\rho_K(u)}{\lambda_o}\right), \phi^\star\left(-\frac{|u_o \cdot u|\rho_K(u)}{\lambda_o}\right)\} \, dV_K^*(u)\\
&\qquad \le \int_{S^{n-1}} \max\{\phi^\star(\rho_K(u)/\lambda_o), \phi^\star(-\rho_K(u)/\lambda_o)\} \, dV_K^*(u)\\
&\qquad \le \int_{S^{n-1}} \max\{\phi^\star(R_K/\lambda_o), \phi^\star(-R_K/\lambda_o)\} \, dV_K^*(u)\\
&\qquad = \max\{\phi^\star(R_K/\lambda_o), \phi^\star(-R_K/\lambda_o)\}.
\end{aligned}$$

But the even function $t \mapsto \max\{\phi^\star(t), \phi^\star(-t)\}$ is monotone increasing on $[0,\infty)$ so we conclude

$$\lambda_o \ \le\ \frac{R_K}{c_{\phi^\star}}.$$

q.e.d.

For $c > 0$, an immediate consequence of definition (2.2) and (1.2) is the fact that

$$\Gamma_\phi c K \ =\ c\,\Gamma_\phi K. \tag{2.7}$$

We next show that the Orlicz centroid operator $\Gamma_\phi : \mathcal{S}_o^n \to \mathcal{S}_o^n$ is continuous.

Lemma 2.4. *Suppose $\phi \in \mathcal{C}$. If $K_i \in \mathcal{S}_o^n$ and $K_i \to K \in \mathcal{S}_o^n$, then $\Gamma_\phi K_i \to \Gamma_\phi K$.*

Proof. Suppose $u_o \in S^{n-1}$. We will show that

$$h_{\Gamma_\phi K_i}(u_o) \to h_{\Gamma_\phi K}(u_o).$$

Let

$$h_{\Gamma_\phi K_i}(u_o) = \lambda_i,$$

and note that Lemma 2.3 gives

$$\frac{\omega_{n-1} r_{K_i}^{n+1}}{n c_{\phi^\star} |K_i|} \ \le\ \lambda_i \ \le\ \frac{R_{K_i}}{c_{\phi^\star}}.$$

Since $K_i \to K \in \mathcal{S}_o^n$, we have $r_{K_i} \to r_K > 0$ and $R_{K_i} \to R_K < \infty$, and thus there exist a, b such that $0 < a \le \lambda_i \le b < \infty$, for all i. To show that the bounded sequence $\{\lambda_i\}$ converges to $h_{\Gamma_\phi K}(u_o)$, we show that every convergent subsequence of $\{\lambda_i\}$ converges to $h_{\Gamma_\phi K}(u_o)$. Denote an arbitrary convergent subsequence of $\{\lambda_i\}$ by $\{\lambda_i\}$ as well, and suppose that for this subsequence we have

$$\lambda_i \to \lambda_*.$$

Obviously, $a \le \lambda_* \le b$. Let $\bar{K}_i = \lambda_i^{-1} K_i$. Since $\lambda_i^{-1} \to \lambda_*^{-1}$ and $K_i \to K$, we have

$$\bar{K}_i \to \lambda_*^{-1} K.$$

Now (2.7), (1.4), and the fact that $h_{\Gamma_\phi K_i}(u_o) = \lambda_i$, shows that $h_{\Gamma_\phi \bar{K}_i}(u_o) = 1$; i.e.

$$\int_{S^{n-1}} \phi^\star((u_o \cdot u)\rho_{\bar{K}_i}(u))\, dV^*_{\bar{K}_i}(u) = 1,$$

for all i. But $\bar{K}_i \to \lambda_*^{-1} K$ and the continuity of $\phi^\star$ now give

$$\int_{S^{n-1}} \phi^\star((u_o \cdot u)\rho_{\lambda_*^{-1} K}(u))\, dV^*_{\lambda_*^{-1} K}(u) = 1,$$

which by Lemma 2.1 gives

$$h_{\Gamma_\phi \lambda_*^{-1} K}(u_o) = 1.$$

This, (2.7) and (1.4) now give

$$h_{\Gamma_\phi K}(u_o) = \lambda_*.$$

This shows that $h_{\Gamma_\phi K_i}(u_o) \to h_{\Gamma_\phi K}(u_o)$ as desired.

