# Orthogonal F-Rectangles For All Even v

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SUMMARY

It is shown how to construct a pair of orthogonal v/2 by 2v rectangles for all even v. Also, it is shown how to construct a set of t pairwise orthogonal v/2 by 2v F-rectangles for all even v for which a set of t pairwise orthogonal Latin squares of order v exists. Some situations where these designs are useful in practice are indicated.

## 1. INTRODUCTION

The class of row-column experiment designs introduced here has usefulness in many areas of experimentation. In a variety of situations, it may be undesirable, or even impossible, to have as many treatment periods as there are treatments, but it is relatively easy to obtain more individuals or organizations for the experiment. For example, in a study of diet and aerobic dance exercise, it was undesirable to subject each individual to more than three exercise-diet treatments, but it was relatively easy, and desirable, to obtain 12 or 24 individuals for the study for the six diet-exercise treatments. It was necessary to have a class of individuals in order to teach aerobic dancing. It was considered essen-

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tial to have every treatment follow every other treatment and vice versa. We give a design for this situation at the end of the paper. Some other situations where these experiment designs may be useful are:

(i) In marketing and other experiments where the 2v sampling units (individuals, organizations, etc.) are used for v/2 time periods for two sets of v treatments,

(ii) In changeover experiments with 4v sequences and v/2 periods partially balanced for residual effects of v treatments,

(iii) In surveys where the order of the v questions (sensitive or otherwise) is to be in a balanced arrangement for residual effects for each set of 4v individuals and each individual answers v/2 questions.

We first give a definition of a pair of k row by b column F-rectangles. Then, we show how to construct a pair of orthogonal (v/2) X v F-rectangle for all even v and how to construct a set of t such pairwise orthogonal F-rectangles for all even v for which t orthogonal latin squares of order v exist.

## 2. CONSTRUCTION OF PAIRWISE ORTHOGONAL F-RECTANGLES

Let $V = \{1,2,\cdots,v\}$ be a set of v distinct symbols. A k X b array filled with the elements of V is said to be an F-rectangle if

(i) Every element of V appears the same number of times, $\bar{r} = bk/v$, in the array,

(ii) The appearance of each element in each row and column is as uniform as possible.

Note that condition (ii) indicates that if $k \leq v$, no element of V appears more than once in each column, and if $b = tv$, each element of V appears t times in each row.

Definition. Let $F_1$ and $F_2$ be two $k \times b$ F-rectangles. Then we say $F_1$ is orthogonal to $F_2$ (denoted by $F_1 \perp F_2$) if upon superposition of $F_1$ on $F_2$ every element of V in $F_1$ appears the same number of times with every element of V in $F_2$.

Example. Let $V = \{1,2,3,4\}$; then, the following $F_1$, $F_2$ and $F_3$ are pairwise orthogonal $2 \times 8$ F-rectangles.

$F_1$

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 1 | 2 | 3 | 4 | 3 | 4 | 1 | 2 |
| 2 | 1 | 4 | 3 | 4 | 3 | 2 | 1 |

,

$F_2$

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 1 | 2 | 3 | 4 | 4 | 3 | 2 | 1 |
| 3 | 4 | 1 | 2 | 2 | 1 | 4 | 3 |

$F_3$

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 1 | 2 | 3 | 4 | 2 | 1 | 4 | 3 |
| 4 | 3 | 2 | 1 | 3 | 4 | 1 | 2 |

We shall now prove the following theorem.

Theorem 1. There exists a pair of orthogonal $(v/2) \times 2v$ F-rectangles for all even $v$.

The proof is by construction. Construct a $(v/2) \times v$ F-rectangle, A, based on V with $1, 2, \cdots, v/2$ as its entries in the first column and fill the remaining cells cyclically. Construct another $(v/2) \times v$ F-rectangle, B, based on V with $(v/2)+1, (v/2)+2, \cdots, v$ as its entries in the first column and fill the remaining cells cyclically. Construct another $(v/2) \times v$ F-rectangle, C, based on V with the odd numbers among $1, 2, \cdots, v$ as its entries in the first column and fill the remaining cells cyclically. Then,

$$F_1 = \boxed{A \mid B} \quad \text{and} \quad F_2 = \boxed{C \mid C}$$

form a pair of orthogonal $(v/2) \times 2v$ F-rectangles. It is obvious that $F_1$ and $F_2$ are F-rectangles. The fact that they are orthogonal follows from (a) the cyclic construction of A, B and C and (b) the property that each element of V in $F_2$ appears once with the element 1 in $F_1$.

