Ornstein–Uhlenbeck operators with time periodic coefficients

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Abstract

We study the realization of the differential operator $u \mapsto u_t - L(t)u$ in the space of continuous time periodic functions, and in L^2 with respect to its (unique) invariant measure. Here $L(t)$ is an Ornstein-Uhlenbeck operator in $\mathbb{R}^n$, such that $L(t+T) = L(t)$ for each $t \in \mathbb{R}$.

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1 Introduction

Let $\{L(t)\}_{t\in\mathbb{R}}$ be a time dependent family of Ornstein-Uhlenbeck operators,

$$L(t)\varphi(x) = \frac{1}{2}\,\mathrm{Tr}\,[B(t)B^*(t)D^2\varphi(x)] + \langle A(t)x + f(t), D\varphi(x)\rangle, \tag{1.1}$$

where the data $A : \mathbb{R} \mapsto \mathscr{L}(\mathbb{R}^n)$, $B : \mathbb{R} \mapsto \mathscr{L}(\mathbb{R}^n)$ and $f : \mathbb{R} \mapsto \mathbb{R}^n$ are continuous and T-periodic, for some $T > 0$.

In this paper we study nonautonomous equations of the type

$$\begin{cases} u_s(s,x) + L(s)u(s,x) = 0, \;\; s < t, \; x \in \mathbb{R}^n, \\ \\ u(t) = \varphi(x), \; x \in \mathbb{R}^n, \end{cases} \tag{1.2}$$

and the associated differential operator $\mathscr{G}$ defined by

$$\mathscr{G}u(t,x) = D_t u(t,x) + L(t)u(t,\cdot)(x), \;\; t \in \mathbb{R}, \; x \in \mathbb{R}^n. \tag{1.3}$$

Even if the contents of this paper is essentially analytic, it is motivated by probabilistic problems that we describe below.

Let us consider the stochastic differential equation in $\mathbb{R}^n$

$$\begin{cases} dX(t) = (A(t)X(t) + f(t))dt + B(t)dW(t), \\ \\ X(s) = x, \end{cases} \tag{1.4}$$

where $W(t)$ is a standard n-dimensional Brownian motion and $x \in \mathbb{R}^n$. Problem (1.4) has obviously a unique mild solution $X(t, s, x)$ given by

$$X(t, s, x) = U(t, s)x + \int_s^t U(t, r)f(r)dr + \int_s^t U(t, r)B(r)dW(r),$$

where $U(t, s)$ is the evolution operator in $\mathbb{R}^n$ associated to the family $\{A(t)\}_{t\in\mathbb{R}}$, that is the solution of

$$\begin{cases} \dfrac{\partial U(t,s)}{\partial t} = A(t)U(t, s), \quad t,\ s \in \mathbb{R}, \\ \\ U(s, s) = I. \end{cases}$$

Consequently, the law of $X(t, s, x)$ is the Gaussian measure $\mathscr{N}_{m(t,s),Q(t,s)}$ with mean and covariance defined respectively by

$$m(t, s) := U(t, s)x + g(t, s), \quad Q(t, s) := \int_s^t U(t, r)B(r)B^*(r)U^*(t, r)dr \tag{1.5}$$

where

$$g(t, s) := \int_s^t U(t, r)f(r)dr. \tag{1.6}$$

The corresponding transition evolution operator is given by

$$P_{s,t}\varphi(x) := \mathbb{E}[\varphi(X(t, s, x))] = \int_{\mathbb{R}^n} \varphi(y)\mathscr{N}_{m(t,s),Q(t,s)}(dy), \quad \varphi \in C_b(\mathbb{R}^n),\ s \le t.$$

By Itô's formula it follows that for all $\varphi \in C_b^2(\mathbb{R}^n)$ the function

$$u(t, s, x) = P_{s,t}\varphi(x), \quad s \le t,\ x \in \mathbb{R}^n,$$

is a strict solution of the Kolmogorov equation (1.2). In fact, it is its unique bounded solution, and this can be proved by analytic arguments too, see section 2.

A big part of the paper is devoted to the asymptotic behaviour of $P_{s,t}\varphi(x)$ both for $s \to -\infty$, with fixed t, and for $t \to +\infty$, with fixed s. As well known, in the autonomous case a fundamental role in this problem is played by invariant measures. In the nonautonomous case it is not natural to have a single invariant measure, but rather a family of Borel probability measures $\{\nu_t\}_{t\in\mathbb{R}}$ such that for all $-\infty < s < t < +\infty$ and $\varphi \in C_b(\mathbb{R}^n)$ we have

$$\int_{\mathbb{R}^n} P_{s,t}\varphi(x)\nu_s(dx) = \int_{\mathbb{R}^n} \varphi(x)\nu_t(dx). \tag{1.7}$$

Such a family is called an *evolution system of measures*, see [3]. Since the coefficients in (1.1) are T-periodic, we require also that $\nu_{t+T} = \nu_t$ for all $t \in \mathbb{R}$.

Existence and uniqueness of a T-periodic evolution system of measures is proved in section 3 under the natural stability assumption that there exist M, $\omega > 0$ such that

$$\|U(t,s)\| \le M e^{-\omega(t-s)}, \quad -\infty < s \le t < +\infty. \tag{1.8}$$

In this case we have an explicit expression of ν_t,

$$\nu_t = \mathscr{N}_{g(t,-\infty),Q(t,-\infty)}, \;\; t \in \mathbb{R}. \tag{1.9}$$

Concerning the asymptotic behaviour of $P_{s,t}\varphi$ we prove in section 6 that for each continuous and bounded φ we have

$$\lim_{s\to-\infty} P_{s,t}\varphi(x) = \int_{\mathbb{R}^n} \varphi(y)\nu_t(dy), \;\; t \in \mathbb{R}, \; x \in \mathbb{R}^n, \tag{1.10}$$

and moreover

$$\lim_{t\to+\infty} \left[P_{s,t}\varphi(x) - \int_{\mathbb{R}^n} \varphi(y)\nu_t(dy) \right] = 0, \;\; s \in \mathbb{R}, \; x \in \mathbb{R}^n. \tag{1.11}$$

Identity (1.11) means that, as $t \to +\infty$, the orbit $t \to P_{s,t}\varphi$ approaches pointwise a periodic function which is independent of s and x.

It is of much help to reduce (1.4) to an autonomous problem, namely

$$\begin{cases} dZ(\tau) = [A(y(\tau))Z(\tau) + f(y(\tau))]d\tau + B(y(\tau))dW(\tau), \\ \\ dy(\tau) = d\tau, \\ \\ Z(0) = x, \quad y(0) = t, \end{cases} \tag{1.12}$$

whose solution (which is also explicit) we denote by $(y(\tau,t,x), Z(\tau,t,x))$. The corresponding transition semigroup is set in the space $C_b^{\#}(\mathbb{R}^{1+n})$, consisting of the continuous and bounded functions $u : \mathbb{R}^{1+n} \mapsto \mathbb{R}$ such that $u(t+T,\cdot) = u(t,\cdot)$ for each $t \in \mathbb{R}$, and it is defined by

$$\mathscr{P}_\tau u(t,x) = \mathbb{E}[u(y(\tau,t,x), Z(\tau,t,x))], \quad \tau > 0, \; (t,x) \in \mathbb{R}^{1+n}.$$

It is easy to see that $\mathscr{P}_\tau$ is a Markov semigroup given by

$$\mathscr{P}_\tau u(t,x) = (P_{t,t+\tau} u(t+\tau,\cdot))(x), \quad u \in C_b^{\#}(\mathbb{R}^{1+n}), \; (t,x) \in \mathbb{R}^{1+n}. \tag{1.13}$$

This procedure has a deterministic counterpart, which comes from the theory of the evolution semigroups, see [2] and the references therein. Namely, given a strongly continuous backward evolution operator $P(s,t)$, $s \le t$, in a Banach space X, the semigroup $\mathscr{T}_\tau$ defined by

$$\mathscr{T}_\tau u(t) = P(t,t+\tau)u(t+\tau), \;\; \tau \ge 0, \; t \in \mathbb{R},$$

is strongly continuous in $C_0(\mathbb{R};X)$ and in $L^p(\mathbb{R};X)$ for $p < \infty$, and this fact gives the possibility of studying several properties of $P(s,t)$ such as exponential dichotomies, through the general theory of semigroups. The above formula coincides with (1.13) if $P(s,t)$ is our $P_{s,t}$. However, we are not able to use any result from the general theory of evolution semigroups, because $P_{s,t}$ is not strongly continuous in $C_b(\mathbb{R}^n)$, and the same difficulty arises even if $C_b(\mathbb{R}^n)$ is replaced by the space $BUC(\mathbb{R}^n)$ of the bounded and uniformly continuous functions.

We show that $\mathscr{P}_\tau$ has a unique invariant measure ν given by

$$\nu(I \times K) = \frac{1}{T} \int_I \nu_t(K)dt, \quad I \in \mathscr{B}(\mathbb{R}),\ K \in \mathscr{B}(\mathbb{R}^n),$$

and then extended in a standard way to all Borel sets in $\mathbb{R}^{1+n}$. In section 5 we study the realization of the semigroup $\mathscr{P}_\tau$ in $L^2_\#(\mathbb{R}^{n+1}, \nu)$, the space of the functions $u : \mathbb{R}^{1+n} \mapsto \mathbb{R}$ such that $u(t + T, \cdot) = u(t, \cdot)$ for a.e. $t \in \mathbb{R}$ and such that $u_{|(0,T)\times\mathbb{R}^n}$ belongs to $L^2((0, T) \times \mathbb{R}^n, \nu)$. We denote by G the infinitesimal generator of $\mathscr{P}_\tau$ in $L^2_\#(\mathbb{R}^{n+1}, \nu)$. It is convenient to introduce the space $\mathscr{E}^\#(\mathbb{R}^{1+n})$, the linear span of all real and imaginary parts of the functions $u_{\phi,h}$ of the form

$$u_{\phi,h}(t, x) = \phi(t)e^{i\langle x,h(t)\rangle}, \quad t \in \mathbb{R},\ x \in \mathbb{R}^n,$$

where $\phi \in C^1(\mathbb{R})$, $h \in C^1(\mathbb{R}, \mathbb{R}^n)$ are T-periodic. We show that $\mathscr{E}^\#(\mathbb{R}^{1+n})$ is a core for G and that, as expected,

$$Gu(t, x) = \mathscr{G}u(t, x) = u_t(t, x) + L(t)u(t, x), \quad u \in \mathscr{E}^\#(\mathbb{R}^{1+n}).$$

Also in this L^2 context we are not able to use any result from the general theory of evolution semigroups, because in the genuinely nonautonomous case our space $L^2_\#(\mathbb{R}^{n+1}, \nu)$ cannot be identified with $L^2((0, T); X)$ for any Banach space X. Indeed, $P_{s,t}$ maps $X_t := L^2(\mathbb{R}^n, \nu_t)$ into $X_s := L^2(\mathbb{R}^n, \nu_s)$, and these spaces do not coincide in general for $t \neq s$.

