Orthogonal Polynomials, Measures and Recurrence Relations

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ORTHOGONAL POLYNOMIALS, MEASURES AND RECURRENCE RELATIONS*

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Abstract. Properties of measures associated with orthogonal polynomials are investigated in terms of the coefficients of the three term recurrence formula satisfied by the orthogonal polynomials.

1. Introduction. Let $d\alpha$ be a positive measure on the real line with finite moments and infinite support, and let $\{p_n\}_{n=0}^{\infty}$, $p_n(x) = \gamma_n x^n + \cdots$, $\gamma_n > 0$, be the system of orthonormal polynomials associated with $d\alpha$. The polynomials p_n satisfy the recurrence formula

$$(1) \qquad xp_n = a_{n+1}p_{n+1} + b_n p_n + a_n p_{n-1}, \qquad n = 0, 1, \cdots,$$

where $p_{-1} = 0$, $p_0 = \gamma_0$, $a_0 = 0$, $a_n = \gamma_{n-1}/\gamma_n$ and

$$b_n = \int_{-\infty}^{\infty} x p_n^2(x) \, d\alpha(x).$$

By J. Favard's theorem [10, p. 60] every system of polynomials generated by (1) where $a_n > 0$ $(n = 1, 2, \cdots)$ and $b_n \in \mathbb{R}$ is in fact a system of orthonormal polynomials. The corresponding measure $d\alpha$ is uniquely determined if and only if the associated moment problem has a unique solution, and the latter holds if, say, both sequences $\{a_n\}$ and $\{b_n\}$ are bounded. Recently there has been an upsurge in research activity concerning the determination of the relationship between orthogonal polynomials, recurrence relations and measures. Several such papers are listed in the references. In particular, R. Askey and M. Ismail [1, p. 102] asked whether it is true that if

$$(2) \qquad a_n = \frac{1}{2} + \frac{c}{n} + O(n^{-2}) \quad \text{and} \quad b_n = 0$$

where $c > 0$ then the absolutely continuous portion of the corresponding measure $d\alpha$ is in Szegö's class which means that $\log \alpha'(\cos t) \in L^1$. One of the main goals of this paper is to show that the Askey–Ismail problem can essentially be solved. More precisely, it follows from Theorem 3 below that if (2) is replaced by

$$a_n = \frac{1}{2} + \frac{c}{n} + \frac{d}{n^2} + o(n^{-2}) \quad \text{and} \quad b_n = 0$$

where $c > 0$ and $d \in \mathbb{R}$ then $\log \alpha'(\cos t) \in L^1$. While we suspect that condition (2) fails to imply the integrability of $\log \alpha'(\cos t)$, we do not have evidence supporting our claim at the present time. Let us point out that Theorem 2 in fact yields $\alpha'(x) \geq \text{const}\sqrt{1 - x^2}$, $|x| < 1$, if only $a_n \downarrow 1/2$ and $b_n = 0$. The latter is quite a surprise if compared with J. Shohat's result [26, p. 50] claiming that if $\text{supp}(d\alpha) = [-1, 1]$ then $\log \alpha'(\cos t) \in L^1$ if

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and only if

$$\sum_{k=1}^{\infty} \left(a_k - \frac{1}{2} \right) < \infty, \qquad \sum_{k=1}^{\infty} b_k < \infty$$

and

$$\sum_{k=1}^{\infty} \left\{ \left(a_k - \frac{1}{2} \right)^2 + b_k^2 \right\} < \infty.$$

The natural way to connect the recursion coefficients with the measure is via Stieltjes transforms (see e.g. [1]). However, it seems that this approach is feasible only when the recursion coefficients are given in terms of explicitly defined expressions such as rational functions of n. The general cases can better be handled by techniques introduced on one side in [5]–[9] and on the other side in [15], [16], [18], [19] and [20].

