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ORTHOGONAL POLYNOMIALS AND CUBATURE FORMULAE ON SPHERES AND ON BALLS*

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Abstract. Orthogonal polynomials on the unit sphere in $\mathbb{R}^{d+1}$ and on the unit ball in $\mathbb{R}^d$ are shown to be closely related to each other for symmetric weight functions. Furthermore, it is shown that a large class of cubature formulae on the unit sphere can be derived from those on the unit ball and vice versa. The results provide a new approach to study orthogonal polynomials and cubature formulae on spheres.

Key words. orthogonal polynomials in several variables, on spheres, on balls, spherical harmonics, cubature formulae

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1. Introduction.

We are interested in orthogonal polynomials in several variables with emphasis on those orthogonal with respect to a given measure on the unit sphere S^d in $\mathbb{R}^{d+1}$. In contrast to orthogonal polynomials with respect to measures defined on the unit ball B^d in $\mathbb{R}^d$, there have been relatively few studies on the structure of orthogonal polynomials on S^d beyond the ordinary spherical harmonics which are orthogonal with respect to the surface (Lebesgue) measure (cf. [2, 3, 4, 5, 6, 8]). The classical theory of spherical harmonics is primarily based on the fact that the ordinary harmonics satisfy the Laplace equation. Recently Dunkl (cf. [2, 3, 4, 5] and the references therein) opened a way to study orthogonal polynomials on the spheres with respect to measures invariant under a finite reflection group by developing a theory of spherical harmonics analogous to the classical one. In this important theory the role of Laplacian operator is replaced by a differential-difference operator in the commutative algebra generated by a family of commuting first-order differential-difference operators (Dunkl's operators). Other than these results, however, we are not aware of any other method of studying orthogonal polynomials on spheres.

A closely related question is constructing cubature formulae on spheres and on balls. Cubature formulae with a minimal number of nodes are known to be related to orthogonal polynomials. Over the years, a lot of effort has been put into the study of cubature formulae for measures supported on the unit ball, or on other geometric domains with nonempty interior in $\mathbb{R}^d$. In contrast, the study of cubature formulae on the unit sphere has been more or less focused on the surface measure on the sphere; there is little work on the construction of cubature formulae with respect to other measures. This is partly due to the importance of cubature formulae with respect to the surface measure, which play a role in several fields in mathematics, and perhaps partly due to the lack of study of orthogonal polynomials with respect to a general measure on the sphere.

One main purpose of this paper is to provide an elementary approach towards the study of orthogonal polynomials on S^d for a large class of measures. This approach is

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based on a close connection between orthogonal polynomials on S^d and those on the unit ball B^d; a prototype of the connection is the following elementary example.

For $d = 1$, the spherical harmonics of degree n are given in the standard polar coordinates by

$$(1.1) \qquad Y_n^{(1)}(x_1, x_2) = r^n \cos n\theta \quad \text{and} \quad Y_n^{(2)}(x_1, x_2) = r^n \sin n\theta.$$

Under the transform $x = \cos\theta$, the polynomials $T_n(x) = \cos n\theta$ and $U_n(x) = \sin n\theta / \sin\theta$ are the Chebyshev polynomials of the first and the second kind, orthogonal with respect to $1/\sqrt{1-x^2}$ and $\sqrt{1-x^2}$, respectively, on the unit ball $[-1, 1]$ in $\mathbb{R}$. Hence, the spherical harmonics on S^1 can be derived from orthogonal polynomials on B^1.

We shall show that for a large class of weight functions on $\mathbb{R}^{d+1}$ we can construct homogeneous orthogonal polynomials on S^d from the corresponding orthogonal polynomials on B^d in a similar way. This allows us to derive properties of orthogonal polynomials on S^d from those on B^d; the latter have been studied much more extensively. Although the approach is elementary and there is no differential or differential-difference operator involved, the result offers a new way to study the structure of orthogonal polynomials on S^d.

Our approach depends on an elementary formula that links the integration on B^d to the integration on S^d. The same formula yields an important connection between cubature formulae on S^d and those on B^d; the result states roughly that a large class of cubature formulae on S^d is generated by cubature formulae on B^d and vice versa. In particular, it allows us to shift our attention from the study of cubature formulae on the unit sphere to the study of cubature formulae on the unit ball; there has been much more understanding towards the structure of the latter one. Although the result is simple and elementary, its importance is apparent. It yields, in particular, many new cubature formulae on spheres and on balls. Because the main focus of this paper is on the relation between orthogonal polynomials and cubature formulae on spheres and those on balls, we will present examples of cubature formulae in a separate paper.

The paper is organized as follows. In section 2 we introduce notation and present the necessary preliminaries, where we also prove the basic lemma. In section 3 we show how to construct orthogonal polynomials on S^d from those on B^d. In section 4 we discuss the relation between cubature formulae on the unit sphere and those on the unit ball.

2. Preliminary and basic lemma. For $\mathbf{x}, \mathbf{y} \in \mathbb{R}^d$ we let $\mathbf{x} \cdot \mathbf{y}$ denote the usual inner product of $\mathbb{R}^d$ and $|\mathbf{x}| = (\mathbf{x} \cdot \mathbf{x})^{1/2}$ the Euclidean norm of $\mathbf{x}$. Let B^d be the unit ball of $\mathbb{R}^d$ and S^d be the unit sphere on $\mathbb{R}^{d+1}$; that is,

$$B^d = \{\mathbf{x} \in \mathbb{R}^d : |\mathbf{x}| \le 1\} \qquad \text{and} \qquad S^d = \{\mathbf{y} \in \mathbb{R}^{d+1} : |\mathbf{y}| = 1\}.$$

Polynomial spaces. Let $\mathbb{N}_0$ be the set of nonnegative integers. For $\alpha = (\alpha_1, \dots, \alpha_d) \in \mathbb{N}_0^d$ and $\mathbf{x} = (x_1, \dots, x_d) \in \mathbb{R}^d$ we write $\mathbf{x}^\alpha = x_1^{\alpha_1} \cdots x_d^{\alpha_d}$. The number $|\alpha|_1 = \alpha_1 + \cdots + \alpha_d$ is called the total degree of $\mathbf{x}^\alpha$. We denote by Π^d the set of polynomials in d variables on $\mathbb{R}^d$ and by Π_n^d the subset of polynomials of total degree at most n. We also denote by $\mathcal{P}_n^d$ the space of homogeneous polynomials of degree n on $\mathbb{R}^d$ and we let $r_n^d = \dim \mathcal{P}_n^d$. It is well known that

$$\dim \Pi_n^d = \binom{n+d}{n} \quad \text{and} \quad r_n^d = \binom{n+d-1}{n}.$$

