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Abstract

The contents of these notes were presented during ten lectures, in November 2011, by Peter Sjögren in Gothenburg. The text was written by Adam Andersson who participated and is improved after the careful reading by Peter Sjögren. Ornstein-Uhlenbeck theory can be described as a model of harmonic analysis in which Lebesgue measure is everywhere replaced by a Gaussian measure. The theory has applications in quantum physics and probability theory. If one passes to infinite dimensions and places the theory in a probabilistic context, one gets the Malliavin calculus. In Chapter 1, the basic theory is developed. This concerns the Hermite polynomials, the Ornstein-Uhlenbeck operator and most importantly its semigroup. The Hermite polynomials form an orthogonal system with respect to the Gaussian measure in Euclidean space. It turns out that they are the eigenfunctions of the Ornstein-Uhlenbeck operator, and since this operator is self-adjoint and positive semidefinite, the semigroup can be defined spectrally. An explicit kernel is derived for the semigroup, known as the Mehler kernel. It will be of central importance in this text. In Chapter 2, boundary convergence for the semigroup is considered, i.e., the limiting behavior of the semigroup as the "time" tends to zero. This is done by introducing a maximal operator for the semigroup and proving that it is of weak type (1,1). This result implies almost everywhere convergence for integrable boundary functions. In Chapter 3, first-order Riesz operators related to the Ornstein-Uhlenbeck operator are treated. Explicit off-diagonal kernels for these operators are found. It is finally proved that the Riesz operators are of weak type (1,1).

Contents

Notation

γ ... Gaussian measure on $\mathbb{R}^d$ or $\mathbb{R}$.
$L^p(\gamma)$ ... L^p with respect to γ.
$L^p(\mathrm{d}x)$ ... L^p with respect to Lebesgue measure.
$C_0^\infty(\mathbb{R}^d)$ ... smooth functions with compact support.
$\mathcal{F}$ or $\widehat{\ }$ ... Fourier transform.
$\int$ ... integration on the whole of $\mathbb{R}^d$ or $\mathbb{R}$.
$\frac{\mathrm{d}^i}{\mathrm{d}x^i}$ and D^i ... derivatives.
∂_i, ∂_{ij}, $\frac{\partial^i}{\partial x^i}$, $\frac{\partial}{\partial x_i \partial x_j}$ ... partial derivatives.
∂_i^* ... $L^2(\gamma)$ adjoint of ∂_i.
α, β ... multi-indices in $\mathbb{N}^d$.
$|\alpha| = \sum_{i=1}^d \alpha_i$.
$\alpha! = \prod_{i=1}^d \alpha_i!$.
$(e_i)_{i=1}^d$ standard basis in $\mathbb{R}^d$.
$\lesssim$ ... less than or equal, up to a constant factor $C = C(d) > 0$.
$\gtrsim$ ... greater than or equal, up to a constant factor $C = C(d) > 0$.
$\sim$ ... relation when both $\lesssim$ and $\gtrsim$ satisfied.
$a \wedge b$ ... minimum of a and b in $\mathbb{R}$.
$a \vee b$ ... maximum of a and b in $\mathbb{R}$.
$B(x, r)$ ... open ball centered at $x \in \mathbb{R}^d$ with radius $r > 0$.
$|B|$ ... Lebesgue measure of a measurable set $B \subset \mathbb{R}^d$.

CHAPTER 1

Basics of Ornstein-Uhlenbeck theory

In this first chapter, the basic framework of the theory will be developed. The Gaussian measure and the Hermite polynomials are introduced in the first section. The Hermite polynomials form a complete orthogonal system in the weighted L^2 space over $\mathbb{R}^d$ with respect to the Gaussian measure. In Section 2, a second-order differential operator, called the Ornstein-Uhlenbeck operator is defined. It plays the role of the Laplace operator in the Gaussian setting. The corresponding Ornstein-Uhlenbeck semigroup is defined spectrally. It turns out that this semigroup has an explicit integral kernel, which was found already 1866 by Mehler. All results of this chapter are well known; nevertheless, the rigorous derivation of these facts seems hard to find in the literature.

1. Gaussian measure and Hermite polynomials

Define on $\mathbb{R}^d$ the normalized Gaussian measure

$$\mathrm{d}\gamma(x) = \frac{1}{\pi^{d/2}} e^{-|x|^2}\,\mathrm{d}x.$$

Consider first the case $d = 1$. The Taylor expansion of e^{-x^2} at the point x, with increment t is

$$e^{-(x-t)^2} = \sum_{n=0}^{\infty} a_n t^n,$$

where

$$a_n = \frac{(-1)^n}{n!}\frac{\mathrm{d}^n}{\mathrm{d}x^n} e^{-x^2} = \text{polynomial} \times e^{-x^2}.$$

This series is convergent for all real or complex values of x and t, since we are dealing with an entire function. Multiply both sides by e^{x^2} to get

$$e^{2xt-t^2} = \sum_{n=0}^{\infty} \frac{1}{n!}(-1)^n e^{x^2}\frac{\mathrm{d}^n}{\mathrm{d}x^n} e^{-x^2} t^n.$$

It is clear that the coefficient of t^n here is a polynomial in x. We define the n:th *Hermite polynomial* H_n by

$$H_n(x) = (-1)^n e^{x^2}\frac{\mathrm{d}^n}{\mathrm{d}x^n} e^{-x^2}.$$

Then

$$e^{2xt-t^2} = \sum_{n=0}^{\infty} \frac{1}{n!} H_n(x) t^n. \tag{1.1}$$

The function e^{2xt-t^2} is called the generating function of $(H_n)_{n=0}^{\infty}$. By differentiation, we see that H_n is of the form

$$H_n(x) = 2^n x^n + \text{lower order terms}$$

and in particular

$$H_0 = 1.$$

PROPOSITION 1.1. *The polynomials* $(H_n)_{n=0}^{\infty}$ *form a complete orthogonal system in* $L^2(\gamma)$, *and* $\|H_n\|_{L^2(\gamma)} = 2^{n/2}\sqrt{n!}$.

PROOF. Let $m \leq n$. Using the definition of H_n and integrating by parts, we get, with $D = \mathrm{d}/\mathrm{d}x$,

$$\begin{aligned}\int H_m(x)H_n(x)\,\mathrm{d}\gamma(x) &= \frac{(-1)^n}{\sqrt{\pi}}\int H_m(x)e^{x^2}(D^n e^{-x^2})e^{-x^2}\,\mathrm{d}x\\ &= \frac{(-1)^n}{\sqrt{\pi}}\int H_m(x)D^n e^{-x^2}\,\mathrm{d}x\\ &= \frac{(-1)^n}{\sqrt{\pi}}(-1)^n\int (D^n H_m(x))e^{-x^2}\,\mathrm{d}x,\end{aligned}$$

and this vanishes if $m < n$. For $m = n$ the same calculation yields

$$\frac{1}{\sqrt{\pi}}\int D^n H_n(x)e^{-x^2}\,\mathrm{d}x = 2^n n!,$$

and thus

$$\|H_n\|_{L^2(\gamma)} = 2^{n/2}\sqrt{n!},$$

as claimed.

It remains to prove the completeness. Since any polynomial can be expressed as linear combinations of Hermite polynomials, it suffices to show that the set of all polynomials is dense in $L^2(\gamma)$. Assume that $f \in L^2(\gamma) \subset L^1(\gamma)$ is orthogonal to all polynomials. If f can be shown to be zero, completeness is proved. The product $f(x)e^{-x^2}$ is in $L^1(\,\mathrm{d}x)$, so it has a well-defined Fourier transform. Calculating this Fourier transform, expanding $e^{i\xi x}$ in a Taylor series and assuming that we can interchange the order of summation and integration, we get that

$$\int e^{i\xi x}f(x)\,\mathrm{d}\gamma(x) = \int\sum_{n=0}^{\infty}\frac{i^n\xi^n}{n!}x^n f(x)\,\mathrm{d}\gamma(x) = \sum_{n=0}^{\infty}\frac{i^n\xi^n}{n!}\int x^n f(x)\,\mathrm{d}\gamma(x) = 0, \quad \forall\xi\in\mathbb{R}. \tag{1.2}$$

We conclude that $f = 0$.

Finally, we must verify that the order of summation and integration in (1.2) can be switched. We shall majorize

$$\sum_{n=0}^{N}\frac{|\xi|^n}{n!}|x|^n|f(x)|$$

by an $L^1(\gamma)$ function, uniformly in $N \in \mathbb{N}$. But

$$\sum_{n=0}^{N}\frac{|\xi|^n}{n!}|x|^n|f(x)| \leq \sum_{n=0}^{\infty}\frac{|\xi|^n}{n!}|x|^n|f(x)| = e^{|\xi||x|}|f(x)|,$$

and by the Cauchy-Schwarz inequality

$$\int e^{|\xi||x|}|f(x)|\,\mathrm{d}\gamma(x) \leq \Big(\int |f(x)|^2\,\mathrm{d}\gamma(x)\Big)^{1/2}\Big(\int e^{2|\xi||x|}\,\mathrm{d}\gamma(x)\Big)^{1/2} < \infty.$$

□

To compute the derivative H_n', we assume for the moment that we can differentiate termwise with respect to x in the series in (1.1), so that

$$2te^{2xt-t^2} = \sum_{n=1}^{\infty}\frac{1}{n!}H_n'(x)t^n. \tag{1.3}$$

Using (1.1) again, we also get

$$2te^{2xt-t^2} = 2\sum_{n=0}^{\infty} \frac{1}{n!} H_n(x) t^{n+1} = 2\sum_{n=1}^{\infty} \frac{1}{(n-1)!} H_{n-1}(x) t^n.$$

By the uniqueness of the Taylor expansion, we can compare terms to see that

$$H_n'(x) = 2nH_{n-1}(x). \tag{1.4}$$

It remains to verify (1.3). For this purpose we use once more the generating function. The Taylor expansion of the function $-2xe^{-x^2} = De^{-x^2}$ at x is

$$-2(x-t)e^{-(x-t)^2} = \sum_{n=0}^{\infty} b_n t^n$$

with

$$\begin{aligned} b_n &= \frac{(-1)^n}{n!} D^{n+1} e^{-x^2} = \frac{(-1)^n}{n!} D\big(e^{-x^2} e^{x^2} D^n e^{-x^2}\big) \\ &= \frac{1}{n!} D(e^{-x^2} H_n(x)) = \frac{1}{n!}\big[(-2x)e^{-x^2} H_n(x) + e^{-x^2} H_n'(x)\big]. \end{aligned}$$

Summing in n and using (1.1) yields

$$\begin{aligned} -2(x-t)e^{2xt-t^2} &= -2x\sum_{n=0}^{\infty} \frac{1}{n!} H_n(x) t^n + \sum_{n=0}^{\infty} \frac{1}{n!} H_n'(x) t^n \\ &= -2xe^{2xt-t^2} + \sum_{n=0}^{\infty} \frac{1}{n!} H_n'(x) t^n. \end{aligned}$$

We have proved (1.3) and thus also (1.4).

Now, let $d \geq 1$. Hermite polynomials over $\mathbb{R}^d$ are defined by

$$H_\alpha(x) = \prod_{i=1}^{d} H_{\alpha_i}(x_i), \quad x = (x_1, \dots, x_d) \in \mathbb{R}^d,$$

where $\alpha \in \mathbb{N}^d$ is a multi-index. Then $H_\alpha = \otimes_{i=1}^d H_{\alpha_i}$ is a polynomial of degree $|\alpha|$. An easy check shows that $(H_\alpha)_{\alpha\in\mathbb{N}^d}$ is a complete orthogonal system in $L^2(\gamma)$. It is a direct consequence of (1.4) that for $d \geq 1$

$$\partial_i H_\alpha(x) = 2\alpha_i H_{\alpha - e_i}(x),$$

where $(e_i)_{i=1}^d$ is the standard basis of vectors in $\mathbb{R}^d$.

Finally, define the normalized Hermite polynomials

$$h_n(x) = \frac{1}{2^{n/2}\sqrt{n!}} H_n(x). \tag{1.5}$$

They will sometimes be used.

2. The Ornstein-Uhlenbeck operator and its semigroup

Let $\partial_i = \partial/\partial x_i$. The operator ∂_i is unbounded on $L^2(\gamma)$. We will explore its adjoint operator ∂_i^* in $L^2(\gamma)$. For this purpose, take $f, g \in C_0^\infty(\mathbb{R}^d)$, i.e., infinitely many times differentiable functions with compact support. Then

$$\begin{aligned} \langle \partial_i f, g\rangle_{L^2(\gamma)} &= \frac{1}{\pi^{d/2}} \int \partial_i f(x) g(x) e^{-|x|^2}\,\mathrm{d}x \\ &= \frac{1}{\pi^{d/2}} \int f(x)[2x_i g(x) - \partial_i g(x)] e^{-|x|^2}\,\mathrm{d}x \\ &= \langle f, (2x_i - \partial_i) g\rangle_{L^2(\gamma)}. \end{aligned}$$

We see that

$$\partial_i^* = 2x_i - \partial_i,$$

where the first term is a multiplication operator. Define a second-order differential operator by

$$L = \frac{1}{2}\sum_{i=1}^{d} \partial_i^* \partial_i = -\frac{1}{2}\Delta + x \cdot \text{grad}.$$

It is positive and symmetric and plays the role of the Laplacian on $L^2(\gamma)$. Symmetry is shown by

$$\langle Lf, g\rangle = \frac{1}{2}\sum_{i=1}^{d}\langle \partial_i^*\partial_i f, g\rangle = \frac{1}{2}\sum_{i=1}^{d}\langle \partial_i f, \partial_i g\rangle = \frac{1}{2}\sum_{i=1}^{d}\langle f, \partial_i^*\partial_i g\rangle = \langle f, Lg\rangle.$$

Positivity follows by setting $f = g$ in the middle expression above. The operator L is called the *Ornstein-Uhlenbeck operator.*

PROPOSITION 2.1. *The Hermite polynomials are eigenvectors for the Ornstein-Uhlenbeck operator. Moreover, for any multi-index $\alpha \in \mathbb{N}^d$,*

$$LH_\alpha = |\alpha| H_\alpha.$$

PROOF. Again consider $d = 1$. We first explore the action of D^* on H_n.

$$\langle D^* H_{n-1}, H_j\rangle = \langle H_{n-1}, DH_j\rangle = 2j\langle H_{n-1}, H_{j-1}\rangle = 0, \quad j \neq n.$$

So, $D^* H_{n-1}$ is a multiple of H_n. Take $j = n$.

$$\langle D^* H_{n-1}, H_n\rangle = 2n\langle H_{n-1}, H_{n-1}\rangle = 2n2^{n-1}(n-1)! = 2^n n! = \langle H_n, H_n\rangle.$$

Thus $D^* H_{n-1} = H_n$ and it follows that $\partial_i^* H_{\alpha - e_i} = H_\alpha$, for $d \geq 1$. We are ready to compute the action of L on H_α:

$$LH_\alpha = \frac{1}{2}\sum_{i=1}^{d}\partial_i^*\partial_i H_\alpha = \frac{1}{2}\sum_{i=1}^{d}\partial_i^* 2\alpha_i H_{\alpha-e_i} = \sum_{i=1}^{d}\alpha_i H_\alpha = |\alpha| H_\alpha.$$

□

We now turn to the Ornstein-Uhlenbeck semigroup, i.e., the semigroup generated by L. For this purpose we use our spectral decomposition of $L^2(\gamma)$. Let $(T_t)_{t\geq 0} = (e^{-tL})_{t\geq 0}$ be the family of bounded linear operators acting on

$$f = \sum_{\alpha\in\mathbb{N}^d} a_\alpha H_\alpha \in L^2(\gamma) \tag{1.6}$$

by

$$e^{-tL} f = \sum_{\alpha\in\mathbb{N}^d} e^{-t|\alpha|} a_\alpha H_\alpha. \tag{1.7}$$

In particular

$$e^{-tL} H_\alpha = e^{-t|\alpha|} H_\alpha. \tag{1.8}$$

It follows that e^{-tL} is a bounded operator on $L^2(\gamma)$ for any $t \geq 0$ and that

$$e^{-tL}e^{-sL} = e^{-(s+t)L}, \quad s, t \geq 0.$$

Since T_0 is the identity, $(T_t)_{t\geq 0}$ forms a semigroup.