But for support functions on S^{n-1} pointwise and uniform convergence are equivalent (see, e.g., Schneider [55, p. 54]). Thus, the pointwise convergence $h_{\Gamma_\phi K_i} \to h_{\Gamma_\phi K}$ on S^{n-1} completes the proof. q.e.d.

We next show that the Orlicz centroid operator is continuous in ϕ as well.

Lemma 2.5. *If $\phi_i \in \mathcal{C}$ and $\phi_i \to \phi \in \mathcal{C}$, then $\Gamma_{\phi_i} K \to \Gamma_\phi K$, for each $K \in \mathcal{S}_o^n$.*

Proof. Suppose $K \in \mathcal{S}_o^n$ and $u_o \in S^{n-1}$. We will show that for the support functions of the convex bodies $\Gamma_{\phi_i} K$ we have

$$h_{\Gamma_{\phi_i} K}(u_o) \to h_{\Gamma_\phi K}(u_o).$$

Let

$$h_{\Gamma_{\phi_i} K}(u_o) = \lambda_i,$$

and note that Lemma 2.3 gives

$$\frac{\omega_{n-1} r_K^{n+1}}{n c_{\phi_i^\star} |K|} \ \le\ \lambda_i \ \le\ \frac{R_K}{c_{\phi_i^\star}}.$$

Since $\phi_i^\star \to \phi^\star \in \mathcal{C}$, we have $c_{\phi_i^\star} \to c_{\phi^\star} \in (0,\infty)$ and thus there exist a, b such that $0 < a \le \lambda_i \le b < \infty$, for all i.

To show that the bounded sequence $\{\lambda_i\}$ converges to $h_{\Gamma_\phi K}(u_o)$, we show that every convergent subsequence of $\{\lambda_i\}$ converges to $h_{\Gamma_\phi K}(u_o)$. Denote an arbitrary convergent subsequence of $\{\lambda_i\}$ by $\{\lambda_i\}$ as well, and suppose that for this subsequence we have,

$$\lambda_i \to \lambda_*.$$

Obviously, $0 < a \le \lambda_* \le b$. Since $h(\Gamma_{\phi_i} K; u_o) = \lambda_i$,

$$1 = \int_{S^{n-1}} \phi_i^\star \left(\frac{u_o \cdot u}{\lambda_i}\, \rho_K(u) \right) dV_K^*(u).$$

This, together with $\phi_i^\star \to \phi^\star \in \mathcal{C}$ and $\lambda_i \to \lambda_*$, gives

$$1 = \int_{S^{n-1}} \phi^\star \left(\frac{u_o \cdot u}{\lambda_*}\, \rho_K(u) \right) dV_K^*(u).$$

But by Lemma 2.1 this gives

$$h_{\Gamma_\phi K}(u_o) = \lambda_*.$$

This shows that $h(\Gamma_{\phi_i} K; u_o) \to h(\Gamma_\phi K; u_o)$ as desired.

Since the support functions $h_{\Gamma_{\phi_i} K} \to h_{\Gamma_\phi K}$ pointwise (on S^{n-1}) they converge uniformly and hence

$$\Gamma_{\phi_i} K \to \Gamma_\phi K.$$

q.e.d.

The operator Γ_ϕ intertwines with elements of $\mathrm{GL}(n)$:

Lemma 2.6. *Suppose $\phi \in \mathcal{C}$. For a star body $K \in \mathcal{S}_o^n$ and a linear transformation $A \in \mathrm{GL}(n)$,*

$$\Gamma_\phi(AK) = A(\Gamma_\phi K). \tag{2.8}$$

Proof. From (2.7) it follows that we may assume, without loss of generality, that $A \in \mathrm{SL}(n)$.