Orthogonal polynomials on B^d. Let W be a nonnegative weight function on B^d and assume $\int_{B^d} W(\mathbf{x}) d\mathbf{x} < \infty$. It is known that for each $n \in \mathbb{N}_0$ the set of polynomials of degree n that are orthogonal to all polynomials of lower degree forms a vector space $\mathcal{V}_n$ whose dimension is r_n^d. We denote by $\{P_k^n\}$, $1 \le k \le r_n^d$ and $n \in \mathbb{N}_0$, one family of orthonormal polynomials with respect to W on B^d that forms a basis of Π_n^d, where the superscript n means that $P_k^n \in \Pi_n^d$. The orthonormality means that

$$\int_{B^d} P_k^n(\mathbf{x}) P_j^m(\mathbf{x}) W(\mathbf{x}) d\mathbf{x} = \delta_{j,k}\delta_{m,n}.$$

A useful notation is $\mathbb{P}_n = (P_1^n, \dots, P_{r_n^d}^n)^T$, which is a vector with P_j^n as components (cf. [22, 24]). For each $n \in \mathbb{N}_0$, the polynomials P_k^n, $1 \le k \le r_n^d$, form an orthonormal basis of $\mathcal{V}_n$. We note that there are many bases of $\mathcal{V}_n$; if Q is an invertible matrix of size r_n^d, then the components of $Q\mathbb{P}_n$ form another basis of $\mathcal{V}_n$ which is orthonormal if Q is an orthogonal matrix. For general results on orthogonal polynomials in several variables, including some of the recent development, we refer to the survey [24] and the references therein. One family of weight functions on B^d whose corresponding orthogonal polynomials have been studied in detail is $(1 - |\mathbf{x}|^2)^{\mu - 1/2}$, $\mu \ge 0$, which we will refer to as classical orthogonal polynomials on B^d (cf. [1, 6, 25]).

Ordinary spherical harmonics. The harmonic polynomials on $\mathbb{R}^d$ are the homogeneous polynomials satisfying the Laplace equation $\Delta P = 0$, where

$$\Delta = \partial_1^2 + \cdots + \partial_d^2 \qquad \text{on} \quad \mathbb{R}^d$$

and ∂_i is the ordinary partial derivative with respect to the ith coordinate. They span a subspace $\mathcal{H}_n^d = \ker \Delta \cap \mathcal{P}_n^d$ of dimension $\dim \mathcal{P}_n^d - \dim \mathcal{P}_{n-2}^d$. The spherical harmonics are the restriction of harmonic polynomials on S^{d-1}. If $Y_n \in \mathcal{H}_n^d$, then Y_n is orthogonal to $Q \in \mathcal{P}_k^d$, $0 \le k < n$, with respect to the surface measure $d\omega$ on S^{d-1}.

Dunkl's h-harmonics. For a nonzero vector $\mathbf{v} \in \mathbb{R}^d$ we define the reflection $\sigma_{\mathbf{v}}$ by

$$\mathbf{x}\sigma_{\mathbf{v}} := \mathbf{x} - 2(\mathbf{x} \cdot \mathbf{v})\mathbf{v}/|\mathbf{v}|^2, \qquad \mathbf{x} \in \mathbb{R}^d.$$

Suppose that G is a finite reflection group on $\mathbb{R}^d$ with the set $\{\mathbf{v}_i : i = 1, 2, ..., m\}$ of positive roots; assume that $|\mathbf{v}_i| = |\mathbf{v}_j|$ whenever σ_i is conjugate to σ_j in G, where we write $\sigma_i = \sigma_{\mathbf{v}_i}$, $1 \le i \le m$. Then G is a subgroup of the orthogonal group generated by the reflections $\{\sigma_{\mathbf{v}_i} : 1 \le i \le m\}$.

The h-harmonics yield orthogonal polynomials on S^{d-1} with respect to $h_\alpha^2 d\omega$, where the weight function h_α is defined by

$$h_\alpha(\mathbf{x}) := \prod_{i=1}^m |\mathbf{x} \cdot \mathbf{v}_i|^{\alpha_i}, \qquad \alpha_i \ge 0, \tag{2.1}$$

with $\alpha_i = \alpha_j$ whenever σ_i is conjugate to σ_j in G. The function h_α is a positively homogeneous G-invariant function of degree $|\alpha|_1 = \alpha_1 + \cdots + \alpha_m$. The key ingredient of the theory is a family of commuting first-order differential-difference operators, $\mathcal{D}_i$ (Dunkl's operators), defined by

$$\mathcal{D}_i f(\mathbf{x}) := \partial_i f(\mathbf{x}) + \sum_{j=1}^m \alpha_j \frac{f(\mathbf{x}) - f(\mathbf{x}\sigma_j)}{\mathbf{x} \cdot \mathbf{v}_j} \mathbf{v}_j \cdot \mathbf{e}_i, \qquad 1 \le i \le d, \tag{2.2}$$

where $\mathbf{e}_1, \dots, \mathbf{e}_d$ are the standard unit vectors of $\mathbb{R}^d$. The h-Laplacian is defined by (see [3])

$$\Delta_h = \mathcal{D}_1^2 + \cdots + \mathcal{D}_d^2, \tag{2.3}$$

which plays the role of Laplacian in the theory of the ordinary harmonics. In particular, the h-harmonics are the homogeneous polynomials satisfying the equation $\Delta_h Y = 0$; in other words, they are the elements of the polynomial subspace $\mathcal{H}_n^d(h^2) := \mathcal{P}_n^d \cap \ker \Delta_h$. The h-spherical harmonics are the restriction of h-harmonics on the sphere.

Basic lemma. We let $d\omega = d\omega_d$ denote the surface measure on S^d, and the surface area

$$\omega = \omega_d = \int_{S^d} d\omega_d = 2\pi^{(d+1)/2}/\Gamma((d+1)/2).$$

The standard change of variables from $\mathbf{x} \in \mathbb{R}^d$ to polar coordinates $r\mathbf{x}'$, $\mathbf{x}' \in S^{d-1}$, yields the following useful formula:

$$\int_{B^d} f(\mathbf{x}) W(\mathbf{x}) d\mathbf{x} = \int_0^1 r^{d-1} \int_{S^{d-1}} f(r\mathbf{x}') W(r\mathbf{x}') d\omega_{d-1} dr. \tag{2.4}$$

This formula connects the integral on B^d to S^{d-1} in a natural way. Our basic formula in the following establishes another relation between integrations over the unit sphere and over the unit ball.