Before continuing, we give a short review of Hilbert-Schmidt integral operators and their kernels. Any $\Phi \in L^2(\gamma \times \gamma)$ defines a bounded linear operator on $L^2(\gamma)$ by

$$Tf(x) = \int \Phi(x,y) f(y) \, d\gamma(y). \tag{1.9}$$

It is not essential here that we work in our Gaussian setting. Any L^2-space would do fine. We verify the boundedness. The Cauchy-Schwarz inequality gives that

$$(Tf(x))^2 \leq \int |\Phi(x,y)|^2 \, d\gamma(y) \int |f(y)|^2 \, d\gamma(y).$$

Integrating both sides in x leads to

$$\|Tf\|^2 \leq \|\Phi\|^2_{L^2(\gamma\times\gamma)} \|f\|^2. \tag{1.10}$$

We now leave the general situation.

The operator T_t, for $t > 0$, is given by a kernel in the sense that

$$T_t f(x) = \int_{\mathbb{R}^d} M_t^\gamma(x,y) f(y) \, d\gamma(y). \tag{1.11}$$

The explicit expression for this kernel was found already in 1866 by Mehler, [**3**]. It is named the Mehler kernel. Using the normalized Hermite polynomials h_α, we shall first verify that the kernel can be expressed in the form

$$M_t^\gamma(x,y) = \sum_{\alpha\in\mathbb{N}^d} e^{-t|\alpha|} h_\alpha(x) h_\alpha(y). \tag{1.12}$$

It is easy to check that this series converges in $L^2(\gamma \times \gamma)$. Consider, for $N \in \mathbb{N}$, the truncated kernel

$$\sum_{|\alpha|<N} e^{-t|\alpha|} h_\alpha(x) h_\alpha(y).$$

For $|\beta| < N$, the corresponding operator acts on H_β as

$$\begin{aligned} &\int \sum_{|\alpha|<N} e^{-t|\alpha|} h_\alpha(x) h_\alpha(y) H_\beta(y) \, d\gamma(y) \\ &\quad = e^{-t|\beta|} \langle h_\beta, H_\beta\rangle h_\beta(x) = e^{-t|\beta|} \|H_\beta\| h_\beta(x) = e^{-t|\beta|} H_\beta = T_t H_\beta. \end{aligned} \tag{1.13}$$

Since the truncated kernels converge in $L^2(\gamma \times \gamma)$, the corresponding operators converge in the operator norm, by (1.10). We conclude that (1.11) holds.

We next want to compute a closed expression for M_t^γ. Let $d = 1$. Since $\mathcal{F}(e^{-\xi^2})(x) = \sqrt{\pi} e^{-x^2/4}$, where $\mathcal{F}$ denotes the Fourier transform, H_n can be written

$$\begin{aligned} H_n(y) &= (-1)^n e^{y^2} \left(\frac{d}{dy}\right)^n e^{-y^2} = (-1)^n e^{y^2} \left(\frac{d}{dy}\right)^n \frac{1}{\sqrt{\pi}} \int e^{2iy\xi-\xi^2} \, d\xi \\ &= (-1)^n e^{y^2} \frac{2^n i^n}{\sqrt{\pi}} \int \xi^n e^{2iy\xi-\xi^2} \, d\xi. \end{aligned}$$

Assuming that the order of summation and integration can be switched, we get using (1.5)

$$\begin{aligned} M_t^\gamma(x,y) &= \sum_{n=0}^\infty e^{-tn} h_n(x) h_n(y) \\ &= \sum_{n=0}^\infty e^{-tn} \frac{1}{2^n n!} H_n(x) (-1)^n e^{y^2} \frac{2^n i^n}{\sqrt{\pi}} \int \xi^n e^{2iy\xi-\xi^2} \, d\xi \\ &= \frac{1}{\sqrt{\pi}} e^{y^2} \int \sum_{n=0}^\infty \frac{1}{n!} (-i\xi e^{-t})^n H_n(x) e^{2iy\xi-\xi^2} \, d\xi. \end{aligned}$$

The expansion (1.1) of the generating function gives that

$$\sum_{n=0}^{\infty} \frac{1}{n!}(-i\xi e^{-t})^n H_n(x) = e^{-i2\xi e^{-t}x+\xi^2 e^{-2t}}.$$

Hence,

$$M_t^\gamma(x,y) = \frac{e^{y^2}}{\sqrt{\pi}} \int e^{2i\xi(y-e^{-t}x)-\xi^2(1-e^{-2t})}\,\mathrm{d}\xi.$$

Let $\xi' = \xi\sqrt{1-e^{-2t}}$. Then, taking the inverse Fourier transform yields

$$\begin{aligned} M_t^\gamma(x,y) &= \frac{1}{\sqrt{\pi}} \frac{e^{y^2}}{\sqrt{1-e^{-2t}}} \int e^{2i\xi' \frac{y-e^{-t}x}{\sqrt{1-e^{-2t}}} - \xi'^2}\,\mathrm{d}\xi' \\ &= \frac{e^{y^2}}{\sqrt{1-e^{-2t}}} e^{-\frac{(y-e^{-t}x)^2}{1-e^{-2t}}}. \end{aligned}$$

This is a closed expression for the kernel, but it remains to verify the switch of order above. For this we use dominated convergence. Introduce $s = -i\xi e^{-t}$. Now, two Taylor expansions give

$$e^{2xs-s^2} = \sum_{k=0}^{\infty} \frac{1}{k!}(2xs)^k \sum_{l=0}^{\infty} \frac{1}{l!}(-s^2)^l = \sum_{k=0}^{\infty}\sum_{l=0}^{\infty} c_{kl}(x)s^{k+2l} = \sum_{n=0}^{\infty} \frac{1}{n!} H_n(x)s^n.$$

for some coefficients c_{kl}. The last equality is nothing but (1.1). For any $N \geq 0$,

$$\begin{aligned} \Big|\sum_{n=0}^{N} \frac{1}{n!} H_n(x)s^n\Big| &\leq \sum_{n=0}^{\infty} \frac{1}{n!} |H_n(x)||s|^n \\ &\leq \sum_{k=0}^{\infty}\sum_{l=0}^{\infty} |c_{kl}(x)||s|^{k+2l} \\ &= \sum_{k=0}^{\infty} \frac{1}{k!}\big(2|x||s|\big)^k \sum_{l=0}^{\infty} \frac{1}{l!}|s|^{2l} \\ &= e^{2|x||s|}e^{|s|^2}. \end{aligned}$$

Then, for all $N \geq 0$,

$$\Big|\sum_{n=0}^{N} \frac{1}{n!}(-i\xi e^{-t})^n H_n(x)\, e^{2i\xi y-\xi^2}\Big| \leq e^{2|x||\xi|e^{-t}+|\xi|^2e^{-2t}-|\xi|^2}.$$

But the right hand side here is in $L^1(\mathrm{d}\xi)$. Thus, a majorizing function of the partial sums has been found. We are done for the one-dimensional case.

Let $d \geq 1$. Then

$$\begin{aligned} M_t^\gamma(x,y) &= \sum_{\alpha\in\mathbb{N}^d} e^{-t|\alpha|} h_\alpha(x)h_\alpha(y) \\ &= \prod_{i=1}^{d} \sum_{\alpha_i=0}^{\infty} e^{-t\alpha_i} h_{\alpha_i}(x_i)h_{\alpha_i}(y_i) \\ &= \prod_{i=1}^{d} \frac{e^{y_i^2}}{\sqrt{1-e^{-2t}}} e^{-\frac{(y_i-e^{-t}x_i)^2}{1-e^{-2t}}} \\ &= \frac{e^{|y|^2}}{(1-e^{-2t})^{d/2}} e^{-\frac{|y-e^{-t}x|^2}{1-e^{-2t}}}. \end{aligned}$$

Notice that M_t^γ is symmetric, since

$$M_t^\gamma(x,y) = \frac{1}{(1-e^{-2t})^{d/2}} e^{\frac{-e^{-2t}|x|^2 + 2e^{-t}\langle x,y\rangle - e^{-2t}|y|^2}{1-e^{-2t}}} = M_t^\gamma(y,x). \tag{1.14}$$

The kernel M_t, for integration against Lebesgue measure, is

$$M_t(x,y) = \frac{1}{\pi^{d/2}(1-e^{-2t})^{d/2}} e^{-\frac{|y-e^{-t}x|^2}{1-e^{-2t}}},$$

in the sense that

$$T_t f(x) = \int M_t(x,y) f(y)\,\mathrm{d}y.$$

Integration against M_t^γ is well defined for $f \in L^1(\gamma)$, so we use (1.11) to extend the domain of T_t to $L^1(\gamma)$, which of course contains $L^p(\gamma)$ for $1 \le p \le \infty$. We summarize the main result of this section in the following theorem.

THEOREM 2.2. *Let $1 \le p \le \infty$. For each $f \in L^p(\gamma)$, $t > 0$ and $x \in \mathbb{R}^d$, the function $T_t f = e^{-tL} f$ is given by*

$$T_t f(x) = \int M_t(x,y) f(y)\,\mathrm{d}y, \tag{1.15}$$

where

$$M_t(x,y) = \frac{1}{\pi^{d/2}(1-e^{-2t})^{d/2}} e^{-\frac{|y-e^{-t}x|^2}{1-e^{-2t}}}.$$

Notice that, since $T_t 1 = T_t H_0 = H_0 = 1$ we have

$$\int M_t^\gamma(x,y)\,\mathrm{d}\gamma(y) = 1 = \int M_t^\gamma(x,y)\,\mathrm{d}\gamma(x).$$

Now, let $f \in L^1(\gamma)$. Then, by Tonelli's theorem and the fact that M_t^γ is positive,

$$\begin{aligned} \|T_t f(x)\|_{L^1(\gamma)} &= \int \Big| \int M_t^\gamma(x,y) f(y)\,\mathrm{d}\gamma(y)\Big|\,\mathrm{d}\gamma(x) \\ &\le \int |f(y)| \int M_t^\gamma(x,y)\,\mathrm{d}\gamma(x)\,\mathrm{d}\gamma(y) \\ &= \|f\|_{L^1(\gamma)}. \end{aligned}$$

Hence, T_t is non-expansive on $L^1(\gamma)$. By duality $T_t = T_t^*$ is also non-expansive on $L^\infty = L^\infty(\gamma)$. For $1 < p < \infty$ we let q be the dual exponent of p. Then by Hölder's inequality for $f \in L^p(\gamma)$

$$\begin{aligned} |T_t f(x)|^p &= \Big| \int M_t^\gamma(x,y)^{1/q+1/p} f(y)\,\mathrm{d}\gamma(y)\Big|^p \\ &\le \Big(\int (M_t^\gamma(x,y)^{1/q})^q\,\mathrm{d}\gamma(y)\Big)^{\frac{p}{q}} \int (M_t^\gamma(x,y)^{1/p})^p |f(y)|^p\,\mathrm{d}\gamma(y) \\ &= T_t|f|^p(x). \end{aligned}$$

Integrating both sides yields

$$\|T_t f\|_{L^p(\gamma)}^p \le \|T_t|f|^p\|_{L^1(\gamma)} \le \|f\|_{L^p(\gamma)}^p.$$

Thus T_t is non-expansive on every $L^p(\gamma)$.

REMARK 2.3. The function

$$u(x,t) = (T_t f)(x)$$

solves the heat equation

$$\frac{\partial}{\partial t}u + Lu = 0, \quad t > 0, \tag{1.16}$$

with

$$u(x,0) = f(x) \tag{1.17}$$

for $f \in L^2(\gamma)$. This will not be of essential interest for us, and we only observe that it follows by termwise differentiation in (1.7) and Proposition 2.1.

We end this section with a remark concerning the case $d = \infty$.

REMARK 2.4. Making the change of variable

$$z = \frac{y - e^{-t}x}{\sqrt{1-e^{-2t}}}$$

in (1.15), we get

$$T_t f(x) = \int M_t(x,y) f(y)\,\mathrm{d}y = \int f(e^{-t}x + z\sqrt{1-e^{-2t}}\,)\,\mathrm{d}\gamma(z).$$

This is sometimes called Mehler's formula. In this form, the semigroup can be defined in infinite dimension. If γ_1 is the Gaussian measure on $\mathbb{R}$, the measure

$$\gamma_\infty = \prod_{i=1}^{\infty} \gamma_1$$

is a well-defined Gaussian measure on $\mathbb{R}^{\mathbb{N}}$. Mehler's formula still holds in this case. Much of the theory developed so far holds in infinite dimension. The Hermite polynomials on $\mathbb{R}^{\mathbb{N}}$ are defined by

$$H_\alpha = \prod_{i=1}^{\infty} H_{\alpha_i},$$

where $\alpha \in \mathbb{N}^{\mathbb{N}}$ is a multi-index with no more than finitely many non-zero elements, i.e., with $|\alpha| < \infty$. This will be all about the case $d = \infty$ in this course.