Then we investigate further properties of the operator G. Since ν is the unique invariant measure of $\mathscr{P}_\tau$, it follows that it is ergodic, and that the kernel of G consists of constant functions. Moreover, since $\mathscr{P}_\tau$ is a contraction semigroup, then G is an m-dissipative operator.

Another important property is the integration by parts formula proved in section 5,

$$\int_{(0,T)\times\mathbb{R}^n} Gu(t, x)\, u(t, x)\, d\nu = -\frac{1}{2} \int_{(0,T)\times\mathbb{R}^n} |B^*(t)D_x u(t, x)|^2 d\nu \qquad (1.14)$$

valid for any $u \in \mathscr{E}^\#(\mathbb{R}^{1+n})$.

Note that if the determinant of $B(t)$ is not zero, then $L(t)$ is a uniformly elliptic differential operator, if det $B(t) = 0$ but $B(t)$ does not vanish then $L(t)$ is a degenerate elliptic operator, if $B(t) = 0$ and $A(t)$ or $f(t)$ do not vanish then $L(t)$ is a first order differential operator. Consequently, $\mathscr{G}$ may be a uniformly parabolic operator, a degenerate parabolic operator, or a first order operator. An interesting intermediate situation is when the determinant of $B(t)$ is zero for some t, and however the determinant of $Q(t, s)$ is nonzero for each $t > s$. This can be considered as a sort of hypoellipticity condition (in fact, in the autonomous case it is equivalent to hypoellipticity in the sense of Hörmander), and in this case the evolution operator $P_{s,t}$ has nice smoothing properties, that will be studied in a forthcoming paper.

In the uniformly elliptic case, that is when det $B(t) \neq 0$ for each t, it is natural that $P_{s,t}$, $\mathscr{P}_\tau$ and G have better properties. In particular, in sect. 5 we show that (1.14) holds for each $u \in D(G)$, and we use it to prove generalizations of the classical Poincaré and log-Sobolev inequalities, namely

$$\int_{(0,T)\times\mathbb{R}^n} (u(t,x) - \overline{u}_t)^2 d\nu \leq const. \int_{(0,T)\times\mathbb{R}^n} |D_x u(t,x)|^2 d\nu,$$

$$\int_{(0,T)\times\mathbb{R}^n} u^2 \log u^2 d\nu \leq const. \int_{(0,T)\times\mathbb{R}^n} |D_x u|^2 d\nu + \frac{1}{T}\int_0^T \overline{u^2}_s \log \overline{u^2}_s ds,$$

where $\overline{v}_s := \int_{\mathbb{R}^n} v(s,\cdot) d\nu_s$.

The Poincaré inequality gives further insight on the asymptotic behavior of $\mathscr{P}_\tau$ and of $P_{s,t}$, whereas is not clear at the moment whether the log-Sobolev inequality is connected to some kind of hypercontractivity properties as in the autonomous case or not. This will be the object of future investigations.

2 The evolution operator $P_{s,t}$

We are given a family of Kolmogorov operators,

$$L(t)\varphi(x) = \frac{1}{2}\,\mathrm{Tr}\,[B(t)B^*(t)D^2\varphi(x)] + \langle A(t)x + f(t), D\varphi(x)\rangle, \tag{2.1}$$

where $t \in \mathbb{R}$, $A\colon \mathbb{R} \to \mathscr{L}(\mathbb{R}^n)$, $B\colon \mathbb{R} \to \mathscr{L}(\mathbb{R}^n)$ and $f\colon \mathbb{R} \to \mathbb{R}^n$ are continuous and T–periodic, $T > 0$ is given.

We denote by $U(t,s)$, $t,\ s \in \mathbb{R}$, the evolution operator in $\mathbb{R}^n$ generated by $A(\cdot)$. We recall that

$$\begin{cases} \dfrac{\partial U(t,s)}{\partial t} = A(t)U(t,s), \quad t,\ s \in \mathbb{R}, \\ \\ U(s,s) = I, \end{cases} \tag{2.2}$$

$$U(t,s)U(s,r) = U(t,r), \qquad s,\ r \in \mathbb{R}, \tag{2.3}$$

and

$$\frac{\partial U(t,s)}{\partial s} = -U(t,s)A(s), \quad t,\ s \in \mathbb{R}. \tag{2.4}$$

Moreover, since $A(\cdot)$ is T–periodic, then

$$U(t+T, s+T) = U(t,s), \quad t,\ s \in \mathbb{R}. \tag{2.5}$$

Let us introduce the Poincaré operator

$$V(t) := U(t+T, t), \quad -\infty < t < +\infty. \tag{2.6}$$

Then $V(\cdot)$ is T–periodic and the spectrum of $V(t)$ is independent of t. Its elements are called *Floquet exponents* of A. If all the Floquet exponents have modulus less than 1 there exist $M > 0$, $\omega > 0$ such that

$$\|U(t,s)\| \leq Me^{-\omega(t-s)}, \quad -\infty < s \leq t < +\infty. \tag{2.7}$$

In this case we say that the family A is *stable*. We assume that (2.7) holds from now on.

We are here concerned with the problem

$$\begin{cases} u_s(s,x) + L(s)u(s,x) = 0, & s < t, \ x \in \mathbb{R}^n, \\ \\ u(t) = \varphi(x), \end{cases} \tag{2.8}$$

where $t \in \mathbb{R}$ is fixed. The representation formula for its solution involves an integral with respect to a Gaussian measure. We recall that for each vector $m \in \mathbb{R}^n$ and for each symmetric nonnegative definite matrix $Q \in \mathscr{L}(\mathbb{R}^n)$, the Gaussian measure $\mathscr{N}_{m,Q}$ is the unique probability measure in $\mathbb{R}^n$ whose Fourier transform is given by

$$\widehat{\mathscr{N}}_{m,Q}(h) = e^{i\langle m,h\rangle - \frac{1}{2}\langle Qh,h\rangle}, \quad h \in \mathbb{R}^n. \tag{2.9}$$

If Q is positive definite, then $\mathscr{N}_{m,Q}$ is absolutely continuous with respect to the Lebesgue measure, and it is given by

$$\mathscr{N}_{m,Q}(dx) = \frac{1}{(2\pi)^{n/2} \det Q^{1/2}} \exp\left(-\frac{\langle Q^{-1}(x-m), x-m\rangle}{2} \right) dx.$$

Proposition 2.1 *Assume that $\varphi \in C_b^2(\mathbb{R}^n)$ and fix $t \in \mathbb{R}$. Then problem* (2.8) *has a unique bounded classical solution $u \in C^{1,2}((-\infty,t] \times \mathbb{R}^n)$, given by the formula*

$$u(s,x) = \int_{\mathbb{R}^n} \varphi(y) \mathscr{N}_{U(t,s)x+g(t,s),Q(t,s)}(dy), \quad -\infty < s \le t < +\infty, \tag{2.10}$$

where g and Q are defined by (1.6) *and* (1.5)*, respectively.*

Proof. — Setting

$$u(s,x) = v(s, U(t,s)x + g(t,s)),$$

we have, recalling (2.4),

$$u_s(s,x) = v_s(s, U(t,s)x+g(t,s)) + \langle(-U(t,s)A(s)x - U(t,s)f(s), D_x v(s, U(t,s)x+g(t,s))\rangle,$$

$$D_x u(s,x) = U^*(t,s) D_x v(s, U(t,s)x + g(t,s)),$$

and

$$D_x^2 u(s,x) = U^*(t,s) D_x^2 v(s, U(t,s)x + g(t,s)) U(t,s).$$

Therefore, u is a solution to (2.8) if and only if v is a solution to

$$\begin{cases} v_s(s,x) + \dfrac{1}{2} \operatorname{Tr}\left[U(t,s)B(s)B^*(s)U(t,s)^* D_x^2 v(s,x)\right] = 0, & s \le t, \ x \in \mathbb{R}^n, \\ \\ v(t,t) = \varphi(x), \quad x \in \mathbb{R}^n. \end{cases} \tag{2.11}$$

(2.11) is a (possibly degenerate) parabolic Cauchy problem with coefficients depending only on t. So, it can be easily solved. Its unique bounded solution is

$$v(s,x) = \int_{\mathbb{R}^n} \varphi(y) \mathscr{N}_{x,Q(t,s)}(dy). \tag{2.12}$$

The conclusion follows. □

Note that the right hand side of (2.10) is well defined for each measurable and bounded φ. So, we define

$$P_{s,t}\varphi(x) := \int_{\mathbb{R}^n} \varphi(y)\mathscr{N}_{U(t,s)x+g(t,s),Q(t,s)}(dy) = \int_{\mathbb{R}^n} \varphi(U(t,s)x+g(t,s)+y)\mathscr{N}_{0,Q(t,s)}(dy), \tag{2.13}$$

for each $s \le t$ and $\varphi \in L^\infty(\mathbb{R}^n)$. Since $\mathscr{N}_{x,0}$ is the Dirac measure at x, we have $P_{t,t}\varphi = \varphi$ for each t.

It follows immediately from the definition that each $P_{s,t}$ maps $C_b(\mathbb{R}^n)$ into itself. Moreover, if $\det B(s) \neq 0$ for each s, the differential operator in the left-hand side of (2.11), $u \mapsto \mathrm{Tr}\ [U(t,s)B(s)B^*(s)U(t,s)^*D_x^2 u]$, is uniformly elliptic and even for $\varphi \in C_b(\mathbb{R}^n)$ formula (2.12) gives a classical solution to (2.11), so that formula (2.10) gives a classical solution to (2.8). The same is true in the case when $\det Q(t,s) \neq 0$ for each $t > s$. But in general $P_{t,s}$ has no smoothing effect with respect to all variables. The worst situation is when $B \equiv 0$, in which case $v \equiv \varphi$, and $P_{t,s}\varphi(x) = \varphi(U(t,s)x + g(t,s))$. In any case, the following proposition holds.