Suppose $x_o \in \mathbb{R}^n \backslash \{0\}$ and

$$h(\Gamma_\phi AK; x_o) = \lambda_o.$$

From Lemma 2.1 and (2.4), the substitution $z = Ay$ and the facts that $|AK| = |K|$ and $dz = |A|dy = dy$, we have

$$\begin{aligned}
1 &= \frac{1}{|AK|}\int_{AK}\phi(x_o\cdot z/\lambda_o)\,dz\\
&= \frac{1}{|K|}\int_{K}\phi(x_o\cdot Ay/\lambda_o)\,dy\\
&= \frac{1}{|K|}\int_{K}\phi(A^t x_o\cdot y/\lambda_o)\,dy.
\end{aligned}$$

But by Lemma 2.1, (2.4) and (1.5) this implies that

$$\lambda_o = h(\Gamma_\phi K; A^t x_o) = h(A\Gamma_\phi K; x_o),$$

giving $h(\Gamma_\phi AK; x_o) = h(A\Gamma_\phi K; x_o)$. q.e.d.

3. Proof of the Orlicz Busemann-Petty centroid inequality

The proof of our theorem makes critical use of:

Lemma 3.1. *Suppose $\phi \in \mathcal{C}$, and $K \in \mathcal{K}_o^n$. If $u \in S^{n-1}$ and $x_1', x_2' \in u^\perp$, then*

$$h(\Gamma_\phi(\mathrm{S}_u K); \tfrac{1}{2}x_1' + \tfrac{1}{2}x_2', 1) \le \frac{1}{2}h(\Gamma_\phi K; x_1', 1) + \frac{1}{2}h(\Gamma_\phi K; x_2', -1). \tag{3.1}$$

Equality in the inequality implies that all of the chords of K parallel to u, whose distance from the origin is less than

$$\frac{r_K}{2\max\{1, |\,x_1'|, |\,x_2'|\}},$$

have midpoints that lie in a subspace.

Proof. In light of Lemma 2.6 we may assume, without loss of generality, that $|K| = 1 = |\,\mathrm{S}_u K|$.

Let $K' = K_u$ denote the image of the projection of K onto the subspace $u^\perp$. For each $y' \in K'$, let $\sigma_{y'}(u) = \sigma_{y'} = |\,K \cap (y' + \mathbb{R}u)|$ be the length of the chord $K \cap (y' + \mathbb{R}u)$, and let $m_{y'} = m_{y'}(u)$ be defined such that $y' + m_{y'}u$ is the midpoint of the chord $K \cap (y' + \mathbb{R}u)$.

For $\lambda_1, \lambda_2 > 0$, we have

$$\begin{aligned}
\int_K \phi\left(\frac{(x_1',1)\cdot y}{\lambda_1}\right)dy &= \int_K \phi\left(\frac{(x_1',1)\cdot(y',s)}{\lambda_1}\right)dy'ds\\
&= \int_{K'} dy' \int_{m_{y'}-\sigma_{y'}/2}^{m_{y'}+\sigma_{y'}/2} \phi\left(\frac{x_1'\cdot y' + s}{\lambda_1}\right)ds\\
&= \int_{K'} dy' \int_{-\sigma_{y'}/2}^{\sigma_{y'}/2} \phi\left(\frac{x_1'\cdot y' + t + m_{y'}}{\lambda_1}\right)dt\\
&= \int_{\mathrm{S}_u K} \phi\left(\frac{x_1'\cdot y' + t + m_{y'}(u)}{\lambda_1}\right)dy'dt,
\end{aligned} \tag{3.2}$$

by making the change of variables $t = -m_{y'} + s$, and

$$\begin{aligned}
\int_K \phi\left(\frac{(x_2',-1)\cdot y}{\lambda_2}\right) dy &= \int_K \phi\left(\frac{(x_2',-1)\cdot(y',s)}{\lambda_2}\right) dy'ds \\
&= \int_{K'} dy' \int_{m_{y'}-\sigma_{y'}/2}^{m_{y'}+\sigma_{y'}/2} \phi\left(\frac{x_2'\cdot y' - s}{\lambda_2}\right) ds \\
&= \int_{K'} dy' \int_{-\sigma_{y'}/2}^{\sigma_{y'}/2} \phi\left(\frac{x_2'\cdot y' + t - m_{y'}}{\lambda_2}\right) dt \\
&= \int_{\mathrm{S}_u K} \phi\left(\frac{x_2'\cdot y' + t - m_{y'}(u)}{\lambda_2}\right) dy'dt,
\end{aligned} \tag{3.3}$$

by making the change of variables $t = m_{y'} - s$.