LEMMA 2.1. *Let H be defined on $\mathbb{R}^{d+1}$. Assume that H is symmetric with respect to x_{d+1}; i.e., $H(\mathbf{x}, x_{d+1}) = H(\mathbf{x}, -x_{d+1})$, where $\mathbf{x} \in \mathbb{R}^d$. Then for any continuous function f defined on S^d,*

$$\int_{S^d} f(\mathbf{y}) H(\mathbf{y}) d\omega_d = \int_{B^d} \left[f(\mathbf{x}, \sqrt{1-|\mathbf{x}|^2}) + f(\mathbf{x}, -\sqrt{1-|\mathbf{x}|^2}) \right] \times H(\mathbf{x}, \sqrt{1-|\mathbf{x}|^2}) d\mathbf{x} \Big/ \sqrt{1-|\mathbf{x}|^2}. \tag{2.5}$$

Proof. For $\mathbf{y} \in S^d$, we write $\mathbf{y} = (\sqrt{1-t^2}\mathbf{x}, t)$, where $\mathbf{x} \in S^{d-1}$ and $-1 \le t \le 1$. Then it follows that (cf. [21, p. 436])

$$d\omega_d = (1-t^2)^{(d-2)/2} dt \, d\omega_{d-1}.$$

Starting from the change of variables $\mathbf{y} = (\sqrt{1-t^2}\mathbf{x}, t)$ in the integral, we get

$$\begin{aligned}
\int_{S^d} f(\mathbf{y}) H(\mathbf{y}) d\omega_d &= \int_{-1}^1 \int_{S^{d-1}} f(\sqrt{1-t^2}\mathbf{x}, t) H(\sqrt{1-t^2}\mathbf{x}, t) d\omega_{d-1} (1-t^2)^{(d-2)/2} dt \\
&= \int_0^1 \int_{S^{d-1}} \left[f(\sqrt{1-t^2}\mathbf{x}, t) + f(\sqrt{1-t^2}\mathbf{x}, -t) \right] H(\sqrt{1-t^2}\mathbf{x}, t) d\omega_{d-1} (1-t^2)^{(d-2)/2} dt \\
&= \int_0^1 \int_{S^{d-1}} \left[f(r\mathbf{x}, \sqrt{1-r^2}) + f(r\mathbf{x}, -\sqrt{1-r^2}) \right] H(r\mathbf{x}, \sqrt{1-r^2}) d\omega_{d-1} r^{d-1} \frac{dr}{\sqrt{1-r^2}} \\
&= \int_{B^d} \left[f(\mathbf{x}, \sqrt{1-|\mathbf{x}|^2}) + f(\mathbf{x}, -\sqrt{1-|\mathbf{x}|^2}) \right] H(\mathbf{x}, \sqrt{1-|\mathbf{x}|^2}) \frac{d\mathbf{x}}{\sqrt{1-|\mathbf{x}|^2}},
\end{aligned}$$

where in the second step we have used the symmetry of H with respect to x_{d+1}, in the third step we have changed the variable $t \mapsto \sqrt{1-r^2}$, and in the last step we have used (2.4). ◻

As a special case of this theorem, we notice that the Lebesgue measure on S^d is related to the Chebyshev weight function $1/\sqrt{1-|\mathbf{x}|^2}$ over B^d.

3. Orthogonal polynomials on spheres. Our main result in this section shows a connection between orthogonal polynomials on B^d and those on S^d, which is the surface of B^{d+1} by definition. To be more precise, we need some notation.

Throughout this section we fix the following notation: for $\mathbf{y} \in \mathbb{R}^{d+1}$, we write

$$\mathbf{y} = (y_1, \dots, y_d, y_{d+1}) = (\mathbf{y}', y_{d+1}) = r\mathbf{x} = r(\mathbf{x}', x_{d+1}), \quad \mathbf{x} \in S^d, \quad \mathbf{x}' \in B^d, \tag{3.1}$$

where $r = |\mathbf{y}| = \sqrt{y_1^2 + \cdots + y_{d+1}^2}$ and $\mathbf{x}' = (x_1, \dots, x_d)$.

Definition 3.1. *A weight function H defined on $\mathbb{R}^{d+1}$ is called S-symmetric if it is symmetric with respect to y_{d+1} and centrally symmetric with respect to the variables $\mathbf{y}' = (y_1, \dots, y_d)$; i.e.,*

$$H(\mathbf{y}', y_{d+1}) = H(\mathbf{y}', -y_{d+1}) \quad \textit{and} \quad H(\mathbf{y}', y_{d+1}) = H(-\mathbf{y}', y_{d+1}). \tag{3.2}$$

For examples of S-symmetric weight functions, we may take $H(\mathbf{y}) = W(\mathbf{y}')h(y_{d+1})$, where W is a centrally symmetric function on $\mathbb{R}^d$ and h is an even function on $\mathbb{R}$. There are many other examples of S-symmetric functions, including

$$H(\mathbf{y}) = \prod_{1 \le i < j \le d+1} |y_i^2 - y_j^2|^{\alpha_{ij}}, \qquad \alpha_{ij} \ge 0,$$

which becomes, when $\alpha_{ij} = \alpha$, an example of reflection invariant weight functions considered by Dunkl (associated with the octahedral group). We note that, however, the weight function $\prod_{i<j} |y_i - y_j|^\alpha$ associated with the symmetric group is not an S-symmetric function, since it is not symmetric with respect to y_{d+1}. Nevertheless, this function is centrally symmetric in $\mathbb{R}^{d+1}$. In fact, it is easy to see that S-symmetry implies central symmetry on $\mathbb{R}^{d+1}$, which we formally state in the following proposition.

Proposition 3.2. *If H is an S-symmetric weight function on $\mathbb{R}^{d+1}$, then it is centrally symmetric on $\mathbb{R}^{d+1}$; that is, $H(\mathbf{y}) = H(-\mathbf{y})$ for all $\mathbf{y} \in \mathbb{R}^{d+1}$.*

In association with a weight function H on $\mathbb{R}^{d+1}$, we define a weight function W_H on B^d by

$$W_H(\mathbf{x}) = H(\mathbf{x}, \sqrt{1 - |\mathbf{x}|^2}), \qquad \mathbf{x} \in B^d. \tag{3.3}$$

If H is S-symmetric, then the assumption that H is centrally symmetric with respect to the first d variables implies that W is centrally symmetric on B^d. We denote by $\{P_k^n\}$ and $\{Q_k^n\}$ systems of orthonormal polynomials with respect to the weight functions

$$W_H^{(1)}(\mathbf{x}) = 2W_H(\mathbf{x})/\sqrt{1 - |\mathbf{x}|^2} \qquad \text{and} \qquad W_H^{(2)}(\mathbf{x}) = 2W_H(\mathbf{x})\sqrt{1 - |\mathbf{x}|^2}, \tag{3.4}$$

respectively, where we keep the convention that the superscript n means that P_k^n and Q_k^n are polynomials in Π_n^d, and the subindex k has the range $1 \le k \le r_n^d$. Keeping in mind the notation (3.1) we define

$$Y_{k,n}^{(1)}(\mathbf{y}) = r^n P_k^n(\mathbf{x}') \qquad \text{and} \qquad Y_{j,n}^{(2)}(\mathbf{y}) = r^n x_{d+1} Q_j^{n-1}(\mathbf{x}'), \tag{3.5}$$

where $1 \le k \le r_n^d$, $1 \le j \le r_{n-1}^d$, and we define $Y_{j,0}^{(2)}(\mathbf{y}) = 0$. These functions are, in fact, homogeneous polynomials in $\mathbb{R}^{d+1}$.

