Lawrence Berkeley Laboratory
UNIVERSITY OF CALIFORNIA

Physics, Computer Science & Mathematics Division

To be submitted for publication

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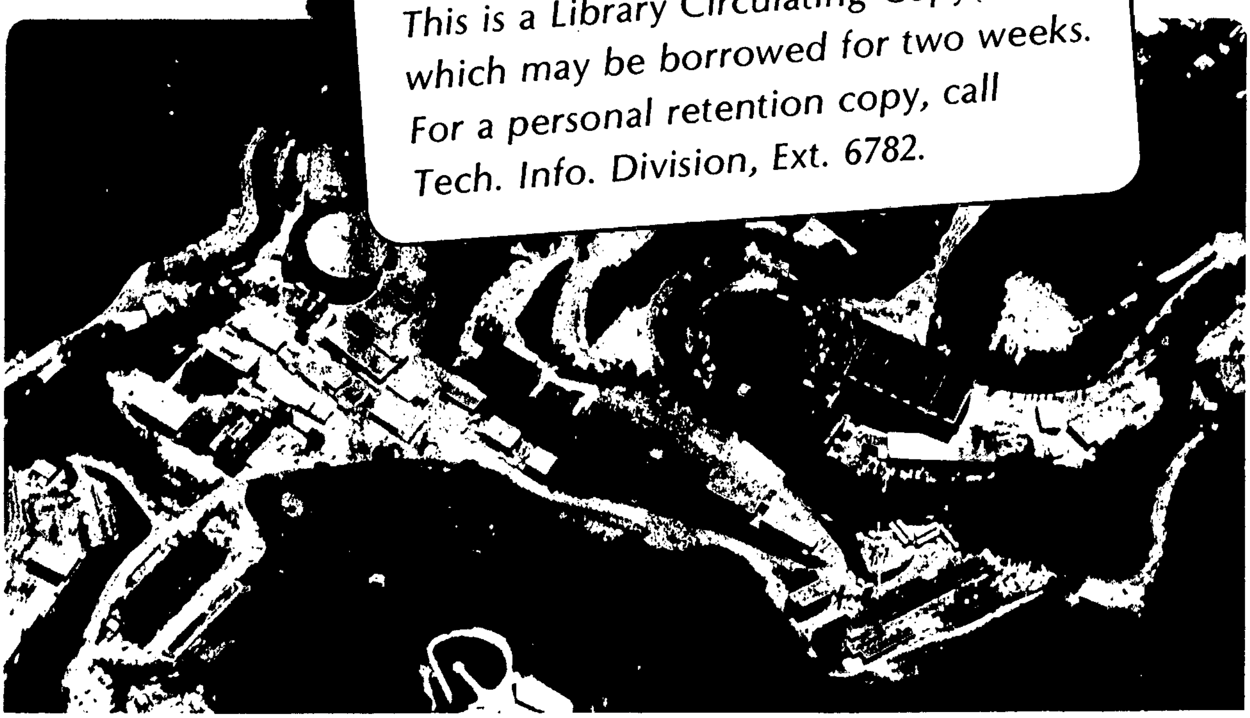
December 1982



Prepared for the U.S. Department of Energy under Contract DE-AC03-76SF00098

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LBL-15652

ORTHOGONAL POLYNOMIALS AND SECOND ORDER DIFFERENTIAL EQUATIONS[1]

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December 1982

[1]The research reported here was supported in part by National Science Foundation Grant MCS81-07086 and by the Director, Office of Basic Energy Sciences, Engineering, Mathematical, and Geosciences Division of the U.S. Department of Energy under contract DE-AC03-76SF00098.

Introduction

Back in 1927 Bochner [1], [2, p.107] and [3, p.150], proved that a sequence of polynomials

$$Y_n \equiv P_n(x) \qquad \deg P_n = n \qquad n = 0,1,2,\ldots$$

satisfy a differential equation of the form

$$a_0(x)\,Y_n'' + a_1(x)\,Y_n'(x) + a_2(x)\,Y_n(x) = \lambda_n Y_n \tag{1}$$

only in a very limited set of cases. After a change of variables you must have the so called "classical orthogonal polynomials" connected with the names of Jacobi, Laguerre, Hermite and Bessel.