Example. Let $V = \{1,2,3,4,5,6\}$. Then,

$$A = \begin{array}{|cccccc|} \hline 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 3 & 4 & 5 & 6 & 1 \\ 3 & 4 & 5 & 6 & 1 & 2 \\ \hline \end{array}, \quad B = \begin{array}{|cccccc|} \hline 4 & 5 & 6 & 1 & 2 & 3 \\ 5 & 6 & 1 & 2 & 3 & 4 \\ 6 & 1 & 2 & 3 & 4 & 5 \\ \hline \end{array}, \quad \text{and} \quad C = \begin{array}{|cccccc|} \hline 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 4 & 5 & 6 & 1 & 2 \\ 5 & 6 & 1 & 2 & 3 & 4 \\ \hline \end{array}.$$

Now form $F_1 = \begin{array}{|c|c|} \hline A & B \\ \hline \end{array}$ and $F_2 = \begin{array}{|c|c|} \hline C & C \\ \hline \end{array}$ as

$$F_1 = \begin{array}{|cccccccccccc|} \hline \textcircled{1} & 2 & 3 & 4 & 5 & 6 & 4 & 5 & 6 & \textcircled{1} & 2 & 3 \\ 2 & 3 & 4 & 5 & 6 & \textcircled{1} & 5 & 6 & \textcircled{1} & 2 & 3 & 4 \\ 3 & 4 & 5 & 6 & \textcircled{1} & 2 & 6 & \textcircled{1} & 2 & 3 & 4 & 5 \\ \hline \end{array}, \quad F_2 = \begin{array}{|cccccccccccc|} \hline \textcircled{1} & 2 & 3 & 4 & 5 & 6 & 1 & 2 & 3 & \textcircled{4} & 5 & 6 \\ 3 & 4 & 5 & 6 & 1 & \textcircled{2} & 3 & 4 & \textcircled{5} & 6 & 1 & 2 \\ 5 & 6 & 1 & 2 & \textcircled{3} & 4 & 5 & \textcircled{6} & 1 & 2 & 3 & 4 \\ \hline \end{array}.$$

The concept of orthogonal F-rectangles is closely related to the concept of orthogonal Latin squares. One such relation in the context of this note is indicated below.

Theorem 2. For v even the existence of t pairwise orthogonal Latin squares of order v implies the existence of t pairwise orthogonal $(v/2) \times 2v$ F-rectangles.

The proof is by construction. If $\{L_1, L_2, \cdots, L_t\}$ is a set of pairwise orthogonal Latin squares of order v (even), then split $L_i$ into halves as

$$L_i = \begin{array}{|c|} \hline A_i \\ \hline B_i \\ \hline \end{array}$$

and let $F_i = \boxed{A_i \mid B_i}$. Then clearly $\{F_1, F_2, \cdots, F_t\}$ forms the required set of pairwise $(v/2) \times 2v$ F-rectangles.

Remark. Theorem 2 cannot be used when $v = 2$ and 6 since there is no pair of orthogonal Latin squares of order 2 and 6. Also, Theorem 2 requires the construction of orthogonal Latin squares, which is not easy for $v \equiv 2 (\text{mod } 4)$. If one is interested only in a pair of orthogonal $(v/2) \times 2v$ F-rectangles then Theorem 1 is useful and easy to implement for all even orders including $v = 6$. For $v = 2$ one may use

$$F_1 = \boxed{1 \quad 2 \quad 1 \quad 2} \qquad \text{and} \qquad F_2 = \boxed{1 \quad 2 \quad 2 \quad 1}$$

to obtain $F_1 \perp F_2$.