CHAPTER 2

Boundary convergence and maximal functions

In this chapter, the action of the semigroup as $t \to 0$ will be explored. The goal is to prove almost everywhere pointwise convergence $T_t f \to f$ as $t \to 0$. The first section contains a somewhat elementary result for smooth functions with compact support. The convergence is in this case pointwise. In the second section, maximal functions are introduced. They will be of central importance in the third section. A lemma, relating boundary convergence and maximal functions, will be proved. Also a convolution inequality involving the Hardy-Littlewood maximal operator is proved. In Section 3, which forms the core of this chapter, it is proved that the the maximal operator for the semigroup is of weak type $(1,1)$. In terms of boundary convergence, this implies almost everywhere convergence for $L^p(\gamma)$-functions, for $1 \leq p \leq \infty$. The proof is carried out by considering a local and a global part of the maximal operator. The first proof was given for $d = 1$ by Muckenhoupt in 1969, [**4**], and for $d < \infty$ by Sjögren in 1983, [**7**]. The proof presented here is from 2003 and due to García-Cuerva, Mauceri, Meda, Sjögren and Torrea, [**2**].

1. Boundary convergence for smooth functions

What happens with $T_t f$ if we let $t \to 0$? Recall that $T_0 = I$. In what sense will $T_t f \to f$? Convergence holds in $L^2(\gamma)$ for $f \in L^2(\gamma)$. To see this, let

$$f = \sum_{\alpha \in \mathbb{N}^d} a_\alpha H_\alpha \in L^2(\gamma).$$

Parseval's identity and dominated convergence imply

$$\|T_t f - f\|^2_{L^2(\gamma)} = \sum_{\alpha \in \mathbb{N}^d} (e^{-t|\alpha|} - 1)^2 |a_\alpha|^2 \|H_\alpha\|^2 \to 0.$$

Lemma 1.1. *Let $f \in C_0^\infty(\mathbb{R}^d)$. Then $T_t f \to f$ pointwise as $t \to 0$.*

Proof. Recall that M_t^γ integrates to one. Fixing x, we write the difference as

$$\begin{aligned} T_t f(x) - f(x) &= \int M_t^\gamma(x,y)(f(y) - f(x))\,\mathrm{d}\gamma(y) \\ &= \Big(\int_{|y-x|<\delta} + \int_{|y-x|\geq\delta} \Big) M_t^\gamma(x,y)(f(y) - f(x))\,\mathrm{d}\gamma(y), \end{aligned}$$

where $\delta > 0$ is chosen as follows. For a given $\epsilon > 0$, take δ so that

$$|f(y) - f(x)| < \epsilon \quad \text{for} \quad |x - y| < \delta.$$

We estimate the first integral.

$$\Big| \int_{|y-x|<\delta} M_t^\gamma(x,y)(f(y) - f(x))\,\mathrm{d}\gamma(y) \Big| \leq \epsilon \int M_t^\gamma(x,y)\,\mathrm{d}\gamma(y) = \epsilon.$$

Now, consider $|y - x| \geq \delta$. For $t > 0$ small enough, one has by the triangle inequality

$$|y - e^{-t}x| = |y - x + x - e^{-t}x| \geq |y - x| - |x|(1 - e^{-t}) \geq \frac{\delta}{2}.$$

Using the explicit expression for the kernel and making the change of variable $z = y - e^{-t}x$ followed by $w = z/\sqrt{1-e^{-2t}}$ we get that

$$\begin{aligned}
&\Big| \int_{|y-x|\geq\delta} M_t(x,y)(f(y)-f(x))\,\mathrm{d}y \Big| \\
&\quad\lesssim \frac{1}{\pi^{d/2}(1-e^{-2t})^{d/2}} \int_{|z|>\frac{\delta}{2}} e^{-\frac{|z|^2}{1-e^{-2t}}}\,\mathrm{d}z\,\|f\|_{L^\infty} \\
&\quad\lesssim \int_{|w|>\frac{\delta}{2\sqrt{1-e^{-2t}}}} e^{-|w|^2}\,\mathrm{d}w\,\|f\|_{L^\infty} \\
&\quad\to 0, \quad \text{as} \quad t\to 0.
\end{aligned}$$

The lemma is proved. □

2. Maximal functions

We define the Ornstein-Uhlenbeck maximal function as

$$T_* f(x) = \sup_{t>0} |T_t f(x)|.$$

for $f \in L^1(\gamma)$.

The operator T_* is sublinear, i.e., for all $\alpha, \beta \geq 0$

$$T_*(\alpha f + \beta g) \leq \alpha T_* f + \beta T_* g.$$

What are the $L^p(\gamma)$-properties of T_*? In order to get sharp results, we need the notion of weak $L^p(\gamma)$. This is the space of all measurable functions f satisfying, for all $\lambda > 0$, the condition

$$\gamma\{x : |f(x)| > \lambda\} \lesssim \frac{1}{\lambda^p}.$$

Weak $L^p(\gamma)$ contains $L^p(\gamma)$, since

$$\lambda^p \gamma\{x : |f| > \lambda\} \leq \int |f|^p\,\mathrm{d}\gamma = \|f\|^p_{L^p(\gamma)}. \tag{2.1}$$

This is nothing but Chebyshev's inequality. For example, in case of Lebesgue measure and $d = 1$, the function $x^{-1} \in \mathrm{weak}L^1 \setminus L^1$ by a simple calculation. Then, for our Gaussian measure, the function $x^{-1}e^{x^2} \in \mathrm{weak}L^1(\gamma) \setminus L^1(\gamma)$. A sublinear operator S is said to be of weak type (p,p) if

$$S : L^p \to \text{weak}\, L^p(\gamma)$$

boundedly, or, differently stated, if for all $f \in L^p(\gamma)$ and $\lambda > 0$

$$\gamma\{x : Sf > \lambda\} \lesssim \frac{\|f\|^p_{L^p(\gamma)}}{\lambda^p},$$

From Chebyshev's inequality, it follows that L^p boundedness implies weak type (p,p). The converse implication is not true in general. For maximal operators in general and for T_* in particular L^1 boundedness does not hold. The sharpest result of this kind we can get for T_* in our setting is weak type $(1,1)$. By the Marcinkiewicz Interpolation Theorem and the easily verified L^∞ boundedness, T_* is bounded on L^p, for $1 < p \leq \infty$. The next lemma gives an implication, in terms of boundary convergence.

Lemma 2.1. *Let $1 \leq p < \infty$. If T_* is of weak type (p,p) for γ, then $T_t f(x) \to f(x)$ for γ-almost all $x \in \mathbb{R}^d$ if $f \in L^p(\gamma)$.*

Proof. Let $f \in L^p(\gamma)$. With $\delta > 0$, we take $g \in C_0^\infty(\mathbb{R}^d)$ such that $\|f-g\|_{L^p(\gamma)} < \delta$. Then

$$T_t f(x) - f(x) = T_t(f-g)(x) + T_t g(x) - g(x) + g(x) - f(x).$$

By Lemma 1.1 and the sublinearity of lim sup, we get

$$\limsup_{t\to 0} |T_t f(x) - f(x)| \le T_*(f-g)(x) + |g(x) - f(x)|.$$

Thus

$$\begin{aligned}&\{x\colon \limsup_{t\to 0} |T_t f(x) - f(x)| > \lambda\}\\ &\qquad \subset \Big\{x\colon T_*(f-g)(x) > \frac{\lambda}{2}\Big\} \cup \Big\{x\colon |g(x)-f(x)| > \frac{\lambda}{2}\Big\}.\end{aligned}$$

The weak type $(1,1)$ assumption for T_* and Chebyshev's inequality lead to

$$\begin{aligned}&\gamma\{x\colon \limsup_{t\to 0} |T_t f(x) - f(x)| > \lambda\}\\ &\qquad \le \gamma\Big\{x\colon T_*(f-g)(x) > \frac{\lambda}{2}\Big\} + \gamma\Big\{x\colon |g(x)-f(x)| > \frac{\lambda}{2}\Big\}\\ &\qquad \lesssim \frac{\|f-g\|_{L^p(\gamma)}^p}{\lambda^p}\\ &\qquad \le \Big(\frac{\delta}{\lambda}\Big)^p.\end{aligned}$$

For any $\lambda > 0$ we can choose $\delta > 0$ so that this quantity is arbitrarily small. It follows that

$$\gamma\{x\colon \limsup_{t\to 0} |T_t f(x) - f(x)| > \lambda\} = 0,$$

for all $\lambda > 0$, and consequently $T_t f \to f$ except on a γ-null set of $\mathbb{R}^d$. □

Before ending this section, we consider $\mathbb{R}^d$ with Lebesgue measure and the Hardy-Littlewood maximal operator M. It is defined, for $f \in L^1_{\mathrm{loc}}(\mathrm{d}x)$, by

$$Mf(x) = \sup_{r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y)|\,\mathrm{d}y.$$

Here $B(x,r)$ is the ball of radius r centered at x and $|B(x,r)|$ its Lebesgue measure. It is known that M is of weak type $(1,1)$.

Theorem 2.2. *Let $\varphi \in L^1(\mathbb{R}^d)$ be non-negative, radial and decreasing, i.e., $\varphi(x) = \psi(|x|)$ for some decreasing function $\psi\colon \mathbb{R}_+ \to \mathbb{R}_+$. Set*

$$\varphi_t(x) = t^{-d}\varphi\Big(\frac{x}{t}\Big).$$

Then

$$\sup_{t>0} \varphi_t * |f| \le \int \varphi(x)\,\mathrm{d}x\; Mf.$$

The operator defined by $M_\varphi f = \sup_{t>0} \varphi_t * |f|$ is of weak type $(1,1)$.

Proof. We will use the fact that, for almost all $|z| > 0$,

$$\varphi(z) = \psi(|z|) = -\int_{|z|}^\infty \mathrm{d}\psi(s).$$

Let $f \geq 0$. Then, by a change of variable and Fubini's theorem, we have

$$\begin{aligned}\varphi_t * f(x) &= \int t^{-d}\varphi\Big(\frac{y}{t}\Big) f(x-y)\,\mathrm{d}y \\ &= \int \varphi(z) f(x-tz)\,\mathrm{d}z \\ &= -\int\int_{|z|}^{\infty} \mathrm{d}\psi(s) f(x-tz)\,\mathrm{d}z \\ &= -\int_0^{\infty}\int_{|z|<s} f(x-tz)\,\mathrm{d}z\,\mathrm{d}\psi(s).\end{aligned}$$

Consider the inner integral here. Changing the variable back to y, we get the estimate

$$\begin{aligned}\int_{|z|<s} f(x-tz)\,\mathrm{d}z = t^{-d}\int_{|y|<st} f(x-y)\,\mathrm{d}y \\ = \frac{|B(x,s)|}{|B(x,st)|}\int_{B(x,st)} f(y)\,\mathrm{d}y \\ \leq |B(x,s)| Mf(x).\end{aligned}$$

Thus

$$\varphi_t * f(x) \leq -\int_0^{\infty} |B(x,s)|\,\mathrm{d}\psi(s)\; Mf(x). \tag{2.2}$$

Repeating the calculations with $f = 1$ yields

$$-\int_0^{\infty} |B(x,s)|\,\mathrm{d}\psi(s) = (\varphi_t * 1)(x) = \int_{\mathbb{R}^d} \varphi(z)\,\mathrm{d}z. \tag{2.3}$$

Combining (2.2) and (2.3) gives the result, since the weak type $(1,1)$ of M_ϕ follows from that of M. □

3. The weak type $(1,1)$ property of the Ornstein-Uhlenbeck maximal operator

THEOREM 3.1. *Let $d < \infty$. The Ornstein-Uhlenbeck maximal operator T_* is of weak type $(1,1)$ for γ, i.e.,*

$$\gamma\{x \colon T_* f(x) > \lambda\} \lesssim \frac{\|f\|_{L^1(\gamma)}}{\lambda}, \quad \lambda > 0,$$

for $f \in L^1(\gamma)$.

For the proof of the weak type estimate we will split T_* into a local and a global part. Local here means that integration is done in suitable "local balls" where the Gaussian density $e^{-|x|^2}$ has constant order of magnitude. This leads to the condition

$$e^{-|x+h|^2} = e^{-|x|^2 - 2\langle x,h\rangle - |h|^2} \sim e^{-|x|^2},$$

which is certainly fulfilled if $|h| < C/|x|$ and $|h| < C$, for some $C > 0$, and thus if $|h| < C(1+|x|)^{-1}$. Define

$$m(x) = \frac{1}{1+|x|}.$$

Our " local balls" will be of the type $B(x, m(x))$ or more generally $B(x, am(x))$, for $a > 0$. In such balls, the Gauss measure is essentially proportional to Lebesgue measure.

Let $|x - x_0| \leq a$, for $a > 0$. Then, since

$$\frac{m(x)}{m(x_0)} = \frac{1+|x_0|}{1+|x|} \leq \frac{1+|x|+a}{1+|x|} \leq \frac{(1+a)(1+|x|)}{1+|x|} = 1+a$$

we have the important relations

$$\frac{1}{1+a} \le \frac{m(x)}{m(x_0)} \le 1+a. \tag{2.4}$$

These will frequently be used. The following lemma will be of importance in the coming proof.

LEMMA 3.2 (Covering Lemma). *Given $a > 0$, one can cover $\mathbb{R}^d$ with a sequence of open balls $B(x_j, am(x_j))$ so that for any $b > 0$ the balls $B(x_j, bm(x_j))$ have bounded overlap, i.e., the function*

$$\sum_j \chi_{B(x_j, bm(x_j))}$$

is uniformly bounded in $\mathbb{R}^d$.

PROOF. For some $\delta > 0$, take a maximal family of pairwise disjoint open balls $B(x_j, \delta m(x_j))$. This is possible since one can squeeze in a maximal number of such balls in any ball $B(0, R)$, for $R > 0$. Then one can double the radius and continue iteratively, adding balls at each step, and finally fill $\mathbb{R}^d$ in the limit.

Then for any point $x \in \mathbb{R}^d$, the ball $B(x, \delta m(x))$ must intersect some $B(x_j, \delta m(x_j))$, by maximality. For such x_j we have, by simple geometry and (2.4), that

$$|x - x_j| \le \delta m(x) + \delta m(x_j) \le \delta(1+\delta)m(x_j) + \delta m(x_j) \le 2\delta(1+\delta)m(x_j).$$

Thus $x \in B(x_j, am(x_j))$, if we choose δ so that $2\delta(1+\delta) \le a$. Hence we have a covering of $\mathbb{R}^d$ with balls $B(x_j, am(x_j))$. Fix $b > 0$ and $x \in \mathbb{R}^d$. For how many x_j can $x \in B(x_j, bm(x_j))$? If $x \in B(x_j, bm(x_j))$ then, using (2.4),

$$B(x_j, \frac{\delta}{1+b} m(x)) \subset B(x_j, \delta m(x_j)) \subset B(x, (b+\delta)m(x_j)) \subset B(x, (b+\delta)(1+b)m(x)).$$

The first inclusion here implies that the balls $B(x_j, \delta(1+b)^{-1}m(x))$ are pairwise disjoint as j varies. The number of such balls that can be included in $B(x, (b+\delta)(1+b)m(x))$ is thus bounded by the volume quotient

$$\left(\frac{(b+\delta)(1+b)}{\delta/(1+b)}\right)^d.$$

□

Define the local maximal operator

$$T_*^{\mathrm{loc}} f(x) = \sup_{t>0} T_t(|f|\,\chi_{B(x,m(x))}) = \sup_{t>0} \int_{B(x,m(x))} M_t(x,y)|f(y)|\,\mathrm{d}y,$$

and the global maximal operator

$$T_*^{\mathrm{glob}} f(x) = \sup_{t>0} T_t(|f|\,\chi_{\mathbb{R}^d \setminus B(x,m(x))}) = \sup_{t>0} \int_{|y-x|>m(x)} M_t(x,y)|f(y)|\,\mathrm{d}y.$$

Theorem 3.1 will be proved by showing that T_*^{loc} and T_*^{glob} are of weak type $(1,1)$ separately.

