ORTHOGONAL POLYNOMIALS ON SEVERAL INTERVALS VIA A POLYNOMIAL MAPPING

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ABSTRACT. Starting from a sequence $\{p_n(x; \mu_0)\}$ of orthogonal polynomials with an orthogonality measure μ_0 supported on $E_0 \subset [-1, 1]$, we construct a new sequence $\{p_n(x; \mu)\}$ of orthogonal polynomials on $E = T^{-1}(E_0)$ (T is a polynomial of degree N) with an orthogonality measure μ that is related to μ_0. If $E_0 = [-1, 1]$, then $E = T^{-1}([-1, 1])$ will in general consist of N intervals. We give explicit formulas relating $\{p_n(x; \mu)\}$ and $\{p_n(x; \mu_0)\}$ and show how the recurrence coefficients in the three-term recurrence formulas for these orthogonal polynomials are related. If one chooses T to be a Chebyshev polynomial of the first kind, then one gets sieved orthogonal polynomials.

I. Introduction. Suppose a sequence of orthogonal polynomials is given with an orthogonality measure supported on a set $E_0 \subset [-1, 1]$. In this paper we will analyze how properties of these orthogonal polynomials, their measure, and the set E_0 are transformed under a polynomial transformation. In particular we will restrict our class of transformations so that the inverse of E_0 under the transformation will be a real set. Although special cases of this problem had been considered by Barrucand and Dickinson [5], the general problem was posed and considered in an important paper by Bessis and Moussa [8]. In their paper many of the algebraic properties of the transformed polynomials and their orthogonality measure were discussed. Here we extend their work and apply it to a number of interesting special cases.

If E_0 is an interval then the inverse image of E_0 (which is assumed to be real) will in general be a finite number of disjoint intervals, and we are led to the problem of polynomials orthogonal with respect to a measure supported on several intervals. Besides being of interest in its own right (Aptekarev [3], Geronimo and Van Assche [12], Geronimus [13-14]), this problem is of interest in numerical analysis when one tries to solve large indefinite linear systems using Richardson iteration (de Boor and Rice [11], Lebedev [18], Saad [23]). Applications of these types of polynomials have also appeared in quantum chemistry (Wheeler [27]) and physics (Pettifor and Weaire [22]). Orthogonal polynomials on several intervals have also appeared in the study of sieved orthogonal polynomials (Al-Salam, Allaway, and Askey [2], Charris and Ismail [9], Ismail [15-17]). Many results involving sieved orthogonal polynomials follow by taking a particular polynomial transformation.

In §II we give an outline of the general procedure of mapping E_0 to $E = T^{-1}(E_0)$, where T is a polynomial of degree N, and describe how a measure μ_0 on E_0 is transformed to a measure μ on E. We also describe how the associated polynomials and their orthogonality measure transform. In §III we give a relationship between the Jacobi matrices for the orthogonal polynomials on E_0 and on E. §IV lists some properties of the new orthogonal polynomial system. In particular, we give explicit formulas for some of the recurrence coefficients in the new system in terms of those of the old system. We also discuss convergence rates of the coefficients in the recurrence formula and growth rates of the polynomials on the spectrum. In §V we describe some of the potential-theoretic properties of the set E and discuss some of the properties of the polynomials associated with the equilibrium measure. In this section we also discuss the Bernstein-Szegő polynomials (Akhiezer [1]). Finally, in §VI we discuss sieved orthogonal polynomials.

II. The general procedure. We start off with a probability measure μ_0 on a set $E_0 \subset [-1, 1]$. If μ_0 has an infinite support, then there exists a unique sequence of orthogonal polynomials $\{p_n(x; \mu_0): n = 0, 1, 2, \dots\}$ for which

$$\int_{E_0} p_n(x; \mu_0) p_m(x; \mu_0) \, d\mu_0(x) = \delta_{m,n}, \qquad m, n \geq 0.$$

We map the interval $[-1, 1]$ to several intervals by a polynomial transformation.

LEMMA 1. *Let T be a polynomial of degree $N \geq 2$ with real and simple zeros $x_1 < x_2 < \cdots < x_N$. Let $y_1 < y_2 < \cdots < y_{N-1}$ be the zeros of T'. If $|T(y_j)| \geq 1$ $(j = 1, 2, \dots, N-1)$, then there exists a unique sequence of closed intervals $\{E_1, E_2, \dots, E_N\}$ such that*
  (a) $T(E_i) = [-1, 1]$,
  (b) $x_i \in E_i$,
  (c) $E_i \cap E_{i+1}$ *contains at most one point* $(i = 1, \dots, N-1)$.

PROOF. The statement is obvious and can most easily be verified from Figure 1. The set $E_i \cap E_{i+1}$ will contain one point if and only if $T(y_i) = \pm 1$. □

In the remainder of this paper we suppose that T satisfies the conditions given in Lemma 1. Following Bessis-Moussa [8], Moussa [20], and Bessis-Geronimo-Moussa [7] we now construct a measure μ on $T^{-1}(E_0)$ starting from a given measure μ_0 on E_0. Let W be a polynomial of degree $N - 1$ such that

$$(2.1) \qquad \frac{W(z)}{T(z) - x} = \sum_{i=1}^{N} \frac{w_i(x)}{z - T_i^{-1}(x)}, \qquad x \in E_0, \ z \in \mathbf{C} \backslash E,$$

$$(2.2) \qquad w_i(x) > 0, \qquad x \in E_0, \ i = 1, 2, \dots, N,$$

$$(2.3) \qquad \sum_{i=1}^{N} w_i(x) = 1, \qquad x \in E_0,$$

where $\{T_i^{-1}: i = 1, 2, \dots, N\}$ is a complete set of inverse branches of T with $T_i^{-1}([-1, 1]) = E_i$. There is always at least one possible choice, namely $W = T'/N$, in which case $w_i(x) = 1/N$ $(i = 1, 2, \dots, N)$. From (2.1) we find