## 3. USE OF ORTHOGONAL F-RECTANGLES IN CONSTRUCTING REPEATED MEASURES DESIGNS TO MEASURE RESIDUAL EFFECTS

Knowing that a pair of orthogonal rectangles exists is important in constructing repeated measures designs to measure residual effects. To do this we start with a Latin square of order v in the form of a repeated measures design balanced for residual effects and use this Latin square of order v to construct an F-rectangle, $F_1 = \boxed{A \mid B}$. Then we form a second F-rectangle, $F_2$, whose first two rows are the first row and $v^{th}$ row of the original Latin square. This allows every treatment (element) to follow and be followed by every other treatment. The remaining rows of $F_2$ are filled to obtain orthogonality. The method is not unique. We illustrate this for $v = 6$ and 8:

$$F_1 = \left[\begin{array}{cccccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 5 & 6 & 1 & 2 & 3 & 4 \\ 6 & 1 & 2 & 3 & 4 & 5 & 3 & 4 & 5 & 6 & 1 & 2 \\ 2 & 3 & 4 & 5 & 6 & 1 & 4 & 5 & 6 & 1 & 2 & 3 \end{array}\right], \quad F_2 = \left[\begin{array}{cccccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 1 & 2 & 3 & 4 & 5 & 6 \\ 4 & 5 & 6 & 1 & 2 & 3 & 4 & 5 & 6 & 1 & 2 & 3 \\ 5 & 6 & 1 & 2 & 3 & 4 & 3 & 4 & 5 & 6 & 1 & 2 \end{array}\right],$$

$$F_1 = \left[\begin{array}{cccccccccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 3 & 4 & 5 & 6 & 7 & 8 & 1 & 2 \\ 8 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 6 & 7 & 8 & 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 4 & 5 & 6 & 7 & 8 & 1 & 4 & 5 & 6 & 7 & 8 & 1 & 2 & 3 \\ 7 & 8 & 1 & 3 & 4 & 5 & 6 & 7 & 5 & 6 & 7 & 8 & 1 & 2 & 3 & 4 \end{array}\right],$$

$$F_2 = \left[\begin{array}{cccccccccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 5 & 6 & 7 & 8 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 1 & 2 & 3 & 4 \\ 4 & 5 & 6 & 7 & 8 & 1 & 2 & 3 & 7 & 8 & 1 & 2 & 3 & 4 & 5 & 6 \\ 8 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \end{array}\right].$$

Note that we could have used

$$\begin{array}{cccccccccccccccc} 6 & 7 & 8 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 4 & 5 & 6 & 7 & 8 & 1 & 6 & 7 & 8 & 1 & 2 & 3 & 4 & 5 \end{array}$$

as the last two rows of $F_2$ for $v = 8$. One should select rows to reduce the confounding between columns and treatments as much as possible.

The above design for $v = 6$ could have been used for the six diet-aerobic dance treatments and for 24 girls. All treatments would precede and follow all other treatments either once or twice. Thus, the design is partially balanced for one-period carry-over effects. One could have used $F_1$ with 12 girls, and all treatments but one would either follow or precede a treatment. The design could also be used with 12 girls and two sets of six treatments. The second set could be a fruit additive to the diet, e.g., grapefruit, orange, raisins, grapes, sweetened peaches, and unsweetened peaches. This set would be orthogonal to those treatments in the first F-rectangle.

## 4. OPTIMAL DESIGNS

The F-rectangle in Theorem 1 formed as $\boxed{C \mid C}$ is not connected. No optimality properties were considered in the previous sections. To obtain a variance-optimal design, we proceed as follows. First, form a cyclic Latin square of even order. Second, order the rows of this square to achieve maximum column-treatment balance. One method of doing this for $v = 4t + 2$ is to use the quadratic residues for $4t + 3$ and the non-quadratic residues omitting the null element to order the rows of A and B in $F = \boxed{A \mid B}$. This will generally not give maximum balance; a few rows may need to be interchanged from A to B to achieve maximum balance. The resulting $F_1 = \boxed{A \mid B}$ will then be as near variance balanced as possible, and consequently variance optimal, because this is a zero-one occurrence design, i.e., either the treatment occurs in a column or it does not.

Thirdly, form $A^* = A + aJ$, mod v, and $B^* = B + bv$, mod v, where a and b are scalars and J is a $v \times v$ matrix of ones. Permute the rows to $A^*$ and $B^*$ such that an $F_2 = \boxed{A^* \mid B^*}$ is orthogonal to $F_1$. If $F_1$ is variance optimal, $F_2$ will also be variance optimal because the addition of a J matrix to A and B does nothing to change the treatment-column balance arrangement. This leads to the following:

Theorem 3. The method of construction outlined above leads to a pair of variance-optimal orthogonal F-rectangles of order $v/2 \times 2v$ with identical information matrices.