If p_n is generated by (1), then we define S_n by

$$(3) \qquad S_n(x) = \sum_{k=0}^{n} \left\{ \left[a_{k+1}^2 - a_k^2 \right] p_k^2(x) + a_k [b_k - b_{k-1}] p_{k-1}(x) p_k(x) \right\}.$$

All of our results are based on the formula

$$(4) \qquad S_n(x) = a_{n+1}^2 \left[p_n^2(x) - \frac{x - b_n}{a_{n+1}} p_n(x) p_{n+1}(x) + p_{n+1}^2(x) \right]$$

proved in [8]. (Caution: the notation in [8] is somewhat different!) The second of us believes that the significance of (4) cannot be overestimated, and it will play a fundamental role in future research on general orthogonal polynomials (see e.g. [22]). The other ingredient of this paper comes from [16] where the necessary spectral analysis was accomplished.

In order not to interrupt our forthcoming discussion, we first prove the following technical proposition. In what follows a_+ denotes the positive part of a and $\log^+$ and $\log^-$ are also defined in the usual way.

LEMMA 1. *Let* $\{ a_n \}$ *and* $\{ b_n \}$ *satisfy* $a_{n+1} \geqq \frac{1}{2}(1 + |b_n|)$ *for* $n > N$ *and let* p_n *and* S_n *be defined by* (1) *and* (3) *respectively. Then*

$$(5) \qquad (1 - x^2) p_{n+1}^2(x) \leqq 4 S_n(x), \qquad |x| \leqq 1,$$

$$(6) \qquad (1 - x^2) p_n^2(x) \leqq 4 S_n(x), \qquad |x| \leqq 1,$$

$$(7) \qquad \max_{|x| \leqq 1} p_{n+1}^2(x) \leqq 4(n+2)^2 \max_{|x| \leqq 1} |S_n(x)|,$$

$$(8) \qquad \max_{|x| \leqq 1} p_n^2(x) \leqq 4(n+1)^2 \max_{|x| \leqq 1} |S_n(x)|,$$

$$(9) \quad 0 \leqq S_{n+1}(x) \leqq S_n(x) \exp \left\{ 4 \frac{\left[a_{n+2}^2 - a_{n+1}^2 \right]_+ + a_{n+1} |b_{n+1} - b_n|}{1 - x^2} \right\}, \qquad |x| \leqq 1,$$

and

$$(10) \quad \max_{|x| \leqq 1} S_{n+1}(x) \max_{|x| \leqq 1} S_n(x) \cdot \exp \left\{ 4(n+2)^2 \left(\left[a_{n+2}^2 - a_{n+1}^2 \right]_+ - a_{n+1} |b_{n+1} - b_n| \right) \right\}$$

hold for $n > N$.

Proof. By (4)

$$S_n(x) = a_{n+1}^2 \left[p_n(x) - \frac{x - b_n}{2 a_{n+1}} p_{n+1}(x) \right]^2 + \frac{1}{4} \left[4 a_{n+1}^2 - (x - b_n)^2 \right] p_{n+1}^2(x)$$

and

$$S_n(x) = a_{n+1}^2 \left[p_{n+1}(x) - \frac{x-b_n}{2a_{n+1}} p_n(x) \right]^2 + \frac{1}{4} \left[4a_{n+1}^2 - (x-b_n)^2 \right] p_n^2(x).$$

If $2a_{n+1} \geqq 1 + |b_n|$ then $4a_{n+1}^2 - (x-b_n)^2 \geqq 1 - x^2$ for $|x| \leqq 1$. Thus (5) and (6) are satisfied. Inequalities (7) and (8) follow from (5), (6) and Bernstein's theorem [17, p. 139]. Writing

$$S_{n+1} = S_n \left[a_{n+2}^2 - a_{n+1}^2 \right] p_{n+1}^2 + a_{n+1} [b_{n+1} - b_n] p_n p_{n+1}$$

and applying (5)–(8), inequalities (9) and (10) follow immediately.

THEOREM 1. *If* $\lim_{n \to \infty} = \frac{1}{2}$, $\lim_{n \to \infty} b_n = 0$ *and*

(11)
$$\sum_{n=0}^{\infty} \left\{ |a_{n+1} - a_n| + |b_{n+1} - b_n| \right\} < \infty$$

then the orthogonal polynomials p_n *generated by* (1) *and the corresponding measure* $d\alpha$ *satisfy*

(12)

$$\sum_{k=0}^{\infty} \left\{ \left[a_{k+1}^2 - a_k^2 \right] p_k^2(x) + a_k [b_k - b_{k-1}] p_{k-1}(x) p_k(x) \right\} = \frac{\sqrt{1-x^2}}{2\pi \alpha'(x)}, \qquad -1 < x < 1,$$

and the convergence is uniform on every closed subinterval of $(-1, 1)$.