PROOF OF THEOREM 3.1, LOCAL PART. The strategy is to estimate the Mehler kernel

$$M_t(x,y) = \frac{1}{\pi^{d/2}(1-e^{-2t})^{d/2}} e^{-\frac{|y-e^{-t}x|^2}{1-e^{-2t}}}$$

in a local ball, by an expression of the type $\varphi_{\sqrt{\tau}}(x-y)$ for $\tau = 1 - e^{-t}$, where φ is a function to which Theorem 2.2 applies. For the first factor we have $1 - e^{-2t} = (1+e^{-t})\tau \sim \tau$.

We continue with the exponent, getting

$$\begin{aligned}\frac{|y-e^{-t}x|^2}{1-e^{-2t}} &= \frac{|y-x+\tau x|^2}{1-e^{-2t}} \\ &= \frac{|y-x|^2}{(1+e^{-t})\tau} + \frac{2\tau\langle y-x,x\rangle}{(1+e^{-t})\tau} + \frac{\tau^2|x|^2}{1-e^{-2t}} \\ &\geq \frac{|y-x|^2}{2\tau} - 2|x|m(x).\end{aligned}$$

The last inequality follows since $y-x \in B(0,m(x))$. With the notation of Theorem 2.2 and using the fact that $|x|m(x) \leq 1$, we conclude that

$$M_t(x,y) \lesssim \frac{1}{\tau^{d/2}} e^{-\frac{|y-x|^2}{2\tau}} = \varphi_{\sqrt{\tau}}(x-y),$$

for $y \in B(x,m(x))$, where $\varphi(z) = e^{-|z|^2/2} \in L^1(\mathrm{d}z)$.

Choose a covering, according to Lemma 3.2, with $a=1$. If $x \in B(x_j,m(x_j))$ and $y \in B(x,m(x))$, then by (2.4),

$$|y-x_j| \leq |y-x| + |x-x_j| \leq m(x) + m(x_j) \leq 2m(x_j) + m(x_j) = 3m(x_j).$$

Thus, the values of $T_*^{\mathrm{loc}} f$ in $B(x_j,m(x_j))$ depends only on the restriction of f to the ball $B(x_j,3m(x_j))$. Take $f \in L^1(\gamma)$ with $f \geq 0$. Then for $x \in B(x_j,m(x_j))$, using Theorem 2.2,

$$\begin{aligned}\int_{B(x,m(x))} M_t(x,y)f(y)\,\mathrm{d}y &\lesssim \int_{B(x_j,3m(x_j))} \varphi_{\sqrt{\tau}}(x-y)f(y)\,\mathrm{d}y \\ &= \varphi_{\sqrt{\tau}} * (f\chi_{B(x_j,3m(x_j))})(x) \\ &\leq \int \varphi(z)\,\mathrm{d}z\, M(f\chi_{B(x_j,3m(x_j))})(x).\end{aligned}$$

This inequality holds for all t and hence for the supremum, i.e., for $T_*^{\mathrm{loc}} f$. Since M is of weak type $(1,1)$,

$$\begin{aligned}&|\{x \in B(x_j,m(x_j))\colon T_*^{\mathrm{loc}} f(x) > \lambda\}| \\ &\quad \leq \left|\left\{x \in B(x_j,m(x_j))\colon \int \varphi(z)\,\mathrm{d}z\, M(f\chi_{B(x_j,3m(x_j))})(x) > \lambda\right\}\right| \\ &\quad \lesssim \frac{1}{\lambda}\int_{B(x_j,3m(x_j))} f(x)\,\mathrm{d}x.\end{aligned}$$

In $B(x_j,3m(x_j))$, we know that γ is essentially proportional to Lebesgue measure. Therefore,

$$\begin{aligned}&\gamma(\{x \in B(x_j,m(x_j))\colon T_*^{\mathrm{loc}} f(x) > \lambda\}) \\ &\quad \lesssim \frac{1}{\lambda}\int_{B(x_j,3m(x_j))} f(x)\,\mathrm{d}\gamma(x).\end{aligned}$$

Summing in j and using the bounded overlap of the $B(x_j,3m(x_j))$ gives us

$$\begin{aligned}&\gamma(\{x \in \mathbb{R}^d\colon T_*^{\mathrm{loc}} f(x) > \lambda\}) \\ &\quad \lesssim \frac{1}{\lambda}\int f(x)\sum_{j\in\mathbb{N}} \chi_{B(x_j,3m(x_j))}(x)\,\mathrm{d}\gamma(x) \\ &\quad \lesssim \frac{1}{\lambda}\int f(x)\,\mathrm{d}\gamma(x).\end{aligned}$$

Thus T_*^{loc} is of weak type $(1,1)$. □

PROOF OF THEOREM 3.1, GLOBAL PART. By moving the supremum inside the integral, we make the crude estimate

$$T_*^{\mathrm{glob}} f(x) \leq \int_{|y-x|>m(x)} \sup_{t>0} M_t^\gamma(x,y)|f(y)| \,\mathrm{d}\gamma(y)$$

Remarkably enough, this estimate is sufficient for our purpose. Let

$$e^{-t} = \frac{1-\tau}{1+\tau} = \frac{2}{1+\tau} - 1.$$

Clearly $t \mapsto \tau(t)$ is an increasing function from $(0,\infty)$ onto $(0,1)$. A simple calculation shows that

$$1 - e^{-2t} = \frac{4\tau}{(1+\tau)^2}.$$

Remember that the kernel is symmetric. We write the kernel in terms of τ as

$$\begin{aligned} M_t^\gamma(x,y) &= \frac{1}{(1-e^{-2t})^{d/2}} \exp\Big(|x|^2 - \frac{|x-e^{-t}y|^2}{1-e^{-2t}}\Big) \\ &= \frac{(1+\tau)^{-d/2}}{2^d} \tau^{-d/2} e^{|x|^2} \exp\Big(-\frac{|(1+\tau)x-(1-\tau)y|^2}{4\tau}\Big). \end{aligned}$$

The first factor satisfies $(1+\tau)^{-d/2}2^{-d} \sim 1$ so that causes no problems. Define

$$Q(x,y,\tau) = \tau^{-d/2} \exp\Big(-\frac{|(1+\tau)x-(1-\tau)y|^2}{4\tau}\Big).$$

Lemma 3.3 and Proposition 3.4 below finish the proof of the global part. □

LEMMA 3.3. *If $|y-x| > m(x)$, then*

$$\sup_{0<\tau<1} Q(x,y,\tau) \lesssim (1+|x|)^d \wedge (|x|\theta)^{-d},$$

where $\theta = \theta(x,y)$ is the angle between x and y.

PROOF. Since, for any $\rho > 0$, there exists a constant $C > 0$ such that $\exp(-a) \leq Ca^{-\rho}$, for $a > 0$, we have that

$$\begin{aligned} \tau^{-d/2} \exp\left(-\frac{|(1+\tau)x-(1-\tau)y|^2}{4\tau}\right) &\lesssim \tau^{-d/2}\left(\frac{4\tau}{|(1+\tau)x-(1-\tau)y|^2}\right)^{d/2} \\ &\lesssim |(1+\tau)x-(1-\tau)y|^{-d}. \end{aligned}$$

Now, let x and y be such that $0 < \theta(x,y) < \pi/2$. Then the length of $(1+\tau)x-(1-\tau)y$ is greater than the length of the projection of $(1+\tau)x$ onto the plane perpendicular to y. This observation leads to

$$|(1+\tau)x-(1-\tau)y| \geq (1+\tau)|x|\sin\theta \gtrsim |x|\theta.$$

Next, let $\pi/2 \leq \theta \leq \pi$. Trivially the distance between $(1+\tau)x$ and $(1-\tau)y$ is then greater than the length of $(1+\tau)x$, i.e.,

$$|(1+\tau)x-(1-\tau)y| \geq (1+\tau)|x| \gtrsim |x|\theta.$$

Thus in both cases, $Q(x,y,\tau) \lesssim (|x|\theta)^{-d}$.

To compare Q with $(1+|x|)^d$, consider first the case $\tau > (1+|x|)^{-2}/4$. Then clearly $Q(x,y,\tau) \leq \tau^{-d/2} \lesssim (1+|x|)^d$. In the opposite case $\tau \leq (1+|x|)^{-2}/4 \leq 1/4$, the triangle inequality yields

$$\begin{aligned} |(1+\tau)x+(1-\tau)y| = |(1-\tau)(y-x) - 2\tau x| &\geq \frac{3}{4}|y-x| - \frac{|x|}{2(1+|x|)^2} \\ &\geq \frac{3}{4(1+|x|)} - \frac{1}{2(1+|x|)} = \frac{1}{4}\frac{1}{1+|x|}. \end{aligned}$$

So $Q(x,y,\tau) \lesssim (1+|x|)^d$ also for small τ. □

PROPOSITION 3.4. *The operator*

$$Sf(x) = e^{|x|^2} \int \Big((1+|x|)^d \wedge (|x|\theta)^{-d}\Big) f(y)\, \mathrm{d}\gamma(y),$$

is of weak type $(1,1)$.

To prove Proposition 3.4 we use the following lemma, dealing with the γ-measure outside large balls around the origin.

LEMMA 3.5. *For* $r_0 > 1$ *it holds that* $\gamma\{x \in \mathbb{R}^d : |x| > r_0\} \sim r_0^{d-2} e^{-r_0^2}$.

PROOF. Let ω_d be the area of the unit sphere in $\mathbb{R}^d$. Then

$$\gamma\{x \in \mathbb{R}^d : |x| > r_0\} = \omega_d \int_{r_0}^{\infty} r^{d-1} e^{-r^2}\, \mathrm{d}r.$$

Making the changes of variable $r = r_0 + s$ and $t = r_0 s$, we get that

$$\begin{aligned}\int_{r_0}^{\infty} r^{d-1} e^{-r^2}\, \mathrm{d}r &= \int_0^{\infty} (r_0+s)^{d-1} e^{-r_0^2 - 2r_0 s - s^2}\, \mathrm{d}s \\ &\lesssim e^{-r_0^2} \int_0^{\infty} (r_0^{d-1} s + s^d) e^{-2r_0 s} \frac{\mathrm{d}s}{s} \\ &\lesssim \Big(e^{-r_0^2} r_0^{d-1} r_0^{-1} + e^{-r_0^2} r_0^{-d}\Big) \\ &\lesssim r_0^{d-2} e^{-r_0^2}.\end{aligned} \tag{2.5}$$

□

PROOF OF PROPOSITION 3.4. Let $f \geq 0$ with $\int f\, \mathrm{d}\gamma = 1$. We shall prove that

$$\gamma\{x : Sf(x) > \lambda\} \lesssim \frac{1}{\lambda} \tag{2.6}$$

for all $\lambda > 0$. For λ small this is trivial, since γ is finite. Clearly,

$$Sf(x) \leq e^{|x|^2} (1+|x|)^d.$$

Thus $Sf(x) \leq e2^d$ for $|x| < 1$. Fix $\lambda > e2^d$, so that the unit ball is disjoint from the level set of (2.6). Let $r_0 > 1$ be the unique positive solution of the equation

$$e^{r^2}(1+r)^d = \lambda.$$

Then $B(0, r_0)$ will not intersect the level set. We will see that we can reduce the region of interest to the ring $r_0 \leq |x| \leq 2r_0$. By the previous lemma

$$\gamma\{x : |x| > 2r_0\} \lesssim r_0^{d-2} e^{-4r_0^2} \lesssim (1+r_0)^{-d} e^{-r_0^2} = \frac{1}{\lambda}.$$

The case $r_0 \leq |x| \leq 2r_0$ requires some more work. Let

$$H = \{x' \in \mathbb{R}^d : |x'| = 1, \{\rho x',\, r_0 < \rho < 2r_0\} \cap \{x : Sf > \lambda\} \neq \emptyset\}.$$

The set H contains the rays which have a non-void intersection in $r_0 \leq |x| \leq 2r_0$ with the level set of (2.6). For $x' \in H$, let

$$r(x') = \inf\{\rho \in (r_0, 2r_0) : Sf(\rho x') > \lambda\}.$$

In words, $r(x')$ is the distance from the origin to the level set in the direction x'. Then, by continuity $Sf(r(x')x') = \lambda$.

For $r_0 < |x| < 2r_0$

$$\begin{aligned} Sf(x) &\sim e^{|x|^2} \int (r_0^d \wedge (r_0\theta)^{-d}) f(y)\,\mathrm{d}\gamma(y) \\ &= e^{|x|^2} r_0^{-d} \int (r_0^{2d} \wedge \theta^{-d}) f(y)\,\mathrm{d}\gamma(y). \end{aligned}$$

It follows that, for $x' \in H$ and θ being the angle between y and x',

$$\lambda = Sf(r(x')x') \sim e^{r(x')^2} r_0^{-d} \int (r_0^{2d} \wedge \theta^{-d}) f(y)\,\mathrm{d}\gamma(y). \tag{2.7}$$

We estimate the measure of the intersection of the level set and the ring $r_0 < |x| < 2r_0$ together with its "shadow", seen from the origin. Let $\mathrm{d}x'$ denote the area measure on the unit sphere. Using (2.5), we get

$$\begin{aligned} &\gamma\{x\colon r_0 < |x| < 2r_0,\ Sf(x) > \lambda\} \\ &\quad \le \gamma\{\rho x'\colon x' \in H,\ \rho > r(x')\} \\ &\quad = \int_H \int_{r(x')}^{\infty} \rho^{d-1} e^{-\rho^2}\,\mathrm{d}\rho\,\mathrm{d}x' \\ &\quad \lesssim \int_H r(x')^{d-2} e^{-|r(x')|^2}\,\mathrm{d}x' \\ &\quad \sim r_0^{-2} \int_H r_0^d e^{-|r(x')|^2}\,\mathrm{d}x' \\ &\quad \sim r_0^{-2} \int_H \frac{1}{\lambda} \int_{|y-x|>m(r(x'))} (r_0^{2d} \wedge \theta^{-d}) f(y)\,\mathrm{d}\gamma(y)\,\mathrm{d}x'. \end{aligned}$$

In the last step we used (2.7). Changing the order of integration and extending the integration to the whole unit sphere S^{d-1} leads to

$$\begin{aligned} &\gamma\{x\colon r_0 < |x| < 2r_0, Sf(x) > \lambda\} \\ &\quad \lesssim r_0^{-2} \frac{1}{\lambda} \int f(y)\,\mathrm{d}\gamma(y) \int_{S^{d-1}} r_0^{2d} \wedge \theta^{-d}\,\mathrm{d}x'. \end{aligned}$$

The last integral here is independent of y, by rotation, and behaves essentially as an integral in $\mathbb{R}^{d-1}$, so let z be a variable in $\mathbb{R}^{d-1}$. Then

$$\int_{S^{d-1}} r_0^{2d} \wedge \theta^{-d}\,\mathrm{d}x' \sim \int_{|z|<\pi} r_0^{2d} \wedge |z|^{-d}\,\mathrm{d}z \sim r_0^2.$$

The last relation easily follows by integration over $\{|z| < r_0^{-2}\}$ and $\{r_0^{-2} < |z| < \pi\}$ separately.

