# Orthogonal Eigenvector Matrix of the Laplacian

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***Abstract*— The orthogonal eigenvector matrix $Z$ of the Laplacian matrix of a graph with $N$ nodes is studied rather than its companion $X$ of the adjacency matrix, because for the Laplacian matrix, the eigenvector matrix $Z$ corresponds to the adjacency companion $X$ of a regular graph, whose properties are easier. In particular, the column sum vector of $Z$ (which we call the fundamental weight vector $w$) is, for a connected graph, proportional to the basic vector $e_N = (0, 0, \ldots, 1)$, so that more information about the specfics of the graph is contained in the row sum of $Z$ (which we call the dual fundamental weight vector $\varphi$). Since little is known about $Z$ (or $X$), we have tried to understand simple properties of $Z$ such as the number of zeros, the sum of elements, the maximum and minimum element and properties of $\varphi$. For the particular class of Erdős-Rényi random graphs, we found that a product of a Gaussian and a super-Gaussian distribution approximates accurately the distribution of $\varphi_U$, a uniformly at random chosen component of the dual fundamental weight vector of $Z$.**

## I. Introduction

Networks abound more than ever before. While many graph metrics have been proposed, that are reviewed e.g. in [1], [2], [3], the eigenvector structure of graph related matrices is hardly understood. A graph on $N$ nodes can be represented by an $N \times N$ adjacency matrix $A$ with $a_{ij} = 1$ if the pair of nodes is connected, otherwise $a_{ij} = 0$. Another graph related matrix is the Laplacian matrix $Q = \Delta - A$, where $\Delta = \text{diag}(d_i)$ is the $N \times N$ diagonal degree matrix and the degree of node $i$ is $d_i = \sum_{j=1}^{N} a_{ij}$ . When confining to an unweighted and undirected graph, the Laplacian matrix $Q$ is symmetric and possess the eigenvalue decomposition $Q = ZMZ^T$. The equality implies that all information at the left-hand side, that we call the topology domain, is also contained in the right-hand side, that we call the spectral domain. Most insight so far in graphs is gained in the topology domain that allows a straightforward drawing of a graph: nodes are interconnected by links and display a typical graph representation, attractive and understandable to humans. The spectral domain, consisting of the set $\{z_1, z_2, \ldots, z_N\}$of eigenvectors of the Laplacian $Q$ and the corresponding set of eigenvalues in $M$, is less intuitive for humans. However, as mentioned in the preface of [4], the spectral decomposition $Q = ZMZ^T$ (or $A = X\Lambda X^T$) represents a transformation of a similar nature as a Fourier transform, which suggests that some information is better or more adequately accessible in one domain and other information in the other domain.

Most spectral results are obtained for eigenvalues, and in particular the largest eigenvalue or spectral radius [5] for the adjacency matrix and the second smallest eigenvalue or the algebraic connectivity [6] for the Laplacian matrix. The spectral radius plays an important role in characterizing the dynamical process on networks, such as SIS (susceptible-infected-susceptible) epidemic spread [7]. The algebraic connectivity [6] plays an important role in bounding the node and link connectivity, i.e. the number of nodes and links that have to be removed to disconnect the graph. Correspondingly, the algebraic connectivity is considered as a robustness measure against node/link failures [8]. The sum of the inverse Laplacian eigenvalues, called the effective graph resistance [9], can be used to improve the robustness of complex networks [10].

While the number of mathematical results on other eigenvalues is already considerably less, results on eigenvectors are relatively scarce [11], [12]. Most results on eigenvectors focus on the principle eigenvector [13], the eigenvector corresponding to the largest eigenvalue of the adjacency matrix of a graph, or the Fiedler vector [6], [14], the eigenvector belonging to the second smallest eigenvalue of the Laplacian matrix.

Here, we approach the challenge of unravelling the "hidden information" in the orthogonal eigenvector matrix $Z$ of the Laplacian matrix by extensive simulations, because the purely mathematical discovery of nice properties of the matrix $Z$ seems of a daunting difficulty. Since many properties of the Erdős-Rényi (ER) graphs $G_p(N)$ are known [15], we concentrate here only on this class of graphs. An ER graph $G_p(N)$ on $N$ nodes and with link density $p$ is generated by randomly connecting a pair of nodes with a probability $p$, independently of any other pair. Although ER graphs are generally not good representatives of real-world networks, we believe that, if we cannot understand this simple class of random graphs, the more realistic (but more complex) classes of graphs are certainly beyond reach. Thus, here, we make a first step to learn about the properties of orthogonal eigenvector matrix $Z$ of the Laplacian by confining to ER graphs. An extra bonus, apart from a computational advantage, is that relatively small sizes $N$ in the class $G_p(N)$, even below $N = 100$, already give a good reflection of the general properties for any $N$.

The paper is organized as follows. Section II presents the definition and the orthogonality properties of the eigenvector matrix of the Laplacian. Section III illustrates the properties of the eigenvector matrix. The dual fundamental weight vector is introduced and the distribution of the dual fundamental weight is studied in Section IV. Section V concludes the paper.

## II. Eigenstructure of the Laplacian $Q$ of a graph

As in [4], we denote by $z_k$ the eigenvector of the $N \times N$ symmetric matrix $Q$ belonging to the eigenvalue $\mu_k$, normalized so that $z_k^T z_k = 1$. The eigenvalues of $Q = Q^T$ are real and can be ordered as $\mu_1 \geq \mu_2 \geq \ldots \geq \mu_N$. The all-one vector $u = (1, 1, \ldots, 1)$ is the eigenvector belonging to $\mu_N = 0$,

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since the row sum is $Qu = 0$ for any Laplacian matrix. Let $Z$ be the orthogonal matrix with the eigenvectors of $Q$ in the columns,

$$Z = \left[ \begin{array}{ccccc} z_1 & z_2 & z_3 & \cdots & z_N \end{array} \right]$$

or explicitly in terms of the $m$-th component $(z_j)_m$ of eigenvector $z_j$,

$$Z = \begin{bmatrix} (z_1)_1 & (z_2)_1 & (z_3)_1 & \cdots & (z_N)_1 \\ (z_1)_2 & (z_2)_2 & (z_3)_2 & \cdots & (z_N)_2 \\ (z_1)_3 & (z_2)_3 & (z_3)_3 & \cdots & (z_N)_3 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ (z_1)_N & (z_2)_N & (z_3)_N & \cdots & (z_N)_N \end{bmatrix} \tag{1}$$

where the element $Z_{ij} = (z_j)_i$. The eigenvalue equation $Qz_k = \mu_k z_k$ translates to the matrix equation $Q = ZMZ^T$, where $M = \text{diag}(\mu_k)$.