Proof. Theorem 1 follows immediately from (3), (4) and

$$\lim_{n \to \infty} \left[p_n^2(x) - \frac{x-b_n}{a_{n+1}} p_n(x) p_{n+1}(x) + p_{n+1}^2(x) \right] = \frac{2\sqrt{1-x^2}}{\pi \alpha'(x)}$$

which holds uniformly on every closed subinterval of $(-1, 1)$ if (11) is satisfied [16].

THEOREM 2. *Let* $\{a_n\}_{n=1}^{\infty}$ *and* $\{b_n\}_{n=0}^{\infty}$ *satisfy* $a_{n+1} \geqq \frac{1}{2}(1 + |b_n|)$ *for* $n > N$, $\lim_{n \to \infty} a_n = \frac{1}{2}$, $\lim_{n \to \infty} b_n = 0$ *and*

$$\sum_{n=1}^{\infty} n^2 \left\{ [a_{n+1} - a_n]_+ + |b_{n+1} - b_n| \right\} < \infty.$$

Then there exist a constant $K > 0$ *such that for the orthogonal polynomials* p_n *defined by* (1) *and for the associated measure* $d\alpha$ *we have*

(13)
$$\sqrt{1-x^2} \, |p_n(x)| \leqq K, \qquad -1 \leqq x \leqq 1,$$

$n = 1, 2, \cdots$, *and*

(14)
$$\alpha'(x) \geqq K^{-1}\sqrt{1-x^2}, \qquad -1 \leqq x \leqq 1.$$

Proof. Repeated application of (10) shows that the sequence $\{S_n\}$ is uniformly bounded in $[-1, 1]$ and then (13) follows from (6) whereas (14) follows from Theorem 1, (3) and (4).

Remark 1. The sharpness of Theorem 2 may best be illustrated by the ultraspherical polynomials which are orthogonal with respect to $d\alpha(x) = (1 - x^2)^\varepsilon dx$ in $[-1, 1]$. For these polynomials

$$a_n = \frac{1}{2} + \frac{1 - 4\varepsilon^2}{16} \frac{1}{n^2} + \frac{\text{const}}{n^3} + O(n^{-4}) \quad \text{and} \quad b_n = 0$$

so that the conditions of Theorem 2 are satisfied if and only if $|\varepsilon| \leqq \frac{1}{2}$ whereas (14) holds if and only if $\varepsilon \leqq \frac{1}{2}$.

THEOREM 3. *Let* $\{a_n\}_{n=1}^{\infty}$ *and* $\{b_n\}_{n=0}^{\infty}$ *satisfy* $a_{n+1} \geqq \frac{1}{2}(1+|b_n|)$ *for* $n > N$, $\lim_{n \to \infty} a_n = \frac{1}{2}$, $\lim_{n \to \infty} b_n = 0$ *and*

$$\sum_{k=1}^{\infty} n\left\{ [a_{n+1} - a_n]_+ + |b_{n+1} - b_n| \right\} < \infty.$$

Let $d\alpha$ *be the measure associated with the orthogonal polynomials defined by* (1). *Then*

(15) $$d\alpha(x) = w(x)\,dx + \text{mass points outside } (-1,1)$$

where w is positive and continuous in $(-1,1)$, *w vanishes outside* $[-1,1]$ *and w belongs to Szegö's class, that is*

(16) $$\int_0^{\pi} |\log w(\cos t)|\,dt < \infty.$$

Remark 2. If $\Sigma n\{|a_n - \frac{1}{2}| + |b_n|\} < \infty$ then the number of mass points in $d\alpha$ is finite and all such mass points are located outside $[-1,1]$ (see [4] and [11]).