To illustrate the above results for $v = 6$, let

$$F_1 = \left[\begin{array}{cccccc|cccccc} 1 & 2 & 3 & 4 & 5 & 6 & 3 & 4 & 5 & 6 & 1 & 2 \\ 2 & 3 & 4 & 5 & 6 & 1 & 5 & 6 & 1 & 2 & 3 & 4 \\ 4 & 5 & 6 & 1 & 2 & 3 & 6 & 1 & 2 & 3 & 4 & 5 \end{array}\right]$$

Note that 1, 2, and 4 are the quadratic residues for 7, and that 3, 5, and 6 are the non-null non-quadratic residues. Let $A^* = A + J$ and $B^* = B + 4J$, to obtain

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 2 | 3 | 4 | 5 | 6 | 1 | 1 | 2 | 3 | 4 | 5 | 6 |
| 3 | 4 | 5 | 6 | 1 | 2 | 3 | 4 | 5 | 6 | 1 | 2 |
| 5 | 6 | 1 | 2 | 3 | 4 | 4 | 5 | 6 | 1 | 2 | 3 |

Permuting the rows of the above in each part, we obtain

$F_2$ =

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 3 | 4 | 5 | 6 | 1 | 2 | 3 | 4 | 5 | 6 | 1 | 2 |
| 5 | 6 | 1 | 2 | 3 | 4 | 4 | 5 | 6 | 1 | 2 | 3 |
| 2 | 3 | 4 | 5 | 6 | 1 | 1 | 2 | 3 | 4 | 5 | 6 |

A form of the variance-covariance matrix for either $F_1$ or $F_2$ is

$$\sigma^2(6I - \tfrac{1}{3}NN' + \tfrac{2}{3}J)^{-1} = 3\sigma^2 \begin{bmatrix} 14 & 0 & 0 & -2 & 0 & 0 \\ 0 & 14 & 0 & 0 & -2 & 0 \\ 0 & 0 & 14 & 0 & 0 & -2 \\ -2 & 0 & 0 & 14 & 0 & 0 \\ 0 & -2 & 0 & 0 & 14 & 0 \\ 0 & 0 & -2 & 0 & 0 & 14 \end{bmatrix}^{-1}$$

For $v = 10$, use the quadratic and non-zero non-quadratic residues of $4t + 3 = 11$, interchanging 8 and 9, and we obtain $F_1$ and $F_2$ as:

$F_1$ =

| | | | | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 0 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 0 | 1 |
| 3 | 4 | 5 | 6 | 7 | 8 | 9 | 0 | 1 | 2 | 6 | 7 | 8 | 9 | 0 | 1 | 2 | 3 | 4 | 5 |
| 4 | 5 | 6 | 7 | 8 | 9 | 0 | 1 | 2 | 3 | 7 | 8 | 9 | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
| 5 | 6 | 7 | 8 | 9 | 0 | 1 | 2 | 3 | 4 | 9 | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
| 8 | 9 | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |

and

$$F_2 = \left[\begin{array}{cccccccccc|cccccccccc}
4 & 5 & 6 & 7 & 8 & 9 & 0 & 1 & 2 & 3 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 0 \\
8 & 9 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 0 & 1 \\
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 0 & 8 & 9 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\
7 & 8 & 9 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 9 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
6 & 7 & 8 & 9 & 0 & 1 & 2 & 3 & 4 & 5 & 4 & 5 & 6 & 7 & 8 & 9 & 0 & 1 & 2 & 3
\end{array}\right]$$

$F_1 = \boxed{A \mid B}$ where A has cyclic rows with the first column being 1, 3, 4, 5, and 8 and B has cyclic rows with the treatments of the first column being 2, 6, 7, 8, 0. These row arrangements make treatment and columns as balanced as possible.