The relation $Z^T Z = I = ZZ^T$ (see e.g. [4, p. 223]) expresses, in fact, double orthogonality. The first equality $Z^T Z = I$ translates to the well-known orthogonality relation

$$z_k^T z_m = \sum_{j=1}^{N} (z_k)_j (z_m)_j = \delta_{km} \tag{2}$$

stating that the eigenvector $z_k$ belonging to eigenvalue $\mu_k$ is orthogonal to any other eigenvector belonging to a different eigenvalue. The second equality $ZZ^T = I$, which arises from the commutativity of the inverse matrix $Z^{-1} = Z^T$ with the matrix $Z$ itself, can be written as $\sum_{j=1}^{N} (z_j)_m (z_j)_k = \delta_{mk}$ and suggests us to define the row vector in $Z$ as

$$y_m = ((z_1)_m, (z_2)_m, \ldots, (z_N)_m) \tag{3}$$

Then, the second orthogonality condition $ZZ^T = I$ implies orthogonality of the vectors

$$y_l^T y_j = \sum_{k=1}^{N} (z_k)_l (z_k)_j = \delta_{lj} \tag{4}$$

The fundamental weight $\omega_k = u^T z_k$ and the dual fundamental weight $\varphi_j = u^T y_j$ have been introduced in [16]. The corresponding vectors $\omega = (\omega_1, \omega_2, \cdots, \omega_N)$ and $\varphi = (\varphi_1, \varphi_2, \cdots, \varphi_N)$ can be written as the column sum and the row sum, respectively, of the orthogonal matrix $Z$

$$\omega = Z^T u \tag{5}$$

and

$$\varphi = Zu \tag{6}$$

Instead of concentrating on the adjacency matrix $A$, we consider here the Laplacian matrix $Q$, mainly because the all-one vector $u$ is always an eigenvector of $Q$, which greatly simplifies the fundamental weight vector $\omega$. Indeed, since the normalized Laplacian eigenvector $z_N = \frac{u}{\sqrt{N}}$ belonging to the smallest eigenvalue $\mu_N = 0$ is orthogonal to all other eigenvectors, it follows from (5) that, in a connected graph,

$$\omega = (0, 0, \cdots, \sqrt{N}) = \sqrt{N} e_N \tag{7}$$

## III. Exploring properties of the orthogonal eigenvector matrix $Z$ of the Laplacian $Q$

Via extensive simulations on Erdős-Rényi (ER) graphs $G_p(N)$, initial insight is gained in the sum of all the elements, the number of zero elements and the maximum and the minimum element in the eigenvector matrix $Z$ of the Laplacian matrix $Q$.

### A. The sum $s_Z$ of the elements in $Z$

Let $s_Z$ be the sum of the elements in the matrix $Z$. Using the definitions (5) and (6) for a connected graph, the sum $s_Z = u^T Z u = u^T \varphi$ as well as $s_Z = \left(Z^T u\right)^T u = \omega^T u = \sqrt{N}$, where (7) has been used. In a disconnected graph $G$, the sum $s_Z$ is

$$s_Z = \sum_{j=1}^{c} \sum_{k=1}^{N} (z_j)_k$$

where $c$ is the number of components in the disconnected graph $G$. For the case $c = 2$, more details are discussed in the Appendix .

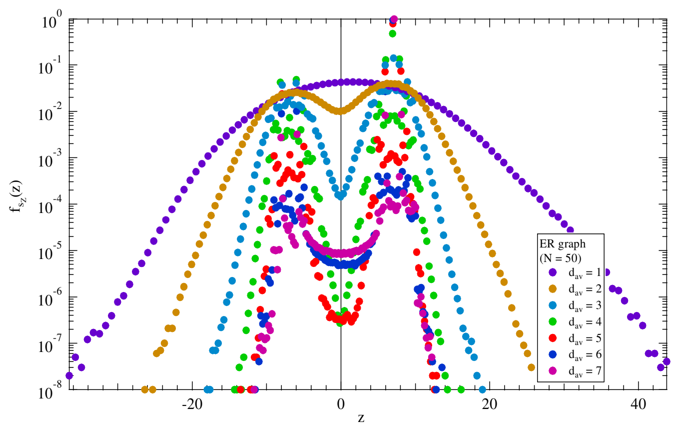


Fig. 1. The probability density function of $s_Z$ in ER random graphs $G_p(N)$ for $N = 50$ and various average degree $d_{av} = p(N-1)$, ranging from $d_{av} = 1$ up to $d_{av} = 7$. The y-axis is in log-scale.

Fig. 1 shows the probability density function $f_{s_Z}(z)$ in ER graphs $G_p(N)$ for $N = 50$ and various average degree $d_{av} = p(N-1)$, ranging from $d_{av} = 1$ up to $d_{av} = 7$. We have generated $10^8$ ER graphs $G_p(50)$. Fig. 1 demonstrates that the maximum value of $f_{s_Z}(z)$ at $z = \sqrt{N}$ increases with the average degree $d_{av}$. For $d_{av} \geq 4$, the maximum value of $f_{s_Z}(z)$ is dominantly high because most generated graphs are connected. Indeed [17], for $N = 50$ and $d_{av} \approx 3.9$, $\Pr[G_p(N)$ is connected$]$ is about 36%. Moreover, ignoring the peak value at $z = \sqrt{N}$, we observe that $f_{s_Z}(z)$ is roughly symmetric around 0.