Proof of Theorem 3. By the conditions $\Sigma(|a_{n+1} - a_n| + |b_{n+1} - b_n|) < \infty$ holds as well so that by the Theorem in [16] and by Blumenthal's result (see e.g. [18, Thm. 3.3.7, p. 23]) formula (15) holds with $w(>0) \in C(-1,1)$ and $\text{supp}(w) = [-1,1]$. Therefore only (16) needs to be proved. Let δ_n be defined by

$$\delta_n = 4\left([a_{n+2}^2 - a_{n+1}^2]_+ + a_{n+1}|b_{n+1} - b_n| \right).$$

Then by the assumptions made

(17) $$\sum_1^{\infty} n\delta_n < \infty,$$

and applying (9) and (10) with $n > N$, we obtain

$$\int_0^{1-(n+1)^2} \frac{\log^+ S_{n+1}(x)}{\sqrt{1-x}}\,dx$$

$$\leqq \int_0^{1-n^{-2}} \frac{\log^+ S_n(x)}{\sqrt{1-x}}\,dx + \delta_n \int_1^{1-n^{-2}} \frac{dx}{(1-x)^{3/2}}$$

$$+ \log^+\left\{ \max_{|x| \leq 1} |S_{n+1}(x)| \right\} \int_{1-n^{-2}}^{1-(n+1)^{-2}} \frac{dx}{\sqrt{1-x}}$$

$$= \int_0^{1-n^{-2}} \frac{\log^+ S_n(s)}{\sqrt{1-x}}\,dx$$

$$+ 2(n-1)\delta_n + 2\left(\frac{1}{n}\right)\frac{1}{n+1}\log^+\left\{ \max_{|x| \leq 1} S_{n+1}(x) \right\}$$

$$\leqq \int_0^{1-n^{-2}} \frac{\log^+ S_n(x)}{\sqrt{1-x}}\,dx + 2(n-1)\delta_n + \frac{2}{n}\log^+\left\{ \max_{|x| \leq 1} S_n(x) \right\}$$

$$- \frac{2}{n+1}\log^+\left\{ \max_{|x| \leq 1} S_{n+1}(x) \right\} + \frac{2(n+2)^2}{n}\delta_n.$$

Therefore

$$\int_0^{1-(n+1)^{-2}} \frac{\log^+ S_{n+1}(x)}{\sqrt{1-x}}\, dx \le \int_0^{1-n^{-2}} \frac{\log^+ S_n(x)}{\sqrt{1-x}}\, dx + 18n\delta_n$$

$$+ \frac{2}{n} \log^+\left\{ \max_{|x|\le 1} S_n(x) \right\} - \frac{2}{n+1} \log^+\left\{ \max_{|x|\le 1} S_{n+1}(x) \right\}$$

from which

$$\int_0^{1-(n+1)^2} \frac{\log^+ S_{n+1}(x)}{\sqrt{1-x}}\, dx \le \int_0^{1-(N+1)^2} \frac{\log^+ S_{n+1}(x)}{\sqrt{1-x}}\, dx + 18 \sum_{k=N+1}^{n} k\delta_k$$

$$+ \frac{2}{N+1} \log^+\left\{ \max_{|x|\le 1} S_{N+1}(x) \right\}$$

follows. Now letting $n \to \infty$ and applying (3), (17), Theorem 1 and Fatou's lemma, we see that

$$\int_0^{\pi/2} \log^- w(\cos t)\, dt > -\infty.$$

By similar arguments

$$\int_{\pi/2}^{\pi} \log^- w(\cos t)\, dt > -\infty$$

holds as well. By Jensen's inequality

$$\int_0^{\pi} \log w(\cos t)\, dt < \infty$$

and thus Theorem 3 has completely been proved.

2. Applications.

A. Ya. L. Geronimus [12]–[14] raised and solved the following problem. Let $m \ge 0$ be a fixed integer and let $\{b_k\}_{k=0}^{\infty}$ and $\{a_k\}_{k=1}^{\infty}$ $(a_k > 0)$ be given sequences such that $b_k = 0$ for $k \ge m$ and $a_k = \frac{1}{2}$ for $k > m$. Let $\{p_n\}_{n=0}^{\infty}$ be the orthogonal polynomial system generated by (1) and let $d\alpha$ be the corresponding measure. The problem is to find $d\alpha$. We will show that on the basis of our results $d\alpha$ can easily be found. It follows from (3) and (4) that

