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Fred Brackx, H. De Schepper, Roman Lávička, Vladimír Souček

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Orthogonal bases of Hermitean monogenic polynomials: an explicit construction in complex dimension 2

F. Brackx*, H. De Schepper*, R. Lávička† and V. Souček†

*Clifford Research Group, Department of Mathematics, Faculty of Engineering, Ghent University, Belgium
†Mathematical Institute, Charles University, Sokolovská 83, 186 75 Praha 8, Czech Republic

Abstract. In this contribution we construct an orthogonal basis of Hermitean monogenic polynomials for the specific case of two complex variables. The approach combines group representation theory, see [5], with a Fischer decomposition for the kernels of each of the considered Dirac operators, see [4], and a Cauchy-Kovalevskaya extension principle, see [3].

Keywords: Hermitean Clifford analysis, orthogonal basis
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BASICS OF HERMITEAN CLIFFORD ANALYSIS

Let $(e_1,\ldots,e_m)$ be an orthonormal basis of $\mathbb{R}^m$, then multiplication in the complex Clifford algebra $\mathbb{C}_m$ is governed by the rule $e_\alpha e_\beta + e_\beta e_\alpha = -2\delta_{\alpha\beta}$, $\alpha,\beta = 1,\ldots,m$, whence $\mathbb{C}_m$ is generated additively by the elements $e_A = e_{j_1}\ldots e_{j_h}$, where $A = \{j_1,\ldots,j_h\} \subset \{1,\ldots,m\}$, with $1 \le j_1 < j_2 < \cdots < j_h \le m$, and $e_\emptyset = 1$.

The framework for Hermitean Clifford analysis is introduced by means of a complex structure, i.e. an $\mathrm{SO}(m;\mathbb{R})$–element J with $J^2 = -\mathbf{1}$ (see [1, 2]). So, the dimension is forced to be even: $m = 2n$. Usually J is chosen to act upon the generators of $\mathbb{C}_{2n}$ as $J[e_j] = -e_{n+j}$ and $J[e_{n+j}] = e_j$, $j = 1,\ldots,n$. By means of the projection operators $\pm\frac{1}{2}(\mathbf{1} \pm iJ)$ associated to J, first the Witt basis elements $(\mathfrak{f}_j,\mathfrak{f}_j^\dagger)_{j=1}^n$ for $\mathbb{C}_{2n}$ are obtained: $\mathfrak{f}_j = \frac{1}{2}(\mathbf{1}+iJ)[e_j] = \frac{1}{2}(e_j - i\,e_{n+j})$ and $\mathfrak{f}_j^\dagger = -\frac{1}{2}(\mathbf{1}-iJ)[e_j] = -\frac{1}{2}(e_j + i\,e_{n+j})$, $j = 1,\ldots,n$, satisfying the relations $\mathfrak{f}_j\mathfrak{f}_k + \mathfrak{f}_k\mathfrak{f}_j = \mathfrak{f}_j^\dagger\mathfrak{f}_k^\dagger + \mathfrak{f}_k^\dagger\mathfrak{f}_j^\dagger = 0$ and $\mathfrak{f}_j\mathfrak{f}_k^\dagger + \mathfrak{f}_k^\dagger\mathfrak{f}_j = \delta_{jk}$, $j,k = 1,\ldots,n$. Next, a vector $(x_1,\ldots,x_n,y_1,\ldots,y_n) \in \mathbb{R}^{2n}$ is identified with $\underline{X} = \sum_{j=1}^n (e_j x_j + e_{n+j} y_j)$, producing the Hermitean variables $\underline{z} = \frac{1}{2}(\mathbf{1}+iJ)[\underline{X}] = \sum_{j=1}^n \mathfrak{f}_j z_j$ and $\underline{z}^\dagger = -\frac{1}{2}(\mathbf{1}-iJ)[\underline{X}] = \sum_{j=1}^n \mathfrak{f}_j^\dagger \bar{z}_j$, expressed in the complex variables $z_j = x_j + iy_j$ and their conjugates $\bar{z}_j = x_j - iy_j$, $j = 1,\ldots,n$. Finally, the Dirac operator $\partial_{\underline{X}} = \sum_{j=1}^n (e_j\,\partial_{x_j} + e_{n+j}\,\partial_{y_j})$ gives rise to the Hermitean Dirac operators $\partial_{\underline{z}}^\dagger = \frac{1}{4}(\mathbf{1}+iJ)[\partial_{\underline{X}}] = \sum_{j=1}^n \mathfrak{f}_j\,\partial_{\bar{z}_j}$ and $\partial_{\underline{z}} = -\frac{1}{4}(\mathbf{1}-iJ)[\partial_{\underline{X}}] = \sum_{j=1}^n \mathfrak{f}_j^\dagger\,\partial_{z_j}$, involving the Cauchy–Riemann operators $\partial_{\bar{z}_j} = \frac{1}{2}(\partial_{x_j} + i\partial_{y_j})$ and their conjugates $\partial_{z_j} = \frac{1}{2}(\partial_{x_j} - i\partial_{y_j})$, $j = 1,\ldots,n$. The Hermitean variables and Dirac operators are isotropic, whence the Laplacian decomposes as $\Delta_{2n} = 4(\partial_{\underline{z}}\partial_{\underline{z}}^\dagger + \partial_{\underline{z}}^\dagger\partial_{\underline{z}})$, while also $\underline{z}\,\underline{z}^\dagger + \underline{z}^\dagger\underline{z} = |\underline{z}|^2$.