$F_2 = \boxed{A^* \mid B^*}$ where $A^*$ is $A + 3J$ with rows permuted and $B^*$ is $B + 2J$ with rows permuted. The variance-covariance matrix for both F-rectangles is:

$$\sigma^2(10I - \tfrac{1}{5}NN' + \tfrac{4}{5}J)^{-1} = 5\sigma^2 \begin{bmatrix}
44 & 0 & 0 & -2 & 0 & 0 & 0 & -2 & 0 & 0 \\
0 & 44 & 0 & 0 & -2 & 0 & 0 & 0 & -2 & 0 \\
0 & 0 & 44 & 0 & 0 & -2 & 0 & 0 & 0 & -2 \\
-2 & 0 & 0 & 44 & 0 & 0 & -2 & 0 & 0 & 0 \\
0 & -2 & 0 & 0 & 44 & 0 & 0 & -2 & 0 & 0 \\
0 & 0 & -2 & 0 & 0 & 44 & 0 & 0 & -2 & 0 \\
0 & 0 & 0 & -2 & 0 & 0 & 44 & 0 & 0 & -2 \\
-2 & 0 & 0 & 0 & -2 & 0 & 0 & 44 & 0 & 0 \\
0 & -2 & 0 & 0 & 0 & -2 & 0 & 0 & 44 & 0 \\
0 & 0 & -2 & 0 & 0 & 0 & -2 & 0 & 0 & 44
\end{bmatrix}^{-1}$$

These F-rectangle designs are near variance optimal, as well as being orthogonal. One can find a method of construction which replaces one of the diagonals of zeros with a diagonal of minus ones and the diagonals of -2 with a diagonal of minus ones. This may be done by interchanging numbers 3 and 4 in the second halves of both $F_1$ and $F_2$ for $v = 6$, and numbers 3 and 4 and numbers 7 and 8 in the second halves of both $F_1$ and $F_2$ for $v = 10$. This procedure results in more balance and produces variance-optimal designs for Theorem 3.

## 5. OTHER F-RECTANGLES

If only two rows (periods) are required, one may easily construct a pair of orthogonal F-rectangles using the Latin square type in section 3. For v = 6,

$F_1$ =

| | | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 2 | 3 | 4 | 5 | 6 | 2 | 3 | 4 | 5 | 6 | 1 | 3 | 4 | 5 | 6 | 1 | 2 |
| 6 | 1 | 2 | 3 | 4 | 5 | 5 | 6 | 1 | 2 | 3 | 4 | 4 | 5 | 6 | 1 | 2 | 3 |

$F_2$ =

| | | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 2 | 3 | 4 | 5 | 6 | 3 | 4 | 5 | 6 | 1 | 2 | 5 | 6 | 1 | 2 | 3 | 4 |
| 4 | 5 | 6 | 1 | 2 | 3 | 4 | 5 | 6 | 1 | 2 | 3 | 1 | 2 | 3 | 4 | 5 | 6 |

$F_2$, although orthogonal to $F_1$, does not have as small a variance as $F_1$. It could perhaps be improved upon by using a construction procedure similar to that in section 4. The procedure generalizes for all even v.

For v odd, one cannot produce F-rectangles of the above sort which are orthogonal. One can produce a pair of F-rectangles which are balanced in the Hedayat et al. (1972) sense, and which are nearly orthogonal. For v = 7, such a pair is

$F_1$ =

| | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 2 | 3 | 4 | 5 | 6 | 0 | 3 | 4 | 5 | 6 | 0 | 1 | 2 |
| 2 | 3 | 4 | 5 | 6 | 0 | 1 | 5 | 6 | 0 | 1 | 2 | 3 | 4 |
| 4 | 5 | 6 | 0 | 1 | 2 | 3 | 6 | 0 | 1 | 2 | 3 | 4 | 5 |

$F_2$ =

| | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 2 | 3 | 4 | 5 | 6 | 0 | 1 | 1 | 2 | 3 | 4 | 5 | 6 | 0 |
| 5 | 6 | 0 | 1 | 2 | 3 | 4 | 2 | 3 | 4 | 5 | 6 | 0 | 1 |
| 3 | 4 | 5 | 6 | 0 | 1 | 2 | 6 | 0 | 1 | 2 | 3 | 4 | 5 |

The variance-covariance matrix is $3\sigma^2 I/14$, and each treatment in $F_1$ occurs once with all but one of the treatments in $F_2$. The procedure generalizes for v = 4t + 3 by using quadratic and non-null nonquadratic residues as the elements of the first and $v+1^{st}$ columns, respectively, of $F_1$.

## LITERATURE CITED

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