### B. The number $z_Z$ of zero elements in $Z$

The number of zero elements in the orthogonal matrix $Z$ is an integer smaller than $N^2 - N$, because each eigenvector is different from the zero vector and, thus, should contain at least one non-zero element. Hence, $0 \leq z_Z \leq N^2 - N$. In the

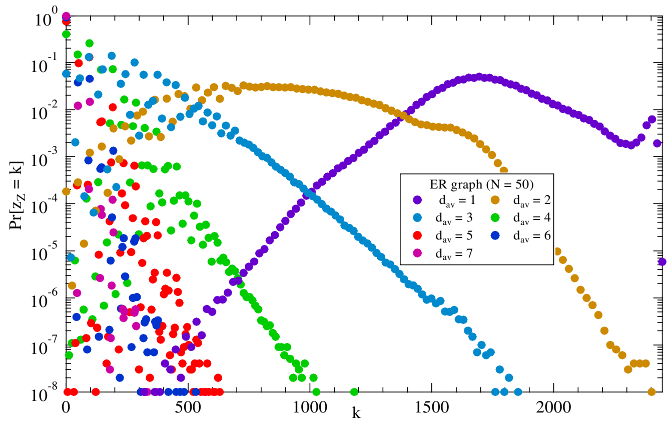


Fig. 2. The probability $\Pr[z_Z = k]$ that the number of zeros in $Z$ equals $k$ in ER random graphs $G_p(N)$ for $N = 50$ and various average degree $d_{av} = p(N-1)$, ranging from $d_{av} = 1$ up to $d_{av} = 7$. The y-axis is in log-scale.

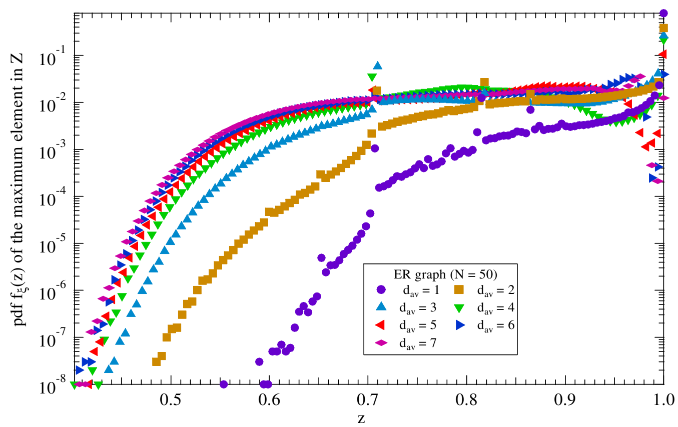


Fig. 4. The probability density function $f_\xi(z)$ of the maximum element in $Z$ in ER random graphs $G_p(N)$ for $N = 50$ and various average degree $d_{av} = p(N-1)$, ranging from $d_{av} = 1$ up to $d_{av} = 7$. The y-axis is in log-scale.

simulations, an element in $Z$ with absolute value smaller than $10^{-10}$ is considered as zero.

Fig. 2 shows that, in ER graphs of $N = 50$ nodes, the average number $E[z_Z]$ of zero elements decreases with the average degree $d_{av}$. The probability $\Pr[z_Z = 0]$ that there is no zero element increases with $d_{av}$. More specifically, for small average degrees, $d_{av} = 1$ and $d_{av} = 2$, the average number $E[z_Z]$ of zero elements is high and the probability that $\Pr[z_Z = 0]$ is small (and almost zero for $d_{av} = 1$). For $d_{av} \geq 4$, the probability $\Pr[z_Z = 0]$ is dominantly high. Moreover, only for $d_{av} \leq 3$, the curve $\Pr[z_Z = k]$ versus $k$ is reasonably stable, but for $d_{av} \geq 4$, large scattering is observed.

### C. The minimum and maximum element in $Z$

We denote the minimum element in the orthogonal matrix $Z$ by $\zeta_Z = \min_{ij} z_{ij}$ and the maximum element by $\xi_Z = \max_{ij} z_{ij}$.

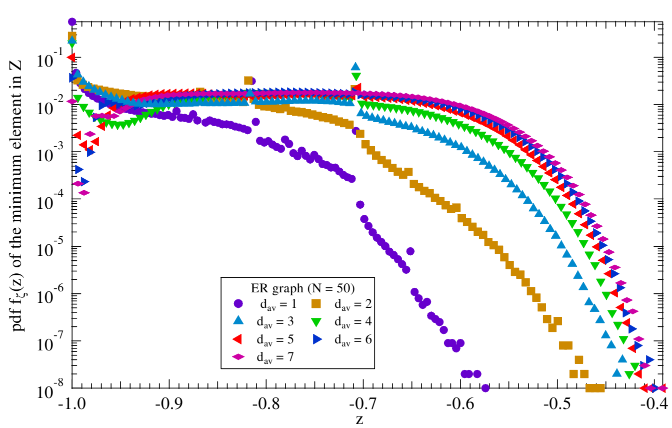


Fig. 3. The probability density function $f_\zeta(z)$ of the minimum element in $Z$ in ER random graphs $G_p(N)$ for $N = 50$ and various average degree $d_{av} = p(N-1)$, ranging from $d_{av} = 1$ up to $d_{av} = 7$. The y-axis is in log-scale.

Figs. 3 and 4 demonstrate that $\xi_Z \overset{d}{=} -\zeta_Z$, where $\overset{d}{=}$ denotes equality in distribution, which is less strong than $\max_{ij} z_{ij} = -\min_{ij} z_{ij}$. Fig. 4 indicates that the lower the average degree $d_{av}$, the higher the probability that the maximum $\xi_Z$ attains the value 1. If only one element is non-zero, then that element must equal $\pm 1$ because of the normalization of eigenvectors.

If the graph is connected, then $z_N = \frac{u}{\sqrt{N}}$ (else, there are $c$ components leading to a different normalization of the $u$ vector, see the Appendix ). The second orthogonality condition (4) requires that the square of a row sum in $Z$ equals one so that, for node $j$,

$$1 = \sum_{k=1}^{N} (z_k)_j^2 = \sum_{k=1}^{N-1} (z_k)_j^2 + \frac{1}{N}$$

implying that $\frac{1}{N} \leq \max_{1\leq k\leq N} (z_k)_j^2 \leq 1 - \frac{1}{N}$. Hence, in any connected graph, we find that $\frac{1}{\sqrt{N}} \leq \xi_Z \leq \sqrt{1-\frac{1}{N}} < 1$ and, similarly, $-\sqrt{1-\frac{1}{N}} \leq \zeta_Z \leq -\frac{1}{\sqrt{N}}$.