Abbreviate

$$x_o' = \tfrac{1}{2}x_1' + \tfrac{1}{2}x_2' \quad \text{and} \quad \lambda_o = \tfrac{1}{2}\lambda_1 + \tfrac{1}{2}\lambda_2,$$

and from the convexity of ϕ, follows

(3.4)
$$2\,\phi\left(\frac{x_o'\cdot y' + t}{\lambda_o}\right) \le \frac{\lambda_1}{\lambda_o}\phi\left(\frac{x_1'\cdot y' + t + m_{y'}}{\lambda_1}\right) + \frac{\lambda_2}{\lambda_o}\phi\left(\frac{x_2'\cdot y' + t - m_{y'}}{\lambda_2}\right).$$

From (3.2) - (3.4), we have

$$\begin{aligned}
&\frac{\lambda_1}{\lambda_o}\int_K \phi\left(\frac{(x_1',1)\cdot y}{\lambda_1}\right) dy + \frac{\lambda_2}{\lambda_o}\int_K \phi\left(\frac{(x_2',-1)\cdot y}{\lambda_2}\right) dy \\
&\quad= \frac{\lambda_1}{\lambda_o}\int_{\mathrm{S}_u K} \phi\left(\frac{x_1'\cdot y' + t + m_{y'}(u)}{\lambda_1}\right) dy'dt \\
&\qquad\qquad + \frac{\lambda_2}{\lambda_o}\int_{\mathrm{S}_u K} \phi\left(\frac{x_2'\cdot y' + t - m_{y'}(u)}{\lambda_2}\right) dy'dt \\
&\quad\ge 2\int_{\mathrm{S}_u K} \phi\left(\frac{(\frac{1}{2}x_1' + \frac{1}{2}x_2')\cdot y' + t}{\frac{1}{2}\lambda_1 + \frac{1}{2}\lambda_2}\right) dy'dt \\
&\quad= 2\int_{\mathrm{S}_u K} \phi\left(\frac{(\frac{1}{2}x_1' + \frac{1}{2}x_2', 1)\cdot(y',t)}{\frac{1}{2}\lambda_1 + \frac{1}{2}\lambda_2}\right) dy'dt \\
&\quad= 2\int_{\mathrm{S}_u K} \phi\left(\frac{(x_o',1)\cdot y}{\lambda_o}\right) dy.
\end{aligned} \tag{3.5}$$

Choose

$$\lambda_1 = h(\Gamma_\phi K; x_1', 1) \quad \text{and} \quad \lambda_2 = h(\Gamma_\phi K; x_2', -1);$$

recall that $|K| = 1$ and we have from Lemma 2.1 and (2.4),

$$\int_K \phi\left(\frac{(x_1',1)\cdot y}{\lambda_1}\right) dy = 1 \quad \text{and} \quad \int_K \phi\left(\frac{(x_2',-1)\cdot y}{\lambda_2}\right) dy = 1.$$

But this in (3.5), and the fact that $|\,\mathrm{S}_u K| = 1$ shows that

$$1 \geq \frac{1}{|\,\mathrm{S}_u K|} \int_{\mathrm{S}_u K} \phi\left(\frac{(\frac{1}{2}x_1' + \frac{1}{2}x_2', 1)\cdot y}{\frac{1}{2}\lambda_1 + \frac{1}{2}\lambda_2}\right) dy,$$

which by definition (2.1) gives

$$h(\Gamma_\phi(\mathrm{S}_u K); \tfrac{1}{2}x_1' + \tfrac{1}{2}x_2', 1) \;\leq\; \tfrac{1}{2}\lambda_1 + \tfrac{1}{2}\lambda_2,$$

with equality forcing (in light of the continuity of ϕ) equality in (3.4) for all $y' \in K'$ and all $t \in [-\sigma_{y'}/2, \sigma_{y'}/2]$.