$\square$

CHAPTER 3

Riesz operators related to the Ornstein-Uhlenbeck operator

In this chapter, we consider first-order Riesz operators for the Ornstein-Uhlenbeck operator. The first section recalls Riesz operators in Euclidean space. Some facts about singular integrals are also stated. In Section 2, kernels are found for Riesz operators. Those kernels are well known, but our detailed derivations seem to be lacking in the literature. In Section 3, we prove that the Riesz operators are of weak type $(1,1)$. This was proved by Muckenhoupt in 1969 for $d=1$, see [**5**]. The case $d<\infty$ was obtained by Fabes, Gutierrez and Scotto in 1994, [**1**], but it is still unknown for $d=\infty$. Riesz operators of higher order was considered by Pérez and Soria in 2000, see [**6**].

As in the previous chapter, the proof of our result will split into a local and a global part.

1. Euclidean Riesz operators and singular integrals

The theory of Riesz operators is a widely studied subject within harmonic analysis. They form useful tools for estimating derivatives related to partial differential and pseudo-differential equations. The first- and second-order Euclidean Riesz operators in $\mathbb{R}^d$ are

$$R_i = \frac{\partial}{\partial x_i}(-\Delta)^{-\frac{1}{2}} \quad \text{and} \quad R_{ij} = \frac{\partial^2}{\partial x_i \partial x_j}(-\Delta)^{-1},$$

where the operators $(-\Delta)^{-\frac{1}{2}}$ and $(-\Delta)^{-1}$ are defined via Fourier multipliers. The operators R_i and R_{ij} are singular integrals. Their actions on $f \in L^p(\mathrm{d}x)$ are

$$R_i f(x) = c_d \ \text{p.v.} \int \frac{\xi_i}{|\xi|^{d+1}} f(x-\xi)\,\mathrm{d}\xi$$

and for $i \neq j$

$$R_{ij} f(x) = \tilde{c}_d \ \text{p.v.} \int \frac{\xi_i \xi_j}{|\xi|^{d+2}} f(x-\xi)\,\mathrm{d}\xi.$$

Here c_d and $\tilde{c}_d$ are constants and the integrals are taken in the principal value sense, i.e.,

$$\text{p.v.} \int \cdot \,\mathrm{d}x = \lim_{\epsilon \to 0} \int_{|x|>\epsilon} \cdot \,\mathrm{d}x.$$

In this way, the non-integrable singularities can be handled, due to cancellations. It is well known that that the Riesz operators are bounded on $L^p(\mathrm{d}x)$ for $1<p<\infty$ and of weak type $(1,1)$.

We now give some brief motivation why Riesz operators are important. Consider the equation

$$-\Delta u = f.$$

Then, at least formally,

$$\partial_i \partial_j u = R_{ij} f.$$

Thus $\partial_i \partial_j u = R_{ij}(-\Delta)u$, and since R_{ij} is bounded on L^p, we have for suitable u, that

$$\|\partial_i \partial_j u\|_{L^p} \lesssim \|(-\Delta)u\|_{L^p}.$$

So the second derivatives can be estimated by means of the Laplacian.

The first-order Riesz operators can be used to obtain the two-sided inequality

$$\|(-\Delta)^{-\frac{1}{2}}u\|_{L^p} \lesssim \|\mathrm{grad}\, u\|_{L^p} \lesssim \|(-\Delta)^{-\frac{1}{2}}u\|_{L^p}.$$

In Section 3 we will need a general result for singular integrals. We state this here. Assume that T is a bounded linear operator on $L^2(\mathbb{R}^d, \mathrm{d}x)$ with off-diagonal kernel k. This means that for $f \in C_0^\infty(\mathbb{R}^d)$,

$$Tf(x) = \int k(x,y)f(y)\,\mathrm{d}y, \qquad x \notin \mathrm{supp}(f).$$

If the kernel k is of class C^1 in $\{(x,y) \colon x \neq y\}$ and satisfies

$$|k(x,y)| \lesssim \frac{1}{|x-y|^d} \tag{3.1}$$

and

$$|\mathrm{grad}_{x,y} k(x,y)| \lesssim \frac{1}{|x-y|^{d+1}}, \tag{3.2}$$

then k is said to satisfy the standard estimates. These holds for R_i and R_{ij}. The following theorem is well known, see for instance [**9**], Chapter II, Section 2, Theorem 1.

Theorem 1.1. *If the operator T is linear and bounded on $L^2(\mathbb{R}^d, \mathrm{d}x)$ and has an off-diagonal kernel satisfying the standard estimates, then T is of weak type $(1,1)$.*

2. First-order Gaussian Riesz operators

A first try to define Riesz operators for the Ornstein-Uhlenbeck operator L would be to let $R_i = \partial_i L^{-\frac{1}{2}}$. But $LH_\alpha = |\alpha|H_\alpha$ and in particular $LH_0 = 0$, so $L^{-\frac{1}{2}}$ does not exist. Let P_0 be the orthogonal projection in $L^2(\gamma)$ onto the span of H_0, i.e., onto the subspace of constant functions. Then $I - P_0$ is the projection onto the orthogonal complement $H_0^\perp$. Define $L^{-\frac{1}{2}}(I-P_0)$ by

$$\sum_{\alpha\in\mathbb{N}^d} a_\alpha H_\alpha \mapsto \sum_{|\alpha|\neq 0} \frac{1}{\sqrt{|\alpha|}} a_\alpha H_\alpha$$

for

$$\sum_{\alpha\in\mathbb{N}^d} |a_\alpha|^2 \|H_\alpha\|^2_{L^2(\gamma)} < \infty.$$

It is well defined and bounded on $L^2(\gamma)$. Recall that $\partial_i H_\alpha = 2\alpha_i H_{\alpha-e_i}$ if $\alpha_i \neq 0$. Define R_i by

$$R_i \colon \sum_{\alpha\in\mathbb{N}^d} a_\alpha H_\alpha \mapsto \sum_{\alpha_i\neq 0} \frac{2\alpha_i}{\sqrt{|\alpha|}} a_\alpha H_{\alpha-e_i}.$$

We first verify that R_i is bounded on $L^2(\gamma)$. Since $\|H_\alpha\|_{L^2(\gamma)} = 2^{|\alpha|}\alpha!$, we have $\|H_{\alpha-e_i}\|_{L^2(\gamma)} = \|H_\alpha\|_{L^2(\gamma)}/2\alpha_i$. Parseval's identity implies

$$\Big\| \sum_{\alpha_i\neq 0} \frac{2\alpha_i}{\sqrt{|\alpha|}} a_\alpha H_{\alpha-e_i} \Big\|^2_{L^2(\gamma)} = \sum_{\alpha_i\neq 0} \frac{4\alpha_i^2}{|\alpha|} |a_\alpha|^2 \frac{1}{2\alpha_i} \|H_\alpha\|^2_{L^2(\gamma)} \le 2 \sum_{\alpha_i\neq 0} |a_\alpha|^2 \|H_\alpha\|^2_{L^2(\gamma)},$$

and the $L^2(\gamma)$ boundedness of R_i follows.

Observe that any function in $L^2(\gamma)$ defines a distribution in $\mathcal{D}'$. Recall that distributions in $\mathcal{D}'$ are those acting on test functions with compact support. Convergence in $L^2(\gamma)$ implies convergence in $\mathcal{D}'$.

Proposition 2.1. *For $f \in L^2(\gamma)$ one has $R_i f = \partial_i L^{-\frac{1}{2}}(I-P_0)$, where $\partial_i = \partial_i^{\mathrm{distr}}$ is taken in the sense of distributions.*

PROOF. Let $f = \sum_\alpha a_\alpha H_\alpha \in L^2(\gamma)$. Then

$$L^{-\frac{1}{2}}(I - P_0)f = \sum_{\alpha\neq 0} \frac{1}{\sqrt{|\alpha|}} a_\alpha H_\alpha$$

with both $L^2(\gamma)$ and $\mathcal{D}'$ convergence. Since differentiation in the distribution sense can always be made termwise, we get that

$$\partial_i^{\text{dist}} L^{-\frac{1}{2}}(I - P_0)f = \sum_{\alpha_i\neq 0} \frac{1}{\sqrt{|\alpha|}} a_\alpha \partial_i^{\text{dist}} H_\alpha = \sum_{\alpha_i\neq 0} \frac{2\alpha_i}{\sqrt{|\alpha|}} a_\alpha H_{\alpha - e_i} = R_i f.$$

□

Our aim for the rest of this section will be to find the kernels of $L^{-\frac{1}{2}}(I - P_0)$ and R_i. Notice that

$$\lambda^{-1/2} = \frac{1}{\sqrt{\pi}} \int_0^\infty t^{-1/2} e^{-t\lambda}\, \mathrm{d}t, \quad \lambda > 0. \tag{3.3}$$

It is tempting to write

$$L^{-1/2} = \frac{1}{\sqrt{\pi}} \int_0^\infty t^{-1/2} e^{-tL}\, \mathrm{d}t, \quad \lambda > 0,$$

with kernel

$$\frac{1}{\sqrt{\pi}} \int_0^\infty t^{-1/2} M_t^\gamma(x,y)\, \mathrm{d}t.$$

But M_t^γ is close to 1 for large t so the integral diverges at infinity. This is not surprising since we saw that $L^{-1/2}$ does not exist. But $(\epsilon + L)^{-1/2}$ exists and is bounded on $L^2(\gamma)$, for $\epsilon > 0$. We also have that

$$(\epsilon + L)^{-1/2} - \epsilon^{-1/2} P_0 \to L^{-1/2}(I - P_0)$$

as $\epsilon \to 0$, in the strong operator topology. This means that

$$\Big((\epsilon + L)^{-1/2} - \epsilon^{-1/2} P_0\Big) f \to L^{-1/2}(I - P_0) f \qquad \text{in} \quad L^2(\gamma)$$

for all $f \in L^2(\gamma)$, which is easily verified from our definitions. Write formally

$$(\epsilon + L)^{-1/2} = \frac{1}{\sqrt{\pi}} \int_0^\infty t^{-1/2} e^{-\epsilon t} e^{-Lt}\, \mathrm{d}t$$

with kernel

$$\frac{1}{\sqrt{\pi}} \int_0^\infty t^{-1/2} e^{-\epsilon t} M_t^\gamma(x,y)\, \mathrm{d}t.$$

The kernel of P_0 is $1 = H_0$ so that the kernel of $\epsilon^{-1/2} P_0$ is

$$\epsilon^{-1/2} = \frac{1}{\sqrt{\pi}} \int_0^\infty t^{-1/2} e^{-\epsilon t}\, \mathrm{d}t.$$

Thus, $(\epsilon + L)^{-1/2} - \epsilon^{-1/2} P_0$ should have the kernel

$$\frac{1}{\sqrt{\pi}} \int_0^\infty t^{-1/2} e^{-\epsilon t} (M_t^\gamma(x,y) - 1)\, \mathrm{d}t.$$

This gives hope since $M_t^\gamma(x,y) - 1 \to 0$ as $t \to \infty$, for all x and y. We conjecture that the kernel of $L^{-1/2}(I - P_0)$ is

$$K(x,y) = \frac{1}{\sqrt{\pi}} \int_0^\infty t^{-1/2} (M_t^\gamma(x,y) - 1)\, \mathrm{d}t \tag{3.4}$$

and that of R_i is

$$k_i^\gamma(x,y) = \frac{1}{\sqrt{\pi}} \int_0^\infty t^{-1/2} \partial_i M_t^\gamma(x,y)\,\mathrm{d}t. \tag{3.5}$$

Here and in the sequel, ∂_i means $\partial/\partial x_i$, also for functions depending on x and y.

Consider first K. We verify that the integral converges. Recall that the Mehler kernel is given by

$$M_t^\gamma(x,y) = \frac{1}{(1-e^{-2t})^{d/2}} e^{|y|^2 - \frac{|y-e^{-t}x|^2}{1-e^{-2t}}}.$$

For t small we have $(1-e^{-2t})^{-d/2} \sim t^{-d/2}$. Also, since $1-e^{-2t} \le 2t$,

$$\begin{aligned} \frac{|y-e^{-t}x|^2}{1-e^{-2t}} &= \frac{|y-x+(1-e^{-t})x|^2}{1-e^{-2t}} \ge \frac{|y-x|^2}{2t} + \frac{2(1-e^{-t})\langle y-x, x\rangle}{1-e^{-2t}} + \frac{(1-e^{-t})^2|x|^2}{1-e^{-2t}} \\ &\ge \frac{|y-x|^2}{2t} - C(x,y), \end{aligned} \tag{3.6}$$

for $x \neq y$, where $C(x,y)$ is bounded on compact sets. Then, for a new $C(x,y)$,

$$|M_t^\gamma(x,y)| \lesssim C(x,y) t^{-\frac{d}{2}} \exp\Big(-\frac{|y-x|^2}{2t}\Big).$$

Making the change of variable $s = |y-x|^2/t$, we get that

$$\begin{aligned} \int_0^1 t^{-1/2} M_t^\gamma(x,y)\,\mathrm{d}t &\lesssim C(x,y) e^{|y|^2} \int_0^1 t^{-\frac{d}{2}-\frac{1}{2}} e^{-c\frac{|y-x|^2}{t}}\,\mathrm{d}t \\ &\lesssim C(x,y) e^{|y|^2} |y-x|^{1-d} \int_0^\infty s^{\frac{d}{2}-\frac{3}{2}} e^{-cs}\,\mathrm{d}s \\ &\lesssim C(x,y) e^{|y|^2} |y-x|^{1-d}. \end{aligned} \tag{3.7}$$

Notice that the integral diverges on the diagonal. For t large enough so that $e^{-t}(|x|^2+|y|^2)$ is small, we use the fact that $\langle x,y\rangle \le |x|^2+|y|^2$ and make a first-order Taylor expansion. Using the expression (1.14) for the kernel we get

$$\begin{aligned} M_t^\gamma(x,y) - 1 &= (1+\mathcal{O}(e^{-2t})) \exp\Big(\frac{-e^{-2t}(|x|^2+|y|^2) + 2e^{-t}\langle x,y\rangle}{1-e^{-2t}}\Big) - 1 \\ &= (1+\mathcal{O}(e^{-2t}))(1+\mathcal{O}(e^{-t}(|x|^2+|y|^2))) - 1 \\ &= \mathcal{O}(e^{-t}(|x|^2+|y|^2)). \end{aligned} \tag{3.8}$$

Thus, $K(x,y)$ exists for $x \neq y$.