## IV. Dual fundamental weight vector $\varphi$

In this section, we study, both numerically and analytically, the distribution of a random component in the dual fundamental weight vector $\varphi$, defined in (6). First, we note [16] that the sum $s_{Z^2}$ of the elements of $Z^2$ is

$$s_{Z^2} = u^T Z^2 u = \omega^T \varphi$$

and with $\omega = \sqrt{N} e_N$, we have for a connected graph,

$$s_{Z^2} = \sqrt{N} \varphi_N$$

where $\varphi_N = \sum_{j=1}^{N} (z_N)_j$ is the $N$-th row sum of $Z$.

### A. Randomly chosen component of the dual fundamental weight vector $\varphi$

As shown in [16], the vector $\omega$ is invariant with respect to a node relabeling transformation, but the dual fundamental weight vector $\varphi$ is not, nor is $s_{Z^2}$. The consequence is that, by generating Erdős-Rényi random graphs, the node labeling is uniformly distributed so that the random variable $s_{Z^2} \overset{d}{=}$

$\sqrt{N}\varphi_U$, where $U \in [1, N]$ is a discrete uniform random variable.

The expectation of a randomly chosen element $\varphi_U$ is

$$E[\varphi_U] = \sum_{k=1}^{N} \varphi_k \Pr[U = k] = \frac{1}{N} \sum_{k=1}^{N} \varphi_k = \frac{1}{N} u^T \varphi$$

Since $u^T \varphi = u^T \omega = \sqrt{N}$ (see [16]), we find that

$$E[\varphi_U] = \frac{1}{\sqrt{N}} \tag{8}$$

The variance of $\varphi_U$, $\text{Var}[\varphi_U] = E[\varphi_U^2] - (E[\varphi_U])^2$ follows, with $\sum_{k=1}^{N} \varphi_k^2 = N$ (see [16]) from

$$E[\varphi_U^2] = \sum_{k=1}^{N} \varphi_k^2 \Pr[U = k] = \frac{1}{N} \sum_{k=1}^{N} \varphi_k^2 = 1$$

so that

$$\text{Var}[\varphi_U] = 1 - \frac{1}{N} \tag{9}$$

Extensive simulations on $\varphi_U$ in Erdős-Rényi random graphs $G_p(N)$ are performed. We simulate ER random graphs for various $N$, where $N = 10,\ 20,\ 30,\ \cdots,\ 100$ and with the link density $p = 0.3$. For each $N$, we have simulated $10^8$ ER random graphs that resulted in $10^8$ realizations of $\varphi_U$. The probability density function $f_{\varphi_U}(z)$ for each $N$ is plotted and fitted.

Next, we show that $\varphi_U$ does not depend on the degree vector $d$ for a regular graph. We start from

$$d^T \varphi = \sum_{k=1}^{N} d_k \varphi_k = N \sum_{k=1}^{N} d_k \varphi_k \Pr[U = k] = N E[d_U \varphi_U]$$

Thus, the correlation coefficient

$$\rho(d_U, \varphi_U) = \frac{1}{N} d^T \varphi - E[d_U] E[\varphi_U] = \frac{1}{N} d^T \varphi - \frac{2L}{N} \frac{1}{\sqrt{N}}$$

and

$$\rho(d_U, \varphi_U) = \frac{1}{N} \left( d^T \varphi - \frac{2L}{\sqrt{N}} \right)$$

The dependence or correlation between the degree vector $d$ and the dual fundamental weight vector $\varphi$ is zero provided $d^T \varphi = \frac{2L}{\sqrt{N}}$. In a regular graph, for example, $d = ru$, $\frac{2L}{\sqrt{N}} = r\sqrt{N}$ and $d^T \varphi = r u^T \varphi = r u^T \omega = r\sqrt{N}$, so that $\rho(d_U, \varphi_U) = 0$. Simulations hint that $\rho(d_U, \varphi_U) \approx 0$ for ER random graphs, too! Fig. 5 demonstrates that the probability density function $f_{\varphi_U}(z)$ is approximately an invariant with respect to the average degree $d_{av}$ (and thus the link density $p$ in $G_p(N)$).

### *B. The product of a Gaussian and a super-Gaussian distribution*

The probability density function $f_{\varphi_U}(z)$ is accurately fitted by the probability density function

$$f_X(z) = c \exp\left[-b(z - z_0)^2\right] \exp\left[-a(z - z_0)^4\right] \tag{10}$$

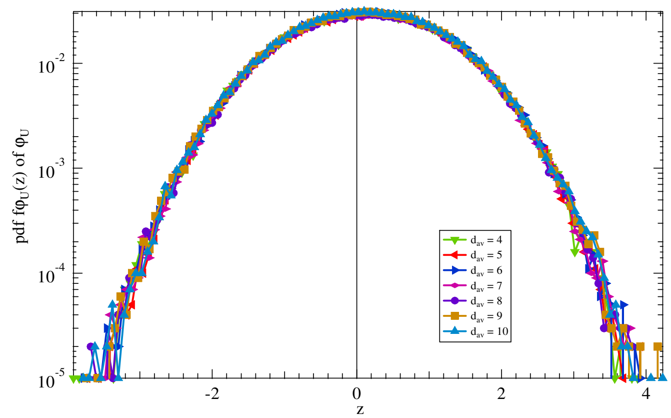


Fig. 5. The probability density function $f_{\varphi_U}(z)$ of $\varphi_U$ for connected ER random graphs $G_p(N)$ for $N = 50$ and vaious average degree $d_{av}$, ranging from $d_{av} = 4$ up to $d_{av} = 10$. The y-axis is in log-scale.

which is a product of a Gaussian and a super-Gaussian distribution. A random variable $Y_m$ possesses a super-Gaussian distribution, defined by

$$f_{Y_m}(z) = A_m \exp\left[-a(z - z_0)^m\right]$$

where $m$ is an even integer and $a > 0$ is a positive real number.