We take functions with values in an irreducible representation $\mathbb{S}_n$ of $\mathbb{C}_{2n}$, called spinor space, which is realized within $\mathbb{C}_{2n}$ using a primitive idempotent $I = I_1\ldots I_n$, with $I_j = \mathfrak{f}_j\mathfrak{f}_j^\dagger$, $j = 1,\ldots,n$. With that choice $\mathbb{S}_n \equiv \mathbb{C}_{2n}I \cong \bigwedge_n^\dagger I$, where $\bigwedge_n^\dagger$ denotes the Grassmann algebra generated by the $\mathfrak{f}_j^\dagger$'s, since $\mathfrak{f}_j I = 0$. Hence $\mathbb{S}_n$ decomposes into homogeneous parts as $\mathbb{S}_n = \bigoplus_{r=0}^n \mathbb{S}_n^{(r)} = \bigoplus_{r=0}^n (\bigwedge_n^\dagger)^{(r)} I$, with $(\bigwedge_n^\dagger)^{(r)} = \mathrm{span}_{\mathbb{C}}(\mathfrak{f}_{k_1}^\dagger \wedge \mathfrak{f}_{k_2}^\dagger \wedge \cdots \wedge \mathfrak{f}_{k_r}^\dagger : \{k_1,\ldots,k_r\} \subset \{1,\ldots,n\})$.

A continuously differentiable function g in an open region Ω of $\mathbb{R}^{2n}$, taking values in $\mathbb{S}_n$, then is called (left) Hermitean monogenic in Ω iff it satisfies in Ω the system $\partial_{\underline{z}}\,g = 0 = \partial_{\underline{z}}^\dagger\,g$. A major difference with Euclidean Clifford analysis concerns the underlying group invariance. Where $\partial_{\underline{X}}$ is invariant under the action of $\mathrm{SO}(m)$, the system invariance of $(\partial_{\underline{z}},\partial_{\underline{z}}^\dagger)$ breaks down to the group $\mathrm{U}(n)$, see e.g. [1, 2]. For this reason $\mathrm{U}(n)$ will play a fundamental role in the construction of an orthogonal basis of Hermitean monogenic polynomials, as explained in [5].

The spaces of homogeneous polynomials on $\mathbb{C}^n$ with bidegree of homogeneity (a,b) in $(\underline{z},\underline{z}^\dagger)$, taking values in $\mathbb{S}_n^{(r)}$, will be denoted by $\mathscr{P}_{a,b}^r(\mathbb{C}^n)$. By $\mathscr{M}_{a,b}(\mathbb{C}^n)$ we denote the space of Hermitean monogenic polynomials of bidegree (a,b) in $(\underline{z},\underline{z}^\dagger)$, and by $\mathscr{M}_{a,b}^r(\mathbb{C}^n)$ its subspace with values in $\mathbb{S}_n^{(r)}$; the latter may be further split as