THEOREM 3.3. *Let H be an S-symmetric weight function defined on $\mathbb{R}^{d+1}$. Assume that W_H in (3.3) is a nonzero weight function on B^d. Then the functions $Y_{k,n}^{(1)}(\mathbf{y})$ and $Y_{k,n}^{(2)}(\mathbf{y})$ defined in (3.5) are homogeneous polynomials of degree n on $\mathbb{R}^{d+1}$ and*

$$\int_{S^d} Y_{k,n}^{(i)}(\mathbf{x})Y_{l,m}^{(j)}(\mathbf{x})H(\mathbf{x})d\omega_d = \delta_{k,l}\delta_{n,m}\delta_{i,j}, \qquad i,j = 1,2.$$

Proof. From the definition of W_H in (3.3), it follows that both $W_H^{(1)}$ and $W_H^{(2)}$ in (3.4) are centrally symmetric weight functions on B^d. As a consequence, the polynomials P_k^n and Q_k^n are even functions if n is even and odd functions if n is odd. In fact, recall the notation $\mathbb{P}_n$ in section 2; it is known (cf. [22]) that there exist proper matrices $D_{n,i}$ and F_n such that

$$\mathbb{P}_{n+1} = \sum_{i=1}^{d} x_i D_{n,i}^T \mathbb{P}_n + F_n \mathbb{P}_{n-1},$$

from which this conclusion follows easily from induction (cf. [23, p. 20]). This allows us to write, for example,

$$P_k^{2n}(\mathbf{x}') = \sum_{j=0}^{n} \sum_{|\alpha|_1=2j} a_\alpha(\mathbf{x}')^\alpha, \qquad a_\alpha \in \mathbb{R}, \quad \mathbf{x}' \in B^d,$$

where $\alpha \in \mathbb{N}_0^d$, which implies that

$$Y_{k,2n}^{(1)}(\mathbf{y}) = r^n P_k^{2n}(\mathbf{x}') = \sum_{k=0}^{n} r^{2n-2k} \sum_{|\alpha|_1=2k} a_\alpha(\mathbf{y}')^\alpha.$$

Since $r^2 = y_1^2 + \cdots + y_{d+1}^2$ and $\mathbf{y}' = (y_1, \ldots, y_d)$, this shows that $Y_{k,2n}^{(1)}(\mathbf{y})$ is a homogeneous polynomial of degree $2n$ in $\mathbf{y}$. Similar proof can be adopted to show that $Y_{k,2n-1}^{(1)}$ is homogeneous of degree $2n-1$ and, using the fact $rx_{d+1} = y_{d+1}$, that $Y_{k,n}^{(2)}$ are homogeneous of degree n.

Since $Y_{k,n}^{(1)}$, when restricted to S^d, is independent of x_{d+1} and $Y_{k,2n}^{(1)}$ contains a single factor x_{d+1}, it follows that

$$\int_{S^d} Y_{k,n}^{(1)}(\mathbf{x})Y_{l,m}^{(2)}(\mathbf{x})H(\mathbf{x})d\omega_d = \int_{S^d} x_{d+1}P_k^n(\mathbf{x}')Q_l^m(\mathbf{x}')H(\mathbf{x})d\omega_d = 0$$

for any (k,n) and (l,m). By the basic formula (2.6),

$$\begin{aligned}\int_{S^d} Y_{k,n}^{(1)}(\mathbf{x})Y_{l,m}^{(1)}(\mathbf{x})H(\mathbf{x})d\omega_d &= 2\int_{B^d} P_k^n(\mathbf{x}')P_l^m(\mathbf{x}')H(\mathbf{x}', \sqrt{1-|\mathbf{x}'|^2})\frac{d\mathbf{x}'}{\sqrt{1-|\mathbf{x}'|^2}} \\ &= \int_{B^d} P_k^n(\mathbf{x}')P_l^m(\mathbf{x}')W_H^{(1)}(\mathbf{x}')d\mathbf{x}' = \delta_{k,l}\delta_{n,m}\end{aligned}$$

and similarly, using the fact that $x_{d+1}^2 = 1-|\mathbf{x}'|^2$,

$$\begin{aligned}\int_{S^d} Y_{k,n}^{(2)}(\mathbf{x})Y_{l,m}^{(2)}(\mathbf{x})H(\mathbf{x})d\omega_d &= 2\int_{B^d} (1-|\mathbf{x}'|^2)Q_k^{n-1}(\mathbf{x}')Q_l^{m-1}(\mathbf{x}')H(\mathbf{x}', \sqrt{1-|\mathbf{x}'|^2}) \\ &\times \frac{d\mathbf{x}'}{\sqrt{1-|\mathbf{x}'|^2}} = \int_{B^d} Q_k^{n-1}(\mathbf{x}')Q_l^{m-1}(\mathbf{x}')W_H^{(2)}(\mathbf{x}')d\mathbf{x}' = \delta_{k,l}\delta_{n,m}.\end{aligned}$$

This completes the proof. ∎

The assumption that H is S-symmetry in Theorem 3.3 is necessary; it is used to show that $Y_{k,n}^{(1)}$ and $Y_{k,n}^{(2)}$ in (3.5) are indeed polynomials in $\mathbf{y}$.

Example 3.4. If $H(\mathbf{y}) = 1$, then $Y_{k,n}^{(1)}$ and $Y_{k,n}^{(2)}$ are orthonormal with respect to the surface measure $d\omega$; they are the ordinary spherical harmonics. According to Theorem 3.3, the harmonics are related to the orthogonal polynomials with respect to the radial weight functions $W_0(\mathbf{x}) = 1/\sqrt{1-|\mathbf{x}|^2}$ and $W_1(\mathbf{x}) = \sqrt{1-|\mathbf{x}|^2}$ on B^d, both of which belong to the family of weight functions $W_\mu(\mathbf{x}) = W_{\mu,d}(\mathbf{x}) = w_\mu(1-|\mathbf{x}|^2)^{\mu-\frac{1}{2}}$, $\mu > -1/2$, whose corresponding orthogonal polynomials have been studied in [1, 6, 25]. For $d = 1$, the spherical harmonics are given in the polar coordinates $(y_1, y_2) = r(x_1, x_2) = r(\cos\theta, \sin\theta)$ by the formula (1.1), which can be written as

$$Y_n^{(1)}(y_1, y_2) = r^n T_n(x_1) \quad \text{and} \quad Y_n^{(2)}(y_1, y_2) = r^n x_2 U_{n-1}(x_1),$$

where, with $t = \cos\theta$, $T_n(t) = \cos n\theta$ and $U_n(t) = \sin n\theta/\sin\theta$ are the Chebyshev polynomials of the first and the second kind, which are orthogonal with respect to $1/\sqrt{1-x^2}$ and $\sqrt{1-x^2}$, respectively. It is this example that motivates our present consideration. ∎

Definition 3.5. *We define a subspace $\mathcal{H}_n^{d+1}(H)$ of $\mathcal{P}_n^{d+1}$ by*

$$\mathcal{H}_n^{d+1}(H) = \operatorname{span}\{Y_{k,n}^{(1)}, \quad 1 \le k \le r_n^d; \quad \textit{and} \quad Y_{j,n}^{(2)}, \quad 1 \le j \le r_{n-1}^d\}.$$