This is a nice result from the standpoint of elegance but it is bad news for applications of orthogonal polynomials to boundary value problems.

In a recent paper prompted by work on a nonlinear differential equation [4], R.Smith [5] relaxes the condition on the degree of polynomials and allows these degrees to advance in steps of length m. He then shows that the differential equation can be reduced to one of the forms below:

$$(1 - x^m)xY_n'' - (k - 1 + \delta x^m)Y_n' + n(n + \delta - 1)x^{m-1}Y_n = 0 \quad , \tag{A}$$

$$xY_n'' - (k - 1 + mx^m)Y_n' + mnx^{m-1}Y_n = 0 \quad . \tag{B}$$

In this note we point out for any value of n a solution $Y_n(x)$ of (A) is given in terms of Jacobi polynomials, namely,

$$Y_n(x) = x^k P_{\frac{n-k}{m}}^{\frac{k}{m}, \frac{\delta+k-m-1}{m}} (1-2x^m) \quad . \tag{2}$$

One can then see that if either n or n-k is an integer multiple of m, $Y_n(x)$ is a polynomial in x.

We notice too that a solution of (B) is given in terms of generalized Laguerre polynomials

$$Y_n(x) = L_{\frac{n}{m}}^{-\frac{k}{m}} (x^m) \tag{3}$$

and again if either n or n-k is an integer multiple of m, $Y_n(x)$ is a polynomial in x.

The result of Smith therefore strengthen those of Bochner, and show that unfortunately in the presence of extra properties like (1), orthogonal polynomials are as useful as they are rare.

Canonical Form for the Differential Equation (A)

Take (A) in the form

$$x^{2-m}(1-x^m)Y'' - (k-1+\delta x^m)x^{1-m}Y' + n(n+\delta-1)Y = 0 \ . \tag{4}$$

Now define a new variable z by means of

$$x^m = \left(\sin \frac{mz}{2}\right)^2 \tag{5}$$

to rewrite (4) in the form

$$\frac{d^2Y}{dz^2} + \frac{\frac{m-2k}{2} - (\delta-1)\sin^2 \frac{mz}{2}}{\sin \frac{mz}{2} \cos \frac{mz}{2}} \frac{dY}{dz} + n(n+\delta-1)Y = 0 \ . \tag{6}$$

Now put

$$r(z) = e^{-\frac{1}{2}\int \beta(z)\,dz}$$

where $\beta(z)$ denotes the coefficient in front of $\frac{dY}{dz}$ in (6).

One obtains

$$r(z) = \left(\cos\frac{mz}{2}\right)^{\frac{2-2\delta+m-2k}{2m}} \left(\sin\frac{mz}{2}\right)^{\frac{2k-m}{2m}}$$

and then (6) can be written as

$$r(z)\left[\frac{d^2}{dz^2}\left(\frac{Y(z)}{r(z)}\right) + V(z)\,\frac{Y(z)}{r(z)} + n(n+\delta-1)\,\frac{Y(z)}{r(z)}\right] = 0$$

where

$$V(z) = \frac{m^2-4k^2}{16\sin^2(\frac{mz}{2})} + \frac{m^2-(2\delta+2k-2m-2)^2}{16\cos^2(\frac{mz}{2})} + \frac{(\delta-1)}{4} .$$

The Jacobi Polynomials

If we set $\theta = mz$ we see that the solutions of (6) are given by

$$Y(z) = r(z)y(z) \tag{7}$$

where y $(=y(u))$ satisfies

$$-\frac{d^2y}{du^2} + \left(\frac{4\left(\frac{k}{m}\right)^2-1}{16\sin^2\frac{u}{2}} + \frac{4\left(\frac{\delta+k-m-1}{m}\right)^2-1}{16\cos^2\frac{u}{2}}\right)y = \left(\frac{2n-\delta-1}{2m}\right)^2 y .$$