$$(2.4) \qquad w_i(x) = \frac{W(T_i^{-1}(x))}{T'(T_i^{-1}(x))}, \qquad i = 1, 2, \dots, N,$$
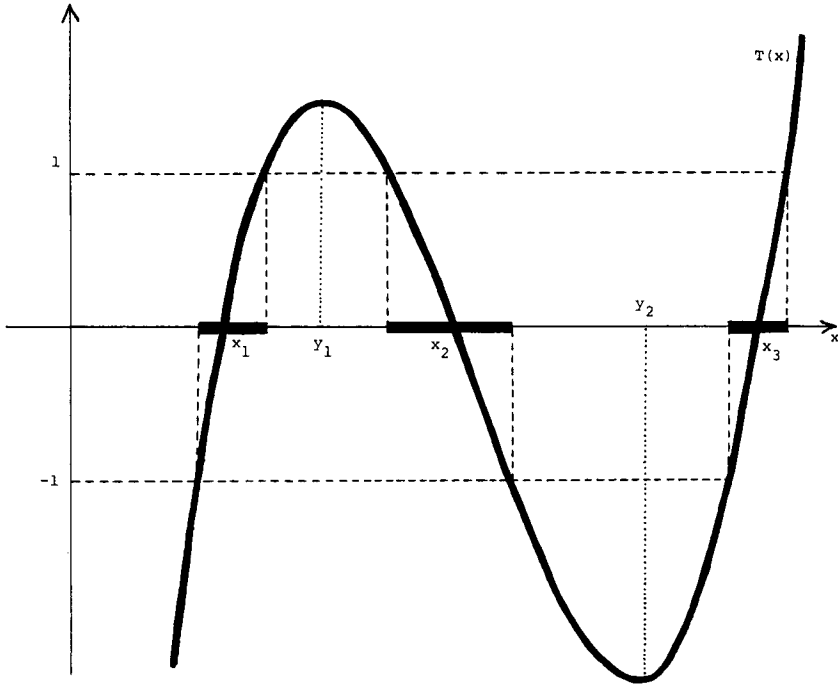
FIGURE 1. The set $E = T^{-1}([-1,1])$ are the intervals indicated by bold lines

which means that (2.2) is satisfied if and only if W has the same sign as T' on every E_i $(i = 1, 2, \ldots, N)$. In particular, this implies that the zeros $z_1 < z_2 < \cdots < z_{N-1}$ of W are such that z_i is between the intervals E_i and E_{i+1}, or possibly at the endpoints of these intervals. We construct a measure μ on $E = T^{-1}(E_0)$ by imposing that for every Borel set A in E_0

$$\mu(T_i^{-1}(A)) = \int_A w_i(x) \, d\mu_0(x).$$

From this it follows that for every $f \in L_1(\mu)$

$$(2.5) \qquad \int_E f(x) \, d\mu = \sum_{i=1}^N \int_{E_0} w_i(x) f(T_i^{-1}(x)) \, d\mu_0(x).$$

REMARK 1. From (2.2) and (2.5) we see that the absolutely continuous, singular continuous, and discrete parts of μ_0 are mapped into, respectively, the absolutely continuous, singular continuous, and discrete parts of μ. Consequently, if σ_0 and σ are, respectively, the absolutely continuous parts of μ_0 and μ then

$$(2.6) \qquad \sigma(x) = |W(x)|\sigma_0(T(x)).$$

REMARK 2. The Stieltjes transform of a measure μ is defined to be

$$S(z; \mu) = \int \frac{d\mu(x)}{z - x}, \qquad z \notin \mathrm{supp}(\mu).$$

Setting $f(x) = 1/(z-x)$ $(z \notin E)$ in (2.5) gives the following relation among Stieltjes transforms:

$$(2.7) \qquad\qquad S(z; \mu) = W(z)S(T(z), \mu_0).$$

Let $\{p_n(x; \mu): n = 0, 1, 2, \dots\}$ be the orthonormal polynomials with orthogonality measure μ on E, i.e.,

$$\int_E p_n(x; \mu)p_m(x; \mu)\, d\mu(x) = \delta_{n,m}, \qquad n, m \geq 0.$$

Then there is an easy relation between $p_{nN}(x; \mu)$ and $p_n(x; \mu_0)$.

THEOREM 1 [7, 8]. *Suppose μ_0 is a measure on $E_0 \subset [-1, 1]$ and μ is the measure obtained from μ_0 by* (2.5). *Then*

$$(2.8) \qquad\qquad p_{nN}(x; \mu) = p_n(T(x); \mu_0), \qquad n = 0, 1, 2, \dots.$$

PROOF. We will show that

$$(2.9) \qquad\qquad \int_E x^m p_n(T(x); \mu)\, d\mu(x) = 0, \qquad m < nN,$$

and

$$(2.10) \qquad\qquad \int_E p_n^2(T(x); \mu_0)\, d\mu(x) = 1.$$

To prove (2.9) we note that from (2.5)

$$\int_E x^m p_n(T(x); \mu_0)\, d\mu(x)$$

$$= \sum_{i=1}^N \int_{E_0} w_i(x)(T_i^{-1}(x))^m p_n(x; \mu_0)\, d\mu_0(x).$$

If we expand (2.1) into powers of $1/z$, then the left-hand side of (2.1) becomes

$$\sum_{n=0}^\infty \frac{1}{z^{n+1}} \sum_{i=1}^N w_i(x)(T_i^{-1}(x))^n$$

and the right-hand side is

$$W(z) \sum_{n=0}^\infty \frac{x^n}{(T(z))^{n+1}}.$$

Since

$$\frac{W(z)}{(T(z))^{n+1}} = O\left(\frac{1}{z^{nN+1}}\right),$$

we have

$$\frac{W(z)}{T(z) - x} = \sum_{n=0}^\infty \frac{1}{z^{n+1}} \Pi_{[n/N]}(x),$$

where Π_m is a polynomial of degree at most m and $[n/N]$ is the integer part of n/N. Hence by comparing the coefficients of z^{-m-1}

$$\int_E x^m p_n(T(x); \mu_0)\, d\mu(x) = \int_{E_0} \Pi_{[m/N]}(x)p_n(x; \mu_0)\, d\mu_0(x)$$

and the latter integral is zero for $m < nN$. For (2.10) we have by (2.5)

$$\int_E p_n^2(T(x); \mu_0)\, d\mu = \sum_{i=1}^{N} \int_{E_0} w_i(x) p_n^2(x; \mu_0)\, d\mu_0(x)$$

and this is equal to one by (2.2) and the orthonormality of the $\{p_n(x; \mu_0)\}$. $\square$