Next, we focus on determining the parameters $a$, $b$ and $c$ in (10). Since $\int_{-\infty}^{\infty} f_X(z) dz = 1$, with $z - z_0 = x$, we have

$$c \int_{-\infty}^{\infty} \exp[-bx^2 - ax^4] dz = 1$$

The integral, proved in [18],

$$\int_0^{\infty} \exp\left[-bu^2 - au^4\right] du = \frac{1}{4} \sqrt{\frac{b}{a}} e^{\frac{b^2}{8a}} K_{\frac{1}{4}}\left(\frac{b^2}{8a}\right)$$

and where $K_s(z)$ is the modified Bessel function of the Second Kind [19], determines $c$ as

$$c = \frac{1}{2\int_0^{\infty} \exp\left[-bu^2 - au^4\right] du} = \sqrt{\frac{a}{b}} \frac{2e^{-\frac{b^2}{8a}}}{K_{\frac{1}{4}}\left(\frac{b^2}{8a}\right)} \tag{11}$$

Since $f_X(z)$ is a symmetric function around $z_0$, all odd centered moments around $z_0$, $E\left[(X - z_0)^k\right] = \int_{-\infty}^{\infty} (x - z_0)^k f_Z(x) dx$, are zero and, thus $E[X] = z_0$. Combination with (8) shows that $z_0 = \frac{1}{\sqrt{N}}$. We can compute the variance $\text{Var}[X] = E\left[(X - z_0)^2\right]$ explicitly as

$$\text{Var}[X] = \frac{1}{2b} h\left(\frac{y^2}{8}\right) \tag{12}$$

with

$$h(t) = 2t\left(\frac{K_{\frac{3}{4}}(t)}{K_{\frac{1}{4}}(t)} + \frac{K_{\frac{5}{4}}(t)}{K_{\frac{1}{4}}(t)} - 2\right) - 1$$

where $y^2 = \frac{b^2}{a}$. Further, $\text{Var}[X]$ is increasing with $y$ from 0 (for $y = 0$) to $\frac{1}{2b}$ (when $y \to \infty$). Using (9) yields

$$b = \frac{h\left(\frac{y^2}{8}\right)}{2\left(1 - \frac{1}{N}\right)} \tag{13}$$

while $y^2 = \frac{b^2}{a}$ then leads to

$$a = \frac{h^2\left(\frac{y^2}{8}\right)}{4y^2\left(1-\frac{1}{N}\right)^2} \tag{14}$$

Hence, (13) and (14) indicate that $b$ increases with $y$ towards $\frac{1}{2\left(1-\frac{1}{N}\right)}$, while $a$ decreases with $y$ towards 0.

### C. Fitting result

Fig. 6 shows the natural logarithm of the probability density function $f\varphi_U(z)$ for $\varphi_U$ from simulations, fitted by the function (10). As observed from Fig. 6, the simulations agree astonishingly well with (10) for all $N$ simulated in this paper.

Fig. 7 shows that the parameter $y^2 = \frac{b^2}{a}$ is approximately linear in $N$,

$$y^2 = 0.5N - 3.85 \tag{15}$$

Substituting the linear function (15) into (14) and (13) determines $a$ and $b$ analytically. As shown in Figs. 8 and 9, $a$ and $b$ (red curve, theory from (14) and (13) with (15)) agree well with simulations of $\varphi_U$ (black dots), after fitting $a$ and $b$ from (10). Fig. 10 shows $c$ from (11) and from fitting function (10) for $f_{\varphi_U}(z)$ for each $N$. Fig. 11 presents $z_0$ from (8) and from the fitting function (10).

As shown in Fig. 8-11, the fitting parameters $a$, $b$, $c$, $z_0$ in (10) from simulations agree well with equations (14), (13), (11), (8), respectively. Thus, our simulations lead us to believe that the distribution of the components of the dual fundamental weight vector $\varphi$ in Erdős-Rényi random graphs is given by (10), which is the product of a Gaussian and a super-Gaussian. Fig. 8 and (14) (with (15)) show that $a$ tends as $O(1/N)$ to zero with $N$, implying that, for large $N$, the super-Gaussian disappears and the expected Gaussian behavior (from random matrix theory) appears. The parameter $a$ in (10) constraints the Gaussian behavior, which is likely due to the orthogonality conditions (2) and (4) that create dependence among the eigenvector components. Indeed, the larger $N$, the less the orthogonality conditions are confining, which suggest that $a$ would decrease inversely proportional to $N$, precisely as observed in Fig. 8.

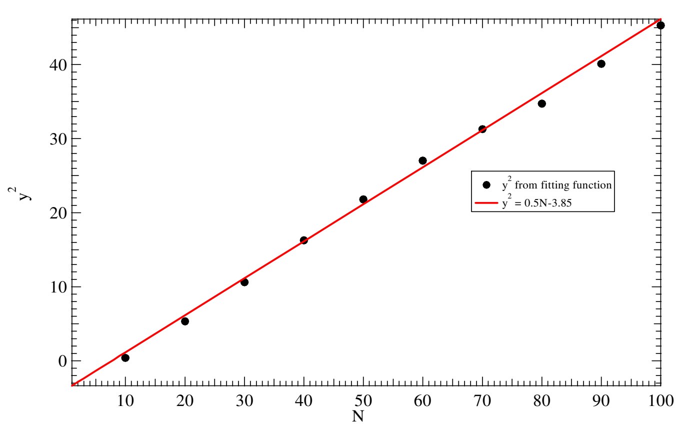


Fig. 7. Fitting parameter $y^2 = \frac{b}{a}$ as a function of $N$ in ER graphs.

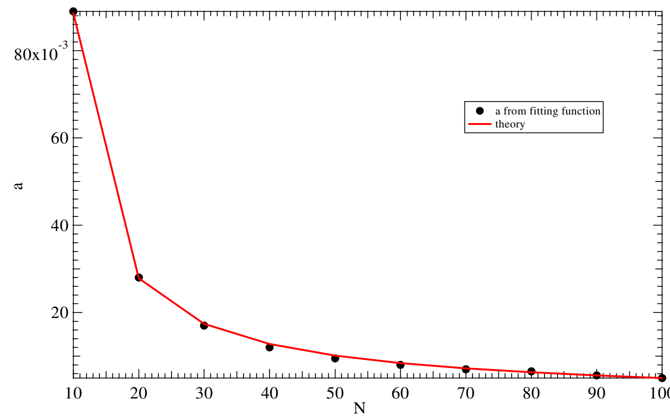


Fig. 8. Fitting parameter $a$ as a function of $N$ in ER graphs.