Theorem 3.6. *Let H be an S-symmetric function on $\mathbb{R}^{d+1}$. For each $n \in \mathbb{N}_0$,*

$$\dim \mathcal{H}_n^{d+1}(H) = \binom{n+d}{d} - \binom{n+d-2}{d} = \dim \mathcal{P}_n^{d+1} - \dim \mathcal{P}_{n-2}^{d+1}.$$

Proof. From the orthogonality in Theorem 3.3, the polynomials in $\{Y_{k,n}^{(1)}, Y_{j,n}^{(2)}\}$ are linearly independent. Hence, it follows readily that

$$\dim \mathcal{H}_n^{d+1}(H) = r_n^d + r_{n-1}^d = \binom{n+d-1}{n} + \binom{n+d-2}{n-1},$$

where we use the convention that $\binom{k}{j} = 0$ if $j < 0$. Using the identity $\binom{n+m}{n} - \binom{n+m-1}{n} = \binom{n+m-1}{n-1}$, it is easy to verify that

$$\dim \mathcal{H}_n^{d+1}(H) = \binom{n+d}{d} - \binom{n+d-2}{d},$$

which is the desired result. ∎

Theorem 3.7. *Let H be an S-symmetric function on $\mathbb{R}^{d+1}$. For $n \in \mathbb{N}_0$,*

$$\mathcal{P}_n^{d+1} = \bigoplus_{k=0}^{[n/2]} |\mathbf{y}|^{2k} \mathcal{H}_{n-2k}^{d+1}(H);$$

that is, if $P \in \mathcal{P}_n^{d+1}$, then there is a unique decomposition

$$P(\mathbf{y}) = \sum_{k=0}^{[n/2]} |\mathbf{y}|^{2k} P_{n-2k}(\mathbf{y}), \qquad P_{n-2k} \in \mathcal{H}_{n-2k}^{d+1}(H).$$

Proof. Since P is homogeneous of degree n, we can write $P(\mathbf{y}) = r^n P(\mathbf{x})$, where we use the notation in (3.1) again. According to the power of y_{d+1} being even or odd and using $x_{d+1}^2 = 1 - |\mathbf{x}'|^2$ whenever possible, we can further write

$$P(\mathbf{y}) = r^n P(\mathbf{x}) = r^n[p(\mathbf{x}') + x_{d+1}q(\mathbf{x}')],$$

where p and q are polynomials of degree at most n and $n-1$, respectively, in $\mathbf{x}' \in B^d$. Moreover, if n is even, then p is even and q is odd; if n is odd, then p is odd and q is even. Since both $\{P_k^n\}$ and $\{Q_k^n\}$ form a basis for Π_n^d and since the weight functions $W_H^{(1)}$ and $W_H^{(2)}$ in (3.4) are centrally symmetric, we have the unique expansions

$$p(\mathbf{x}') = \sum_{k=0}^{[n/2]} \sum_j a_{j,k} P_j^{n-2k}(\mathbf{x}') \quad \text{and} \quad q(\mathbf{x}') = \sum_{k=0}^{[(n-1)/2]} \sum_j b_{j,k} Q_j^{n-2k-1}(\mathbf{x}'),$$

where $1 \le j \le r_{n-2k}^d$. Therefore, by the definition of $Y_{k,n}^{(1)}$ and $Y_{k,n}^{(2)}$, we have

$$P(\mathbf{y}) = \sum_{k=0}^{[n/2]} r^{2k} \sum_j a_{j,k} Y_{j,n-2k}^{(1)}(\mathbf{y}) + \sum_{k=0}^{[(n-1)/2]} r^{2k-1} \sum_j b_{j,k} Y_{j,n-2k+1}^{(2)}(\mathbf{y}),$$

which is the desired decomposition. The uniqueness follows from the orthogonality in Theorem 3.3. ∎

For the spherical harmonics or h-harmonics, the above theorem is usually established using the differential or differential-difference operator (cf. [19, 2]). The importance of the results in this section lies in the fact that they provide an approach to studying orthogonal polynomials on S^d with respect to a large class of measures. For example, one of the essential ingredients in the recent work of orthogonal polynomials in several variables (cf. [22, 24]) is a three-term relation in a vector-matrix form,

$$x_i \mathbb{P}_n = A_{n,i}\mathbb{P}_{n+1} + B_{n,i}\mathbb{P}_n + A_{n-1,i}^T \mathbb{P}_{n-1},$$

where $A_{n,i}$ and $B_{n,i}$ are proper matrices, which also plays a decisive role in the study of common zeros of $\mathbb{P}_n$ and cubature formulae; the results in Theorem 3.3 show that the h-spherical harmonic polynomials that are even (or odd) in x_{d+1} also satisfy such a three-term relation.

It is worthwhile to point out that the relation between orthogonal polynomials on B^d and those on S^d goes both ways. In fact, the following result holds.

THEOREM 3.8. *Let H be a weight function defined on $\mathbb{R}^{d+1}$ which is symmetric with respect to y_{d+1}. Assume that W_H in* (3.3) *is a nonzero weight function on B^d. Let $Y_{k,n}^{(1)}$ be the orthonormal polynomials of degree n with respect to $H(\mathbf{y})d\omega$ on S^d that are even in y_{d+1}, and write the orthonormal polynomials that are odd in y_{d+1} as $y_{d+1}Y_{k,n-1}^{(2)}$. Then*

$$P_k^n(\mathbf{x}) = Y_{k,n}^{(1)}(\mathbf{x}, \sqrt{1-|\mathbf{x}|^2}) \quad \text{and} \quad Q_k^n(\mathbf{x}) = Y_{k,n}^{(2)}(\mathbf{x}, \sqrt{1-|\mathbf{x}|^2})$$

are orthonormal polynomials of degree n in $\mathbf{x} \in B^d$ with respect to $W_H^{(1)}$ and $W_H^{(2)}$ defined in (3.4)*, respectively.*

Proof. The orthogonality follows easily from Lemma 2.1 as in the proof of Theorem 3.3. We show that the assumption on $Y_{k,n}^{(i)}$ is justified. Since H is symmetric with respect to y_{d+1}, we can pick the orthogonal polynomials with respect to $Hd\omega$ on S^d as either even in y_{d+1} or odd in y_{d+1} (recall the nonuniqueness of orthonormal bases). Indeed, if Y_n is a polynomial of degree n orthogonal to lower degree polynomials with respect to $Hd\omega$, so is the polynomial $Y_n(\mathbf{y}', -y_{d+1})$ by the symmetry of H with respect to y_{d+1}. Hence, if n is even, then the polynomial $Y_n(\mathbf{y}) + Y_n(\mathbf{y}', -y_{d+1})$ is an orthogonal polynomial of degree n which is even in y_{d+1}; if n is odd, then we consider $Y_n(\mathbf{y}) - Y_n(\mathbf{y}', -y_{d+1})$ instead. Therefore, the polynomials P_k^n and Q_k^n are well defined on B^d. ∎

It should be noted that there is no need to assume that H is S-symmetric in the above theorem; consequently, there is no assurance that $Y_{k,n}^{(i)}$ are homogeneous.