This establishes the desired inequality.

Suppose there is equality. Hence there is equality in (3.4) for all $y' \in K'$ and all $t \in [-\sigma_{y'}/2, \sigma_{y'}/2]$.

From definition (1.6) of r_K we see that if $|\,y'| < r_K/2$ then

$$\left(-\frac{r_K}{2}, \frac{r_K}{2}\right) \subset \left(m_{y'} - \frac{\sigma_{y'}}{2}, m_{y'} + \frac{\sigma_{y'}}{2}\right) \tag{3.6a}$$

and therefore also

$$\left(-\frac{r_K}{2}, \frac{r_K}{2}\right) \subset \left(-m_{y'} - \frac{\sigma_{y'}}{2}, -m_{y'} + \frac{\sigma_{y'}}{2}\right). \tag{3.6b}$$

Suppose y' is such that

$$|\,y'| < \frac{r_K}{2\max\{1, |\,x_1'|, |\,x_2'|\}}.$$

Then,

$$x_1' \cdot y' \in \left(-\frac{r_K}{2}, \frac{r_K}{2}\right) \qquad \text{and} \qquad x_2' \cdot y' \in \left(-\frac{r_K}{2}, \frac{r_K}{2}\right)$$

and from (3.6) it follows that

$$x_1' \cdot y' + m_{y'} \in \left(-\frac{\sigma_{y'}}{2}, \frac{\sigma_{y'}}{2}\right) \qquad \text{and} \qquad x_2' \cdot y' - m_{y'} \in \left(-\frac{\sigma_{y'}}{2}, \frac{\sigma_{y'}}{2}\right).$$

Thus, the linear functions

$$t \mapsto\; x_1' \cdot y' + t + m_{y'} \quad \text{and} \quad t \mapsto\; x_2' \cdot y' + t - m_{y'}$$

both have their root in $(-\sigma_{y'}/2, \sigma_{y'}/2)$. Thus, they either (1) have their root at the same $t = t_{y'} \in (-\sigma_{y'}/2, \sigma_{y'}/2)$ or (2) there will exist a $t = t^\star_{y'} \in (-\sigma_{y'}/2, \sigma_{y'}/2)$ at which these functions have opposite signs.

Consider case (2) first. The fact that

$$x_1' \cdot y' + t^\star_{y'} + m_{y'} \quad \text{and} \quad x_2' \cdot y' + t^\star_{y'} - m_{y'}$$

have opposite signs tells us that

$$x_1' \cdot y' + t + m_{y'} \quad \text{and} \quad x_2' \cdot y' + t - m_{y'}$$

have opposite signs for all $t \in (t^\star_{y'} - \delta_{y'}, t^\star_{y'} + \delta_{y'})$ for some $\delta_{y'} > 0$. This and the fact that there is equality in (3.4) together with the fact that ϕ can not be linear in a neighborhood of the origin give

$$\frac{x_1'\cdot y' + t + m_{y'}}{\lambda_1} = \frac{x_2'\cdot y' + t - m_{y'}}{\lambda_2},$$

for all $t \in (t^\star_{y'} - \delta_{y'}, t^\star_{y'} + \delta_{y'})$ which contradicts the assumption that the linear functions have opposite signs.

In case (1) the linear functions

$$t \mapsto\ x_1' \cdot y' + t + m_{y'} \quad \text{and} \quad t \mapsto\ x_2' \cdot y' + t - m_{y'}$$

have a root at the same $t = t_{y'} \in (-\sigma_{y'}/2, \sigma_{y'}/2)$ and this immediately yields

$$(x_2' - x_1') \cdot y' = 2m_{y'}.$$

But this means that for $|\,y'| < r_K/\max\{2, 2|\,x_1'|, 2|\,x_2'|\}$, the midpoints

$$\{(y', m_{y'}) : y' \in K'\}$$

of the chords of K parallel to u lie in the subspace that contains

$$\{(y', \tfrac{1}{2}(x_2' - x_1') \cdot y') : y' \in K'\}$$

of $\mathbb{R}^n$. q.e.d.