Theorem 2.2. *(a) For any $f \in L^2(\gamma)$ the map $y \mapsto K(x,y)f(y)$ is in $L^1(\gamma)$ for a.a. x, where K is given by (3.4), and the integral*

$$\int K(x,y) f(y)\,\mathrm{d}\gamma(y)$$

defines a bounded linear operator on $L^2(\gamma)$.

(b) This operator coincides with $L^{-\frac{1}{2}}(I-P_0)$.

Proof. With $T = T(x,y) = \max(10, \log(|x|^2+|y|^2))$, we write K as

$$\begin{aligned} \sqrt{\pi}\,K(x,y) &= \int_0^{10} t^{-\frac{1}{2}} M_t^\gamma(x,y)\,\mathrm{d}t + \int_{10}^{T(x,y)} t^{-\frac{1}{2}} M_t^\gamma(x,y)\,\mathrm{d}t \\ &\quad - \int_0^{T(x,y)} t^{-\frac{1}{2}}\,\mathrm{d}t + \int_{T(x,y)}^\infty t^{-\frac{1}{2}} (M_t^\gamma(x,y) - 1)\,\mathrm{d}t \\ &= K_1 + K_2 + K_3 + K_4, \end{aligned}$$

where $K_i = K_i(x,y)$ for each i. We shall prove that each K_i satisfies (a).

Let $f \in L^2(\gamma)$. For K_1 we apply Tonelli's theorem and Minkowski's inequality, getting

$$\begin{aligned}\Big\| \int K_1(x,y)|f(y)|\,\mathrm{d}\gamma(y)\Big\|_{L^2(\gamma)} &= \Big\| \int_0^{10} t^{-\frac12} \int M_t^\gamma(x,y)|f(y)|\,\mathrm{d}\gamma(y)\,\mathrm{d}t\Big\|_{L^2(\gamma)}\\ &\le \int_0^{10} t^{-\frac12} \Big\| \int M_t^\gamma(x,y)|f(y)|\,\mathrm{d}\gamma(y)\Big\|_{L^2(\gamma)}\,\mathrm{d}t\\ &= \int_0^{10} t^{-\frac12} \|T_t|f|\|_{L^2(\gamma)}\\ &\lesssim \|f\|_{L^2(\gamma)}\end{aligned}$$

Thus (a) holds for K_1.

In K_2 we have $t > 10$ and thus $1-e^{-2t} \sim 1$ and $M_t^\gamma(x,y) \lesssim \exp(4e^{-t}|x||y|)$ in view of (1.14). Then

$$\begin{aligned}0 \le K_2(x,y) &\lesssim \int_{10}^{T(x,y)} \exp(4e^{-t}|x||y|)\,\mathrm{d}t \le \int_{10}^{T(x,y)} \exp\Big(4e^{-10}\frac{|x|^2+|y|^2}{2}\Big)\,\mathrm{d}t\\ &\le \exp\Big(\frac{|x|^2}{4}+\frac{|y|^2}{4}\Big)\big(10 \vee \log(|x|^2+|y|^2)\big) \in L^2(\gamma\times\gamma).\end{aligned}$$

Therefore, K_2 defines a Hilbert-Schmidt operator in $L^2(\gamma)$, and (a) follows for K_2.

The kernel K_3 is $2\sqrt{T(x,y)}$, which is another Hilbert-Schmidt kernel.

To deal with K_4, we apply (3.8) to get

$$|K_4(x,y)| \le \int_{T(x,y)}^\infty t^{-\frac12}|M_t^\gamma(x,y)-1|\,\mathrm{d}t \lesssim (|x|^2+|y|^2)\Big|\int_{T(x,y)}^\infty e^{-t}\,\mathrm{d}t\Big| = (|x|^2+|y|^2)e^{-T(x,y)} \le 1.$$

With this we have proved (a).

The proof of (a) also shows that the $L^2(\gamma\times\gamma)$ norm of $\int_{t'}^\infty t^{-\frac12}(M_t^\gamma(x,y)-1)\,\mathrm{d}t$ tends to 0 as $t' \to \infty$. The operator obtained in (a), call it momentarily A, is thus the limit in the operator norm as $t' \to \infty$ of the operator defined by the kernels

$$\frac{1}{\sqrt\pi}\int_0^{t'} t^{-\frac12}(M_t^\gamma(x,y)-1)\,\mathrm{d}t.$$

To prove (b), it is enough to verify that A coincides with $L^{-\frac12}(I-P_0)$ on each Hermite polynomial H_α. Now AH_α is the limit as $t' \to \infty$ of

$$\int \frac{1}{\sqrt\pi}\int_0^{t'} t^{-\frac12}(M_t^\gamma(x,y)-1)\,\mathrm{d}t\; H_\alpha(y)\,\mathrm{d}\gamma(y).$$

Here we can apply Fubini's theorem, in view of the above estimates. Using also (1.8), we see that the expression equals

$$\begin{aligned}&\frac{1}{\sqrt\pi}\Big[\int_0^{t'} t^{-\frac12}\int M_t^\gamma(x,y)H_\alpha(y)\,\mathrm{d}\gamma(y)\,\mathrm{d}t - \frac{1}{\sqrt\pi}\int_0^{t'} t^{-\frac12}\int H_\alpha(y)\,\mathrm{d}\gamma(y)\,\mathrm{d}t\Big]\\ &\quad= \frac{1}{\sqrt\pi}\Big[\int_0^{t'} t^{-\frac12}e^{-t|\alpha|}\,\mathrm{d}t\; H_\alpha(x) - \int_0^{t'} t^{-\frac12}\,\mathrm{d}t\;\langle H_\alpha, 1\rangle\Big].\end{aligned}$$

For $\alpha \neq 0$, the second term in this last expression vanishes, and the first term tends to $|\alpha|^{-1/2}H_\alpha(x)$. If $\alpha = 0$, the whole expression vanishes. Thus the limit, as $t' \to \infty$, is $L^{-1/2}(I-P_0)H_\alpha(x)$ in both cases. □

THEOREM 2.3. *The off-diagonal kernel of R_i is $k_i^\gamma(x,y) = \partial_i K(x,y)$, in the sense that for $f \in C_0^\infty(\mathbb{R}^d)$ and $x \notin \operatorname{supp}(f)$*

$$R_i f(x) = \int k_i^\gamma(x,y) f(y)\,\mathrm{d}\gamma(y).$$

Moreover, k_i^γ is given by (3.5), for $x \neq y$.

PROOF. We start by verifying that

$$\partial_i K(x,y) = \frac{1}{\sqrt{\pi}} \int_0^\infty t^{-\frac{1}{2}} \partial_i M_t^\gamma(x,y)\,\mathrm{d}t. \tag{3.9}$$

Differentiation gives us

$$\partial_i M_t^\gamma(x,y) = \frac{1}{(1-e^{-2t})^{\frac{d}{2}}} \frac{2e^{-t}(y_i - e^{-t}x_i)}{1-e^{-2t}} \exp\Big(|y|^2 - \frac{|y-e^{-t}x|^2}{1-e^{-2t}}\Big). \tag{3.10}$$

Thus,

$$\begin{aligned} |\partial_i M_t^\gamma(x,y)| &\le \frac{2e^{-t}}{(1-e^{-2t})^{\frac{d}{2}+\frac{1}{2}}} \frac{|y-e^{-t}x|}{\sqrt{1-e^{-2t}}} \exp\Big(|y|^2 - \frac{|y-e^{-t}x|^2}{1-e^{-2t}}\Big) \\ &\lesssim \frac{e^{-t}}{(1-e^{-2t})^{\frac{d}{2}+\frac{1}{2}}} \exp\Big(|y|^2 - \frac{1}{2}\frac{|y-e^{-t}x|^2}{1-e^{-2t}}\Big). \end{aligned} \tag{3.11}$$

For small t we have

$$|\partial_i M_t^\gamma(x,y)| \lesssim C(x,y) t^{-\frac{d}{2}-\frac{1}{2}} e^{-c\frac{|y-x|^2}{t}},$$

where $C(x,y)$ is locally bounded; cf. (3.6). As in (3.7), for a new $C(x,y)$,

$$\int_0^1 t^{-\frac{1}{2}} |\partial_i M_t^\gamma(x,y)|\,\mathrm{d}t \lesssim C(x,y) \int_0^1 t^{-\frac{d}{2}-1} e^{-c\frac{|y-x|^2}{t}}\,\mathrm{d}t \lesssim C(x,y)|x-y|^{-d}.$$

For t large, (3.11) implies $|\partial_i M_t^\gamma(x,y)| \lesssim e^{-t}(|x|+|y|)e^{|y|^2}$, and so

$$\int_1^\infty t^{-\frac{1}{2}} |\partial_i M_t^\gamma(x,y)|\,\mathrm{d}t \lesssim C(x,y).$$

Thus

$$\int_0^\infty t^{-\frac{1}{2}} |\partial_i M_t^\gamma(x,y)|\,\mathrm{d}t \lesssim C(x,y)|x-y|^{-d} < \infty, \tag{3.12}$$

for $x \neq y$, and so the integral in (3.9) is well defined.

The equality (3.9) follows if

$$\int_{a_0}^{a_1} \frac{1}{\sqrt{\pi}} \int_0^\infty t^{-\frac{1}{2}} \partial_i M_t^\gamma(x,y)\,\mathrm{d}t\,\mathrm{d}x_i = \big[K(x,y)\big]_{x_i=a_0}^{a_1}, \tag{3.13}$$

by differentiation with respect to a_1. Here a_0 and a_1 must be chosen so that y is not in the segment $\{(x_1,\dots,x_{i-1},a,x_{i+1},\dots,x_d)\colon a_0 \le a \le a_1\}$. Indeed, if Fubini can be applied, then the left-hand side becomes, after a suitable choice of primitive function,

$$\frac{1}{\sqrt{\pi}} \int_0^\infty t^{-\frac{1}{2}} \big[M_t(x,y) - 1\big]_{x_i=a_0}^{a_1}\,\mathrm{d}t = \big[K(x,y)\big]_{x_i=a_0}^{a_1}.$$

To verify the use of Fubini, we apply (3.12) to see that

$$\int_0^\infty t^{-\frac{1}{2}} \int_{a_0}^{a_1} |\partial_i M_t^\gamma(x,y)|\,\mathrm{d}x_i\,\mathrm{d}t \sim (a_1-a_0) \int_0^\infty \sup_x t^{-\frac{1}{2}} |\partial_i M_t^\gamma(x,y)|\,\mathrm{d}t < \infty,$$

where the supremum is taken over x in the segment mentioned above.

We now prove that, for $x \notin \operatorname{supp} f$,

$$\int k_i^\gamma(x,y) f(y)\,\mathrm{d}\gamma(y) = \partial_i \int K(x,y) f(y)\,\mathrm{d}\gamma(y),$$

or equivalently,

$$\begin{aligned}\int_{a_0}^{a_1} \int \int_0^\infty & t^{-\frac{1}{2}} \partial_i M_t^\gamma(x,y)\,\mathrm{d}t\, f(y)\,\mathrm{d}\gamma(y)\,\mathrm{d}x_i \\ &= \left[\int \int_0^\infty t^{-\frac{1}{2}} (M_t^\gamma(x,y) - 1)\,\mathrm{d}t\, f(y)\,\mathrm{d}\gamma(y) \right]_{x_i = a_0}^{a_1},\end{aligned}$$

for a_0 and a_1 such that $(x_1, \ldots, x_{i-1}, a, x_{i+1}, \ldots, x_d) \notin \operatorname{supp}(f)$ for all $a_0 \le a \le a_1$. This follows from Fubini by the same calculation and the same justification as above.

We now know that

$$\int k_i^\gamma(x,y) f(y)\,\mathrm{d}\gamma(y) = \partial_i L^{-\frac{1}{2}}(I - P_0) f(x),$$

for $x \notin \operatorname{supp} f$. But $R_i = \partial_i^{\mathrm{distr}} L^{-\frac{1}{2}}(I - P_0)$. □

3. The weak type (1, 1) property

THEOREM 3.1. *For $d < \infty$, each R_i is of weak type $(1,1)$, with respect to γ.*

We start with some proof preparations. Using (3.11) and making the change of variable $r = e^{-t}$ yields

$$\begin{aligned}|k_i^\gamma(x,y)| &\lesssim \int_0^\infty t^{-\frac{1}{2}} \frac{e^{-t}}{(1-e^{-2t})^{\frac{d}{2}+\frac{1}{2}}} \frac{|y - e^{-t}x|}{\sqrt{1-e^{-2t}}} \exp\left(|y|^2 - \frac{|y-e^{-t}x|^2}{1-e^{-2t}}\right)\mathrm{d}t \\ &= \int_0^1 \left(\frac{1-r^2}{\log r^{-1}}\right)^{\frac{1}{2}} \frac{1}{(1-r^2)^{\frac{d}{2}+1}} \frac{|y-rx|}{\sqrt{1-r^2}} \exp\left(|y|^2 - \frac{|y-rx|^2}{1-r^2}\right)\mathrm{d}r.\end{aligned} \tag{3.14}$$

Let

$$v(r) = \frac{|y-rx|^2}{1-r^2}.$$

Since $(1-r^2)/\log(1/r) \le 2$ for $0 < r < 1$, we get

$$|k_i^\gamma(x,y)| \lesssim \int_0^1 \frac{\sqrt{v(r)}}{(1-r^2)^{\frac{d}{2}+1}} \exp(|y|^2 - v(r))\,\mathrm{d}r = p_i^\gamma(x,y). \tag{3.15}$$

We will also need the kernels k_i and p_i for integration against Lebesgue measure, i.e., with the factor $e^{-|y|^2}$ removed.

We will now split R_i into a global and a local part as in Chapter 2. For this purpose a smooth cutoff function is needed, since we are working with derivatives. Take a function $0 \le N(x,y) \in C^\infty(\mathbb{R}^d \times \mathbb{R}^d)$ satisfying

$$N(x,y) = \begin{cases} 1 & \text{if} \quad |x-y| \le m(x) \\ 0 & \text{if} \quad |x-y| \ge 2m(x) \end{cases}$$

and for all y the condition

$$|\mathrm{grad}_{x,y} N(x,y)| \lesssim \frac{1}{m(x)} = 1 + |x|.$$

We hint how N can be constructed. Let $0 \le \phi \in C^\infty(\mathbb{R}^+)$ such that $\phi(x) = 1$ for $0 \le x \le 1$, decreasing for $1 < x < 2$ and with $\phi(x) = 0$ for $x \ge 2$. Then $N(x,y) = \phi(|x-y|/m(x))$ is such a function except that it lacks differentiability at $x = 0$. By a smoothing argument this can be overcome.