In an effort to understand Dunkl's theory of h-harmonics, we study the orthogonal polynomials on S^d associated to $h(\mathbf{y}) = |y_1|^{\alpha_1} \cdots |y_{d+1}|^{\alpha_{d+1}}$ in detail in [26]. In particular, making use of the product structure of the measure, an orthonormal basis of h-harmonics is given in terms of the orthonormal polynomials of one variable with respect to the measure $(1-t^2)^\lambda |t|^{2\mu}$ on $[-1, 1]$ (which in turn can be written in terms of Jacobi polynomials). By Theorem 3.8, we can then derive an explicit basis of orthogonal polynomials with respect to $W_H(\mathbf{x}) = |x_1|^{\alpha_1} \cdots |x_d|^{\alpha_d}(1 - |\mathbf{x}|^2)^{\alpha_{d+1}}$.

The theory of the h-harmonics developed by Dunkl recently is a rich one; it has found applications in a number of fields. For numerical work, one essential problem in dealing with h-harmonics is the construction of a workable orthonormal basis for $\mathcal{H}_n^{d+1}(h^2)$. So far, such a basis has been constructed only in the case of $h(\mathbf{y}) = |y_1|^{\alpha_1} \cdots |y_{d+1}|^{\alpha_{d+1}}$, associated to the reflection group $Z_2 \times \cdots \times Z_2$. Theorem 3.3 indicates that an explicit construction of such a basis may be difficult for the reflection invariant weight functions h associated with most of other reflection groups. We illustrate by the following example.

Example 3.9. Consider the weight function h on $\mathbb{R}^3$ defined by

$$h(y_1, y_2, y_3) = |(y_1^2 - y_2^2)(y_1^2 - y_3^2)(y_2^2 - y_3^2)|^\mu,$$

which is associated to the octahedral group; the group is generated by the reflections in $y_i = 0$, $1 \le i \le 3$, and $y_i \pm y_j = 0$, $1 \le i, j \le 3$; it is the Weyl group of type B_3. This weight function is one of the simplest nonproduct weight functions on S^2. According to Theorem 3.1, the h-harmonics associated to the function $H(\mathbf{y}) = h^2(\mathbf{y})$ are related to the orthogonal polynomials on the disc $B^2 \subset \mathbb{R}^2$ with respect to the weight function $W_H^{(1)}$ and $W_H^{(2)}$ in (3.4), where the weight function $W_H^{(1)}$ is given by

$$W_H^{(1)}(x_1, x_2) = 2|(x_1^2 - x_2^2)(1 - 2x_1^2 - x_2^2)(1 - x_1^2 - 2x_2^2)|^{2\mu} / \sqrt{1 - x_1^2 - x_2^2}, \quad (x_1, x_2) \in B^2.$$

An explicit basis for the h-harmonics will mean an explicit basis for orthogonal polynomials with respect to $W_H^{(1)}$ and vice versa. However, the form of $W_H^{(1)}$ given above indicates that it may be difficult to find a closed formula for such a basis. ∎

4. Cubature formula on spheres and on balls. In this section we discuss the connection between cubature formulae on spheres and on balls. For a given integral $\mathcal{L}(f) := \int f d\mu$, where $d\mu$ is a nonnegative measure with support set on B^d, a cubature formula of degree M is a linear functional

$$\mathcal{I}_M(f) = \sum_{k=1}^N \lambda_k f(\mathbf{x}_k), \quad \lambda_k > 0, \quad \mathbf{x}_k \in \mathbb{R}^d,$$

defined on Π^d, such that $\mathcal{L}(f) = \mathcal{I}_M(f)$ whenever $f \in \Pi_M^d$, and $\mathcal{L}(f^*) \neq \mathcal{I}_M(f^*)$ for at least one $f^* \in \Pi_{M+1}^d$. When the measure is supported on S^d, we need to replace Π_M^d by $\bigcup_{k=0}^M \mathcal{P}_k^{d+1}$ in the above formulation and require $\mathbf{x}_k \in S^d$. The points $\mathbf{x}_1, \ldots, \mathbf{x}_N$ are called *nodes* and the numbers $\lambda_1, \ldots, \lambda_N$ are called *weights*. Such a formula is called minimal if N, the number of nodes, is minimal among all cubature formulae of degree M.

Cubature formulae on the unit sphere have important applications in numerical integration and in areas ranging from coding theory to isometric embeddings between classical Banach spaces (cf. [11, 15, 16] and the references therein). Over years, construction of cubature formulae on the unit sphere with respect to the surface measure $d\omega$ has attracted a lot of attention. For example, starting from the pioneer work of Sobolev (cf. [17]), the Russian school of mathematicians have constructed various cubature formulae on S^d that are invariant under finite groups (cf. [14, 10] and the references therein). There are also important studies on Chebyshev cubature formulae, which are formulae with equal weights (cf. [9, 11, 15] and the references therein). Nevertheless, the simple results we present below on the connection between cubature formula on balls and on spheres do not seem to have been noticed before.

THEOREM 4.1. *Let H defined on $\mathbb{R}^{d+1}$ be symmetric with respect to y_{d+1}. Suppose that there is a cubature formula of degree M on B^d for W_H defined in* (3.3),

$$\int_{B^d} g(\mathbf{x}) W_H(\mathbf{x}) \frac{d\mathbf{x}}{\sqrt{1-|\mathbf{x}|^2}} = \sum_{i=1}^N \lambda_i g(\mathbf{x}_i), \qquad g \in \Pi_M^d, \tag{4.1}$$

whose N nodes lie inside the unit ball B^d; that is, $|\mathbf{x}_i| \le 1$. Then there is a cubature formula of degree M on the unit sphere S^d,

$$\begin{aligned} \int_{S^d} f(\mathbf{y}) H(\mathbf{y}) d\omega \\ = \sum_{i=1}^N \lambda_i \Big[f(\mathbf{x}_i, \sqrt{1-|\mathbf{x}_i|^2}) + f(\mathbf{x}_i, -\sqrt{1-|\mathbf{x}_i|^2}) \Big], \quad f \in \bigcup_{k=0}^M \mathcal{P}_k^{d+1}. \end{aligned} \tag{4.2}$$