A sequence $\{p_n(x): n = 0, 1, 2, \dots\}$ of orthogonal polynomials on the real line satisfies a three-term recurrence formula

$$(2.11) \qquad x p_n(x) = a_{n+1} p_{n+1}(x) + b_n p_n(x) + a_n p_{n-1}(x), \qquad n = 0, 1, \dots,$$

with $p_0(x) = 1$, $p_{-1}(x) = 0$, $a_n > 0$, and $b_n \in \mathbf{R}$. The kth associated polynomials $\{p_n^{(k)}(x), n = 0, 1, 2, \dots\}$ $(k \in \mathbf{N})$ satisfy the "shifted recurrence relation"

$$x p_n^{(k)}(x) = a_{n+k+1} p_{n+1}^{(k)}(x) + b_{n+k} p_n^{(k)}(x) + a_{n+k} p_{n-1}^{(k)}(x), \qquad n = 0, 1, 2, \dots,$$

with $p_0^{(k)} = 1$ and $p_{-1}^{(k)}(x) = 0$. Let $\{a_{n+1}^0, b_n^0 : n = 0, 1, 2, \dots\}$ be the recurrence coefficients for $\{p_n(x; \mu_0)\}$ and $\{a_{n+1}, b_n : n = 0, 1, 2, \dots\}$ be those for $\{p_n(x; \mu)\}$. Then we have

THEOREM 2. *With the same notation as in Theorem 1, we have*

$$(2.12) \qquad p_{nN-1}^{(1)}(x; \mu) = \frac{a_1}{a_1^0} W(x) p_{n-1}^{(1)}(T(x); \mu_0), \qquad n = 0, 1, 2, \dots.$$

PROOF. It is well known (and one can easily check) that

$$p_{n-1}^{(1)}(x; \mu) = a_1 \int_E \frac{p_n(x; \mu) - p_n(y; \mu)}{x - y}\, d\mu(y).$$

By (2.5) and (2.8) we have

$$p_{nN-1}^{(1)}(x; \mu) = a_1 \int_E \frac{p_n(T(x); \mu_0) - p_n(T(y); \mu_0)}{x - y}\, d\mu(y)$$

$$= a_1 \sum_{i=1}^{N} \int_{E_0} w_i(y) \frac{p_n(T(x); \mu_0) - p_n(y; \mu_0)}{x - T_i^{-1}(y)}\, d\mu_0(y).$$

Using (2.1) gives

$$p_{nN-1}^{(1)}(x; \mu) = a_1 W(x) \int_E \frac{p_n(T(x); \mu_0) - p_n(y; \mu_0)}{T(x) - y}\, d\mu_0(y)$$

from which the desired result follows. $\square$

Let $\mu^{(1)}$ and $\mu_0^{(1)}$ be the orthogonality measures associated with $\{p_n^{(1)}(x; \mu)\}$ and $\{p_n^{(1)}(x; \mu_0)\}$ respectively. The next theorem describes how the Stieltjes transforms associated with $\mu^{(1)}$ and $\mu_0^{(1)}$ are related and gives a relation analogous to (2.5) among these two measures. A result similar to (2.13) when $W(z) = T'(z)/N$ was given by Barnsley, Geronimo, and Harrington [**4**].

THEOREM 3. *Let $S(z, \mu^{(1)})$ and $S(z, \mu_0^{(1)})$ be the Stieltjes transforms associated with $\mu^{(1)}$ and $\mu_0^{(1)}$ respectively. Then*

$$(2.13) \qquad S(z; \mu^{(1)}) = \left(\frac{a_1^0}{a_1}\right)^2 \frac{S(T(z); \mu_0^{(1)})}{W(z)} + \frac{\hat{p}_{N-2}^{(2)}(z; \mu)}{\hat{p}_{N-1}^{(1)}(z; \mu)},$$

where $\hat{p}_{N-2}^{(2)}$ and $\hat{p}_{N-1}^{(1)}$ are monic orthogonal polynomials. Let $\{z_i,\ i = 1, 2, \ldots, N - 1\}$ be the zeros of $W(z)$ and suppose that $S(T(z_i), \mu_0^{(1)})$ exists. Then (2.13) implies that for every bounded continuous function f,
(2.14)

$$\int f(x)\, d\mu^{(1)}(x) = \left(\frac{a_1^0}{a_1}\right)^2 \sum_{i=1}^{N} \int \frac{w_i(x)}{|W(T_i^{-1}(x))|^2} f(T_i^{-1}(x))\, d\mu_0^{(1)}(x)$$

$$+ \sum_{i=1}^{N-1} \left\{ \left(\frac{a_1^0}{a_1}\right)^2 S(T(z_i); \mu_0^{(1)}) + c\hat{p}_{N-2}^{(2)}(z_i; \mu) \right\} \frac{f(z_i)}{W'(z_i)},$$

where c is the leading coefficient of T and W (see (3.6)).

PROOF. A well-known relation between $S(z; \mu)$ and $S(z; \mu^{(1)})$ is given by

(2.15)
$$S(z; \mu) = \frac{1}{z - b_0 - a_1^2 S(z; \mu^{(1)})}.$$

If we use this relation for μ_0 and $\mu_0^{(1)}$ with z replaced by $T(z)$, we find

(2.16)
$$S(T(z); \mu_0) = \frac{1}{T(z) - b_0^0 - (a_1^0)^2 S(T(z), \mu_0^{(1)})}.$$

Multiply both sides of this equation by $W(z)$. Then by (2.7)

$$S(z; \mu) = \frac{W(z)}{T(z) - b_0^0 - (a_1^0)^2 S(T(z), \mu_0^{(1)})}.$$

Substituting this result into (2.16) and then solving the resulting equation for $S(z, \mu^{(1)})$ yields

$$S(z, \mu^{(1)}) = \left(\frac{a_1^0}{a_1}\right)^2 \frac{S(T(z); \mu_0^{(1)})}{W(z)} + \frac{(z - b_0)W(z) - T(z) + b_0^0}{a_1^2 W(z)}.$$