Proposition 2.2 *For each $\varphi \in C_b(\mathbb{R}^n)$, the function $(s,t,x) \mapsto P_{s,t}\varphi(x)$ is continuous in $\Lambda := \{(s,t,x) \in \mathbb{R}^{2+n} :\ s \le t\}$. Moreover for each $t \in \mathbb{R}$ and $x \in \mathbb{R}^n$ we have*

$$\lim_{s\to-\infty} P_{s,t}\varphi(x) = \int_{\mathbb{R}^n} \varphi(y)\mathscr{N}_{g(t,-\infty),Q(t,-\infty)}(dy), \tag{2.14}$$

where $g(t,-\infty)$, $Q(t,-\infty)$ are defined by (1.6), (1.5) with $s = -\infty$, i.e.

$$g(t,-\infty) := \int_{-\infty}^t U(t,s)f(s)ds, \quad Q(t,-\infty) := \int_{-\infty}^t U(t,r)B(r)B^*(r)U^*(t,r)dr. \tag{2.15}$$

Proof. — Let $(s_k, t_k, x_k) \in \Lambda$ go to (s,t,x) as $k \to \infty$. Then

$$P_{s_k,t_k}\varphi(x_k) - P_{s,t}\varphi(x) =$$

$$\int_{\mathbb{R}^n} \varphi(y)\mathscr{N}_{U(t_k,s_k)x_k+g(t_k,s_k),Q(t_k,s_k)}(dy) - \int_{\mathbb{R}^n} \varphi(y)\mathscr{N}_{U(t,s)x+g(t,s),Q(t,s)}(dy).$$

Formula (2.9) implies that the Fourier transform $\widehat{\mathscr{N}}_{U(t_k,s_k)x_k+g(t_k,s_k),Q(t_k,s_k)}$ goes to the Fourier transform $\widehat{\mathscr{N}}_{U(t,s)x+g(t,s),Q(t,s)}$ pointwise as $k \to \infty$. The statement follows now from the Lévy Theorem, see e.g. [6, §9.8].

The same argument works for $s = -\infty$, with a sequence $(s_k, t_k, x_k) \to (-\infty, t, x)$ as $k \to +\infty$. □

We notice for further use that

$$g(t+T, s+T) = g(t,s), \ \ Q(t+T, s+T) = Q(t,s), \quad -\infty < s \le t < +\infty. \tag{2.16}$$

It follows that for each $t \in \mathbb{R}$ we have

$$g(t+T,-\infty) = g(t,-\infty), \quad Q(t+T,-\infty) = Q(t,-\infty), \tag{2.17}$$

i.e. $g(\cdot,-\infty)$ and $Q(\cdot,-\infty)$ are T-periodic. Moreover, for any $t\geq s\geq r\geq -\infty$,

$$g(t,s)=\int_r^t U(t,\sigma)f(\sigma)d\sigma-\int_r^s U(t,\sigma)f(\sigma)d\sigma=g(t,r)-U(t,s)\int_r^s U(s,\sigma)f(\sigma)d\sigma,$$

so that

$$g(t,s)=g(t,r)-U(t,s)g(s,r),\quad t\geq s\geq r\geq -\infty. \tag{2.18}$$

Arguing similarly we get

$$Q(t,s)=Q(t,r)-U(t,s)Q(s,r)U^*(t,s),\quad t\geq s\geq r\geq -\infty. \tag{2.19}$$

In particular, for any $t\in\mathbb{R}$, taking $r=-\infty$ we get

$$\begin{cases} g(t+T,t)=(I-V(t))g(t,-\infty),\\ \\ Q(t+T,t)=Q(t,-\infty)-V(t)Q(t,-\infty)V^*(t). \end{cases} \tag{2.20}$$

Since our operators $P_{s,t}$ are defined through Gaussian measures, it is convenient to use exponential functions. For any $h\in\mathbb{R}^n$ we define

$$\varphi_h(x)=e^{i\langle x,h\rangle},\quad x\in\mathbb{R}^n.$$

We denote by $\mathscr{E}(\mathbb{R}^n)$ the linear span of all real and imaginary parts of the functions φ_h.

The following result is well known.

Proposition 2.3 *For all $\varphi\in C_b(\mathbb{R}^n)$, there exists a sequence $\{\varphi_k\}\subset\mathscr{E}(\mathbb{R}^n)$ such that*

(i) $\|\varphi_k\|_\infty\leq\|\varphi\|_\infty$,

(ii) $\lim_{k\to\infty}\varphi_k(x)=\varphi(x),\quad \forall\, x\in\mathbb{R}^n$.

Now we are able to show that $\{P_{s,t}:\ s\leq t\}$ is a backward evolution operator in $C_b(\mathbb{R}^n)$.

Proposition 2.4 *For each $s\leq s\leq t\in\mathbb{R}$ we have*

$$P_{r,s}P_{s,t}=P_{r,t}.$$

Proof. — By proposition 2.3, it is enough to show that $P_{r,s}P_{s,t}\varphi=P_{r,t}\varphi$ for $\varphi=\varphi_h(x)=e^{i\langle h,x\rangle}$, where $h\in\mathbb{R}^n$ is arbitrary.

Recalling the Fourier transform of a Gaussian measure (2.9), we get, for each $x\in\mathbb{R}^n$,

$$P_{s,t}\varphi_h(x)=e^{i\langle g(t,s),h\rangle-\frac12\langle Q(t,s)h,h\rangle}\varphi_{U^*(t,s)h}(x),$$

and

$$\begin{aligned} P_{r,s}\varphi_{U^*(t,s)h}(x) &= e^{i\langle g(s,r),U^*(t,s)h\rangle-\frac12\langle Q(s,r)U^*(t,s)h,U^*(t,s)h\rangle}\varphi_{U^*(s,r)U^*(t,s)h}(x)\\ &= e^{i\langle U(t,s)g(s,r),h\rangle-\frac12\langle U(t,s)Q(s,r)U^*(t,s)h,h\rangle}\varphi_{U^*(t,r)h}(x). \end{aligned}$$

(2.18) and (2.19) yield

$$P_{r,s}P_{s,t}\varphi_h(x)=e^{i\langle g(t,r),h\rangle-\frac12\langle Q(t,r)h,h\rangle}\varphi_{U^*(t,r)h}(x)=P_{r,t}\varphi_h(x),\quad x\in\mathbb{R}^n,$$

and the statement holds. □

3 T–periodic evolution systems of measures

In the autonomous case $A(t) \equiv A$, $B(t) \equiv B$, $f(t) \equiv 0$, we have $U(t,s) = e^{(t-s)A}$, $g(t,s) = 0$, $Q(t,s) = R(t-s)$ where $R(t) = \int_0^t e^{rA}BB^*e^{rA^*}\,dr$, and hence $P_{s,t} = T(t-s)$ where $T(t)$ is the well known Ornstein-Uhlenbeck semigroup,

$$T(t)\varphi(x) = \int_{\mathbb{R}^n} \varphi(e^{tA}x + y)\mathscr{N}_{0,R(t)}(dy), \;\; t \geq 0.$$

Our stability assumption (2.7) implies that the Gaussian measure $\mathscr{N}_{0,R(+\infty)}$ is the unique invariant measure for $T(t)$, i.e. the unique probability measure μ such that

$$\int_{\mathbb{R}^n} (T(t)\varphi)(y)\mu(dy) = \int_{\mathbb{R}^n} \varphi(y)\mu(dy), \;\; t \geq 0,$$

for all continuous and bounded φ. See e.g. [4, Ch. 11]. In the genuine nonautonomous case we cannot expect to have an invariant measure, but rather a family of Borel probability measures ν_t, $t \in \mathbb{R}$, such that

$$\int_{\mathbb{R}^n} P_{s,t}\varphi(x)\nu_s(dx) = \int_{\mathbb{R}^n} \varphi(x)\nu_t(dx), \;\; s \leq t, \tag{3.1}$$

for all continuous and bounded φ. Such family is called *evolution system of measures*, see [3]. An evolution system of measures $\{\nu_t : \; t \in \mathbb{R}\}$, is called T–periodic if $\nu_{t+T} = \nu_t$ for all $t \in \mathbb{R}$. In this case we have

$$\int_{\mathbb{R}^n} P_{t,t+T}\varphi(x)\nu_t(dx) = \int_{\mathbb{R}^n} \varphi(x)\nu_t(dx), \quad t \in \mathbb{R}, \tag{3.2}$$

for all φ continuous and bounded.

Proposition 3.1 *The measures defined by*

$$\nu_t = \mathscr{N}_{g(t,-\infty),Q(t,-\infty)}, \quad t \in \mathbb{R}, \tag{3.3}$$

are a T-periodic system of measures. Conversely, if a family $\{\nu_t : \; t \in \mathbb{R}\}$ of Borel probability measures on $\mathbb{R}^n$, satisfies (3.2)*, then they are the measures defined by* (3.3)*.*

Proof. — *Existence.* Let $t \in \mathbb{R}$ be fixed and let ν_t be given by (3.3). We claim that (3.1) holds. In view of proposition 2.3 it is enough to show that (3.1) is fulfilled for any $\varphi_h(x) = e^{i\langle x,h\rangle}$ where $h \in \mathbb{R}^n$. In this case by (2.9) we have

$$\int_{\mathbb{R}^n} \varphi_h(x)\nu_t(dx) = e^{i\langle g(t,-\infty),h\rangle - \frac{1}{2}\,\langle Q(t,-\infty)h,h\rangle}, \tag{3.4}$$

and

$$P_{s,t}\varphi_h(x) = e^{i\langle g(t,s),h\rangle - \frac{1}{2}\,\langle Q(t.s)h,h\rangle}\varphi_{U^*(t.s)h}(x).$$

Integrating and using once again (2.9) we get

$$\begin{aligned}\int_{\mathbb{R}^n} P_{s,t}\varphi_h(x)\nu_s(dx) &= e^{i\langle g(t,s),h\rangle - \frac{1}{2}\,\langle Q(t,s)h,h\rangle} \\ &\times \;\; e^{i\langle g(s,-\infty),U^*(s,t)h\rangle - \frac{1}{2}\,\langle Q(s,-\infty)U^*(t,s)h,U^*(t,s)h\rangle},\end{aligned} \tag{3.5}$$

and the right hand side is equal to $e^{i\langle g(t,-\infty),h\rangle-\frac{1}{2}\,\langle Q(t,-\infty)h,h\rangle}$ by formulae (2.18) and (2.19) with $r=-\infty$.