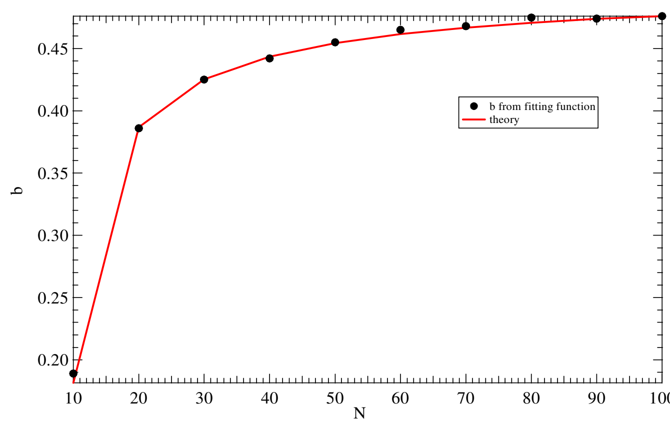


Fig. 9. Fitting parameter $b$ as a function of $N$ in ER graphs.

### D. Very small sizes of N

We observe that when $N < 8$ (obtained at the point $y^2 < 0$ in (15)), the simulation result is better fitted by a Gaussian distribution, instead of the product of a Gaussian and a super-Gaussian.

As shown in Fig. 12, the product of a Gaussian and super-Gaussian distribution does not precisely fit the simulations at the tail. When $N$ is decreased to 6 in Fig. 13, the simulation is fitted by a Gaussian distribution.

## V. Conclusion

We have studied the eigenvector matrix $Z$ of the Laplacian matrix $Q$ for a graph $G$ with the aim to understand how properties of $Z$ contain information about the structure of $G$. We find that the sum $s_Z$ of all the elements in $Z$ increases with the size of the graph as $O\left(\sqrt{N}\right)$. The higher the average degree in a graph, the lower the number of zeros in the eigenvector matrix. Moreover, the distribution of the maximum element in the eigenvector matrix is the same as the distribution of the minimum element.

The row sum of the eigenvector matrix $Z$ of the Laplacian $Q$, coined the dual fundamental weight $\varphi$, in Erdős-Rényi random graphs follows closely the product of a Gaussian and a super-Gaussian distribution.

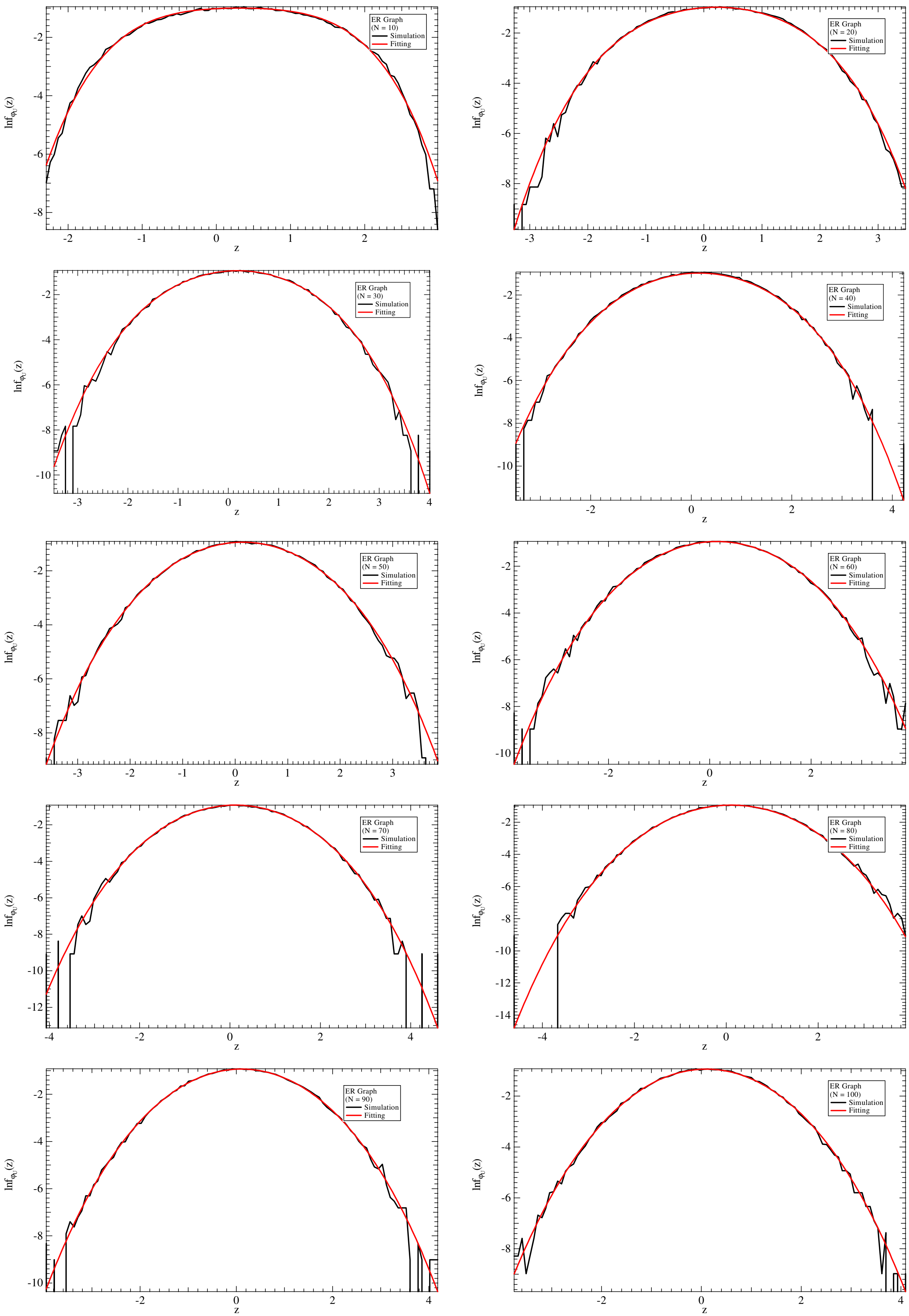


Fig. 6. Natural logarithm $\ln(f\varphi_U(z))$ of the probability density function $f_{\varphi_U}(z)$ for ER graphs with $p = 0.3$ and various $N$, ranging from $N = 10$ to $N = 100$.