$$\mathbb{S}_n^{(r)} \equiv (\textstyle\bigwedge_n^\dagger)^{(r)} I = (\bigwedge_{n-1}^\dagger)^{(r)}(\mathfrak{f}_1^\dagger,\ldots,\mathfrak{f}_{n-1}^\dagger)\,I \bigoplus (\bigwedge_{n-1}^\dagger)^{(r-1)}(\mathfrak{f}_1^\dagger,\ldots,\mathfrak{f}_{n-1}^\dagger)\,\mathfrak{f}_n^\dagger I$$

whence we can decompose polynomials in $\mathscr{M}^r_{a,b}(\mathbb{C}^n)$ as $p_{a,b} = p^0_{a,b} I + p^1_{a,b} \mathfrak{f}^\dagger_n I$, with $p^0_{a,b}$ taking values in $(\bigwedge^\dagger_{n-1})^{(r)}(\mathfrak{f}^\dagger_1, \ldots, \mathfrak{f}^\dagger_{n-1})$ and $p^1_{a,b}$ taking values in $(\bigwedge^\dagger_{n-1})^{(r-1)}(\mathfrak{f}^\dagger_1, \ldots, \mathfrak{f}^\dagger_{n-1})$. Note that for $r = 0$ or $r = n$ one of these components becomes trivial. In the same order of ideas we single out the variables $(z_n, \bar{z}_n)$ and rewrite the Hermitean variables as $\underline{z} = \widetilde{\underline{z}} + \mathfrak{f}_n z_n$ and $\underline{z}^\dagger = \widetilde{\underline{z}}^\dagger + \mathfrak{f}^\dagger_n \bar{z}_n$, and the Hermitean Dirac operators as $\partial_{\underline{z}} = \widetilde{\partial}_{\underline{z}} + \mathfrak{f}^\dagger_n \partial_{z_n}$ and $\partial^\dagger_{\underline{z}} = \widetilde{\partial}_{\underline{z}}^\dagger + \mathfrak{f}_n \partial_{\bar{z}_n}$. We will consider restrictions to $\{z_n = 0 = \bar{z}_n\}$, identified with $\mathbb{C}^{n-1}$. The following results were then proven in [3].

Proposition 1. *(i) Given the polynomial $p^0_{a,b-j} I \in Ker(\widetilde{\partial}_{\underline{z}})$ on $\mathbb{C}^{n-1}$ ($j = 0, \ldots, b$), there exists a unique polynomial $M^0_{a,b,j} \in \mathscr{M}_{a,b}(\mathbb{C}^n)$, given by*

$$M^0_{a,b,j} = \bar{z}_n{}^j \left(\sum_{k=0}^{\min(2a+1, 2(b-j))} \frac{1}{\lfloor \frac{k}{2} \rfloor !} \frac{1}{\lfloor \frac{k+1}{2} \rfloor !} \left(z_n \widetilde{\partial}_{\underline{z}} \mathfrak{f}_n + \bar{z}_n \widetilde{\partial}_{\underline{z}}^\dagger \mathfrak{f}^\dagger_n \right)^k p^0_{a,b-j} I \right)$$

such that $\partial^j_{\bar{z}_n} M^0_{a,b,j}|_{\mathbb{C}^{n-1}} = p^0_{a,b-j} I$ and all other derivatives w.r.t. $\bar{z}_n$ vanish in $\mathbb{C}^{n-1}$.

(ii) Given the polynomial $p^1_{a-i,b} \mathfrak{f}^\dagger_n I \in Ker(\widetilde{\partial}_{\underline{z}}^\dagger)$ on $\mathbb{C}^{n-1}$ ($i = 0, \ldots, a$), there exists a unique polynomial $M^1_{a,b,i} \in \mathscr{M}_{a,b}(\mathbb{C}^n)$, given by

$$M^1_{a,b,i} = z_n{}^i \left(\sum_{k=0}^{\min(2a, 2b+1)} \frac{1}{\lfloor \frac{k}{2} \rfloor !} \frac{1}{\lfloor \frac{k+1}{2} \rfloor !} \left(z_n \widetilde{\partial}_{\underline{z}} \mathfrak{f}_n + \bar{z}_n \widetilde{\partial}_{\underline{z}}^\dagger \mathfrak{f}^\dagger_n \right)^k p^1_{a-i,b} \mathfrak{f}^\dagger_n I \right)$$

such that $\partial^i_{z_n} M^1_{a,b,i}|_{\mathbb{C}^{n-1}} = p^1_{a-i,b} \mathfrak{f}^\dagger_n I$ and all other derivatives w.r.t. z_n vanish in $\mathbb{C}^{n-1}$.