We notice finally that ν_t is T-periodic by (2.17).

Uniqueness. Let $\{\nu_t:\ t\in\mathbb{R}\}$ be a family of Borel measures satisfying (3.2). Denote by $\widehat{\nu_t}(h)$ the Fourier transform of ν_t. Then we have

$$\int_{\mathbb{R}^n} P_{t,t+T}\varphi_h(x)\nu_t(dx)=e^{i\langle g(t+T,t),h\rangle-\frac{1}{2}\,\langle Q(t+T,t)h,h\rangle}\widehat{\nu_t}(U^*(t+T,t)h)$$

and

$$\int_{\mathbb{R}^n}\varphi_h(x)\nu_t(dx)=\widehat{\nu_t}(h).$$

Consequently by (3.2) the following identity holds,

$$\widehat{\nu_t}(h)=e^{i\langle g(t+T,t),h\rangle-\frac{1}{2}\,\langle Q(t+T,t)h,h\rangle}\widehat{\nu_t}(U^*(t+T,t)h),\quad h\in\mathbb{R}^n.$$

Using (2.20) we get

$$\widehat{\nu_t}(h)=e^{i\langle g(t,-\infty)-V(t)g(t,-\infty),h\rangle-\frac{1}{2}\,\langle Q(t,-\infty)-V(t)Q(t,-\infty)V^*(t)h,h\rangle}\widehat{\nu_t}(V^*(t)h)$$

$$=e^{i\langle g(t,-\infty),h\rangle-\frac{1}{2}\,\langle Q(t,-\infty)h,h\rangle}e^{-i\langle g(t,-\infty),V^*(t)h\rangle+\frac{1}{2}\,\langle Q(t,-\infty)V^*(t)h,V^*(t)h\rangle}\widehat{\nu_t}(V^*(t)h)$$

so that, by (2.9),

$$\frac{\widehat{\nu_t}(h)}{\widehat{\mathscr{N}}_{g(t,-\infty),Q(t,-\infty)}(h)}=\frac{\widehat{\nu_t}(V^*(t)h)}{\widehat{\mathscr{N}}_{g(t,-\infty),Q(t,-\infty)}(V^*(t)h)},\quad h\in\mathbb{R}^n,$$

and iterating

$$\frac{\widehat{\nu_t}(h)}{\widehat{\mathscr{N}}_{g(t,-\infty),Q(t,-\infty)}(h)}=\frac{\widehat{\nu_t}((V^*(t))^kh)}{\widehat{\mathscr{N}}_{g(t,-\infty),Q(t,-\infty)}((V^*(t))^kh))},\quad h\in\mathbb{R}^n,$$

for each $k\in\mathbb{N}$. The stability assumption (2.7) implies that $(V^*(t))^kh\to 0$ as $k\to\infty$, so we get

$$\frac{\widehat{\nu_t}(h)}{\widehat{\mathscr{N}}_{g(t,-\infty),Q(t,-\infty)}(h)}=\frac{\widehat{\nu_t}(0)}{\widehat{\mathscr{N}}_{g(t,-\infty),Q(t,-\infty)}(0)}=1,\quad h\in\mathbb{R}^n.$$

Since the Fourier transform is injective, it follows that ν_t coincides with the Gaussian measure $\mathscr{N}_{g(t,-\infty),Q(t,-\infty)}$. □

We define a function ν on the set $\mathscr{R}$ of all subsets of $\mathbb{R}\times\mathbb{R}^n$ of the type $I\times K$ where $I\in\mathscr{B}(\mathbb{R})$ and $K\in\mathscr{B}(\mathbb{R}^n)$, setting

$$\nu(I\times K)=\frac{1}{T}\int_I\nu_t(K)dt \tag{3.6}$$

It is easy to check that ν is σ–additive on $\mathscr{R}$. Moreover, by a standard argument ν can be uniquely extended to a σ–additive function on the algebra of all measurable pluri–rectangles. By the Caratheodory Theorem ν can be uniquely extended to a Borel measure on $\mathscr{B}(\mathbb{R}\times\mathbb{R}^n)$, that we still call ν.

4 The evolution semigroup in spaces of continuous functions

Since our data are T-periodic in time, it is reasonable to work in spaces of functions that are T-periodic in time. In this section we consider the space $C_b^{\#}(\mathbb{R}^{1+n})$, consisting of the continuous and bounded functions $u : \mathbb{R}^{1+n} \mapsto \mathbb{R}$ such that $u(t+T, \cdot) = u(t, \cdot)$ for each $t \in \mathbb{R}$.

For any T-periodic $\phi \in C^1(\mathbb{R})$ and any $h \in C^1(\mathbb{R}; \mathbb{R}^n)$ we consider the function

$$u_{\phi,h}(t,x) = \phi(t) e^{i\langle x, h(t)\rangle}, \quad t \in \mathbb{R},\ x \in \mathbb{R}^n,$$

and we denote by $\mathscr{E}_{\#}(\mathbb{R}^{1+n})$ the linear span of all real and imaginary parts of the functions $u_{\phi,h}$.

The following lemma will be useful.

Lemma 4.1 *For every $u \in C_b^{\#}(\mathbb{R}^{1+n})$ there is a sequence of functions $u_k \in \mathscr{E}_{\#}(\mathbb{R}^{1+n})$ that converge pointwise to u and such that $\|u_k\|_\infty \leq 2\|u\|_\infty$ for each k.*

Proof. — Let θ be a smooth function such that $0 \leq \theta(x) \leq 1$ for each $x \in \mathbb{R}^n$, $\theta \equiv 1$ in $B(0,1)$, $\theta \equiv 0$ outside $B(0,2)$. The functions $(t,x) \mapsto u(t,x)\theta(x/R)$ converge pointwise to u as $R \to \infty$ and their sup norms do not exceed the sup norm of u. In its turn, each of these functions may be approximated in the sup norm by a sequence of functions that are linear combinations of products $\psi(t)\varphi(x)$, where ψ is T-periodic and continuously differentiable, and φ has compact support. We can assume that the sup norm of the approaching functions do not exceed $2\|u\|_\infty$. The functions $\psi(t)\varphi_k(x)$, where φ_k is given by proposition 2.3, belong to $\mathscr{E}_{\#}(\mathbb{R}^{1+n})$ converge pointwise to $\psi(t)\varphi(x)$, and their sup norms do not exceed $2\|u\|_\infty$. □

We define a semigroup of linear operators $\mathscr{P}_\tau$ in $C_b^{\#}(\mathbb{R}^{1+n})$ by

$$\begin{aligned} \mathscr{P}_\tau u(t,x) &:= (P_{t,t+\tau} u(t+\tau, \cdot))(x) \\ &= \int_{\mathbb{R}^n} u(\tau + t, U(t+\tau,t)x + g(t+\tau,t) + y) \mathscr{N}_{0,Q(t+\tau,t)}(dy) \end{aligned} \tag{4.1}$$

($\mathscr{P}_\tau$ is a semigroup as an immediate consequence of proposition 2.4). This formula is borrowed from the general theory of evolution semigroups, see e.g [2] and the references quoted there. Such a theory has been developed for strongly continuous forward evolution operators. Our evolution operator $P_{s,t}$ is backward and not strongly continuous, even if we replace $C_b(\mathbb{R}^n)$ by $BUC(\mathbb{R}^n)$, the space of the bounded and uniformly continuous functions. While the extension of the general theory to backward evolution operators is straightforward, its extension to not strongly continuous evolution operators is not obvious, so that we are not able to use any result of the general theory.

By proposition 2.2, $\mathscr{P}_\tau$ maps $C_b^{\#}(\mathbb{R}^{1+n})$ into itself. Since $\|P_{s,t}\varphi\|_\infty \leq \|\varphi\|_\infty$ for each $s \leq t$ and for each continuous bounded φ, then $\mathscr{P}_\tau$ is a contraction semigroup in $C_b^{\#}(\mathbb{R}^{1+n})$. But, since $P_{s,t}$ is not strongly continuous in $C_b(\mathbb{R}^n)$, then $\mathscr{P}_\tau$ is not strongly continuous in $C_b^{\#}(\mathbb{R}^{1+n})$. However, the following proposition holds.

Proposition 4.2 *Let $\mathscr{P}_\tau$ be the semigroup of linear operators in $C_b^{\#}(\mathbb{R}^{1+n})$ defined by* (4.1)*, and let ν be the probability measure defined by* (3.6)*. Then $\mathscr{P}_\tau$ is a Markov semigroup and ν is its unique invariant measure.*

Proof. — The Markov property follows easily from the definition.

Let us prove that ν is invariant for $\mathscr{P}_\tau$, that is

$$\int_{(0,T)\times\mathbb{R}^n} \mathscr{P}_\tau u(t,x)d\nu = \int_{(0,T)\times\mathbb{R}^n} u(t,x)d\nu, \quad \tau > 0, \tag{4.2}$$

for all $u \in C_b^{\#}(\mathbb{R}^{1+n})$. This is a consequence of proposition 3.1. Indeed, using (3.1), for each $t \in \mathbb{R}$ and $\tau > 0$ we get

$$\int_{\mathbb{R}^n} \mathscr{P}_\tau u(t,x)\nu_t(dx) = \int_{\mathbb{R}^n} P_{t,t+\tau}u(t+\tau,\cdot)(x)\nu_t(dx) = \int_{\mathbb{R}^n} u(t+\tau,x)(x)\nu_{t+\tau}(dx)$$

so that, integrating with respect to t,

$$\int_{(0,T)\times\mathbb{R}^n} \mathscr{P}_\tau u(t,x)d\nu = \frac{1}{T}\int_0^T \int_{\mathbb{R}^n} u(t+\tau,x)(x)\nu_{t+\tau}(dx)\,dt$$

$$= \frac{1}{T}\int_\tau^{T+\tau} \int_{\mathbb{R}^n} u(s,x)(x)\nu_s(dx)\,ds,$$

which implies (4.2) by the periodicity of u and of ν_t.