Define

$$R_i^{\mathrm{glob}} f(x) = \int k_i^{\gamma}(x,y)(1 - N(x,y)) f(y)\, \mathrm{d}\gamma(y).$$

and

$$R_i^{\mathrm{loc}} = R_i - R_i^{\mathrm{glob}}.$$

We will prove separately that R_i^{loc} and R_i^{glob} are of weak type $(1,1)$. For the local part, we use the results stated in Section 1 about singular integrals. For the global part, we use techniques like those applied to the maximal operator in Chapter 2.

PROOF OF THEOREM 3.1, LOCAL PART. The off-diagonal kernel of R_i^{loc} is $k_i(x,y)N(x,y)$, which is supported in $\{(x,y)\colon |x-y| \le 2m(x)\}$. Notice here that we work with the kernel k_i for integration against Lebesgue measure. The strategy for the local proof is to verify the standard estimates (3.1) and (3.2) for $k_i(x,y)N(x,y)$ and then prove the $L^2(\mathrm{d}x)$ boundedness of R_i^{loc}, in order to conclude, by Theorem 1.1, the weak type $(1,1)$ property with respect to Lebesgue measure. The result will then follow in our Gaussian setting.

We shall estimate the kernel $k_i(x,y)N(x,y)$, so assume that $|y-x| \le 2m(x)$. By (3.15)

$$|k_i(x,y)| \lesssim \int_0^1 \frac{\sqrt{v(r)}}{(1-r^2)^{\frac{d}{2}+1}} e^{-v(r)}\, \mathrm{d}r \lesssim \int_0^1 \frac{1}{(1-r^2)^{\frac{d}{2}+1}} e^{-\frac{1}{2}v(r)}\, \mathrm{d}r.$$

Using the fact that $1 - r \le 1 - r^2 \le 2(1-r)$, we estimate

$$\begin{aligned} v(r) = \frac{|y-x-(1-r)x|^2}{1-r^2} &\ge \frac{|y-x|^2}{1-r^2} - \frac{2(1-r)}{1-r^2}\langle y-x, x\rangle \\ &\ge \frac{|y-x|^2}{2(1-r)} - 2|y-x||x| \ge \frac{|y-x|^2}{2(1-r)} - 4, \end{aligned} \tag{3.16}$$

the last step since $|x-y| \le 2m(x)$. Then the change of variable $s = |x-y|^2/(1-r)$ yields

$$\begin{aligned} |k_i(x,y)N(x,y)| &\lesssim \int_0^1 \frac{1}{(1-r)^{\frac{d}{2}}} \exp\Big(-\frac{|x-y|^2}{4(1-r)}\Big)\frac{\mathrm{d}r}{1-r} \\ &\lesssim |x-y|^{-d} \int_0^\infty s^{\frac{d}{2}} e^{-\frac{s}{4}} \frac{\mathrm{d}s}{s} \\ &\lesssim |x-y|^{-d}. \end{aligned} \tag{3.17}$$

Hence the first standard estimate is satisfied. It remains to bound the gradient by $|x-y|^{-d-1}$. In view of (3.10) and (3.14) a second differentiation gives

$$|\partial_{x_j} k_i(x,y)| \lesssim \int_0^1 \left(\frac{1-r^2}{\log(\frac{1}{r})}\right)^{\frac{1}{2}} \left| -\frac{r\delta_{ij}}{(1-r^2)^{\frac{d}{2}+\frac{3}{2}}} + \frac{2r(y_i - rx_i)(y_j - rx_j)}{(1-r^2)^{\frac{d}{2}+\frac{5}{2}}} \right| e^{-v(r)}\, \mathrm{d}r.$$

This can be verified using Fubini as in (3.13). Clearly, using (3.16) and our usual change of variable, we get as before

$$|\partial_{x_j} k_i(x,y)| \lesssim \int \frac{1}{(1-r)^{\frac{d}{2}+\frac{1}{2}}} \exp\Big(-\frac{|x-y|^2}{4(1-r)}\Big)\frac{\mathrm{d}r}{(1-r)} \lesssim |x-y|^{-d-1}.$$

The derivatives $\partial_{y_j} k_i$ satisfy the same estimate. Now,

$$|\mathrm{grad}_{x,y} k_i N| \le N|\mathrm{grad}_{x,y} k_i| + |k_i||\mathrm{grad}_{x,y} N|.$$

But, by the properties of on N and since $|y-x| \le 2m(x)$,

$$|\mathrm{grad}_{x,y} N(x,y)| \lesssim \frac{1}{m(x)} \lesssim |x-y|^{-1},$$

and so

$$|\mathrm{grad}_{x,y} k_i N(x,y)| \lesssim |x-y|^{-d-1}.$$

The standard estimates are satisfied.

Aiming at the L^2 estimate, we take a covering $B_j = B(x_j, m(x_j))$ according to Lemma 3.2 of Chapter 2. Then the balls $\tilde{B}_j = B(x_j, 5m(x_j))$ have bounded overlap. Let $x \in B_j$, so that $2^{-1}m(x_j) < m(x) < 2m(x_j)$ in view of (2.4). Take $f \in L^2(\gamma)$ with compact support. We have that

$$R_i^{\mathrm{loc}} f(x) - R_i^{\mathrm{loc}}(f\chi_{\tilde{B}_j})(x) = R_i^{\mathrm{loc}}(f\chi_{\mathbb{R}^d \setminus \tilde{B}_j})(x) = \int_{\mathbb{R}^d \setminus \tilde{B}_j} k_i(x,y) N(x,y) f(y)\,\mathrm{d}y = 0.$$

The last conclusion follows since for $y \notin \tilde{B}_j$

$$|y-x| \geq |y-x_j| - |x_j - x| \geq 5m(x_j) - m(x_j) \geq 2m(x)$$

and so $N(x,y) = 0$. Thus $R_i^{\mathrm{loc}} f = R_i^{\mathrm{loc}}(f\chi_{\tilde{B}_j})$ in B_j, and then

$$\int_{B_j} \left|R_i^{\mathrm{loc}} f(x)\right|^2 \mathrm{d}x \lesssim \int_{B_j} \left|R_i(f\chi_{\tilde{B}_j})(x)\right|^2 \mathrm{d}x + \int_{B_j} \left|R_i(f\chi_{\tilde{B}_j})(x) - R_i^{\mathrm{loc}} f(x)\right|^2 \mathrm{d}x. \tag{3.18}$$

The difference $D(x) = R_i(f\chi_{\tilde{B}_j})(x) - R_i^{\mathrm{loc}} f(x)$ appearing here satisfies

$$D(x) = R_i(f\chi_{\tilde{B}_j})(x) - R_i^{\mathrm{loc}}(f\chi_{\tilde{B}_j})(x) = R_i^{\mathrm{glob}}(f\chi_{\tilde{B}_j})(x) = \int_{\tilde{B}_j} k_i(x,y)(1 - N(x,y)) f(y)\,\mathrm{d}y.$$

For $x \in B_j$ and $1 - N(x,y) \neq 0$ it holds that $|x-y| \geq m(x) > 2^{-1}m(x_j)$. So we have, using (3.17), that $|k_i(x,y)| \lesssim m(x_j)^{-d} \sim |\tilde{B}_j|^{-1}$. Hence

$$|D(x)| \lesssim \frac{1}{|\tilde{B}_j|} \int_{\tilde{B}_j} |f(y)|\,\mathrm{d}y,$$

and by the Cauchy-Schwarz inequality

$$\int_{B_j} |D(x)|^2\,\mathrm{d}x \lesssim \int_{\tilde{B}_j} |f(y)|^2\,\mathrm{d}y.$$

Thus (3.18) implies

$$\int_{B_j} \left|R_i^{\mathrm{loc}} f(x)\right|^2 \mathrm{d}x \lesssim \int_{B_j} \left|R_i(f\chi_{\tilde{B}_j})(x)\right|^2 \mathrm{d}x + \int_{\tilde{B}_j} |f(y)|^2\,\mathrm{d}y.$$

Since γ is essentially proportional to Lebesgue measure in our local balls, the same inequality holds with the three integrals taken with respect to γ. But R_i is bounded on $L^2(\gamma)$, as verified in the beginning of Section 2, and it follows that

$$\begin{aligned}\int_{B_j} \left|R_i^{\mathrm{loc}} f(x)\right|^2 \mathrm{d}\gamma(x) &\lesssim \int |f\chi_{\tilde{B}_j}(x)|^2\,\mathrm{d}\gamma(x) + \int_{\tilde{B}_j} |f(y)|^2\,\mathrm{d}\gamma(y) \\ &\lesssim \int_{\tilde{B}_j} |f(y)|^2\,\mathrm{d}\gamma(y).\end{aligned}$$

Switching back to Lebesgue measure, summing over all j and using the bounded overlap we conclude

$$\int |R_i^{\mathrm{loc}} f(x)|^2\,\mathrm{d}x \lesssim \int |f(x)|^2\,\mathrm{d}x.$$

Thus R_i^{loc} is of weak type $(1,1)$ for Lebesgue measure, by Theorem 1.1. On the local balls, we thus have that

$$|\{x \in B_j \colon |R_i^{\mathrm{loc}} f(x)| > \lambda\}| = |\{x \in B_j \colon |R_i^{\mathrm{loc}}(f\chi_{\tilde{B}_j})(x)| > \lambda\}| \lesssim \frac{1}{\lambda} \int_{\tilde{B}_j} |f|\,\mathrm{d}x$$

and hence

$$\gamma\{x \in B_j \colon |R_i^{\mathrm{loc}} f(x)| > \lambda\} \lesssim \frac{1}{\lambda} \int_{\tilde{B}_j} |f| \, \mathrm{d}\gamma(x).$$

Summation in j finishes the proof. □

PROOF OF THEOREM 3.1, GLOBAL PART. We shall estimate $p_i^\gamma(x,y)$, defined in (3.15), for $|x-y| > m(x)$. For this purpose we first determine the minimum of $v(r)$ in $0 < r < 1$. Write

$$v(r) = \frac{|x|^2 + |y|^2 - 2r\langle x, y\rangle}{1 - r^2} - |x|^2.$$

Differentiation gives

$$v'(r) = \frac{-2\langle x,y\rangle r^2 + 2(|x|^2 + |y|^2)r - 2\langle x,y\rangle}{(1-r^2)^2}.$$

If $\langle x,y\rangle \le 0$ then $v'(r) > 0$ for all $0 \le r \le 1$. Assume that $\langle x,y\rangle > 0$. Then $v'(r) = 0$ for

$$r = r_{0,1} = \frac{|x|^2 + |y|^2 \mp \sqrt{(|x|^2+|y|^2)^2 - 4\langle x,y\rangle^2}}{2\langle x,y\rangle},$$

and r_0 is a minimum of v. Notice also that $r_0 r_1 = 1$, so that $0 < r_0 < 1 < r_1$. An easy calculation shows that

$$|x+y|^2|x-y|^2 = (|x|^2+|y|^2)^2 - 4\langle x,y\rangle^2, \tag{3.19}$$

and thus

$$r_0 = \frac{|x|^2 + |y|^2 - |x+y||x-y|}{2\langle x,y\rangle}.$$

In order to compute $v(r_0)$ we calculate, using (3.19) and the relationship $r_0 r_1 = 1$,

$$\begin{aligned} 1 - r_0^2 &= \frac{4\langle x,y\rangle^2 - (|x|^2+|y|^2)^2 - |x+y|^2|x-y|^2 + 2(|x|^2+|y|^2)|x+y||x-y|}{4\langle x,y\rangle^2} \\ &= 2|x+y||x-y| \frac{|x|^2+|y|^2 - |x+y||x-y|}{4\langle x,y\rangle^2} = \frac{2|x+y||x-y| r_0}{2\langle x,y\rangle} \\ &= \frac{2|x+y||x-y|}{2\langle x,y\rangle r_1} = \frac{2|x+y||x-y|}{|x|^2+|y|^2+|x+y||x-y|}. \end{aligned}$$

For $\langle x,y\rangle > 0$, one has $|x+y|^2 \sim |x|^2 + |y|^2 \sim |x|^2+|y|^2+|x+y||x-y|$, so that

$$1 - r_0^2 \sim \frac{|x+y||x-y|}{|x|^2+|y|^2} \sim \frac{|x-y|}{|x+y|}. \tag{3.20}$$

Using the exact expression for $1 - r_0^2$, we get that

$$v(r_0) = \frac{-|x|^2 + |y|^2 + |x+y||x-y|}{2}.$$

So

$$p_i^\gamma(x,y) = \int_0^1 \frac{\sqrt{v(r)}}{(1-r^2)^{\frac{d}{2}+1}} e^{-(v(r)-v(r_0))} \, \mathrm{d}r \, \exp\Big(\frac{|x|^2+|y|^2-|x+y||x-y|}{2}\Big).$$

The weak type $(1,1)$ of R_i^{glob} is a consequence of the following theorem, in view Proposition 3.4 of Chapter 2.