The polynomial $M^0_{a,b,j}$ (respectively $M^1_{a,b,i}$) is called the Hermitean Cauchy-Kovalevskaya extension of the initial polynomial $p^0_{a,b-j} I$ (respectively the initial polynomial $p^1_{a-i,b} \mathfrak{f}_n I$). This CK extension will play an important role in the construction of the desired orthogonal basis. Indeed, introducing, as in [5], the following spaces of initial polynomials:

$$\begin{aligned} \mathscr{A}^r_{a,b-j} &= \left\{ p^0_{a-i,b} I \mid p^0_{a-i,b} I \in \mathrm{Ker}(\widetilde{\partial}_{\underline{z}}) \cap \mathscr{P}^r_{a,b-j}(\mathbb{C}^{n-1}) \right\} \\ \mathscr{B}^r_{a-i,b} &= \left\{ p^1_{a-i,b} \mathfrak{f}^\dagger_n I \mid p^1_{a-i,b} I \in \mathrm{Ker}(\partial^\dagger_{\underline{z}}) \cap \mathscr{P}^{r-1}_{a-i,b}(\mathbb{C}^{n-1}) \right\} \end{aligned}$$

the CK extension map is an isomorphism from $\oplus_{j=0}^b \mathscr{A}^r_{a,b-j} \oplus \oplus_{i=0}^a \mathscr{B}^r_{a-i,b}$ to $\mathscr{M}^r_{a,b}$, commuting with the action of $\mathrm{U}(n-1)$, whence it yields a splitting of $\mathscr{M}^r_{a,b}$ into a direct sum of $\mathrm{U}(n-1)$ invariant subspaces. Since the initial polynomials on $\mathbb{C}^{n-1}$ for the CK extension have to be submit to the compatibility condition of being either in the kernel of $\widetilde{\partial}_{\underline{z}}$ or in the kernel of $\widetilde{\partial}_{\underline{z}}^\dagger$, the so-called Fischer decomposition of these kernels in terms of Hermitean monogenics will also be involved. Under the action of $\mathrm{U}(n-1)$, see [4], the space $\mathrm{Ker}^r_{a,b}(\widetilde{\partial}_{\underline{z}}) \equiv \mathrm{Ker}(\widetilde{\partial}_{\underline{z}}) \cap \mathscr{P}^r_{a,b}(\mathbb{C}^{n-1})$ has the multiplicity free irreducible decomposition

$$\mathrm{Ker}^r_{a,b}(\widetilde{\partial}_{\underline{z}}) = \mathscr{M}^r_{a,b} \bigoplus_{j=0}^{\min(a,b-1)} |\underline{z}|^{2j} \underline{z}^\dagger \mathscr{M}^{r-1}_{a-j,b-j-1} \bigoplus_{j=0}^{\min(a-1,b-1)} |\underline{z}|^{2j} (\underline{z}^\dagger \underline{z} + \frac{(a-j-1+r)}{(a+r)} \underline{z}\, \underline{z}^\dagger) \mathscr{M}^r_{a-j-1,b-j-1} \quad (1)$$

and the space $\mathrm{Ker}^{r-1}_{a,b}(\widetilde{\partial}_{\underline{z}}^\dagger) \equiv \mathrm{Ker}(\widetilde{\partial}_{\underline{z}}^\dagger) \cap \mathscr{P}^{r-1}_{a,b}(\mathbb{C}^{n-1})$ has the multiplicity free irreducible decomposition

$$\mathrm{Ker}^{r-1}_{a,b}(\widetilde{\partial}_{\underline{z}}^\dagger) = \mathscr{M}^{r-1}_{a,b} \bigoplus_{j=0}^{\min(a-1,b)} |\underline{z}|^{2j} \underline{z} \mathscr{M}^r_{a-j-1,b-j} \bigoplus_{j=0}^{\min(a-1,b-1)} |\underline{z}|^{2j} (\underline{z}\underline{z}^\dagger + \frac{(b-j-1+n-r+1)}{(b+n-r+!)} \underline{z}\, \underline{z}^\dagger) \mathscr{M}^{r-1}_{a-j-1,b-j-1} \quad (2)$$