$$(18) \qquad\qquad 4S_n = p_n^2 - 2xp_n p_{n+1} + p_{n+1}^2, \qquad n \ge m,$$

and

$$(19) \qquad\qquad\qquad S_n = S_m, \qquad n \ge m.$$

Thus by Theorem 1 and 3 and by Remark 2,

$$(20) \quad d\alpha(x) = \frac{2}{\pi} \frac{\sqrt{1-x^2}}{p_m^2(x) - 2xp_m(x)p_{m+1}(x) + p_{m+1}^2(x)} \chi(x) + \sum_{l=1}^{M} J_l \delta(x - x_l)$$

where χ is the characteristic function of $[-1,1]$ and x_l's are the mass points with mass $J_l > 0$. It is well known [27] that for an arbitary system of orthogonal polynomials if the associated moment problem has a unique solution then x is a mass point for $d\alpha$ with

mass J if and only if $\sum p_k^2(x) < \infty$ and then

(21)

$$\frac{1}{J} = \sum_{k=0}^{\infty} p_k^2(x) = \lim_{k \to \infty} a_{k+1} \left[p_{k+1}'(x) p_k(x) - p_{k+1}(x) p_k'(x) \right]$$

$$= \lim_{k \to \infty} a_{k+1} \left[p_{k+1}(x) / p_k(x) \right]' p_k^2(x) = - \lim_{k \to \infty} a_{k+1} \left[p_k(x) / p_{k+1}(x) \right]' p_{k+1}^2(x).$$

Therefore by (18) and (19) the mass points x_l are zeros of S_m and hence $M \leq 2m+2$. However not all zeros of S_m are in fact mass points. Applying the recurrence formula (1) and (3)–(4) we obtain

(22) $$4 S_m = p_{n+1}^2 - p_n p_{n+2}, \qquad n \geq m$$

so that

(23) $$\frac{p_{n+1}}{p_n} = \frac{p_{m+1}}{p_m}, \quad n \geq m, \quad \text{if } S_m = 0.$$

On the other hand, by (18)

(24) $$\frac{p_{m+1}}{p_m} = x \pm \sqrt{x^2 - 1} \quad \text{if } S_m = 0$$

(here $\sqrt{x^2 - 1} > 0$ if $x > 1$ and $\sqrt{x^2 - 1} < 0$ if $x < -1$). Thus by (21) x is a mass point if and only if $S_m(x) = 0$ and $|p_{m+1}(x)| < |p_m(x)|$. By (22)

$$4 S_m' = - (p_n / p_{n+1})' p_{n+1}^2 \frac{p_{n+2}}{p_{n+1}} - (p_{n+2} / p_{n+1})' p_{n+1}^2 \frac{p_n}{p_{n+1}}, \qquad n \geq m,$$

if $S_m = 0$ so that by (21), (23) and (24)

$$2 S_m'(x_l) = J_l^{-1} \left(x_l - \sqrt{x_l^2 - 1} \right) - J_l^{-1} \left(x_l + \sqrt{x_l^2 - 1} \right) = - 2 J_l^{-1} \sqrt{x_l^2 - 1}.$$

Thus the mass points x_l in (20) are those zeros of $p_m^2 - 2 x p_m p_{m+1} + p_{m+1}^2$ for which $|p_{m+1}(x_l)| < |p_m(x_l)|$ and the corresponding mass J_l is given by

$$J_l = \frac{- \sqrt{x_l^2 - 1}}{S_m'(x_l)} = \left| \frac{\sqrt{x_l^2 - 1}}{S_m'(x_l)} \right|.$$

B. The previous analysis can be applied to the case when

$$\lim_{n \to \infty} \left| a_n - \frac{1}{2} \right|^{1/n} = 0 \quad \text{and} \quad \lim_{n \to \infty} |b_n|^{1/n} = 0.$$

Without going into details we point out that in this case one can prove

$$\limsup_{n \to \infty} | p_n(x) |^{1/n} < \infty$$

uniformly on every compact set in the complex plane, and thus

$$S(x) = \lim_{n \to \infty} S_n(x)$$

is an entire function. The corresponding measure $d\alpha$ can be written as

$$d\alpha(x) = \frac{1}{2\pi} \frac{\sqrt{1-x^2}}{S(x)} \chi(x)\,dx + \sum_{x_l \in Z} \left| \frac{\sqrt{x_l^2-1}}{S'(x_l)} \right| \delta(x-x_l)$$

where χ is the characteristic function of $[-1,1]$ and Z is the collection of those zeros x_l of S for which $\lim_{n \to \infty} |p_{n+1}(x_l)/p_n(x_l)| < 1$. Of course, Z is a finite set.