As an aside, observe that the inequality of Lemma 3.1 could have been presented as:

$$h(\Gamma_\phi(\mathrm{S}_u K); \tfrac{1}{2}x_1' + \tfrac{1}{2}x_2', -1) \le \frac{1}{2}h(\Gamma_\phi K; x_1', 1) + \frac{1}{2}h(\Gamma_\phi K; x_2', -1).$$

If ϕ is assumed to be strictly convex, then the equality conditions of the inequality in Lemma 3.1 are simple.

Lemma 3.2. *Suppose ϕ is strictly convex and $K \in \mathcal{K}^n_o$. If $u \in S^{n-1}$ and $x_1', x_2' \in u^\perp$, then*

$$h(\Gamma_\phi(\mathrm{S}_u K); \tfrac{1}{2}x_1' + \tfrac{1}{2}x_2', 1) \le \frac{1}{2}h(\Gamma_\phi K; x_1', 1) + \frac{1}{2}h(\Gamma_\phi K; x_2', -1),$$

and

$$h(\Gamma_\phi(\mathrm{S}_u K); \tfrac{1}{2}x_1' + \tfrac{1}{2}x_2', -1) \le \frac{1}{2}h(\Gamma_\phi K; x_1', 1) + \frac{1}{2}h(\Gamma_\phi K; x_2', -1).$$

Equality in either inequality, implies

$$h(\Gamma_\phi K; x_1', 1) = h(\Gamma_\phi K; x_2', -1)$$

and that all of the midpoints of the chords of K parallel to u lie in a subspace.

Proof. Observe that equality forces equality in (3.4) for all $y' \in K'$ and all $t \in [-\sigma_{y'}/2, \sigma_{y'}/2]$. But since ϕ is strictly convex this means that we must have

$$\frac{x_1' \cdot y' + t + m_{y'}}{\lambda_1} = \frac{x_2' \cdot y' + t - m_{y'}}{\lambda_2}, \tag{3.7}$$

for all $t \in (-\sigma_{y'}/2, \sigma_{y'}/2)$, where $\lambda_1 = h(\Gamma_\phi K; x_1', 1)$ and $\lambda_2 = h(\Gamma_\phi K; x_2', -1)$. Equation (3.7) immediately gives

$$h(\Gamma_\phi K; x_1', 1) = \lambda_1 = \lambda_2 = h(\Gamma_\phi K; x_2', -1),$$

and

$$(x_2' - x_1') \cdot y' = 2m_{y'},$$

for all $y' \in K'$. But this means that the midpoints $\{(y', m_{y'}) : y' \in K'\}$ of the chords of K parallel to u lie in the subspace that contains

$$\left\{ \left(y', \frac{1}{2}(x_2' - x_1') \cdot y' \right) : y' \in K' \right\}$$

of $\mathbb{R}^n$. q.e.d.

The theorem will be proved using:

Lemma 3.3. *Suppose $\phi \in \mathcal{C}$ and $K \in \mathcal{K}_o^n$. If $u \in S^{n-1}$, then*

$$\Gamma_\phi(\mathrm{S}_u K) \subset \mathrm{S}_u(\Gamma_\phi K). \tag{3.8}$$

If the inclusion is an identity then all of the chords of K parallel to u, whose distance from the origin is less than

$$\frac{r_K \, r_{\Gamma_\phi K}}{4R_{\Gamma_\phi K}},$$

have midpoints that lie in a subspace.