It now becomes clear that, once the desired bases have been constructed in dimension $n-1$, these results can been used as building blocks in the above Fischer decompositions, yielding bases for the spaces $\mathscr{A}^r_{a,b-j}$ and $\mathscr{B}^r_{a-i,b}$ of initial polynomials. Subsequent application of the CK extension procedure, will then produce a basis for the space $\mathscr{M}^r_{a,b}$ in dimension n, which, by construction, will be orthogonal w.r.t. any $\mathrm{U}(n)$ invariant inner product.

We will now follow this general procedure as explained above, and, in more detail, in [5], to explicitly obtain orthogonal bases for the spaces $\mathscr{M}^r_{a,b}(\mathbb{C}^2)$, $r = 0, 1, 2$, $(a,b) \in \mathbb{N}^2$. Since the procedure is inductive, we need however to start with the case $n = 1$.

THE CASE $n=1$

In this case we are considering polynomials $f(z_1,\bar{z}_1)$ defined in the complex plane and taking values in the spinor space $\mathbb{S}_1 = \mathrm{span}_{\mathbb{C}}\{1,\mathfrak{f}_1^\dagger\}I$. The Hermitean Dirac operators are simply $\widetilde{\partial}_{\underline{z}} = \mathfrak{f}_1^\dagger\,\partial_{z_1}$ and $\widetilde{\partial}_{\underline{z}}^{\,\dagger} = \mathfrak{f}_1\,\partial_{\bar{z}_1}$, whence Hermitean monogenicity means nothing else but anti-holomorphy in the case $r=0$ and holomorphy in the case $r=1$. The symmetry group here is $\mathrm{U}(1) \simeq \mathrm{SO}(2)$.

For $r=0$ the U(1) modules $\widetilde{\mathscr{M}}^0_{0,b}$ are given by span $\left\{\frac{\bar{z}_1{}^b}{b!}I\right\}$, $b=0,1,2,\ldots$. They have highest weight $(-b)$.

For $r=1$ the U(1) modules $\widetilde{\mathscr{M}}^1_{a,0}$ are given by span $\left\{\frac{z_1{}^a}{a!}\mathfrak{f}_1^\dagger I\right\}$, $a=0,1,2,\ldots$. They have highest weight $(a+1)$.

THE CASE $n=2$

Now we consider polynomials $f(z_1,\bar{z}_1,z_2,\bar{z}_2)$ taking values in the spinor space $\mathbb{S}_2 = \mathrm{span}_{\mathbb{C}}\{1,\mathfrak{f}_1^\dagger,\mathfrak{f}_2^\dagger,\mathfrak{f}_1^\dagger\mathfrak{f}_2^\dagger\}I$. If $r=0$ or $r=2$ we are again confronted with (anti-)holomorphy, see [2], so we will focus on the interesting case $r=1$.

The dimension of the U(2) module $\mathscr{M}^1_{a,b}$ is $a+b+2$, see [3]. Each of the spaces of initial polynomials $\mathscr{A}^1_{a,b-j}$, $j=0,\ldots,b$ and $\mathscr{B}^1_{a-i,b}$, $i=0,\ldots,a$, is one-dimensional. The general theory of the CK extension procedure, see [3], predicts that the compatibility conditions imposed on these initial polynomials will be trivially fulfilled, so they simply are all homogeneous polynomials in the variables z_1 ans $\bar{z}_1$ of the appropriate bidegree, which is moreover confirmed by the Fischer decompositions (1)–(2):

$$\mathscr{A}^1_{a,b-j} = \mathrm{span}_{\mathbb{C}}\left\{(-1)^{b-j}\frac{z_1{}^a}{a!}\frac{\bar{z}_1{}^{b-j}}{(b-j)!}\mathfrak{f}_1^\dagger I\right\}, \qquad j=0,\ldots,b$$

$$\mathscr{B}^1_{a-i,b} = \mathrm{span}_{\mathbb{C}}\left\{(-1)^{b}\frac{z_1{}^{a-i}}{(a-i)!}\frac{\bar{z}_1{}^{b}}{b!}\mathfrak{f}_2^\dagger I\right\}, \qquad i=0,\ldots,a$$