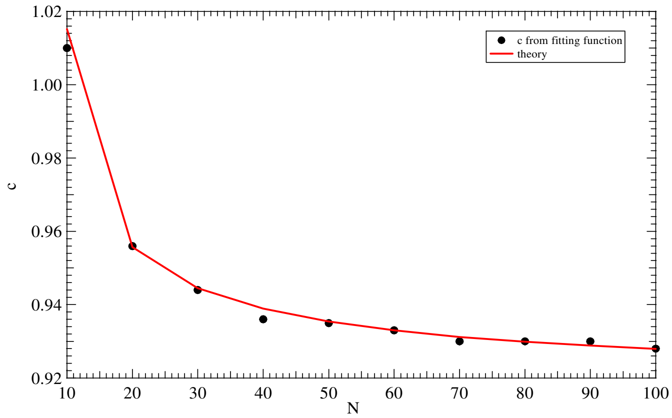


Fig. 10. Fitting parameter $c$ as a function of $N$ in ER graphs.

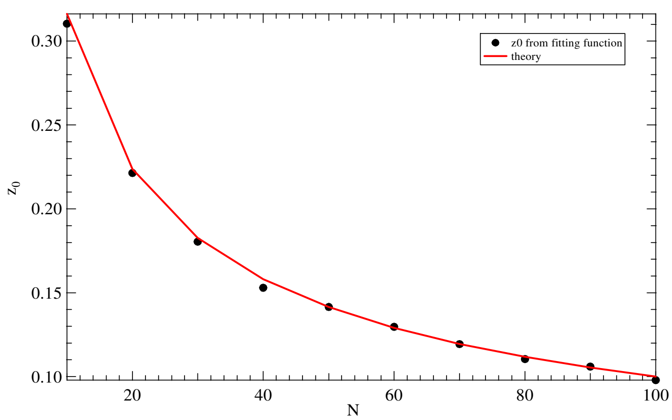


Fig. 11. Fitting parameter $z_0$ as a function of $N$ in ER graphs.


## Acknowledgment

This research is supported by the China Scholarship Council (CSC).


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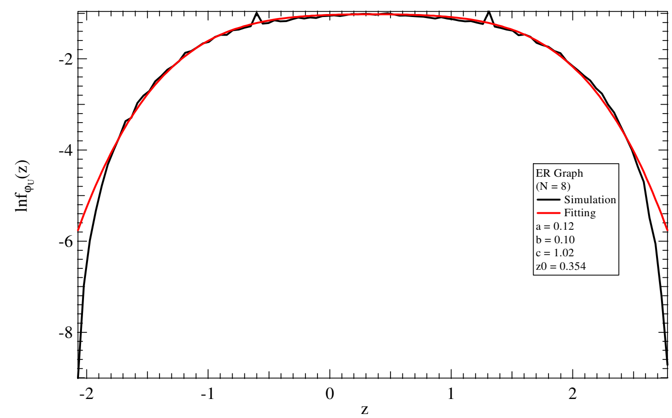


Fig. 12. Natural logarithm $\ln(f\varphi_U(z))$ of the probability density function $f_{\varphi_U}(Z)$ for $10^8$ ER graphs with $p = 2\log(N)/N$ (to make sure the graph is connected) and $N = 8$.

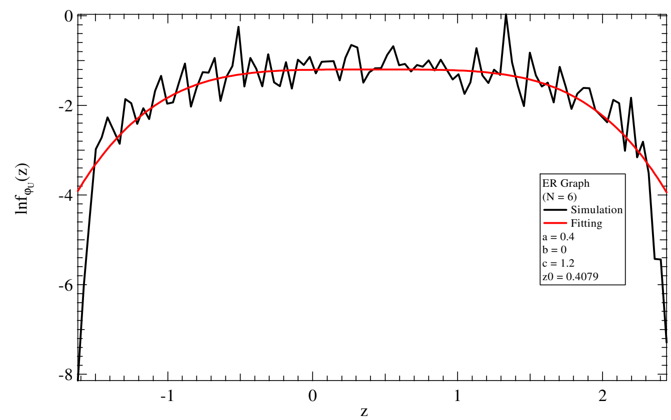


Fig. 13. Natural logarithm $\ln(f\varphi_U(z))$ of the probability density function $f_{\varphi_U}(Z)$ for $10^8$ ER graphs with $p = 2\log(N)/N$ (to make sure the graph is connected) and $N = 6$.


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## Appendix

We write the $N\times N$ symmetric matrix $A$ as a block matrix

$$A=\left[\begin{array}{cc} A_1 & B \\ B^T & A_2 \end{array}\right]$$

where $A_1$ is an $(N-m)\times(N-m)$ symmetric matrix and $A_2$ is a $m\times m$ symmetric matrix with[1] $0\leq m<\frac{N}{2}$. For example, for a graph $G$, $A_1$ and $A_2$ are the adjacency matrices of two subgraphs $G_1$ and $G_2$ of $G$, $B$ represents the interconnection matrix of the links between $G_1$ and $G_2$. The eigenvalue equation $Ax=\lambda(A)\,x$ is written as the linear block set, with the eigenvector $x^T=\left[\begin{array}{cc} v_{(N-m)\times 1} & w_{m\times 1}\end{array}\right]^T$,

$$\begin{cases} A_1v+Bw=\lambda(A)\,v \\ B^Tv+A_2w=\lambda(A)\,w \end{cases}$$

where we choose the normalization $x^Tx=1$, equivalent to $v^Tv+w^Tw=1$. If the coupling matrix $B=0$, then the set simplifies to

$$\begin{cases} A_1v=\lambda(A)\,v \\ A_2w=\lambda(A)\,w \end{cases}$$

which illustrates that $v$ and $w$ are eigenvectors (satisfying $v^Tv+w^Tw=1$) belonging to the eigenvalue $\lambda(A)$, which is also an eigenvalue of at least one matrix, $A_1$ or $A_2$, because an eigenvector $x$ is different from the zero vector, so that not both $v$ and $w$ can be the zero vector.