From (2.12) with $n = 1$ and the definition of $W(z)$ we have that $\hat{p}_{N-1}^{(1)}(z) = W(z)/c$. Furthermore (2.8) with $n = 1$ tells us that

$$\hat{p}_N(z) = \frac{T(z) - b_0^0}{c}.$$

The result now follows from the fact that $a_1^2 \hat{p}_{N-2}^{(2)}(z) = (z - b_0)\hat{p}_{N-1}^{(1)}(z) - \hat{p}_N(z)$ (see also (3.10)). To prove (2.14) it is sufficient to take $f(x) = 1/(z - x)$. Utilizing the fact that

$$\frac{1}{W(z)} \frac{1}{T(z) - x} = \sum_{i=1}^{N} \frac{1}{W(T_i^{-1}(x))} \frac{1}{T'(T_i^{-1}(x))} \frac{1}{z - T_i^{-1}(x)}$$

$$+ \sum_{i=1}^{N-1} \frac{1}{W'(z_i)} \frac{1}{T(z_i) - x} \frac{1}{z - z_i}$$

and (2.4), in (2.13) gives (2.14) for $f(x) = 1/(z - x)$. $\square$

REMARK 3. We note that if one wishes to start with a measure $\mu_0^{(1)}$ one must check that $\operatorname{supp} \mu_0$ is contained in $[-1, 1]$. (2.13) shows that mass points at the

zeros of $W(z)$ may appear in the measure $\mu^{(1)}$. Letting $\sigma^{(1)}$ be the absolutely continuous part of $\mu^{(1)}$ and $\sigma_0^{(1)}$ be the absolutely continuous part of $\mu_0^{(1)}$, we see that

$$(2.17) \qquad \sigma^{(1)}(x) = \left(\frac{a_1^0}{a_1}\right)^2 \frac{1}{|W(x)|} \sigma_0^{(1)}(T(x)), \qquad x \in T^{-1}(-1,1).$$

From (2.8) we see that there is a simple relation between the zeros of $p_{nN}(x;\mu)$ and $p_n(x;\mu_0)$. Let $\{x_{i,n}^0, i = 1, 2, \ldots, n\}$ denote the zeros of $p_n(x;\mu_0)$. Then the zeros $\{x_{i,j,nN}, \ i = 1, 2, \ldots, N, \ j = 1, 2, \ldots, n\}$ of $p_{nN}(x;\mu)$ are given by

$$x_{i,j,nN} = T_i^{-1}(x_{j,n}^0), \qquad 1 \le j \le n, \ 1 \le i \le N.$$

Recall that the Christoffel numbers $\{\lambda_{j,n} : 1 \le j \le n, \ n = 1, 2, \ldots\}$ for a system of orthogonal polynomials $\{p_n(x)\}$ with spectral measure μ are the Côtes numbers for the Gauss-Jacobi quadrature

$$\int \pi(x)\, d\mu(x) = \sum_{j=1}^{n} \lambda_{j,n} \pi(x_{j,n}),$$

which holds for every polynomial π of degree at most $2n - 1$. These Christoffel numbers are also equal to the residues of

$$\frac{\hat{p}_{n-1}^{(1)}(x)}{\hat{p}_n(x)} = \sum_{j=1}^{n} \frac{\lambda_{j,n}}{x - x_{j,n}}.$$

THEOREM 4. *Let $\{\lambda_{j,n}^0\}$ be the Christoffel numbers for the spectral mesure μ_0 on E_0 and $\{\lambda_{j,n}\}$ those for the measure μ on E. Then*

$$\lambda_{i,j,nN} = \lambda_{j,n}^0 w_i(x_{j,n}^0), \qquad 1 \le j \le n, \ 1 \le i \le N.$$

PROOF. From (2.8) and (2.12) we see that the $[(nN-1)/nN]$ Padé approximant to $S(z,\mu)$ is

$$\frac{\hat{p}_{nN-1}^{(1)}(x;\mu)}{\hat{p}_{nN}(x;\mu)} = W(x) \frac{\hat{p}_{n-1}^{(1)}(T(x);\mu_0)}{\hat{p}_n(T(x);\mu_0)},$$

where $\hat{p}_i$ denotes the monic orthogonal polynomial of degree i. Evaluating the residues of the above rational fractions and using the relation between the zeros of $\hat{p}_{nN}(x;\mu)$ and $\hat{p}_n(x;\mu_0)$ given above gives the result. $\square$

If one uses Chebyshev polynomials of the first kind on $[-1,1]$, then it is well known that $\lambda_{j,n}^0 = 1/n \ (j = 1, \ldots, n)$. Choosing $W(x) = T'(x)/N$ gives Christoffel numbers $\lambda_{j,nN} = 1/nN$ and the Gauss-Jacobi quadrature uses equal weights under these conditions.

III. Jacobi matrices. Let $\{p_n(x) : n = 0, 1, 2, \ldots\}$ be a sequence of orthogonal polynomials that satisfy the recurrence relation (2.11); then closely related to these polynomials is the Jacobi matrix

$$J = \begin{pmatrix} b_0 & a_1 & & & 0 \\ a_1 & b_1 & a_2 & & \\ 0 & a_2 & b_2 & a_3 & \\ & & \ddots & \ddots & \ddots \end{pmatrix}.$$

Let l_2 be the square summable sequences $\{\psi = (\psi(0), \psi(1), \psi(2), \dots): \sum_{n=0}^{\infty} \psi^2(n) < \infty\}$. Then J acts as an operator from l_2 to l_2. There is an easy relationship between the resolvents of the Jacobi matrices of $\{p_n(x; \mu_0)\}$ and $\{p_n(x; \mu)\}$:

THEOREM 5 [6]. *Let* $\{a_{n+1}^0, b_n^0: n = 0, 1, 2, \dots\}$ *be the recurrence coefficients for the measure* μ_0 *on* $E_0 \subset [-1, 1]$ *with Jacobi matrix* J_0 *and* $\{a_{n+1}, b_n: n = 0, 1, 2, \dots\}$ *the recurrence coefficients for* μ *on* $E = T^{-1}(E_0)$ *with Jacobi matrix* J. *Then*

$$(3.1) \qquad W(z)(T(z) - J_0)^{-1} = D(z - J)^{-1}D^*, \qquad z \notin E,$$

where $D: l_2 \to l_2$ *acts as* $(D\psi)(n) = \psi(nN)$ *and* D^* *is the adjoint (transpose) of* D.