Proof. Assuming (4.1), to prove (4.2) it suffices to prove, by Lemma 2.1, that

$$\begin{aligned} \int_{B^d} \Big[f(\mathbf{x}, \sqrt{1-|\mathbf{x}|^2}) + f(\mathbf{x}, -\sqrt{1-|\mathbf{x}|^2}) \Big] W_H(\mathbf{x}) \frac{d\mathbf{x}}{\sqrt{1-|\mathbf{x}|^2}} \\ = \sum_{i=1}^N \lambda_i \Big[f(\mathbf{x}_i, \sqrt{1-|\mathbf{x}_i|^2}) + f(\mathbf{x}_i, -\sqrt{1-|\mathbf{x}_i|^2}) \Big] \end{aligned} \tag{4.3}$$

for all polynomials $f \in \Pi_M^d$. We consider the basis of $\bigcup_{k=0}^M \mathcal{P}_k^{d+1}$ consisting of monomial $\{f_\alpha\}_{|\alpha|_1 \le M}$, where $f_\alpha(\mathbf{y}) = \mathbf{y}^\alpha$ and $\alpha \in \mathbb{N}^{d+1}$. If f_α is an odd function in y_{d+1}, then both the left side and the right side of (4.3) are zero, so the equality holds. If f_α is even in y_{d+1}, $|\alpha|_1 \le M$, then the function

$$f_\alpha(\mathbf{x}, \sqrt{1-|\mathbf{x}|^2}) = \mathbf{x}^{\alpha'} (1-|\mathbf{x}|^2)^{\alpha_{d+1}/2},$$

where we write $\alpha = (\alpha', \alpha_{d+1})$, is a polynomial of degree at most M in $\mathbf{x}$. Hence, it follows from the cubature formula (4.1) that

$$\int_{B^d} f(\mathbf{x}, \pm\sqrt{1-|\mathbf{x}|^2}) W_H(\mathbf{x}) \frac{d\mathbf{x}}{\sqrt{1-|\mathbf{x}|^2}} = \sum_{i=1}^N \lambda_i f(\mathbf{x}_i, \pm\sqrt{1-|\mathbf{x}_i|^2})$$

holds. Adding the above equations for $f(\mathbf{x}, \sqrt{1-|\mathbf{x}|^2})$ and for $f(\mathbf{x}, -\sqrt{1-|\mathbf{x}|^2})$ together proves (4.3). ∎

The theorem states that each cubature formula on the unit ball B^d leads to a cubature formula on the unit sphere S^d. The converse of this result is also true.

THEOREM 4.2. *Let H be a weight function on $\mathbb{R}^{d+1}$ which is symmetric with respect to x_{d+1}. Suppose that there is a cubature formula of degree M on the sphere S^d*

$$\int_{S^d} f(\mathbf{y})H(\mathbf{y})d\omega = \sum_{i=1}^{N} \lambda_i f(\mathbf{y}_i), \qquad f \in \bigcup_{k=0}^{M} \mathcal{P}_k^{d+1} \tag{4.4}$$

whose nodes are all located on S^d. Then there is a cubature formula of degree M on the unit ball B^d

$$2\int_{B^d} g(\mathbf{x})W_H(\mathbf{x})\frac{d\mathbf{x}}{\sqrt{1-|\mathbf{x}|^2}} = \sum_{i=1}^{N} \lambda_i g(\mathbf{x}_i), \qquad g \in \Pi_M^d, \tag{4.5}$$

where $\mathbf{x}_i \in B^d$ are the first d components of $\mathbf{y}_i$; that is, $\mathbf{y}_i = (\mathbf{x}_i, x_{d+1,i})$.

Proof. By Lemma 2.1, the cubature formula (4.4) is equivalent to

$$\int_{B^d} \Big[f(\mathbf{x}, \sqrt{1-|\mathbf{x}|^2}) + f(\mathbf{x}, -\sqrt{1-|\mathbf{x}|^2})\Big] W_H(\mathbf{x})\frac{d\mathbf{x}}{\sqrt{1-|\mathbf{x}|^2}} = \sum_{i=1}^{N} \lambda_i f(\mathbf{y}_i). \tag{4.6}$$

If we write $\mathbf{y} = (\mathbf{x}, x_{d+1}) \in \mathbb{R}^{d+1}$, where $\mathbf{x} \in \mathbb{R}^d$, then for every monomial $g_\alpha(\mathbf{x}) = \mathbf{x}^\alpha \in \Pi_M^d$ the function f_α defined by $f_\alpha(\mathbf{y}) = g_\alpha(\mathbf{x})$ is a polynomial in $\mathcal{P}_k^{d+1}$, where $|\alpha|_1 = k \le M$. We can apply cubature formula (4.4) to it. Since f so defined is apparently even in x_{d+1}, the cubature (4.6) becomes cubature formula (4.5). ∎

Although these theorems are simple to state, they have important implications. They allow us to fit a large class of cubature formula on spheres into the structure of cubature formulae on balls, which suggests an alternative approach to study and construct cubature formulae.

Example 4.3. In the case $d = 1$, the formula (4.1) under the change of variable $x = \cos\theta$ becomes

$$\int_0^\pi g(\cos\theta)W_H(\cos\theta)d\theta = \sum_{i=1}^{N} \lambda_i g(\cos\theta_i).$$

On the other hand, we can write the integral over S^1 in the polar coordinates as

$$\int_{S^1} f(\mathbf{y})H(\mathbf{y})d\omega = \int_0^{2\pi} f(\cos\theta, \sin\theta)H(\cos\theta, \sin\theta)d\theta.$$

Since H is symmetric with respect to x_2, it follows that $W_H(\cos\theta) = H(\cos\theta, \sin\theta)$ in the notation of (3.3). Hence, (4.2) becomes

$$\int_0^{2\pi} f(\cos\theta, \sin\theta)W_H(\cos\theta)d\theta = \sum_{i=1}^{N} \lambda_i \Big[f(\cos\theta_i, \sin\theta_i) + f(\cos\theta_i, -\sin\theta_i)\Big].$$

From these formulae the relation between (4.1) and (4.2) is evident. ∎

In a separate paper we will present a number of examples on S^2 that are obtained using this approach. Here we concentrate on the theoretic side of the matter. What we are interested in is the minimal cubature formula, or cubature formula whose number of nodes is close to minimal.

We state the lower bounds on the number of nodes of cubature formulae, which are used to test whether a given cubature is minimal. Let us denote by N_{B^d} the number of nodes for a cubature formula on B^d, and by N_{S^d} the number of nodes for a cubature formula on S^d. It is well known (cf. [7, 18]) that

$$N_{B^d} \geq \dim \Pi_n^d = \binom{n+d}{n}, \qquad M = 2n \text{ or } M = 2n+1, \tag{4.7}$$

and

$$N_{S^d} \geq \sum_{k=0}^{n} \dim \mathcal{H}_k^{d+1} = \binom{n+d}{n} + \binom{n+d-1}{n-1}, \quad M = 2n \text{ or } M = 2n+1, \tag{4.8}$$

where the equal sign in (4.8) follows from the formula for $\dim \mathcal{H}_k^{d+1}$ (cf. Theorem 3.6 with $H = 1$) and simple computation. Moreover, for centrally symmetric weight functions there are improved lower bounds for cubature formula of odd degree, due to Möller for N_{B^d} and to Mysovskikh for N_{S^d} (cf. [13, 14]), which states that

$$N_{B^2} \geq \binom{n+2}{n} + \left[\frac{n+1}{2}\right], \qquad M = 2n+1, \tag{4.9}$$

and

$$N_{S^d} \geq 2\binom{n+d}{n}, \qquad M = 2n+1, \tag{4.10}$$

where, for simplicity, we have restricted the lower bound of N_{B^d} to the case $d = 2$.