In the case of the Laplacian $Q$ of $G$, where $u$ is an eigenvector of $Q_1$, $Q_2$ and $Q$ belonging to eigenvalue $\mu=0$, then it holds that

$$\begin{cases} Q_1v=0 \\ Q_2w=0 \end{cases}$$

where $v=\alpha u$ and $w=\beta u$ with $1=\alpha^2(N-m)+\beta^2m$. The latter is the equation of an ellipse with the two main axes $\frac{1}{\sqrt{N-m}}$ and $\frac{1}{\sqrt{m}}$,

$$\frac{\alpha^2}{\left(\frac{1}{\sqrt{N-m}}\right)^2}+\frac{\beta^2}{\left(\frac{1}{\sqrt{m}}\right)^2}=1 \tag{16}$$

and any set $(\alpha,\beta)$ with both $\alpha\neq 0$ and $\beta\neq 0$ on the ellipse is a solution. Hence[2], for $m>0$, there exists infinitely many normalizations of the eigenvector of $Q$ belonging to the eigenvalue $\mu_N=0$. When $m\to 0$ (and hence $\beta=0$), the ellipse degenerates into the points $\alpha=\pm\frac{1}{\sqrt{N}}$. Moreover, we can construct two orthogonal eigenvectors (since the multiplicity of $\mu=0$ is two). Let $x_1^T=\left[\begin{array}{cc}\alpha u & \beta u\end{array}\right]^T$ and $x_2^T=\left[\begin{array}{cc}\gamma u & \delta u\end{array}\right]^T$, where $(\gamma,\delta)$ is also a point on the above ellipse. Orthogonality requires that

$$0=x_1^Tx_2=\left[\begin{array}{cc}\alpha u & \beta u\end{array}\right]^T\left[\begin{array}{c}\gamma u \\ \delta u\end{array}\right]=\alpha\gamma(N-m)+\beta\delta m$$

leading to

$$\gamma=-\frac{\beta m}{\alpha(N-m)}\delta$$

[1]If $m\geq\frac{N}{2}$, we can interchange subgraph $G_1$ and $G_2$ so that $m<\frac{N}{2}$.

[2]When there are $c$ disconnected subgraphs in $G$, the normalization procedure results in $c$-dimensional ellipsoid leading to $c-1$ degrees of freedom to normalize the $c$ eigenvectors belonging to eigenvalue $\mu_N=0$ of $Q$.

but also $1=\gamma^2(N-m)+\delta^2m$. Combined yields $\delta=\pm\frac{1}{\sqrt{\left(\frac{\beta m}{\alpha\sqrt{N-m}}\right)^2+m}}$ and, after using $1=\alpha^2(N-m)+\beta^2m$, we find

$$\delta=\pm\frac{\alpha\sqrt{N-m}}{\sqrt{m}} \tag{17}$$

and

$$\gamma=\mp\frac{\beta\sqrt{m}}{\sqrt{N-m}} \tag{18}$$

In conclusion, with each choice of $(\alpha,\beta)$ as a point on the ellipse, there correspond two points $(\gamma,\delta)$ (with oppositive sign) on the same ellipse, for which we obtain two orthogonal vectors $(\alpha\beta=-\gamma\delta)$. All other eigenvectors are orthogonal on $x_1$ and $x_2$. Thus, $x_k^T=\left[\begin{array}{cc}v_k & w_k\end{array}\right]^T$ obeys $x_k^Tx_1=0$ and $x_k^Tx_2=0$,

$$\begin{cases}\alpha v_k^Tu+\beta w_k^Tu=0 \\ \gamma v_k^Tu+\delta w_k^Tu=0\end{cases}$$

or

$$\left[\begin{array}{cc}\alpha & \beta \\ \gamma & \delta\end{array}\right]\left[\begin{array}{c}v_k^Tu \\ w_k^Tu\end{array}\right]=0$$

which only has the zero solution $v_k^Tu=w_k^Tu=0$ because $\det\left[\begin{array}{cc}\alpha & \beta \\ \gamma & \delta\end{array}\right]=\frac{1}{\sqrt{(N-m)m}}>0$. Since all other eigenvectors $x_k$ are orthogonal to $u$ (with $\sum_{j=1}^{N}(x_k)_j=u^Tx_k=0$), the sum of the elements in $Z$ equals the sum of the elements in $x_1$ and $x_2$:

$$s_Z=(\alpha+\gamma)(N-m)+(\beta+\delta)\,m$$

Introducing the expression (18) for $\gamma$ and (17) for $\delta$ into $s_Z$ gives us

$$s_Z=\alpha N+(\alpha-\beta)\sqrt{m}\left(\sqrt{N-m}-\sqrt{m}\right)$$

From $1=\alpha^2(N-m)+\beta^2m$, we eliminate $\alpha=\sqrt{\frac{1-\beta^2m}{N-m}}$ and, after substitution, we have

$$s_Z=N\sqrt{\frac{1-\beta^2m}{N-m}}+\left(\sqrt{\frac{1-\beta^2m}{N-m}}-\beta\right)\sqrt{m}\left(\sqrt{N-m}-\sqrt{m}\right)$$

illustrating that, if $m=0$ and the graph is connected, then $s_Z=\sqrt{N}$. Moreover, $s_Z$ is a function of the integer $m$ and the real number $\beta$. For the case $1\leq m<\frac{N}{2}$, it is convenient to denote $y=\beta^2m\in(0,1)$ and write

$$s_Z(m,y)=N\sqrt{\frac{1-y}{N-m}}+\left(\sqrt{\frac{1-y}{N-m}}-\frac{\sqrt{y}}{\sqrt{m}}\right)\sqrt{m}\left(\sqrt{N-m}-\sqrt{m}\right)$$

For $y=0$, we have $s_Z(m,0)=\sqrt{N-m}+\sqrt{m}$. Since $\left(\sqrt{N-m}+\sqrt{m}\right)^2=N+2\sqrt{m}\sqrt{N-m}>N$, we find that $s_Z(m,0)>\sqrt{N}$. The other extremum $s_Z(m,1)=-\left(\sqrt{N-m}-\sqrt{m}\right)$is smaller than $s_Z(m,1)<0<\sqrt{N}$. Since $y$ is a continuous real variable and $s_Z(m,y)$ is monotonously decreasing in $y$, there must exist, for each integer $m\in[1,\frac{N}{2})$, a $y^*\in(0,1)$ for which $s_Z(m,y^*)=\sqrt{N}$. In summary, we have demonstrated the following Theorem:

*Theorem 1:* If the graph $G$ is connected, then the number $s_Z$ of elements in the orthogonal matrix $Z$ of the Laplacian of the graph $G$ equals $s_Z=\sqrt{N}$. The converse, "if $s_Z=\sqrt{N}$, then the graph $G$ is connected" is not always true.