It remains to show uniqueness of the invariant measure. Assume that ζ is a probability measure in $\mathbb{R}^{1+n}$ such that

$$\int_{(0,T)\times\mathbb{R}^n} \mathscr{P}_\tau u(t,x)\zeta(dt,dx) = \int_{(0,T)\times\mathbb{R}^n} u(t,x)\zeta(dt,dx), \tag{4.3}$$

for all $u \in C_b^{\#}(\mathbb{R}^{1+n})$.

Denote by ζ_1 the marginal of ζ,

$$\zeta_1(I) = \zeta(I\times\mathbb{R}^n), \quad I \in \mathscr{B}(\mathbb{R})$$

and let $(\zeta_t)_{t\in\mathbb{R}}$ be a disintegration of ζ (see e.g. [6, Theorem10.2.1]), so that

$$\int_{(0,T)\times\mathbb{R}^n} u(t,x)\zeta(dt,dx) = \int_0^T \left[\int_{\mathbb{R}^n} u(t,x)\zeta_t(dx)\right]\zeta_1(dt), \tag{4.4}$$

for every $u \in C_b^{\#}(\mathbb{R}^{1+n})$.

Choose in particular $u(t,x) = \rho(t)$ independent of x. Then

$$\mathscr{P}_\tau u(t,x) = \rho(t+\tau),\ \ \tau > 0,\ t\in\mathbb{R},\ x\in\mathbb{R}^n,$$

so that (4.4) yields

$$\int_0^T \rho(t+\tau)\zeta_1(dt) = \int_0^T \rho(t)\zeta_1(dt).$$

This means that the measure ζ_1 is invariant by translation (modulus T), so that it must coincide with the Lebesgue measure dt. Therefore we have

$$\zeta(dt,dx) = \zeta_t(dx)dt.$$

Now we prove that $\zeta_t(dx) = \nu_t(dx)$ for each t. By assumption, for every $\varphi \in C_b(\mathbb{R}^n)$ and for every T-periodic and continuous ρ we have

$$\int_0^T \rho(t+\tau) \int_{\mathbb{R}^n} P_{t,t+\tau}\varphi(x)\zeta_t(dx)dt = \int_0^T \rho(t) \int_{\mathbb{R}^n} \zeta_t(dx)dt, \quad \tau > 0.$$

Choosing $\tau = T$ we get

$$\int_0^T \rho(t) \int_{\mathbb{R}^n} P_{t,t+T}\varphi(x)\zeta_t(dx)dt = \int_0^T \rho(t) \int_{\mathbb{R}^n} \zeta_t(dx)dt,$$

for each continuous and T-periodic ρ. Therefore,

$$\int_{\mathbb{R}^n} P_{t,t+T}\varphi(x)\zeta_t(dx) = \int_{\mathbb{R}^n} \zeta_t(dx),$$

for each continuos and bounded φ. By the uniqueness part of proposition 3.1, $\zeta_t = \nu_t$. □

The following technical lemma shows that $\mathscr{P}_\tau$ enjoys some properties of the weakly continuous semigroups and of the π-semigroups treated in [1, 8]. However, in general it does not preserve uniform continuity, which is one of the assumptions of [1, 8]. For instance, if $B \equiv 0$, $f \equiv 0$ we have $\mathscr{P}_\tau u(t,x) = u(t+\tau, U(t+\tau,t)x)$ which is not uniformly continuous for each uniformly continuous and bounded u. Therefore we cannot use the theories of [1, 8].

Lemma 4.3 *Let (u_k) be a bounded sequence in $C_b^{\#}(\mathbb{R}^{1+n})$, that converges pointwise to $u \in C_b^{\#}(\mathbb{R}^{1+n})$ as $k \to \infty$. Then:*

(a) for each $\tau > 0$, $(\mathscr{P}_\tau u_k)$ converges pointwise to $\mathscr{P}_\tau u$ as $k \to \infty$;

(b) if $u_k \to u$ uniformly on each compact set as $k \to \infty$, then for each compact set $K \in \mathbb{R}^n$ we have

$$\lim_{k\to\infty} \sup\{|\mathscr{P}_\tau(u_k - u)(t,x)| : \tau > 0,\ 0 \le t \le T,\ x \in K\} = 0.$$

Proof. — Statement (a) is an easy consequence of the dominated convergence theorem.

Let us prove (b). For every $R > 0$ we have

$$|\mathscr{P}_\tau(u_k - u)(t,x)| =$$

$$\left| \int_{\mathbb{R}^n} (u_k - u)(\tau + t, U(t+\tau,t)x + g(t+\tau,t) + y)\mathscr{N}_{0,Q(t+\tau,t)}(dy) \right|$$

$$\le \int_{\mathbb{R}^n \setminus B(0,R)} \|u_k - u\|_\infty \mathscr{N}_{0,Q(t+\tau,t)}(dy)$$

$$+ \int_{B(0,R)} |(u_k - u)(\tau + t, U(t+\tau,t)x + g(t+\tau,t) + y)|\mathscr{N}_{0,Q(t+\tau,t)}(dy) := I_1 + I_2.$$

Note that for each t, $s \in \mathbb{R}$ we have

$$\int_{\mathbb{R}^n \setminus B(0,R)} \mathscr{N}_{0,Q(t,s)}(dy) \leq \frac{1}{R^2} \int_{\mathbb{R}^n} |y|^2 \mathscr{N}_{0,Q(t,s)}(dy) = \frac{1}{R^2} \operatorname{Tr} Q(t,s),$$

so that

$$I_1 \leq \frac{1}{R^2}(\|u\|_\infty + \|u_k\|_\infty) \sup_{t,s\in\mathbb{R}} \operatorname{Tr} Q(t,s) \leq \frac{C_1}{R^2}.$$

Fixed any compact set $K \subset \mathbb{R}^n$, for each $y \in B(0,R)$ and $x \in K$ we have

$$|U(t+\tau,t)x + g(t+\tau,t) + y| \leq M|x| + M\|f\|_\infty/\omega + R \leq C_2 + R, \quad t \in \mathbb{R}, \ \tau > 0.$$

Therefore,

$$I_2 \leq \|u_k - u\|_{L^\infty(\mathbb{R}\times B(0,C_2+R))}.$$

Choosing R large enough and then k large enough the statement follows. $\square$

5 The evolution semigroup in $L^2_{\#}(\nu)$

Fixed any $p \in [1,+\infty)$ we introduce the Banach space $L^p_{\#}(\nu)$, consisting of the Borel measurable functions u such that $u(t,x) = u(t+T,x)$ a.e. and $u_{|(0,T)\times\mathbb{R}^n} \in L^p((0,T)\times\mathbb{R}^n,\nu)$.

In this section we consider the realization of the semigroup $\mathscr{P}_\tau$ defined in (4.1) in the spaces $L^p_{\#}(\nu)$, with particular attention to the case $p=2$.

Lemma 5.1 *The space $\mathscr{E}_{\#}(\mathbb{R}^{1+n})$ is dense in $L^p_{\#}(\nu)$ for each $p \in [1,+\infty)$.*

Proof. — Every $u \in L^p_{\#}(\nu)$ may be approximated in $L^p_{\#}(\nu)$ by a sequence of functions belonging to $C^{\#}_b(\mathbb{R}^{1+n})$. In its turn, by lemma 4.1 each $v \in C^{\#}_b(\mathbb{R}^{1+n})$ is the pointwise limit of a sequence $(v_k) \subset \mathscr{E}_{\#}(\mathbb{R}^{1+n})$ that such that $\|v_k\|_\infty \leq 2\|v\|_\infty$ for each k. By dominated convergence, $v_k \to v$ in $L^p_{\#}(\nu)$ as $k \to \infty$, and the statement follows. $\square$

Proposition 5.2 *For each $p \in [1,+\infty)$, $\mathscr{P}_\tau$ is a strongly continuous contraction semigroup in $L^p_{\#}(\nu)$, that leaves $\mathscr{E}_{\#}(\mathbb{R}^{1+n})$ invariant. Formula (4.2) holds for each $u \in L^p_{\#}(\nu)$.*

Proof. — Let $u = u_{\phi,h} \in \mathscr{E}_{\#}(\mathbb{R}^{1+n})$, that is $u(t,x) = \phi(t)e^{i\langle x,h(t)\rangle}$ where $h \in C^1(\mathbb{R};\mathbb{R}^n)$ and $\phi \in C^1(\mathbb{R})$ are T-periodic. For such u we have

$$\begin{aligned} \mathscr{P}_\tau u(t,x) &= \int_{\mathbb{R}^n} \phi(t+\tau) e^{i\langle U(t+\tau,t)x + g(t+\tau,t)+y, h(t+\tau)\rangle} \mathscr{N}_{0,Q(t+\tau+t)}(dy) \\ &= \phi(t+\tau) e^{i\langle g(t+\tau,t),h(t+\tau)\rangle} e^{i\langle U(t+\tau,t)x,h(t+\tau)\rangle} e^{-\frac{1}{2}\langle Q(t+\tau,t)h(t+\tau),h(t+\tau)\rangle} \end{aligned} \tag{5.1}$$

which can be written as $u_{\psi,k}$, with $\psi(t) = \phi(t+\tau)e^{i\langle g(t+\tau,t),h(t+\tau)\rangle}e^{-\frac{1}{2}\langle Q(t+\tau,t)h(t+\tau),h(t+\tau)\rangle}$, and $k(t) = U^*(t+\tau,t)h(t+\tau)$. Therefore, $\mathscr{P}_\tau$ preserves $\mathscr{E}_{\#}(\mathbb{R}^{1+n})$.

Formula (5.1) also implies that $\mathscr{P}_\tau$ is a strongly continuous semigroup in $\mathscr{E}_{\#}(\mathbb{R}^{1+n})$, with respect to the $L^p_{\#}(\nu)$-norm.