C. The Pollaczek polynomials satisfy (1) with

$$a_n^2 = \frac{1}{4} \frac{(n+c)(n+2\lambda+c-1)}{(n+\lambda+a+c)(n+\lambda+a+c-1)}, \qquad n = 1, 2, \cdots,$$

and

$$b_n = - \frac{b}{n+\lambda+a+c}, \qquad n = 0, 1, 2, \cdots,$$

where the parameters a, b, c and λ are chosen so that $b_n \in \mathbb{R}$ and $a_n > 0$. Pollaczek [23] (see also [3]) investigated the case when either $a > |b|$, $2\lambda+c > 0$, $c \geq 0$ or $a > |b|$, $2\lambda+c \geq 1$, $c > -1$ and determined that $d\alpha$ is absolutely continuous. Hence $d\alpha$ is completely described by (16). The explicit expression for α' [3, p. 185] shows that α' is not Szegő's class in this case. Since

$$a_n = \frac{1}{2} - \frac{a}{2n} + \frac{\text{const}}{n^2} + O(n^{-3})$$

holds, we see that the conditions of Theorem 2 are satisfied provided that $a < 0$, $b = 0$, and consequently

$$\alpha'(x) \geq K^{-1}\sqrt{1-x^2}, \qquad -1 \leq x \leq 1,$$

with a suitably chosen positive constant K if $a < 0$ and $b = 0$. In particular, $\log \alpha'(\cos t) \in L^1$ in this case. Examples of Pollaczek polynomials with not necessarily absolutely continuous measures have been investigated in [1], [2], [24], and [29].

D. Let $\{a_n\}_{n=1}^{\infty}$ satisfy $0 < a_n \leq \frac{1}{2}$, $\lim_{n \to \infty} a_n = \frac{1}{2}$ and

$$\sum_{k=1}^{\infty} |a_{k+1} - a_k| < \infty,$$

and let $b_n = 0$ for every n. Let $d\alpha$ be the measure associated with the orthogonal polynomials p_n which are defined by (1). It is well known that in this case $\text{supp}(d\alpha) = [-1]$ (see e.g. [18, Thm. 3.3.7, p. 23]). Let us show that ± 1 are not mass points of $d\alpha$. If x is a mass point, then by (21) $\lim_{n \to \infty} |p_n(x)| = 0$ so that there exists n_0 such that $|p_n(x)| \leq p_{n_0}(x)$ for every n and $|p_n(x)| < |p_{n_0}(x)|$ for $n < n_0$. By the recurrence formula

$$\left| x p_{n_0}(x) \right| \leq a_{n_0+1} \left| p_{n_0+1}(x) \right| + a_{n_0} \left| p_{n_0-1}(x) \right| < \left| p_{n_0}(x) \right|$$

and hence $|x| < 1$, that is $x \neq \pm 1$. It has been shown in both [8] and [16] that $d\alpha$ is absolutely continuous in $(-1,1)$. Therefore $d\alpha$ is absolutely continuous on the whole real line and by Theorem 1

$$d\alpha(x) = \frac{1}{2\pi} \frac{\sqrt{1-x^2}}{\sum_{k=0}^{\infty} [a_{k+1}^2 - a_k^2] p_k^2(x)} \chi(x)\,dx$$

where χ is the characteristic function of $[-1,1]$. By the previously quoted theorem of Shohat [26, p. 50] $\log \alpha'(\cos t) \in L^1$ if and only if $\sum(a_k - \frac{1}{2}) < \infty$.

E. If there exists N such that $a_n > a_{n+1}$ for $n > N$ and $\lim_{n \to \infty} a_n = \frac{1}{2}$ and if $b_n = 0$ for every n, then the conditions of Theorem 2 are satisfied. Hence $\alpha'(x) \geq K^{-1}\sqrt{1-x^2}$, $|x| \leq 1$, holds in this case.

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