PROOF. The (n, m)-entry of $(z - J)^{-1}$ is given by

$$(3.2) \qquad [(z - J)^{-1}]_{n,m} = \int_E \frac{p_n(x; \mu)p_m(x; \mu)}{z - x}\, d\mu(x)$$

(Stone [24, p. 546]). By (2.8) we then have

$$[D(z - J)^{-1}D^*]_{n,m} = [(z - J)^{-1}]_{nN, mN}$$
$$= \int_E \frac{p_n(T(x); \mu_0)p_m(T(x); \mu_0)}{z - x}\, d\mu(x).$$

Next use (2.5), (2.1), and (2.8) in the integral to find

$$[D(z - J)^{-1}D^*]_{n,m} = W(z) \int_{E_0} \frac{p_n(x; \mu_0)p_m(x; \mu_0)}{T(z) - x}\, d\mu_0(x),$$

and the last integral is the (n, m)-entry of $(T(z) - J_0)^{-1}$. □

The resolvent formula (3.1) uniquely determines the recurrence coefficients $\{a_{n+1}, b_n: n = 0, 1, 2, \dots\}$ when $\{a_{n+1}^0, b_n^0: n = 0, 1, 2, \dots\}$ are given. This follows by looking at the $(0, 0)$-entries in (3.1).

If one wants to solve (3.1) for J, then one runs into problems since D is not invertible. In order to get around that problem we introduce the projections

$$P = D^*D, \quad Q = I - P.$$

Following Bellissard [6], we arrive at the following Schur decomposition.

LEMMA 2. *Let* $\tilde{J} = QJQ$. *Then for* $z \in \mathbf{C} \backslash (E \cup \{0\})$

$$(3.3) \qquad D(z - J)^{-1}D^* = [z - DJD^* - DJQ(z - \tilde{J})^{-1}QJD^*]^{-1}.$$

PROOF. We will show that

$$D(z - J)^{-1}D^*[z - DJD^* - DJQ(z - \tilde{J})^{-1}QJD^*] = I.$$

The left-hand side of this equation is equal to

$$(3.4) \quad zD(z - J)^{-1}D^* - D(z - J)^{-1}PJD^* - D(z - J)^{-1}PJQ(z - \tilde{J})^{-1}QJD^*.$$

By the definition of Q

$$\tilde{J}_{m,n} = [QJQ]_{m,n} = 0, \qquad m = 0 \pmod{N} \text{ or } n = 0 \pmod{N}$$

so that

$$[(z - \tilde{J})^{-1}]_{mN,n} = 0, \qquad n \neq nM.$$

We then easily find

$$P(z - \tilde{J})^{-1}Q = 0.$$

Using this we can rewrite the last term in (3.4) as

$$D(z - J)^{-1}[PJQ + Pz](z - \tilde{J})QJD^*.$$

Since $P = I - Q$ and $DQ = 0$ this is equal to

$$D(z - J)^{-1}[(z - \tilde{J}) - (z - J)Q](z - \tilde{J})^{-1}QJD^* = D(z - J)^{-1}QJD^*.$$

Therefore (3.4) becomes

$$zD(z - J)^{-1}D^* - D(z - J)^{-1}PJD^* - D(z - J)^{-1}QJD^*$$
$$= D(z - J)^{-1}[z - PJ - QJ]D^* = DD^*$$

and since $DD^* = I$ this proves the result. $\square$

A combination of (3.1) and (3.3) gives

(3.5) $\qquad T(z) - J_0 = W(z)[z - DJD^* - DJQ(z - \tilde{J})^{-1}QJD^*].$

This equation then leads to

THEOREM 6 [8]. *Let*

(3.6) $\qquad \begin{aligned} T(z) &= cz^N + \alpha_1 z^{N-1} + \alpha_2 z^{N-2} + \cdots + \alpha_N, \\ W(z) &= cz^{N-1} + \beta_1 z^{N-2} + \cdots + \beta_{N-1}. \end{aligned}$

Then

(3.7) $\qquad b_{nN} = \dfrac{\beta_1 - \alpha_1}{c}, \qquad n = 0, 1, 2, \ldots,$

(3.8) $\qquad \displaystyle\prod_{i=1}^{N} a_{nN+i} = \frac{1}{c} a_{n+1}^0, \qquad n = 0, 1, 2, \ldots,$

(3.9) $\qquad \hat{p}_{N-1}^{(nN+1)}(z; \mu) = \dfrac{1}{c}W(z), \qquad n = 0, 1, 2, \ldots,$

(3.10) $\qquad \begin{aligned} & a_{nN}^2 \hat{p}_{N-2}^{(nN-N+1)}(z; \mu) + a_{nN+1}^2 \hat{p}_{N-2}^{(nN+2)}(z; \mu) \\ & = \frac{1}{c}\left\{ b_n^0 - T(z) + \left(z - \frac{\beta_1 - \alpha_1}{c}\right) W(z) \right\}, \qquad n = 0, 1, 2, \ldots, \end{aligned}$

where $a_0 = 0$ and $\hat{p}_n^{(k)}(z; \mu)$ is the monic associated polynomial of order k (i.e., with leading coefficient one).

PROOF. If one compares the coefficients of z^{N-1} in (3.5), then

(3.11) $\qquad (\beta_1 - \alpha_1)I = cDJD^*$

and since

$$[DJD^*]_{n,n} = J_{nN,nN} = b_{nN}$$

this gives (3.7). From (3.5) we also find, using (3.11),

$$(3.12) \qquad W(z)[DJQ(z - \tilde{J})^{-1}QJD^*]_{n,n+1} = a_{n+1}^0.$$

Let $J^{(n)}$ $(n = 0, 1, 2, \dots)$ be a $(N-1) \times (N-1)$ Jacobi matrix defined by

$$J^{(n)} = \begin{pmatrix} b_{nN+1} & a_{nN+2} & & & & \\ a_{nN+2} & b_{nN+2} & a_{nN+3} & & & \\ & a_{nN+3} & b_{nN+3} & & & \\ & & & & & a_{nN+N-1} \\ & & & & a_{nN+N-1} & b_{nN+N-1} \end{pmatrix}.$$