By CK extension each of the spaces of initial polynomials thus gives rise to exactly one Hermitean monogenic basis polynomial, together yielding an orthogonal basis for $\mathscr{M}^1_{a,b}$, see [5]. These basis polynomials are respectively given by

$$\begin{aligned} M^0_{a,b,j} &= \sum_{k=0}^{\min(a,b-j)} (-1)^{b-j-k}\frac{z_2{}^k}{k!}\frac{\bar{z}_2{}^{k+j}}{(k+j)!}\frac{z_1{}^{a-k}}{(a-k)!}\frac{\bar{z}_1{}^{b-j-k}}{(b-j-k)!}\mathfrak{f}_1^\dagger I \\ &+ \sum_{k=0}^{\min(a,b-j-1)} (-1)^{b-j-k-1}\frac{z_2{}^k}{k!}\frac{\bar{z}_2{}^{k+j+1}}{(k+j+1)!}\frac{z_1{}^{a-k}}{(a-k)!}\frac{\bar{z}_1{}^{b-j-k-1}}{(b-j-k-1)!}\mathfrak{f}_2^\dagger I, \qquad j=0,\ldots,b \\ M^1_{a,b,i} &= \sum_{k=0}^{\min(a-i,b)} (-1)^{b-k}\frac{z_2{}^{k+i}}{(k+i)!}\frac{\bar{z}_2{}^{k}}{k!}\frac{z_1{}^{a-i-k}}{(a-i-k)!}\frac{\bar{z}_1{}^{b-k}}{(b-k)!}\mathfrak{f}_2^\dagger I \\ &+ \sum_{k=0}^{\min(a-i-1,b)} (-1)^{b-k}\frac{z_2{}^{k+i+1}}{(k+i+1)!}\frac{\bar{z}_2{}^{k}}{k!}\frac{z_1{}^{a-i-k-1}}{(a-i-k-1)!}\frac{\bar{z}_1{}^{b-k}}{(b-k)!}\mathfrak{f}_1^\dagger I, \qquad i=0,\ldots,a \end{aligned}$$

The following properties may then be verified right away.

Property 1. *Under derivation with respect to the "new" variables $(z_2,\bar{z}_2)$, the orthogonal basis polynomials of $\mathscr{M}^1_{a,b}$ act as follows:*

$$\begin{aligned} &\partial_{z_2} M^1_{a,b,i} = M^1_{a-1,b,i-1} \quad &&\partial_{\bar{z}_2} M^1_{a,b,i} = M^1_{a,b-1,i+1} \quad &&i=1,\ldots,a \\ &\partial_{z_2} M^0_{a,b,j} = M^0_{a-1,b,j+1} \quad &&\partial_{\bar{z}_2} M^0_{a,b,j} = M^0_{a,b-1,j-1} \quad &&j=1,\ldots,b \\ &\partial_{z_2} M^1_{a,b,0} = M^0_{a-1,b,0} \quad &&\partial_{\bar{z}_2} M^0_{a,b,0} = M^1_{a,b-1,0} && \end{aligned}$$

Property 2. *Under derivation with respect to the "old" variables $(z_1,\bar{z}_1)$, the orthogonal basis polynomials of $\mathscr{M}^1_{a,b}$ act as follows:*

$$\begin{aligned} &\partial_{z_1} M^1_{a,b,i} = M^1_{a-1,b,i} \quad &&-\partial_{\bar{z}_1} M^1_{a,b,i} = M^1_{a,b-1,i} \quad &&i=0,\ldots,a \\ &\partial_{z_1} M^0_{a,b,j} = M^0_{a-1,b,j} \quad &&-\partial_{\bar{z}_1} M^0_{a,b,j} = M^0_{a,b-1,j} \quad &&j=0,\ldots,b \end{aligned}$$

Remark 1. *Property 1 holds in any dimension n, whereas Property 2 is specific for the case* $n = 2$.