THEOREM 3.2. *Assume that* $|x-y| > m(x)$.

a: *Let* $\langle x,y\rangle > 0$. *Then*

$$\int_0^1 \frac{\sqrt{v(r)}}{(1-r^2)^{\frac{d}{2}+1}} e^{v(r_0)-v(r)}\,\mathrm{d}r \lesssim \left(\frac{|x+y|}{|x-y|}\right)^{\frac{d}{2}}$$

and

$$\left(\frac{|x+y|}{|x-y|}\right)^{\frac{d}{2}} e^{\frac{1}{2}(|x|^2+|y|^2-|x-y||x+y|)} \lesssim e^{|x|^2}\Big((1+|x|)^d \wedge (|x|\theta)^{-d}\Big),$$

where $\theta = \theta(x,y)$ *is the angle between* x *and* y.

b: *Let* $\langle x,y\rangle \le 0$. *Then*

$$p_i^\gamma(x,y) \lesssim 1 \lesssim e^{|x|^2}\Big((1+|x|)^d \wedge (|x|\theta)^{-d}\Big).$$

Proof. To prove **a**, we assume $\langle x,y\rangle > 0$. First we notice that (3.19) implies

$$-|x|^2+|y|^2 = (|x|+|y|)(-|x|+|y|) \le |x+y||x-y| \tag{3.21}$$

and this last quantity stays away from 0, in view of the globality assumption $|x-y| > m(x)$. Then

$$v(r_0) = \frac{|x+y||x-y| - |x|^2+|y|^2}{2} \lesssim |x+y||x-y|$$

and

$$\begin{aligned}\sqrt{v(r)}e^{-(v(r)-v(r_0))} &\le (\sqrt{v(r_0)} + \sqrt{v(r)-v(r_0)}\,)e^{-(v(r)-v(r_0))}\\ &\lesssim \sqrt{|x+y||x-y|}\,(1+\sqrt{v(r)-v(r_0)}\,)e^{-(v(r)-v(r_0))}\\ &\lesssim \sqrt{|x+y||x-y|}\,e^{-\frac{1}{2}(v(r)-v(r_0))}.\end{aligned}$$

So we see that, in order to prove the first inequality in **a**, it suffices to verify that

$$\sqrt{|x+y||x-y|}\int_0^1 \frac{1}{(1-r^2)^{\frac{d}{2}+1}} e^{-\frac{1}{2}(v(r)-v(r_0))}\,\mathrm{d}r \lesssim \left(\frac{|x+y|}{|x-y|}\right)^{\frac{d}{2}}. \tag{3.22}$$

Consider the case $r_0 > 1/2$. Then

$$\frac{1}{2} < r_0 = \frac{1}{r_1} = \frac{2\langle x,y\rangle}{|x|^2+|y|^2+|x+y||x-y|} \le \frac{2\langle x,y\rangle}{|x|^2+|y|^2}.$$

So $\langle x,y\rangle \ge (|x|^2+|y|^2)/4$ and

$$|x-y|^2 = |x|^2+|y|^2 - 2\langle x,y\rangle \le \frac{1}{2}(|x|^2+|y|^2) \le \frac{1}{2}|x+y|^2.$$

Writing x and y as $(x+y\pm(x-y))/2$, we conclude that $|x|\sim|y|\sim|x+y|$. It is easy to see that also $\langle x,y\rangle \sim |x+y|^2$.

Now, consider the case $r_0 \le 1/2$. Then $3/4 \le 1-r_0^2 \sim |x-y|/|x+y|$ by (3.20), and hence $|x-y| \gtrsim |x+y|$.

We will write

$$v(r)-v(r_0) = \int_{r_0}^r v'(s)\,\mathrm{d}s \tag{3.23}$$

and use the fact that

$$v'(r) = -2\langle x, y\rangle \frac{(r-r_0)(r-r_1)}{(1-r^2)^2} \sim \langle x, y\rangle \frac{(r-r_0)(r_1-r)}{(1-r^2)^2}.$$

To prove (3.22), we divide the interval of integration into three parts. Consider first

$$I_0 = [0,1] \cap [r_0 - \frac{1-r_0}{2}, r_0 + \frac{1-r_0}{2}] = [r_-, r_+],$$

where $r_- = max(0, 2r_0 - 1)$ and $r_+ = (1+r_0)/2$. For $r \in I_0$, we have $1 - r \sim 1 - r_0$ and $r_1 - r \sim 1 - r_0$. If also $r_0 > 1/2$, then $r_1 - r \sim 1 - r_0$, so that

$$v'(r) \sim |x+y|^2 \frac{r-r_0}{1-r_0}. \tag{3.24}$$

Let instead $r_0 \le 1/2$. Then $r_1 > 2$ and for $r \in I_0$

$$r_1 - r \sim r_1 \sim \frac{|x|^2+|y|^2}{\langle x, y\rangle} \sim \frac{|x+y|^2}{\langle x, y\rangle}.$$

Thus,

$$v'(r) \sim |x+y|^2 \frac{r-r_0}{(1-r_0)^2} \sim |x+y|^2 \frac{r-r_0}{1-r_0},$$

and we have (3.24) again.

Now (3.23), (3.24) and (3.20) give us

$$v(r) - v(r_0) \sim |x+y|^2 \frac{1}{1-r_0}(r-r_0)^2 \sim \frac{|x+y|^3}{|x-y|}(r-r_0)^2. \tag{3.25}$$

The change of variable $s = \sqrt{|x+y|^3/|x-y|}(r-r_0)$ yields the estimate

$$\begin{aligned}
&\sqrt{|x+y||x-y|} \int_{I_0} \frac{1}{(1-r^2)^{\frac{d}{2}+1}} e^{-\frac{1}{2}(v(r)-v(r_0))}\,\mathrm{d}r \\
&\lesssim \sqrt{|x+y||x-y|} \frac{1}{(1-r_0)^{\frac{d}{2}+1}} \int_{-\infty}^{\infty} \exp\Big(-c\frac{|x+y|^3}{|x-y|}(r-r_0)^2\Big)\,\mathrm{d}r \\
&\sim \sqrt{|x+y||x-y|} \Big(\frac{|x+y|}{|x-y|}\Big)^{\frac{d}{2}+1} \Big(\frac{|x-y|}{|x+y|^3}\Big)^{\frac{1}{2}} \\
&\sim \Big(\frac{|x+y|}{|x-y|}\Big)^{\frac{d}{2}},
\end{aligned}$$

for some $c = c(d) > 0$.

Next, consider the left interval $I_- = [0, r_-]$, assuming $r_0 \ge 1/2$ since otherwise I_- is empty. For $r \in I_-$ it is easy to see that $r_0 - r \sim 1 - r$ and $r_1 - r \sim 1 - r$. So

$$-v'(r) \sim |x+y|^2 \frac{(1-r)^2}{(1-r)^2} = |x+y|^2$$

and thus

$$v(r) - v(r_-) = -\int_r^{r_-} v'(s)\,\mathrm{d}s \sim |x+y|^2(r_- - r).$$

From (3.25) we have, since $1 - r_0 \sim r_0 - r_-$,

$$v(r_-) - v(r_0) \sim |x+y|^2(r_0 - r_-).$$

Thus, for $r \in I_-$,

$$\begin{aligned}
v(r) - v(r_0) &= [v(r) - v(r_-)] + [v(r_-) - v(r_0)] \\
&\gtrsim |x+y|^2(r_0 - r) \\
&\sim |x+y|^2(1-r).
\end{aligned}$$

Now, making the change of variable $s = |x+y|^2(1-r)$, we get

$$\begin{aligned}
&\sqrt{|x+y||x-y|}\int_{I_-}\frac{1}{(1-r^2)^{\frac{d}{2}+1}}e^{-\frac{1}{2}(v(r)-v(r_0))}\,\mathrm{d}r\\
&\lesssim \sqrt{|x+y||x-y|}\int_{I_-}\frac{1}{(1-r)^{\frac{d}{2}}}e^{-c|x+y|^2(1-r)}\,\frac{\mathrm{d}r}{1-r}\\
&\lesssim \sqrt{|x+y||x-y|}\;|x+y|^d\int_{c'|x+y||x-y|}^{\infty}s^{-\frac{d}{2}}e^{-cs}\,\frac{\mathrm{d}s}{s}\\
&\lesssim \sqrt{|x+y||x-y|}\;|x+y|^d e^{-c''|x+y||x-y|}\\
&\lesssim \sqrt{|x+y||x-y|}\;|x+y|^d(|x+y||x-y|)^{-\frac{d}{2}-\frac{1}{2}}\\
&= \Big(\frac{|x+y|}{|x-y|}\Big)^{\frac{d}{2}},
\end{aligned}$$

c, c', c'' denoting positive constants.

Here it is essential that $|x+y||x-y|$ stays away from zero. This is ensured by the globality assumption.

Finally consider $I_+ = [r_+, 1]$. For $r \in I_+$ we get

$$v'(r) \gtrsim |x+y|^2\frac{(1-r_0)^2}{(1-r)^2}$$

when $r_0 > 1/2$ and also when $r_0 \le 1/2$, since then $r_1 - r \sim |x+y|^2/\langle x, y\rangle$. It follows that

$$v(r) - v(r_+) \gtrsim |x+y|^2(1-r_0)^2\Big(\frac{1}{1-r} - \frac{1}{1-r_+}\Big).$$

From (3.25) we know that

$$v(r_+) - v(r_0) \gtrsim |x+y|^2\frac{(r_+-r_0)^2}{1-r_0} \sim |x+y|^2(1-r_0)^2\frac{1}{1-r_+}.$$

Thus, for $r \in I_+$

$$\begin{aligned}
v(r) - v(r_0) &= [v(r) - v(r_+)] + [v(r_+) - v(r_0)]\\
&\gtrsim |x+y|^2(1-r_0)^2\frac{1}{1-r}\\
&\sim \frac{|x-y|^2}{1-r}.
\end{aligned}$$

Then the change of variable $s = |x-y|^2/(1-r)$ yields

$$\begin{aligned}
&\sqrt{|x+y||x-y|}\int_{I_+}\frac{1}{(1-r^2)^{\frac{d}{2}+1}}e^{-\frac{1}{2}(v(r)-v(r_0))}\,\mathrm{d}r\\
&\lesssim \sqrt{|x+y||x-y|}\int_{I_+}\frac{1}{(1-r)^{\frac{d}{2}}}\exp\Big(-c\frac{|x-y|^2}{1-r}\Big)\frac{\mathrm{d}r}{1-r}\\
&\lesssim \sqrt{|x+y||x-y|}\;|x-y|^{-d}\int_{c|x+y||x-y|}^{\infty}s^{\frac{d}{2}}e^{-cs}\,\frac{\mathrm{d}s}{s}\\
&\lesssim |x-y|^{-d}.
\end{aligned}$$

Now, by the globality assumption,

$$|x-y| \ge \frac{1}{1+|x|} \ge \frac{1}{1+|x|+|y|} \sim \frac{1}{|x+y|}.$$

So

$$|x-y|^{-d} \lesssim \Big(\frac{|x+y|}{|x-y|}\Big)^{\frac{d}{2}},$$

and the first inequality of **a** is proved.

The second inequality of **a** can be written

$$\Big(\frac{|x+y|}{|x-y|}\Big)^{\frac{d}{2}} e^{-\frac{A}{2}} \lesssim (1+|x|)^d \wedge (|x|\theta)^{-d}$$

where

$$\begin{aligned} A &= |x+y||x-y| - (|y|^2-|x|^2) = \frac{|x+y|^2|x-y|^2-(|y|^2-|x|^2)^2}{|x+y||x-y|+|y|^2-|x|^2} \\ &= \frac{(|x|^2+|y|^2)^2-4\langle x,y\rangle^2-(|y|^2-|x|^2)^2}{|x+y||x-y|+|y|^2-|x|^2} = \frac{4|x|^2|y|^2-4|x|^2|y|^2\cos^2\theta}{|x+y||x-y|+|y|^2-|x|^2} \\ &= \frac{4|x|^2|y|^2\sin^2\theta}{|x+y||x-y|+|y|^2-|x|^2}. \end{aligned}$$

From (3.21) we see that $A \geq 0$ and that

$$A \gtrsim \frac{|x|^2|y|^2\sin^2\theta}{|x+y||x-y|},$$

and thus

$$\begin{aligned} \Big(\frac{|x+y|}{|x-y|}\Big)^{\frac{d}{2}} e^{-\frac{A}{2}} &\lesssim \Big(\frac{|x+y|}{|x-y|}\Big)^{\frac{d}{2}} \Big(\frac{|x+y||x-y|}{|x|^2|y|^2\sin^2\theta}\Big)^{\frac{d}{2}} \\ &\lesssim \frac{1}{(|x|\sin\theta)^d}\frac{|x+y|^d}{|y|^d} \\ &\lesssim \frac{1}{(|x|\theta)^d}, \end{aligned}$$

except when $|y| << |x|$. But in that exceptional case the first expression for A implies $A \sim |x|^2$, and then $|x+y|/|x-y| \sim 1$, so that

$$\Big(\frac{|x+y|}{|x-y|}\Big)^{\frac{d}{2}} e^{-\frac{A}{2}} \lesssim |x|^{-d} \lesssim (|x|\theta)^{-d}.$$

Next, by the globality assumption we have that

$$\Big(\frac{|x+y|}{|x-y|}\Big)^{\frac{d}{2}} e^{-\frac{A}{2}} \lesssim (|x+y|(1+|x|))^{\frac{d}{2}} \lesssim (1+|x|)^d$$

except when $|y| >> |x|$, in which case

$$\Big(\frac{|x+y|}{|x-y|}\Big)^{\frac{d}{2}} e^{-A/2} \lesssim 1 \lesssim (1+|x|)^d.$$

Hence also the second inequality in **a** holds.

For **b** let $\langle x,y\rangle \leq 0$. We need only prove the first inequality, since the second is trivial. Recalling (3.15), we shall prove that

$$\int_0^1 \frac{|rx-y|}{(1-r^2)^{\frac{d}{2}+\frac{3}{2}}} \exp\Big(|y|^2 - \frac{|rx-y|^2}{1-r^2}\Big)\,\mathrm{d}r \lesssim 1.$$

In this situation $|rx-y|^2 \geq r^2|x|^2+|y|^2$ and $|rx-y| \leq |x|+|y|$. Thus

$$|y|^2 - \frac{|rx-y|^2}{1-r^2} \leq |y|^2 - \frac{r^2|x|^2+|y|^2}{1-r^2} = -\frac{r^2(|x|^2+|y|^2)}{1-r^2}$$

and

$$p_i^\gamma(x,y) \lesssim \int_0^1 \frac{|x|+|y|}{(1-r^2)^{\frac{d}{2}+\frac{3}{2}}} \exp\Big(-\frac{r^2(|x|^2+|y|^2)}{1-r^2}\Big)\,\mathrm{d}r.$$

We split the integral in two: first

$$\begin{aligned}\int_0^{\frac12} \frac{|x|+|y|}{(1-r^2)^{\frac{d}{2}+\frac{3}{2}}} \exp\Big(-\frac{r^2(|x|^2+|y|^2)}{1-r^2}\Big)\,\mathrm{d}r \\ \lesssim (|x|+|y|)\int_{-\infty}^{\infty} e^{-r^2(|x|^2+|y|^2)}\,\mathrm{d}r \\ \lesssim \frac{|x|+|y|}{(|x|^2+|y|^2)^{\frac12}} \\ \sim 1,\end{aligned}$$

and then

$$\begin{aligned}&\int_{\frac12}^1 \frac{|x|+|y|}{(1-r^2)^{\frac{d}{2}+\frac{3}{2}}} \exp\Big(-\frac{r^2(|x|^2+|y|^2)}{1-r^2}\Big)\,\mathrm{d}r \\ &\quad\lesssim (|x|+|y|)\int_{\frac12}^1 \frac{1}{(1-r)^{\frac{d}{2}+\frac12}} \exp\Big(-\frac{|x|^2+|y|^2}{4(1-r)}\Big)\frac{\mathrm{d}r}{1-r} \\ &\quad\lesssim (|x|+|y|)(|x|^2+|y|^2)^{-\frac{d}{2}-\frac12}\int_0^\infty s^{+\frac{d}{2}+\frac12}e^{-s/4}\,\frac{\mathrm{d}s}{s} \\ &\quad\lesssim 1,\end{aligned}$$

where we made the change of variable $s=(|x|^2+|y|^2)/(1-r)$. □

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