Next, let us prove the estimate

$$\|\mathscr{P}_\tau u\|_{L^p_\#(\nu)} \leq \|u\|_{L^p_\#(\nu)}, \quad \tau > 0, \ u \in \mathscr{E}_\#(\mathbb{R}^{1+n}). \tag{5.2}$$

To this aim we use the inequality

$$|\mathscr{P}_\tau u(t,x)|^p \leq (\mathscr{P}_\tau(|u|^p))(t,x), \ \ \tau > 0, \ t \in \mathbb{R}, \ x \in \mathbb{R}^n. \tag{5.3}$$

For $p = 1$, (5.3) follows immediately from the definition of $\mathscr{P}_\tau$. For $p > 1$ we use the Hölder inequality as follows,

$$|\mathscr{P}_\tau u(t,x)|^p = \left| \int_{\mathbb{R}^n} u(t+\tau, U(t+\tau,t)x + y + g(t+\tau,t)) \mathscr{N}_{0,Q(t+\tau,t)}(dy) \right|^p$$

$$\leq \int_{\mathbb{R}^n} |u(t+\tau, U(t+\tau,t)x + y + g(t+\tau,t))|^p \mathscr{N}_{0,Q(t+\tau,t)}(dy) = (\mathscr{P}_\tau(|u|^p))(t,x).$$

Integrating (5.3) with respect to ν over $(0,T) \times \mathbb{R}^n$ and using (4.2) with u replaced by $|u|^p$ we get, for each $\tau > 0$,

$$\int_{(0,T)\times\mathbb{R}^n} |\mathscr{P}_\tau u|^p \nu(dt,dx) \leq \int_{(0,T)\times\mathbb{R}^n} \mathscr{P}_\tau(|u|^p) \, \nu(dt,dx) = \int_{(0,T)\times\mathbb{R}^n} |u|^p \nu(dt,dx),$$

and (5.2) is proved. Since $\mathscr{E}_\#(\mathbb{R}^{1+n})$ is dense in $L^p_\#(\nu)$, then $\mathscr{P}_\tau$ is a strongly continuous semigroup in $L^p_\#(\nu)$, and formulae (4.2) and (5.2) hold for each $u \in L^p_\#(\nu)$. $\square$

Let us define the differential operator

$$\mathscr{G}u(t,x) = u_t(t,x) + \frac{1}{2} \text{Tr} \, [B(t)B^*(t)D_x^2 u(t,x)] + \langle A(t)x + f(t), D_x u(t,x) \rangle, \tag{5.4}$$

and let us consider its realization in $L^2_\#(\nu)$ with domain $\mathscr{E}_\#(\mathbb{R}^{1+n})$.

Lemma 5.3 *For all $u \in \mathscr{E}_\#(\mathbb{R}^{1+n})$ we have*

$$\int_{(0,T)\times\mathbb{R}^n} \mathscr{G}u(t,x) \, \nu(dt,dx) = 0, \tag{5.5}$$

and

$$\int_{(0,T)\times\mathbb{R}^n} \mathscr{G}u(t,x) \; u(t,x) \nu(dt,dx) = -\frac{1}{2} \int_{(0,T)\times\mathbb{R}^n} |B^*(t)D_x u(t,x)|^2 \nu(dt,dx). \tag{5.6}$$

Proof. — Identity (5.5) follows from a direct verification. Moreover, since $u^2 \in \mathscr{E}_\#(\mathbb{R}^{1+n})$ and

$$(\mathscr{G}u^2)(t,x) = 2u(t,x)(\mathscr{G}u)(t,x) + |B^*(t)D_x u(t,x)|^2,$$

integrating this identity over $(0,T) \times \mathbb{R}^n$ and taking into account (5.5) yields (5.6). $\square$

From lemma 5.3 it follows that $(\mathscr{G}, \mathscr{E}_\#(\mathbb{R}^{1+n}))$ is dissipative in $L^2_\#(\nu)$ and consequently it is closable. Its closure is the infinitesimal generator of $\mathscr{P}_\tau$ in $L^2_\#(\nu)$, as the next proposition states.

Proposition 5.4 *The closure G of $(\mathscr{G}, \mathscr{E}_\#(\mathbb{R}^{1+n}))$ is the infinitesimal generator of the semigroup $\mathscr{P}_\tau$ in $L^2_\#(\nu)$. The space $\mathscr{E}_\#(\mathbb{R}^{1+n})$ is a core for G. For each $u \in D(G)$, the integral $\int_{(0,T)\times\mathbb{R}^n} Gu\, d\nu$ vanishes.*

Proof. — The space $\mathscr{E}_\#(\mathbb{R}^{1+n})$ is contained in the domain of the infinitesimal generator L of $\mathscr{P}_\tau$, because for $u = \phi(t)e^{i\langle x, h(t)\rangle}$ we have by (5.1)

$$\begin{aligned}\left(\frac{d}{d\tau}\mathscr{P}_\tau u_{|\tau=0}\right)(t,x) &= (\phi'(t) + i\phi(t)\langle x, h'(t)\rangle)e^{i\langle x,h(t)\rangle} \\ &\quad + \left[-\tfrac{1}{2}\,|B^*(t)h|^2 + i\langle A(t)x + f(t), h(t)\rangle\right] u(t,x), \\ &= (Gu)(t,x).\end{aligned}$$

Since $\mathscr{E}_\#(\mathbb{R}^{1+n})$ is invariant under $\mathscr{P}_\tau$ and dense in $L^2_\#(\nu)$, then it is a core for L, which means that it is dense in $D(L)$ for the graph norm. Then L is the closure of $(\mathscr{G}, \mathscr{E}_\#(\mathbb{R}^{1+n}))$, and the last statement follows from (5.5). $\square$

Some spectral properties of G are now easily available.

Corollary 5.5 *For any $z \in \sigma(G)$ and $k \in \mathbb{Z}$, $z + 2\pi ki/T \in \sigma(G)$. Moreover 0 is a simple eigenvalue of G.*

Proof. — For every $k \in \mathbb{Z}$ let us consider the unitary operator T_k in $L^2_\#(\nu)$ defined by $T_k u(t,x) = e^{2k\pi it/T}u(t,x)$. Since the spectrum of G is equal to the spectrum of $(T_k)^{-1}GT_k = G + (2k\pi i/T)I$, the first statement follows.

Since the invariant measure for $\mathscr{P}_\tau$ is unique, then it is ergodic, see [5, Thm. 3.2.6]. In its turn, ergodicity of ν is equivalent to the following property: for each $u \in L^2_\#(\nu)$ such that $\mathscr{P}_\tau u = u$ ν-a.e. for every $\tau > 0$, u is constant ν-a.e. See [5, Thm. 3.2.4]. The functions u such that $\mathscr{P}_\tau u = u$ for every $\tau > 0$ are just the elements of the kernel of G. Therefore, the kernel of G is one-dimensional and it consists of constant functions. Let now $u \in \operatorname{Ker} G^2$. Then $Gu \in \operatorname{Ker} G$, so that there is a constant c such that $Gu \equiv c$. Integrating over $(0,T)\times\mathbb{R}^n$ and recalling proposition 5.4 we get $c = 0$, so that $u \in \operatorname{Ker} G$. Therefore, $\operatorname{Ker} G^2 = \operatorname{Ker} G$, so that 0 is a simple eigenvalue. $\square$

Besides $\mathscr{E}_\#(\mathbb{R}^{1+n})$, other important subspaces of $L^2_\#(\nu)$ are invariant under $\mathscr{P}_\tau$ for each $\tau > 0$. The first one is the space $H^{0,1}_\#(\nu)$, defined by

$$H^{0,1}_\#(\nu) = \{u \in L^2_\#(\nu) :\ \exists D_{x_i}u \in L^2_\#(\nu),\ i = 1, \dots, n\},$$

with norm

$$\|u\|_{H^{0,1}_\#(\nu)} = \int_{(0,T)\times\mathbb{R}^n} u^2\nu(dt,dx) + \int_{(0,T)\times\mathbb{R}^n} |D_x u|^2\nu(dt,dx).$$

Lemma 5.6 *For each $\tau > 0$, $\mathscr{P}_\tau$ maps $H^{0,1}_\#(\nu)$ into itself, and*

$$|D_x\mathscr{P}_\tau u(t,x)|^2 \le M^2 e^{-2\omega\tau}\mathscr{P}_\tau(|D_x u|^2)(t,x),\ \ (t,x) \in \mathbb{R}^{1+n},\ u \in H^{0,1}_\#(\nu), \tag{5.7}$$

so that

$$\| \, |D_x \mathscr{P}_\tau u| \, \|_{L^2_\#(\nu)} \leq M e^{-\omega\tau} \| \, |D_x u| \, \|_{L^2_\#(\nu)}. \tag{5.8}$$

Here M, ω are the constants in assumption (1.8).

Proof. — For each $u \in H^{0,1}_\#(\nu)$, the representation formulae (2.10) and (4.1) yield

$$(D_x \mathscr{P}_\tau u)(t,x) = U^*(t+\tau,t)(\mathscr{P}_\tau(D_x u))(t,x), \;\; \tau > 0, \; (t,x) \in \mathbb{R}^{1+n}, \tag{5.9}$$

so that

$$\begin{aligned} &|D_x \mathscr{P}_\tau u(t,x)|^2 \leq M^2 e^{-2\omega\tau} |\mathscr{P}_\tau(D_x u)(t,x)|^2 \\ &\leq \left(\int_{\mathbb{R}^n} |D_x u(t+\tau, U(t+\tau,t)x + g(t+\tau,t) + y)| \mathscr{N}_{0,Q(t+\tau,t)}(dy) \right)^2 \end{aligned} \tag{5.10}$$

and (5.7) follows using the Hölder inequality. In its turn, (5.7) yields (5.8) integrating with respect to ν and using the invariance of $\mathscr{P}_\tau$. $\square$

In the elliptic case further information is available.