Proof. Suppose $y' \in \operatorname{relint}(\Gamma_\phi K)_u$. By Lemma 1.2 there exist $x_1' = x_1'(y')$ and $x_2' = x_2'(y')$ in $u^\perp$ such that

$$\begin{aligned} \overline{\ell}_u(\Gamma_\phi K, y') &= h_{\Gamma_\phi K}(x_1', 1) - x_1' \cdot y', \\ \underline{\ell}_u(\Gamma_\phi K, y') &= h_{\Gamma_\phi K}(x_2', -1) - x_2' \cdot y'. \end{aligned} \tag{3.9}$$

Now by (1.8), (3.9), followed by Lemma 3.1, and then Lemma 1.2 we have

(3.10)

$$\begin{aligned}\overline{\ell}_u(\mathrm{S}_u(\Gamma_\phi K);y') &= \frac{1}{2}\overline{\ell}_u(\Gamma_\phi K;y') + \frac{1}{2}\underline{\ell}_u(\Gamma_\phi K;y')\\ &= \frac{1}{2}\left(h_{\Gamma_\phi K}(x_1',1) - x_1'\cdot y'\right) + \frac{1}{2}\left(h_{\Gamma_\phi K}(x_2',-1) - x_2'\cdot y'\right)\\ &= \frac{1}{2}h_{\Gamma_\phi K}(x_1',1) + \frac{1}{2}h_{\Gamma_\phi K}(x_2',-1) - (\tfrac{1}{2}x_1' + \tfrac{1}{2}x_2')\cdot y'\\ &\geq h_{\Gamma_\phi(\mathrm{S}_u K)}(\tfrac{1}{2}x_1' + \tfrac{1}{2}x_2', 1) - (\tfrac{1}{2}x_1' + \tfrac{1}{2}x_2')\cdot y'\\ &\geq \min_{x'\in u^\perp}\left\{h_{\Gamma_\phi(\mathrm{S}_u K)}(x',1) - x'\cdot y'\right\}\\ &= \overline{\ell}_u(\Gamma_\phi(\mathrm{S}_u K);y'),\end{aligned}$$

and

$$\begin{aligned}\underline{\ell}_u(\mathrm{S}_u(\Gamma_\phi K);y') &= \frac{1}{2}\overline{\ell}_u(\Gamma_\phi K;y') + \frac{1}{2}\underline{\ell}_u(\Gamma_\phi K;y')\\ &= \frac{1}{2}\left(h_{\Gamma_\phi K}(x_1',1) - x_1'\cdot y'\right) + \frac{1}{2}\left(h_{\Gamma_\phi K}(x_2',-1) - x_2'\cdot y'\right)\\ &= \frac{1}{2}h_{\Gamma_\phi K}(x_1',1) + \frac{1}{2}h_{\Gamma_\phi K}(x_2',-1) - (\tfrac{1}{2}x_1' + \tfrac{1}{2}x_2')\cdot y'\\ &\geq h_{\Gamma_\phi(\mathrm{S}_u K)}(\tfrac{1}{2}x_1' + \tfrac{1}{2}x_2', -1) - (\tfrac{1}{2}x_1' + \tfrac{1}{2}x_2')\cdot y'\\ &\geq \min_{x'\in u^\perp}\left\{h_{\Gamma_\phi(\mathrm{S}_u K)}(x',-1) - x'\cdot y'\right\}\\ &= \underline{\ell}_u(\Gamma_\phi(\mathrm{S}_u K);y').\end{aligned}$$

This establishes the inclusion.

Now suppose

$$\Gamma_\phi(\mathrm{S}_u K) = \mathrm{S}_u(\Gamma_\phi K).$$

Then by Lemma 1.2, for each $y' \in (\Gamma_\phi K)_u \cap (r_{\Gamma_\phi K}/2)B$, there exist $x_1' = x_1'(y')$ and $x_2' = x_2'(y')$ in $u^\perp$ such that

$$\begin{aligned}\overline{\ell}_u(\Gamma_\phi K, y') &= h_{\Gamma_\phi K}(x_1',1) - x_1'\cdot y',\\ \underline{\ell}_u(\Gamma_\phi K, y') &= h_{\Gamma_\phi K}(x_2',-1) - x_2'\cdot y',\end{aligned} \tag{3.11}$$

and since $\Gamma_\phi(\mathrm{S}_u K) = \mathrm{S}_u(\Gamma_\phi K)$, from (3.10) we see that

$$h_{\Gamma_\phi(\mathrm{S}_u K)}(\tfrac{1}{2}x_1' + \tfrac{1}{2}x_2', 1) = \frac{1}{2}h_{\Gamma_\phi K}(x_1',1) + \frac{1}{2}h_{\Gamma_\phi K}(x_2',-1). \tag{3.12}$$