Then $\tilde{J}$ is a block diagonal matrix with the $J^{(n)}$ matrices on the diagonal separated in the (nN, nN)-entries by zeros:

$$\tilde{J} = \begin{pmatrix} 0 & & & & & & \\ & J^{(0)} & & & & & \\ & & 0 & & & & \\ & & & J^{(1)} & & & \\ & & & & 0 & & \\ & & & & & J^{(2)} & \\ & & & & & & 0 & \\ & & & & & & & \ddots \end{pmatrix}$$

and therefore

$$(z - \tilde{J})^{-1} = \begin{pmatrix} 1/z & & & & & \\ & (z - J^{(0)})^{-1} & & & & \\ & & 1/z & & & \\ & & & (z - J^{(1)})^{-1} & & \\ & & & & 1/z & \\ & & & & & \ddots \end{pmatrix}.$$

Some simple algebraic manipulations show that

$$[DJQ(z - \tilde{J})^{-1}QJD^*]_{n,n+1} = [JQ(z - \tilde{J})^{-1}QJ]_{nN,nN+N}$$
$$= a_{nN+1}a_{nN+N}[(z - \tilde{J})^{-1}]_{nN+1,nN+N-1}.$$

Now

$$[(z - \tilde{J})^{-1}]_{nN+1,nN+N-1} = [(z - J^{(n)})^{-1}]_{1,N-1} = \frac{\prod_{i=2}^{N-1} a_{nN+i}}{\hat{p}_{N-1}^{(nN+1)}(z; \mu)};$$

the last equation follows by calculating $(z - J^{(n)})^{-1}$ using cofactors. If we insert this in (3.12), we find

$$W(z) \prod_{i=1}^{N} a_{nN+i}/\hat{p}_{N-1}^{(nN+1)}(z; \mu) = a_{n+1}^0,$$

and this implies both (3.8) and (3.9). In a similar way we find from (3.5), using (3.11),

$$W(z)[DJQ(z - \tilde{J})^{-1}QJD^*]_{n,n} = b_n^0 - T(z) + \left(z - \frac{\beta_1 - \alpha_1}{c} \right) W(z).$$

Some easy algebra gives

$$[DJQ(z-\tilde{J})^{-1}QJD^*]_{n,n} = [JQ(z-\tilde{J})^{-1}QJ]_{nN,nN}$$
$$= a_{nN}^2[(z-\tilde{J})^{-1}]_{nN-1,nN-1}$$
$$+ a_{nN+1}^2[(z-\tilde{J})^{-1}]_{nN+1,nN+1}$$

where for $n=0$ we set $a_0 = 0$. Now

$$[(z-\tilde{J})^{-1}]_{nN-1,nN-1} = [(z-J^{(n-1)})^{-1}]_{N-1,N-1}$$
$$= \frac{\hat{p}_{N-2}^{(nN-N+1)}(z;\mu)}{\hat{p}_{N-1}^{(nN-N+1)}(z;\mu)},$$

$$[(z-\tilde{J})^{-1}]_{nN+1,nN+1} = [(z-J^{(n)})^{-1}]_{1,1} = \frac{\hat{p}_{N-2}^{(nN+2)}(z;\mu)}{\hat{p}_{N-1}^{(nN+1)}(z;\mu)},$$

and (3.10) follows by using (3.8) and (3.9). □

If one uses the formulas

$$\left(z - \frac{\beta_1 - \alpha_1}{c}\right)\frac{W(z)}{c} = (z - b_{nN})\hat{p}_{N-1}^{(nN+1)}(z;\mu)$$

and

$$(z - b_{nN})\hat{p}_{N-1}^{(nN+1)}(z;\mu) = \hat{p}_N^{(nN)}(z;\mu) + a_{nN+1}^2\hat{p}_{N-2}^{(nN+2)}(z;\mu),$$

then (3.10) becomes

(3.13)
$$\frac{T(z) - b_n^0}{c} = \hat{p}_N^{(nN)}(z;\mu) - a_{nN}^2\hat{p}_{N-2}^{(nN-N+1)}(z;\mu).$$

IV. Properties of the new orthogonal polynomials. So far we have a simple relation between the polynomials $p_{nN}(x;\mu)$ and $p_n(x;\mu_0)$ given by Theorem 1, and a similar relation between $p_{nN-1}^{(1)}(x;\mu)$ and $p_{n-1}^{(1)}(x;\mu_0)$ given by Theorem 2. A natural question then is to find a formula for $p_{nN+j}(x;\mu)$ and $p_{nN+j-1}^{(1)}(x;\mu)$ in terms of the initial system $p_n(x;\mu_0)$ and $p_{n-1}^{(1)}(x;\mu_0)$. We will give such formulas by using some Wronskian formulas. If $\{u_n\}$ and $\{v_n\}$ are two solutions of the recurrence relation (2.11), then their Wronskian is defined to be

$$W(u_n, v_n) = a_{n+1}\begin{vmatrix} u_n & v_n \\ u_{n+1} & v_{n+1} \end{vmatrix}$$
$$= a_{n+1}(u_n v_{n+1} - u_{n+1}v_n).$$

One can easily verify that $W(u_n, v_n)$ is independent of n.

THEOREM 7. *Let j be any integer such that $0 \le j < N$. Then*

(4.1)
$$p_{nN+j}(x;\mu) = \frac{1}{W(x)}\frac{a_{n+1}^0}{a_{nN+1}}\left\{ \frac{a_{nN+1}}{a_{nN+j+1}}p_{N-j-1}^{(nN+j+1)}(x;\mu)p_n(T(x);\mu_0) \right.$$
$$\left. + p_{j-1}^{(nN+1)}(x;\mu)p_{n+1}(T(x);\mu_0) \right\}$$

and

$$(4.2) \quad p^{(1)}_{nN+j-1}(x;\mu) = \frac{a_1}{a_1^0}\frac{a_{n+1}^0}{a_{nN+1}} \left\{ \frac{a_{nN+1}}{a_{nN+j+1}} p^{(nN+j+1)}_{N-j-1}(x;\mu) p^{(1)}_{n-1}(T(x);\mu_0) \right.$$
$$\left. + p^{(nN+1)}_{j-1}(x;\mu) p^{(1)}_{n}(T(x);\mu_0) \right\}.$$