Proposition 5.7 *Assume that* $\det B(t) \neq 0$ *for each t. Then $D(G) \subset H^{0,1}_\#(\nu)$, and for each $u \in D(G)$ we have*

$$\begin{aligned} &\int_{(0,T)\times\mathbb{R}^n} Gu(t,x) \; u(t,x) \nu(dt,dx) \\ &= -\frac{1}{2} \int_{(0,T)\times\mathbb{R}^n} |B^*(t) D_x u(t,x)|^2 \nu(dt,dx). \end{aligned} \tag{5.11}$$

Proof. — For $u \in D(G)$ let $(u_k) \subset \mathscr{E}_\#(\mathbb{R}^{1+n})$ be a sequence such that

$$\lim_{k\to\infty} u_k = u, \quad \lim_{k\to\infty} Gu_k = Gu \quad \text{in} \;\; L^2_\#(\nu).$$

By (5.6) it follows that for any $m, k \in \mathbb{N}$,

$$\begin{aligned} &\int_{\mathbb{R}^{1+n}} G(u_k(t,x) - u_m(t,x)) \; (u_k(t,x) - u_m(t,x)) \nu(dt,dx) \\ &= -\frac{1}{2} \int_{\mathbb{R}^{n+1}} |B^*(t)(D_x u_k(t,x) - D_x u_m(t,x))|^2 \nu(dt,dx). \end{aligned}$$

Then the sequence $(|B^*(t) D_x u_k|)$ is a Cauchy sequence in $L^2_\#(\nu)$, and so is $(|D_x u_k|)$. Since ν is locally equivalent to the product measure $dt \times dx$, then for a.e. $t \in \mathbb{R}$ the function $u(t,\cdot)$ is in $H^1_{loc}(\mathbb{R}^n, dx)$, and (5.11) follows. $\square$

6 Asymptotic behaviour of $\mathscr{P}_\tau$ and $P_{s,t}$

In this section we want to investigate the asymptotic behaviour of $P_{s,t}\varphi$ when $t \to +\infty$ and s is fixed. It will be deduced both directly, from the expression of $P_{s,t}$, and from the asymptotic behavior of $\mathscr{P}_\tau u$ as $\tau \to +\infty$, for $u \in L^2_\#(\nu)$, or $u \in D(G)$.

It is convenient to set

$$\overline{u}_t := \int_{\mathbb{R}^n} u(t,x)\nu_t(dx), \quad u \in L^2_\#(\nu),\ t \in \mathbb{R}. \tag{6.1}$$

In particular, if $u(t,x) = \varphi(x)$ is independent of time, we set

$$\overline{\varphi}_t := \int_{\mathbb{R}^n} \varphi(x)\nu_t(dx), \quad t \in \mathbb{R}.$$

An immediate consequence of uniqueness of the invariant measure for $\mathscr{P}_\tau u$ is in the next proposition.

Proposition 6.1 *Assume that $\varphi : \mathbb{R}^n \mapsto \mathbb{R}$ belongs to $L^2(\mathbb{R}^n, \nu_t)$ for each $t \in \mathbb{R}$. Then we have*

$$\lim_{\tau\to+\infty} \frac{1}{\tau} \int_0^\tau (P_{t,t+\sigma}\varphi)(x)d\sigma = \int_{(0,T)\times\mathbb{R}^n} \varphi(y)d\nu \quad \text{in } L^2_\#(\nu). \tag{6.2}$$

Proof. — Since the measure ν is the unique invariant measure for $\mathscr{P}_\tau$, then it is ergodic, see e.g. [5, Thm. 3.2.6]. Through the Von Neumann Theorem, ergodicity is also equivalent to the fact that for any $u \in L^2_\#(\nu)$ we have

$$\lim_{\tau\to+\infty} \frac{1}{\tau} \int_0^\tau (\mathscr{P}_\sigma u)(t,x)d\sigma = \int_{(0,T)\times\mathbb{R}^n} u(s,y)\nu(ds,dy) \quad \text{in } L^2_\#(\nu). \tag{6.3}$$

See e.g. [5, Thm. 3.2.4]. Setting in (6.3) $u(t,x) = \varphi(x)$ and recalling (4.1) the conclusion follows. □

The following proposition shows that $P_{s,t}\varphi(x)$ approaches, as $t \to +\infty$, a periodic function in t which is independent both from x and from s.

Proposition 6.2 *For every $u \in C^\#_b(\mathbb{R}^{1+n})$ and for each compact set $K \subset \mathbb{R}^n$ we have*

$$\lim_{\tau\to+\infty} \sup\{|(\mathscr{P}_\tau u)(t,x) - \overline{u}_{t+\tau}| :\ t \in \mathbb{R},\ x \in K\} = 0. \tag{6.4}$$

It follows that for every $\varphi \in C_b(\mathbb{R}^n)$ we have

$$\lim_{t\to+\infty} (P_{s,t}\varphi)(x) - \int_{\mathbb{R}^n} \varphi(y)\nu_t(dy) = 0, \quad \forall s \in \mathbb{R},\ x \in \mathbb{R}^n, \tag{6.5}$$

and the convergence is uniform when (s,x) varies in any compact set.

Proof. — As a first step, we show that (6.5) holds for functions u which are Lipschitz continuous in x, uniformly with respect to t. In that case, recalling that $\overline{u}_{t+\tau} = \int_{\mathbb{R}^n} P_{t,t+\tau}u(t+\tau,\cdot)(y)\nu_t(dy)$ by (3.1), and using (5.9), we get

$$|(\mathscr{P}_\tau u)(t,x) - \overline{u}_{t+\tau}| = \left|\int_{\mathbb{R}^n} (P_{t,t+\tau}u(t+\tau,\cdot)(x) - P_{t,t+\tau}u(t+\tau,\cdot)(y))\nu_t(dy)\right|$$

$$\leq \|\,|D_x P_{t,t+\tau}u(t+\tau,\cdot)|\,\|_\infty \int_{\mathbb{R}^n} |x-y|\nu_t(dy) \leq Me^{-\omega\tau} \sup_{s\in\mathbb{R}} \|\,|D_x u(s,\cdot)|\,\|_\infty \int_{\mathbb{R}^n} |x-y|\nu_t(dy).$$

Since

$$\int_{\mathbb{R}^n} |x-y|\nu_t(dy) \leq \left(\int_{\mathbb{R}^n} |x-y|^2\nu_t(dy)\right)^{1/2} = (|x-g(t,-\infty)|^2 + \operatorname{Tr} Q(t,-\infty))^{1/2},$$

and $g(t,-\infty)$, $Q(t,-\infty)$ are bounded by constants independent of t, (6.4) follows.

Fixed any $u \in C_b^{\#}(\mathbb{R}^{1+n})$, set

$$u_k(t,x) = \frac{1}{k^n} \int_{\mathbb{R}^n} u(t,y)\varphi\left(\frac{x-y}{k}\right) dy,$$

where φ is any mollifier. The functions u_k belong to $C_b^{\#}(\mathbb{R}^{1+n})$, they are Lipschitz continuous in x, uniformly with respect to t, and they converge to u uniformly on each compact set. By lemma 4.3,

$$\lim_{k\to\infty} \sup_{\tau>0} |\mathscr{P}_\tau u_k(t,x) - \mathscr{P}_\tau u(t,x)| \to 0.$$

The same argument used in the proof of lemma 4.3 shows also that $\overline{u_k}_{\,t} - \overline{u}_t$ goes to zero as $k\to\infty$, uniformly in t. Splitting $(\mathscr{P}_\tau u)(t,x) - \overline{u}_{t+\tau}$ in the sum

$$\mathscr{P}_\tau(u-u_k)(t,x) + (\mathscr{P}_\tau u_k(t,x) - \overline{u_k}_{\,t+\tau}) + (\overline{u_k}_{\,t+\tau} - \overline{u}_{t+\tau}),$$

(6.4) follows.

If $u(t,x) = \varphi(x)$ is independent of time, (6.4) reduces to

$$\lim_{\tau\to+\infty} (P_{t,t+\tau}\varphi)(x) - \overline{\varphi}_{t+\tau} = 0,$$

uniformly in $\mathbb{R}\times K$. Setting $t+\tau = \xi$ we get

$$\lim_{\xi\to+\infty} (P_{t,\xi}\varphi)(x) - \overline{\varphi}_\xi = 0,$$

uniformly for (t,x) in any compact set. $\square$

It is important to know the speed of the convergence in (6.4). As in the autonomous case, in the elliptic case important information may be obtained from the Poincaré inequality, which is also interesting from its own.

Theorem 6.3 *Assume that* $\det B(t) \neq 0$ *for all* $t\in\mathbb{R}$. *Then for any* $u \in D(G)$ *we have*

$$\int_{(0,T)\times\mathbb{R}^n} (u(t,x) - \overline{u}_t)^2 \nu(dt,dx) \leq \frac{M^2C^2}{2\omega} \int_{(0,T)\times\mathbb{R}^n} |D_x u(t,x)|^2 \nu(dt,dx), \qquad (6.6)$$

where ω, M *are the constants in* (2.7), *and* $C = \sup_{0\leq t\leq T} \|B(t)\|$.

Proof. — It is enough to prove that (6.6) holds for each $u \in \mathscr{E}_\#(\mathbb{R}^{1+n})$. Then the statement will follow from propositions 5.4 and 5.7.

So, let $u \in \mathscr{E}_\#(\mathbb{R}^{1+n})$. Using (5.6) and then (5.8) we obtain, for each $\tau > 0$,

$$\frac{d}{d\tau}\int_{(0,T)\times\mathbb{R}^n} (\mathscr{P}_\tau u)^2 d\nu = -\int_{(0,T)\times\mathbb{R}^n} |B^*(t)D_x\mathscr{P}_\tau u|^2 d\nu$$

$$\geq -C^2\int_{(0,T)\times\mathbb{R}^n} |D_x\mathscr{P}_\tau u|^2 d\nu \geq -M^2C^2e^{-2\omega\tau}\int_{(0,T)\times\mathbb{R}^n} |D_x u|^2 d\nu.$$

Integrating with respect to τ yields

$$\int_{(0,T)\times\mathbb{R}^n} (\mathscr{P}_\tau u)^2 d\nu - \int_{(0,T)\times\mathbb{R}^n} u^2 d\nu \geq -M^2C^2\frac{1-e^{-2\omega\tau}}{2\omega}\int_{(0,T)\times\mathbb{R}^n} |D_x u|^2 d\nu. \qquad (6.7)$$

On the other hand, proposition 6.2 and the dominated convergence theorem imply that

$$\lim_{\tau\to+\infty}\int_{(0,T)\times\mathbb{R}^n} [(\mathscr{P}_\tau u)^2 - (\overline{u}_{t+\tau})^2] d\nu = 0.$$

Since $\int_{(0,T)\times\mathbb{R}^n} (\overline{u}_{t+\tau})^2 d\nu = \int_{(0,T)\times\mathbb{R}^n} (\overline{u}_t)^2 d\nu$ for each $\tau > 0$, then

$$\lim_{\tau\to+\infty}\int_{(0,T)\times\mathbb{R}^n} [(\mathscr{P}_\tau u)^2 - (\overline{u}_t)^2] d\nu = 0. \qquad (6.8)$$

Letting $\tau \to \infty$ in (6.7) and using (6.8) we get

$$\int_{(0,T)\times\mathbb{R}^n} ((\overline{u}_t)^2 - u^2) d\nu \geq -\frac{M^2C^2}{2\omega}\int_{(0,T)\times\mathbb{R}^n} |D_x u|^2 d\nu.$$

Since the left hand side is just $-\int_{(0,T)\times\mathbb{R}^n} (u(t,x) - \overline{u}_t)^2 d\nu$, the statement follows. □

Theorem 6.3 gives further information on the asymptotic behaviour of $\mathscr{P}_\tau u$ and of $P_{s,t}\varphi$. Set

$$C^* = (\sup_{0\leq t\leq T} \|(B(t))^{-1}\|)^{-1}.$$

Proposition 6.4 *Assume that* $\det B(t) \neq 0$ *for all* $t \in \mathbb{R}$. *Then for any* $u \in L^2_\#(\nu)$ *and* $\tau > 0$ *we have*

$$\int_{(0,T)\times\mathbb{R}^n} (\mathscr{P}_\tau(u - \overline{u}_t))^2 d\nu \leq e^{-c_0\tau}\int_{(0,T)\times\mathbb{R}^n} (u - \overline{u}_t)^2 d\nu, \qquad (6.9)$$

where $c_0 = 2\omega C^{*2}/M^2C^2$.