From Lemma 1.3 and (3.11), it follows that both

$$|\,x_1'|, |\,x_2'| \ \leq\ \frac{2R_{\Gamma_\phi K}}{r_{\Gamma_\phi K}}.$$

But now (3.12) and the equality conditions of Lemma 3.1 show that all of the chords of K parallel to u, whose distance from the origin is less than

$$\frac{r_K\, r_{\Gamma_\phi K}}{4R_{\Gamma_\phi K}},$$

have midpoints that lie in a subspace. q.e.d.

As a direct consequence of Lemma 1.1, we now have:

Corollary. *Suppose $\phi \in \mathcal{C}$ and $K \in \mathcal{K}_o^n$. If $u \in S^{n-1}$, then*

$$\Gamma_\phi(\mathrm{S}_u K) \subset \mathrm{S}_u(\Gamma_\phi K).$$

If the inclusion is an identity for all u, then K is an ellipsoid centered at the origin.

The Corollary and a standard Steiner symmetrization argument now yield:

Theorem. *If $\phi \in \mathcal{C}$ and $K \in \mathcal{K}_o^n$, then the volume ratio $|\,\Gamma_\phi K|/|K|$ is minimized if and only if K is an ellipsoid centered at the origin.*

4. Open Problems

The class-reduction technique introduced in [33] can be used to show that once the Busemann-Petty centroid inequality and its equality conditions have been established for convex bodies (in fact for a much smaller class of bodies) then one can easily extend the inequality and its equality conditions to all star bodies. It was shown in [37] that this is also the case for L_p centroid bodies. Does there exist a similar class-reduction technique that is applicable for Orlicz centroid bodies?

Conjecture. *If $\phi \in \mathcal{C}$ and $K \in \mathcal{S}_o^n$, then the volume ratio $|\,\Gamma_\phi K|/|K|$ is minimized only by ellipsoids.*

In [45], the Orlicz projection body $\Pi_\phi K$ of a convex body $K \in \mathcal{K}_o^n$, was defined as the convex body whose support function is given by

$$(4.1)\quad h_{\,\Pi_\phi K}(x) = \inf\left\{\lambda > 0 : \int_{S^{n-1}} \phi(\tfrac{1}{\lambda}(x\cdot u)\rho_{K^*}(u))\, dV_K(u) \le 1\right\},$$

where V_K, the volume-normalized *conical measure*, is defined by

$$|K| dV_K = \frac{1}{n} h_K\, dS_K,$$

and S_K is the classical Aleksandrov-Fenchel-Jessen surface area measure of K. Let $\Pi_\phi^* K = (\Pi_\phi K)^*$ denote the polar Orlicz projection body of K. Compare definition (4.1) with definition (2.2).

In [45], the following inequality was established.

Orlicz Petty projection inequality. *Suppose $\phi \in \mathcal{C}$ is strictly convex. If $K \in \mathcal{K}_o^n$ then the volume ratio*

$$|\Pi_\phi^* K|/|K|$$

is maximal if and only if K is an ellipsoid centered at the origin.

The technique introduced in [33] shows that once the Petty Projection inequality has been established then one can easily derive the Busemann-Petty centroid inequality as a consequence, and vice versa. The same is true of the L_p Busemann-Petty centroid inequality and the L_p Petty projection inequality. Is there an easy road from the Orlicz Petty projection inequality to the Orlicz Busemann-Petty centroid inequality? Is there an easy road from the Orlicz Busemann-Petty centroid inequality to the Orlicz Petty projection inequality?

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