PROOF. Let n, N and j be fixed integers. Then the sequences $p_{(n+1)N+k}(x;\mu)$, $p^{(nN+1)}_{N+k-1}(x;\mu)$, and $p^{(nN+j+1)}_{N+k-j-1}(x;\mu)$ are solutions of the recurrence relation

$$xu_k = a_{nN+k+1}u_{k+1} + b_{nN+k}u_k + a_{nN+k}u_{k-1}.$$

Since this is a linear recurrence relation of second order, these three solutions have to be linearly dependent and therefore there exist A, B, and C (independent of k) such that

$$(4.3) \quad Ap^{(nN+1)}_{N+k-1}(x;\mu) = Bp^{(nN+j+1)}_{N+k-j-1}(x;\mu) + Cp_{(n+1)N+k}(x;\mu).$$

Setting k equal to $j - N$ gives

$$Ap^{(nN+1)}_{j-1}(x;\mu) = Cp_{nN+j}(x;\mu),$$

and setting k equal to $j - N + 1$ gives

$$Ap^{(nN+1)}_{j}(x;\mu) = B + Cp_{nN+j+1}(x;\mu).$$

Therefore

$$C = A\frac{p^{(nN+1)}_{j-1}(x;\mu)}{p_{nN+j}(x;\mu)}$$

and

$$B = \frac{A}{p_{nN+j}(x;\mu)}\{p^{(nN+1)}_{j}(x;\mu)p_{nN+j}(x;\mu) - p^{(nN+1)}_{j-1}(x;\mu)p_{nN+j+1}(x;\mu)\}$$
$$= \frac{A}{p_{nN+j}(x;\mu)}\frac{1}{a_{nN+j+1}}W(p_{nN+k}, p^{(nN+1)}_{k-1}).$$

We can evaluate the Wronskian by choosing $k = 0$, giving

$$(4.4) \quad W(p_{nN+k}, p^{(nN+1)}_{k-1}) = a_{nN+1}p_{nN}(x;\mu)$$

so that (4.3) becomes

$$p_{nN+j}(x;\mu)p^{(nN+1)}_{N+k-1}(x;\mu) = \frac{a_{nN+1}}{a_{nN+j+1}}p^{(nN+j+1)}_{N+k-j-1}(x;\mu)p_{nN}(x;\mu)$$
$$+ p^{(nN+1)}_{j-1}(x;\mu)p_{(n+1)N+k}(x;\mu).$$

Let $k = 0$ in this formula and use (3.8) and (3.9); then (4.1) follows. Similar reasoning using the solutions $p^{(1)}_{(n+1)N+k-1}(x;\mu), p^{(nN+1)}_{N+k-1}(x;\mu)$, and $p^{(nN+j-1)}_{N+k-j-1}(x;\mu)$ will lead to (4.2). □

The next theorem shows us how to relate a_{nN} to a_n^0.

THEOREM 8. *Let* $(\mathbf{z}^j)^t = (z_1^j, z_2^j, \ldots, z_{N-1}^j)$, *where* $\{z_i : i = 1, 2, \ldots, N-1\}$ *are the zeros of* $W(z)$ *and let* $(\gamma_n)^t = (\gamma_{1,n}, \gamma_{2,n}, \ldots, \gamma_{N-1,n})$, *where*

$$(4.5) \qquad \gamma_{i,n} = \frac{-a_{n+1}^0}{c} \frac{p_n(T(z_i); \mu_0)}{p_{n+1}(T(z_i); \mu_0)}, \qquad i = 1, 2, \ldots, N-1.$$

Then

$$(4.6) \qquad a_{(n+1)N}^2 = \frac{\det[\mathbf{1}, \mathbf{z}, \mathbf{z}^2, \ldots, \gamma_n]}{\det[\mathbf{1}, \mathbf{z}, \mathbf{z}^2, \ldots, \mathbf{z}^{N-2}]}$$

and

$$\hat{p}_{N-2}^{(nN+1)}(z; \mu) = z^{N-2} + \sum_{i=0}^{N-3} c_i z^i,$$

where

$$(4.7) \quad c_i = \frac{\det[\mathbf{1}, \mathbf{z}, \mathbf{z}^2, \ldots, \mathbf{z}^{i-1}, \gamma_n, \mathbf{z}^{i+1}, \ldots, \mathbf{z}^{N-2}]}{\det[\mathbf{1}, \mathbf{z}, \mathbf{z}^2, \ldots, \mathbf{z}^{N-3}, \gamma_n]}, \qquad i = 0, 1, 2, \ldots, N-3.$$

Furthermore this allows us to calculate $\{a_{nN+i} : i = 1, 2, \ldots, N-1\}$ *and* $\{b_{nN+i} : i = 1, 2, \ldots, N-1\}$ *in terms of ratios* $p_n(z; \mu_0)/p_{n+1}(z; \mu_0)$.

PROOF. Consider the Wronskian (4.4) for $k = N - 1$. Then

$$a_{nN+1} p_{nN}(x; \mu) = a_{(n+1)N} \{ p_{(n+1)N-1}(x; \mu) p_{N-1}^{(nN+1)}(x; \mu)$$
$$- p_{(n+1)N}(x; \mu) p_{N-2}^{(nN+1)}(x; \mu) \}.$$

If we evaluate the above Wronskian at a zero z_i of $W(z)$ and use (3.9), we obtain

$$a_{nN+1} p_{nN}(z_i; \mu) = -a_{(n+1)N} p_{N-2}^{(nN+1)}(z_i; \mu) p_{(n+1)N}(z_i; \mu).$$

The fact that

$$p_{N-1}^{(nN+1)}(z; \mu) = \frac{1}{a_{nN+2} a_{nN+3} \cdots a_{nN+N-1}} \hat{p}_{N-2}^{(nN+1)}(z; \mu)$$
$$= c \frac{a_{nN+1} a_{(n+1)N}}{a_{n+1}^0} \hat{p}_{N-2}^{(nN+1)}(z; \mu)$$

(see (3.8)) and Theorem 1 give

$$a_{(n+1)N}^2 \hat{p}_{N-2}^{(nN+1)}(z_i; \mu) = \frac{-a_{n+1}^0}{c} \frac{p_n(T(z_i); \mu_0)}{p_{n+1}(T(z_i); \mu_0)} = \gamma_{i,n}, \qquad i = 1, 2, \ldots, N-1.$$

If we write

$$a_{(n+1)N}^2 \hat{p}_{N-2}^{(nN+1)}(z; \mu) = \sum_{i=0}^{N-2} k_i z^i$$

and then solve for k_i using Cramer's rule, we find

$$k_i = \frac{\det[\mathbf{1}, \mathbf{z}, \ldots, \mathbf{z}^{i-1}, \gamma_n, \mathbf{z}^{i+1}, \ldots, \mathbf{z}^{N-2}]}{\det[\mathbf{1}, \mathbf{z}, \mathbf{z}^2, \ldots, \mathbf{z}^{N-2}]}.$$

From this we find (4.6) when $i = N - 1$ and (4.7) since $k_i = a_{(n+1)N}^2 c_i$.