Remark 2. *Since the orthogonal Hermitean monogenic basis polynomials are determined only up to a constant, the final expressions may always be normalized, according to some preferred behaviour or property. Here, we have in fact normalized all initial data by requiring that*

(i) if $p^0_{a,b} I \in \mathscr{A}^1_{a,b}$, *then* $\partial^a_{z_1} (-\partial_{\bar{z}_1})^b [p^0_{a,b} I] = \mathfrak{f}^\dagger_1 I$;

(ii) if $p^1_{a,b} \mathfrak{f}^\dagger_2 I \in \mathscr{B}^1_{a,b}$, *then* $\partial^a_{z_1} (-\partial_{\bar{z}_1})^b [p^1_{a,b} \mathfrak{f}^\dagger_2 I] = \mathfrak{f}^\dagger_2 I$.

These normalization conditions are reflected in the eventual orthogonal basis as follows:

$$\partial^a_{z_1} (-\partial_{\bar{z}_1})^{b-j} \left[M^0_{a,b,j} \right] = \frac{\bar{z}_2{}^j}{j!} \mathfrak{f}^\dagger_1 I, \qquad \partial^{a-i}_{z_1} (-\partial_{\bar{z}_1})^b \left[M^1_{a,b,i} \right] = \frac{z_2{}^i}{i!} \mathfrak{f}^\dagger_2 I$$

Finally let us check the branching rules for the space $\mathscr{M}^1_{a,b}$ with highest weight $\lambda = (a+1, -b)$. From group representation theory, see e.g. [6], we know that when restricting the symmetry to U(1), the irreducible U(2) module $\mathscr{M}^1_{a,b}$ decomposes into irreducible U(1) modules as

$$\mathscr{M}^1_{a,b} = \bigoplus_{\mu \succ \lambda} V_\mu = \bigoplus_{k=-b}^{a+1} V_k$$

where each summand appears with multiplicity one; this decomposition is orthogonal w.r.t. any U(2) invariant scalar product on $\mathscr{M}^1_{a,b}$. On the other hand the Fischer decompositions (1)–(2) produce the U(1) irreducible components of the spaces of initial data $\mathscr{A}^1_{a,b-j}$ $(j = 0, \ldots, b)$ and $\mathscr{B}^1_{a-i,b}$ $(i = 0, \ldots, a)$. Assuming that $a > b$ (the cases $a \leq b$ being similar) we have in fact that $\mathscr{A}^1_{a,b-j}$ is a shifted version of the U(1) module $\widetilde{\mathscr{M}}^1_{a-b+j,0}$ with highest weight $(a-b+j+1)$, for all $j = 0, \ldots, b$. Similarly, $\mathscr{B}^1_{a-i,b}$ is a shifted version of the U(1) module $\widetilde{\mathscr{M}}^1_{a-i-b-1,0}$ with highest weight $(a-i-b)$, for all $i = 0, \ldots, a-b-1$, while $\mathscr{B}^1_{a-i,b}$ is a shifted version of $\widetilde{\mathscr{M}}^0_{0,b-a+i}$ with highest weight $(a-i-b)$, for all $i = a-b, \ldots, a$. As the CK extension map is an isomorphism between the initial data space $\oplus^b_{j=0} \mathscr{A}^1_{a,b-j} \oplus \oplus^a_{i=0} \mathscr{B}^1_{a-i,b}$ and the space $\mathscr{M}^1_{a,b}$, which commutes with the action of U(1), our construction of the orthogonal basis of $\mathscr{M}^1_{a,b}$ exactly yields the above splitting of $\mathscr{M}^1_{a,b}$ into the direct sum of $a+b+2$ U(1) invariant subspaces V_k, $k = -b, \ldots, a+1$.

Remark 3. *For completeness we mention here the cases* $r = 0$ *and* $r = 2$. *For* $r = 0$ *the orthogonal basis of* $\mathscr{M}^0_{0,b}$ *consists of all homogeneous anti-holomorphic polynomials in* $(\bar{z}_1, \bar{z}_2)$, *i.e.* $\frac{\bar{z}_1{}^{b-j}}{(b-j)!} \frac{\bar{z}_2{}^j}{j!}$, $j = 0, \ldots, b$, *while for* $r = 2$, *the orthogonal basis of* $\mathscr{M}^2_{a,0}$ *consists of all homogeneous holomorphic polynomials in* (z_1, z_2), *i.e.* $\frac{z_1{}^{a-i}}{(a-i)!} \frac{z_2{}^i}{i!}$, $i = 0, \ldots, a$.

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