Now a glance at Szego [2, p.67] shows that we can take

$$y(\theta) = \left(\sin \frac{\theta}{2}\right)^{\alpha+\frac{1}{2}} \left(\cos \frac{\theta}{2}\right)^{\beta+\frac{1}{2}} P^{\alpha,\beta}_{\frac{n-k}{m}} (\cos \theta)$$

with

$$\alpha = \frac{k}{m} \quad , \qquad \beta = \frac{\delta + k - m - 1}{m} \quad ,$$

and $P^{\alpha,\beta}_{\nu}(\cos\theta)$ the standard Jacobi polynomials. Therefore we obtain, in terms of z,

$$Y(z) = \left(\sin \frac{mz}{2}\right)^{\frac{2k}{m}} P^{\alpha,\beta}_{\frac{n-k}{m}} (\cos mz) \quad .$$

Recall now some of the expressions for $P^{\alpha,\beta}_{\nu}$ in terms of hypergeometric functions from [2, p.62] and [6, p.212], namely

$$P^{\alpha,\beta}_{\nu}(\cos mz) = \binom{\nu+\alpha}{\nu} {}_2F_1\left(-\nu,\ \nu+\alpha+\beta+1,\ \alpha+1,\ \frac{1-\cos mz}{2}\right) \tag{8}$$

and

$$P^{\alpha,\beta}_{\nu}(\cos mz) = \binom{\nu+\beta}{\nu} \left(\frac{1-x}{2}\right)^{-\alpha} (-1)^{\nu} {}_2F_1\left(\beta+\nu+1,\ -\nu-\alpha,\ \beta+1,\ \frac{\cos mz + 1}{2}\right) . \tag{9}$$

Notice that

$$\frac{1-\cos mz}{2} = x^m \quad , \qquad \frac{1+\cos mz}{2} = 1 - x^m$$

A look at (5), (7') and (8) shows that except for multiplicative constants

$$Y_n(x) = x^k {}_2F_1\left(\frac{k-n}{m}, \frac{k+\delta-1+n}{m}, \frac{k+m}{m}, x^m\right) \tag{10}$$

and when

$$\frac{n-k}{m} = \text{nonnegative integer} \quad ,$$

the second factor is a polynomial in x^m.

A look at (5), (7') and (9) show that except for a multiplicative factor

$$Y_n(x) = {}_2F_1\left(\frac{n+\delta-1}{m}, -\frac{n}{m}, \frac{\delta+k-1}{m}, 1-x^m\right) \tag{11}$$

and we get a simple polynomial in $1-x^m$ any time that

$$\frac{n}{m} = \text{nonnegative integer} \quad .$$

Expressions (8) and (9) are equivalent, and the choice of one or the other is a matter of convenience.

As an illustration we give the polynomials obtained using (10) and (11) in the case (treated in [4])

$$m = 4 \ , \qquad k = 1 \ , \qquad \delta = 6 \ .$$

From (10) we obtain

$$Y_1(x) = x \ , \qquad Y_5(x) = (5-11x^4)\frac{x}{5} \ , \qquad Y_9(x) = (3-18x^4+19x^8)\frac{x}{3} \ .$$

From (11) we obtain

$$Y_0(x) = 1 \ , \qquad Y_4(x) = \frac{3x^4-1}{2} \ , \qquad Y_8(x) = \frac{221x^8-182x^4+21}{60} \ .$$

<u>Canonical Form for the Differential Equation (B)</u>

Take (B) kn the form

$$x^{2-m}Y'' - (k-1+mx^{m})x^{1-m}Y' + mnY = 0 \quad . \tag{12}$$

Define a new variable z by means of

$$z = \frac{2}{m}x^{m/2}$$

to rewrite (3) as

$$\frac{d^2Y}{dz^2} + \frac{(m-2k) + \frac{m^2x^2}{2}}{mx}\frac{dY}{dz} + mnY = 0 \quad . \tag{13}$$

Now put, as before,

$$r(z) = e^{-\frac{1}{2}\int \beta(z)dz} = e^{\frac{m^2z^2}{8}} z^{\frac{k}{m}-\frac{1}{2}}$$

and equation (3) is written as

$$r(z)\left[\frac{d^2}{dz^2}\left(\frac{Y(z)}{r(z)}\right) + V(z)\frac{Y(z)}{r(z)} + mn\frac{Y(z)}{r(z)}\right] = 0 \tag{14}$$

where

$$V(z) = -\left(\frac{m^4z^2}{16} + \frac{(\frac{k}{m})^2 - \frac{1}{4}}{z^2} + \frac{km - m^2}{2}\right) \quad .$$