Several characterizations of cubature formulas on B^d that attain the lower bound in (4.7), or (4.9), are known. For example, there is a cubature formula that attains the bound (4.7) for $M = 2n+1$ if, and only if, the corresponding orthogonal polynomials $P_1^{n+1}, \ldots, P_{r_{n+1}^d}^{n+1}$ of degree $n+1$ have $\dim \Pi_n^d$ many distinct real common zeros. The characterization for the case (4.7) with $M = 2n$ and the case (4.9) for centrally symmetric weight functions will involve common zeros of quasi-orthogonal polynomials. For these characterizations and extensions of them we refer to [12, 14, 23, 24] and the references therein. In view of the results in section 3, we see that when H is centrally symmetric, we can relate these characterizations to orthogonal polynomials on spheres.

Let us consider the number of nodes of the cubature formulae in (4.2) and (4.5).

Remark 4.4. In Theorem 4.1, the number of nodes in the cubature formula (4.2) may be less than $2N$, since if one of the nodes of (4.1), say $\mathbf{x}_i$, lies on the boundary $\partial B^d = S^{d-1}$, then $|\mathbf{x}_i| = 1$ and two nodes $(\mathbf{x}_i, \sqrt{1-|\mathbf{x}_i|^2})$ and $(\mathbf{x}_i, -\sqrt{1-|\mathbf{x}_i|^2})$ in (4.2) become one. That is,

$$\text{number of nodes of (4.2)} = 2N - \text{number of } \mathbf{x}_i \text{ on } S^{d-1}. \tag{4.11}$$

Similarly, in Theorem 4.2, the number of nodes in the cubature formula (4.5) may be less than N, since different $\mathbf{y}_i \in S^d$ may have the same first d components, which happens when $\mathbf{y}_i$ and $\mathbf{y}_j$ form a *symmetric pair* with respect to the last component; i.e., $\mathbf{y}_i = (\mathbf{x}_i, x_{d+1})$ and $\mathbf{y}_j = (\mathbf{x}_i, -x_{d+1})$ with $x_{d+1} \neq 0$. We conclude that

(4.12) number of nodes of (4.5) $= N -$ number of symmetric pairs among $\mathbf{y}_i$.

Clearly, the number of nodes in (4.5) satisfies a lower bound $N/2$, which is attained when the nodes of (4.4) consist of only symmetric pairs. ∎

It is important to remark that even if the cubature formula (4.1) on B^d in Theorem 4.1 attains the lower bound (4.7) or (4.9), the cubature formula (4.2) on S^d may not attain the lower bound (4.8) or (4.10), respectively. For example, when $d = 1$, the formula (4.1) for $M = 2n+1$ attains the lower bound (4.7) with $N_{B^1} = n+1$, which is the classical Gaussian quadrature formula. On the other hand, the corresponding formula in (4.2) attains the lower bound (4.8) with $N_{S^1} = 2n+1$ only when $x = 1$ or $x = -1$ is a node of (4.1), which does not hold in general since the nodes of a Gaussian quadrature formula on $[-1, 1]$ are zeros of orthogonal polynomials and are located in $(-1, 1)$.

As an immediate consequence of these lower bounds and Theorems 4.1 and 4.2, we formulate a corollary that seems to be of independent interest.

COROLLARY 4.5. *Let H be an S-symmetric weight function on $\mathbb{R}^3$. If there is a cubature formula of degree $2n+1$ with respect to H on S^2 that attains the lower bound in* (4.10)*, then it contains at least* $2[(n+1)/2]$ *nodes which are not symmetric with respect to x_3.*

Proof. Assume that a cubature formula with respect to H on S^2 exists which attains the lower bound in (4.10). Let E be the number of symmetric pairs among the nodes of the cubature. By Theorem 4.2 and (4.12), there is a cubature formula on B^2 with $N_{S^2} - E = 2\binom{n+2}{n} - E$ many nodes. Moreover, the weight function associated with the new cubature formula is centrally symmetric on B^2. Hence, by (4.9), we have the inequality that

$$N_{S^2} - E = 2\binom{n+2}{n} - E \geq \binom{n+2}{n} + \left[\frac{n+1}{2}\right],$$

from which we get an upper bound for E. Evidently, the number of nodes that do not contain symmetric pairs is equal to $N_{S^2} - 2E$. Hence, the upper bound for E leads to a lower bound on the number of nodes that are not symmetric with respect to x_3, which gives the desired result. ∎

In particular, if the cubature formula on S^2 is symmetric with respect to x_3, which means that the node of the cubature always contains the pair (x_1, x_2, x_3) and $(x_1, x_2, -x_3)$ whenever $x_3 > 0$, then there are at least $2[(n+1)/2]$ many nodes on the largest circle $x_1^2 + x_2^2 = 1$ which is perpendicular to x_3 axis. Results as such provide necessary conditions on the minimal cubature formulae; they may provide insight in the construction of the minimal formula or can be used to prove that such a formula does not exist. An analogue of Corollary 4.5 is as follows.

COROLLARY 4.6. *If there is a cubature formula on B^2 that attains the lower bound in* (4.9) *with all nodes in B^2, then it can have no more than* $2[(n+1)/2]$ *points on the boundary $\partial B^2 = S^1$.*

Proof. If a cubature formula on B^2 as stated exists which attains the lower bound (4.9), then by Theorem 4.1 and (4.11) there is a cubature formula on S^2 with

$$N_{S^2} = 2\binom{n+2}{n} + 2\left[\frac{n+1}{2}\right] - \text{number of nodes on } S^1.$$

The desired result then follows from the lower bound (4.10). □

We conclude this paper with another simple application dealing with Chebyshev cubature formulae, which are cubature formulae with equal weights. It is proved in [9] that the number of nodes of a Chebyshev cubature formula of degree M with respect to $1/\sqrt{1-|\mathbf{x}|^2}$on B^2 is of order $\mathcal{O}(M^3)$. Furthermore, it is conjectured there that the number of nodes of a Chebyshev cubature formula of degree M with respect to the surface measure on S^2 is of order $\mathcal{O}(M^2)$.

COROLLARY 4.7. *If there is a Chebyshev cubature formula of degree M with respect to the surface measure on S^2 whose number of nodes is of order $\mathcal{O}(M^2)$, then its nodes cannot all be symmetric with respect to a plane that contains one largest circle of S^2 and none of the nodes.*

Indeed, if such a cubature formula exists, we may assume that the plane is the coordinate plane perpendicular to the x_3 axis since the integral is invariant under rotation. Then there are an even number of nodes and all nodes form symmetric pairs. Therefore, by Theorem 4.2, there would be a Chebyshev cubature formula of degree M with respect to $1/\sqrt{1-|\mathbf{x}|^2}$ on B^2 with the number of nodes in the order of $\mathcal{O}(M^2)$, which leads to a contradiction.

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