Consequently, for every $\varphi \in C_b(\mathbb{R}^n)$ *and* $\tau > 0$ *we have*

$$\int_{(0,T)\times\mathbb{R}^n} (P_{t,t+\tau}\varphi - \overline{\varphi}_t)^2 d\nu \leq e^{-c_0\tau}\int_{(0,T)\times\mathbb{R}^n} \varphi^2(x)\, d\nu. \qquad (6.10)$$

Proof. — If a function $v = v(t) \in L^2(0,T)$ does not depend on x, then $v \in L^2_\#(\nu)$ and also $\mathscr{P}_\tau v(t,x) = P_{t,t+\tau}v(t+\tau)$ is independent of x. In particular, for each $u \in D(G)$, $\mathscr{P}_\tau \overline{u}_t$ is independent of x.

So, for each $\tau > 0$ we have

$$\frac{d}{d\tau}\int_{(0,T)\times\mathbb{R}^n}(\mathscr{P}_\tau(u-\overline{u}_t))^2 d\nu = -\int_{(0,T)\times\mathbb{R}^n}|B^*(t)D_x\mathscr{P}_\tau(u-\overline{u}_t)|^2 d\nu$$

$$\leq -C^{*2}\int_{(0,T)\times\mathbb{R}^n}|D_x\mathscr{P}_\tau(u-\overline{u}_t)|^2 d\nu = -C^{*2}\int_{(0,T)\times\mathbb{R}^n}|D_x\mathscr{P}_\tau u|^2 d\nu.$$

By the Poincaré inequality (6.6),

$$\frac{d}{d\tau}\int_{(0,T)\times\mathbb{R}^n}(\mathscr{P}_\tau(u-\overline{u}_t))^2 d\nu \leq -c_0\int_{(0,T)\times\mathbb{R}^n}(\mathscr{P}_\tau u - \overline{\mathscr{P}_\tau u}_t)^2 d\nu. \tag{6.11}$$

Since $\{\nu_t :\ t \in \mathbb{R}\}$ is an evolution system of measures, we have

$$\overline{\mathscr{P}_\tau u}_t = \int_{\mathbb{R}^n} P_{t,t+\tau}u(t+\tau,x)d\nu_t = \int_{\mathbb{R}^n} u(t+\tau,x)d\nu_{t+\tau},$$

so that

$$\overline{\mathscr{P}_\tau u}_t = \overline{u}_{t+\tau} = \mathscr{P}_\tau\overline{u}_t, \ \ \forall \tau > 0.$$

Therefore, (6.11) reads as

$$\frac{d}{d\tau}\int_{(0,T)\times\mathbb{R}^n}(\mathscr{P}_\tau(u-\overline{u}_t))^2 d\nu \leq -c_0\int_{(0,T)\times\mathbb{R}^n}(\mathscr{P}_\tau(u-\overline{u}_t))^2 d\nu, \ \ \forall\tau > 0$$

and (6.9) follows. Since $D(G)$ is dense in $L^2_\#(\nu)$, then (6.9) holds true for each $u \in L^2_\#(\nu)$. □

We consider now the log-Sobolev inequality. We cannot expect to have an estimate completely similar to the classical one (see [7]), because it would imply that $\mathscr{P}_\tau$ is hypercontractive, which is obviously false because of the translation in time.

For its proof we need a lemma.

Lemma 6.5 *For any $u \in \mathscr{E}_\#(\mathbb{R}^{1+n})$ and $\gamma \in C^1(\mathbb{R})$ we have*

$$\int_{(0,T)\times\mathbb{R}^n}\gamma(u)Gu\,d\nu = -\frac{1}{2}\int_{(0,T)\times\mathbb{R}^n}\gamma'(u)|B^*(t)D_x u|^2 d\nu.$$

Proof. — Let Γ be a primitive of γ. Then the function $\Gamma(u)$ belongs to $D(G)$, and

$$G(\Gamma(u)) = 1/2\,\Gamma''(u)|B^*(t)Du|^2 + \Gamma'(u)Gu.$$

Integrating with respect to ν and recalling proposition 5.4 the statement follows. □

Theorem 6.6 *For any $u \in \mathscr{E}_\#(\mathbb{R}^{1+n})$ we have*

$$\int_{(0,T)\times\mathbb{R}^n}u^2\log u^2 d\nu \leq \frac{M^2C^2}{\omega}\int_{(0,T)\times\mathbb{R}^n}|D_x u|^2 d\nu + \frac{1}{T}\int_0^T\overline{u^2}_s\log\overline{u^2}_s ds, \tag{6.12}$$

where ω, M are the constants in (2.7), and $C = \sup_{0\leq t\leq T}\|B(t)\|$.

Proof. — Since ν is invariant for $\mathscr{P}_\tau$, then

$$\frac{d}{d\tau}\int_{(0,T)\times\mathbb{R}^n}\mathscr{P}_\tau u^2\,\log(\mathscr{P}_\tau u^2)=\int_{(0,T)\times\mathbb{R}^n}(G\mathscr{P}_\tau u^2\,\log(\mathscr{P}_\tau u^2)+G\mathscr{P}_\tau u^2)d\nu$$

$$=\int_{(0,T)\times\mathbb{R}^n}\mathscr{P}_\tau u^2\,\log(\mathscr{P}_\tau u^2)d\nu.$$

By lemma 6.5 applied to the function $\mathscr{P}_\tau u^2$ we get

$$\frac{d}{d\tau}\int_{(0,T)\times\mathbb{R}^n}\mathscr{P}_\tau u^2\,\log(\mathscr{P}_\tau u^2)d\nu=-\frac{1}{2}\int_{(0,T)\times\mathbb{R}^n}\frac{1}{\mathscr{P}_\tau u^2}\,|B^*(t)D_x\mathscr{P}_\tau u^2|^2d\nu$$
$$\geq-\frac{C^2}{2}\int_{(0,T)\times\mathbb{R}^n}\frac{1}{\mathscr{P}_\tau u^2}\,|D_x\mathscr{P}_\tau u^2|^2d\nu. \tag{6.13}$$

Formula (5.10) yields

$$|D_x\mathscr{P}_\tau u^2|^2\leq M^2e^{-2\omega\tau}(\mathscr{P}_\tau(|D_xu^2|))^2=M^2e^{-2\omega\tau}(\mathscr{P}_\tau(2|u|\,|D_xu|))^2$$

$$\leq 4M^2e^{-2\omega\tau}\mathscr{P}_\tau u^2\,\mathscr{P}_\tau(|D_xu|^2).$$

Substituting this inequality in (6.13) and using again the invariance of ν for $\mathscr{P}_\tau$ we get

$$\frac{d}{d\tau}\int_{(0,T)\times\mathbb{R}^n}\mathscr{P}_\tau u^2\,\log(\mathscr{P}_\tau u^2)\geq-2C^2M^2e^{-2\omega\tau}\int_{(0,T)\times\mathbb{R}^n}\mathscr{P}_\tau(|D_xu|^2)d\nu$$

$$=-2C^2M^2e^{-2\omega\tau}\int_{(0,T)\times\mathbb{R}^n}|D_xu|^2d\nu,\quad\tau>0.$$

Therefore, integrating with respect to τ,

$$\int_{(0,T)\times\mathbb{R}^n}\mathscr{P}_\tau u^2\,\log(\mathscr{P}_\tau u^2)d\nu-\int_{(0,T)\times\mathbb{R}^n}u^2\,\log(u^2)d\nu$$

$$\geq-\frac{C^2M^2}{\omega}(1-e^{-2\omega\tau})\int_{(0,T)\times\mathbb{R}^n}|D_xu|^2d\nu.$$

Recalling that

$$\lim_{\tau\to+\infty}\int_{(0,T)\times\mathbb{R}^n}\mathscr{P}_\tau u^2\,\log(\mathscr{P}_\tau u^2)d\nu=\int_{(0,T)\times\mathbb{R}^n}\overline{u^2}_t\log\overline{u^2}_td\nu,$$

letting $\tau\to+\infty$ we obtain

$$\int_{(0,T)\times\mathbb{R}^n}\overline{u^2}_t\log\overline{u^2}_td\nu-\int_{(0,T)\times\mathbb{R}^n}u^2\log u^2d\nu\geq-\frac{C^2M^2}{\omega}\int_{(0,T)\times\mathbb{R}^n}|D_xu|^2d\nu,$$

which coincides with (6.12). □

Using formula (6.12) for the function $v=u^{p/2}$, $p\geq 2$, we get the following corollary.

Corollary 6.7 *For any nonnegative $u\in\mathscr{E}_\#(\mathbb{R}^{1+n})$ and $p\geq 2$ we have*

$$\int_{(0,T)\times\mathbb{R}^n}u^p\log u^pd\nu\leq\frac{M^2C^2p^2}{4\omega}\int_{(0,T)\times\mathbb{R}^n}u^{p-2}|D_xu|^2d\nu+\frac{1}{T}\int_0^T\overline{u^p}_s\log\overline{u^p}_sds. \tag{6.14}$$

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