The Generalized Laguerre Polynomials

If we set

$$\frac{mz}{z} = u \qquad (= x^{m/2})$$

we see that the solutions of (14) are given by

$$Y(z) = r(z)y(z)$$

where y $(= y(u))$ satisfies

$$-\frac{d^2y}{du^2} + \left(u^2 + \frac{4(k/m)\ \ -1}{4u^2}\right)y = \left(4\frac{n}{m} - 2\frac{k}{m} + 2\right)y \ .$$

Now this has for solutions

$$y(u) = e^{-u^2/2} u^{\alpha+\frac{1}{2}} L^{(\alpha)}_{n/m}(u^2) \tag{15}$$

with $\alpha = -k/m$ and L^{α}_{ν} the generalized Laguerre polynomials.

To get $Y(u)$ we need to multiply by

$$r(u) = e^{u^2/2}\left(\frac{2u}{m}\right)^{\frac{k}{m}-\frac{1}{2}}$$

and we get, except for a multiplicative constant,

$$Y_n(u) = L^{-k/m}_{n/m}(u^2) \quad . \tag{16}$$

From (16) it is clear that if n is an integer multiple of m, $Y_n(u)$ is a polynomial in x^m.

Using the expression

$$L_{\nu}^{\alpha}(v) = v^{-\alpha}\ {}_1F_1(-\nu-\alpha,\ 1-\alpha,\ v) \tag{17}$$

involving the Kummer function, [6, p.243], we see that

$$Y_n(u) = u^{\frac{2k}{m}}\ {}_1F_1\left(-\frac{n}{m}+\frac{k}{m},\ 1+\frac{k}{m},\ u^2\right)$$

$$= x^k\ {}_1F_1\left(-\frac{n}{m}+\frac{k}{m},\ 1+\frac{k}{m},\ x^m\right) .$$

This last expression makes it clear that if $\frac{n-k}{m}$ is a nonnegative integer, Y_n is the product of x^k times a polynomial in x^m.

As an illustration, take

$$m = 2 , \qquad k = 1$$

and using (16) and (17), obtain

$$Y_0 = 1 , \qquad Y_1 = x , \qquad Y_2 = \tfrac{1}{2} - x^2$$

$$Y_3 = x(1 - \tfrac{2}{3} x^2) , \qquad Y_4 = \tfrac{1}{2}(\tfrac{3}{4} - 3x^2 + x^4) .$$

Notice in closing that except for multiplicative constants we have exactly the Hermite polynomials, namely,

$$H_n(x) \cong L_{n/2}^{-\frac{1}{2}}(x^2) = Y_n . \tag{18}$$

Thus, in spite of Smith's statement [5], $\{H_n(x)\}$ is obtained from a single family of generalized Laguerre polynomials, not withstanding the familiar relations which involve two such families, $L_n^{\frac{1}{2}}$ and $L_n^{-\frac{1}{2}}$.

Conclusion

A slight elaboration of the results in [5] shows that orthogonal polynomials connected with second order differential equations (1) have simple and elegant expressions in terms of Jacobi (2) or generalized Laguerre polynomials (3).

References

[1] Bochner, S. Über Sturm-Liouvillesche Polynomsysteme. Math. Zeit. 29, 730-736 (1929).

[2] Szegö, G. Orthogonal Polynomials. American Mathematical Society, Providence, Rhode Island (1978).

[3] Chihara, T. An Introduction to Orthogonal Polynomials. Gordon & Breach, New York (1978).

[4] Smith, R. Similarity solutions of a non-linear diffusion equation. IMA Jour. of Appl. Math. 28, 149-160 (1982).

[5] Smith, R. An abundance of orthogonal polynomials. IMA Jour. of Appl. Math. 28, 161-167 (1982).

[6] Magnus, W., Oberhettinger, F. and Soni, R. Formulas and Theorems for the Special Functions of Mathematical Physics. Springer Verlag, New York (1966).

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