To show the last assertion we write

$$(4.8) \qquad \hat{p}_i^{(nN+1)}(z) = Q_i(z) = z^i + \sum_{j=0}^{i-1} q_{i,j} z^j, \qquad i = 1, 2, \ldots, N-2.$$

Since

$$(4.9) \quad a^2_{nN+i-1}Q_{i-3}(z) = (z - b_{nN+i-1})Q_{i-2}(z) - Q_{i-1}(z), \qquad i = 3, \ldots, N,$$

we have the following formulas which follow by equating like powers of z:

$$(4.10) \qquad\qquad b_{nN+i-1} = -q_{i-1,i-2} + q_{i-2,i-3},$$

$$(4.11) \qquad\qquad a^2_{nN+i-1} = q_{i-2,i-4} - q_{i-1,i-3} - b_{nN+i-1}q_{i-2,i-3},$$

and

$$(4.12) \qquad q_{i-3,j} = \frac{q_{i-2,j-1} - q_{i-1,j} - b_{nN+i-1}q_{i-2,j}}{a^2_{nN+i-1}}, \qquad j = 0, 1, \ldots, i-2.$$

Since $q_{N-1,j}$ and $q_{N-2,j}$ are known, (4.10) gives us b_{nN+N-1} from which follows a^2_{nN+N-1} by (4.11). We then calculate $q_{N-3,j}$. Continuing this procedure reducing N by one each time gives the result. $\square$

REMARK 4. The denominator in (4.6) is a Vandermonde determinant which for some classical polynomials is also referred to as a discriminant. Explicit formulas are available for classical polynomials (Szegö [25, p. 143]).

Let $\phi(T,W)$ denote the mapping that maps polynomials orthonormal on E_0 to polynomials orthonormal on E given by (2.5) or (3.1). The above theorem allows us to determine some classes of polynomials that are mapped into each other under ϕ.

A system of orthonormal polynomials $\{p_n(x)\}$ with orthogonality set Δ is in $M_k(\Delta)$ iff there exists $\{\tilde{a}_n\}_{n=1}^\infty$ and $\{\tilde{b}_n\}_{n=0}^\infty$ with $\tilde{a}_{n+k} = \tilde{a}_n > 0$, and $\tilde{b}_{n+k} = \tilde{b}_n$ real and

$$\lim_{n\to\infty} |a_n - \tilde{a}_n| = 0 \quad \text{and} \quad \lim_{n\to\infty} |b_n - \tilde{b}_n| = 0,$$

where the $\{a_n\}$ and $\{b_n\}$ are the coefficients in (2.11). A system of orthonormal polynomials $\{p_n(x)\}$ with orthogonality set Δ is in $M_k^i(\Delta) \subset M_k(\Delta)$ $(i = 0, 1)$ iff

$$(4.13) \qquad \sum_{n=1}^\infty n^i \{|a_n - \tilde{a}_n| + |b_{n-1} - \tilde{b}_{n-1}|\} < \infty \qquad (i = 0, 1).$$

THEOREM 9. *Suppose the zeros of $W(z)$ are not in $E = T^{-1}(E_0)$ and suppose E_0 supports a class M_k for some k. Then*

$$(4.14) \qquad\qquad \phi(T,W): M_k(E_0) \to M_{Nk}(E).$$

If E_0 is an interval, then (4.14) is true even if the zeros of $W(z)$ are in E. Furthermore in this case

$$(4.15) \qquad\qquad \phi(T,W): M_1^i(E_0) \to M_N^i(E) \qquad (i = 0, 1).$$

PROOF. We shall begin with the case when E_0 is an interval which we shall take without loss of generality, as $[-1, 1]$. We shall also allow that the zeros of $W(z)$ are in E. Since $E_0 = [-1, 1]$, $\tilde{a}_n^0 = \frac{1}{2}$, $\tilde{b}_n^0 = 0$, and $|T(z_i)| \geq 1$, and we have (Chihara [10, Theorems (III.6.4), (IV.2.1), and (IV.2.4)])

$$\lim_{n\to\infty} \frac{p_n(T(z_i); \mu_0)}{p_{n+1}(T(z_i); \mu_0)} = \frac{1}{T(z_i) + \sqrt{T(z_i)^2 - 1}},$$

where the square root is such that $|x + \sqrt{x^2 - 1}| > 1$ when $x \in \mathbf{C}\backslash[-1, 1]$. Consequently we have from (4.5), (4.6), and (4.7) that

$$\lim_{n \to \infty} a_{(n+1)N} = \tilde{a}_N$$

and

$$\lim_{n \to \infty} p_{N-2}^{(nN+1)}(z) \to q_{N-2}^{(1)}(z),$$

where the above limit is uniform on compact subsets of $\mathbf{C}$. Here $q_{N-2}^{(1)}$ is some polynomial of degree $N - 2$. Since $\hat{p}_{N-1}^{(nN+1)}(z) = \hat{p}_{N-1}^{(1)}(z) = \hat{q}_{N-1}^{(1)}$, we can use (4.10), (4.11), and (4.12) to calculate $\tilde{a}_i$, $i = 2, 3, \ldots, N - 1$, and $\tilde{b}_i$, $i = 1, 2, \ldots, N - 1$. That $\tilde{b}_0 = \tilde{b}_{nN}$ follows from (3.7) and we find, by equating coefficients of z^{N-2} in (3.10) then taking the limit, that

$$\tilde{a}_1^2 = \frac{1}{c} \left\{ \beta_2 - \alpha_2 - \frac{(\beta_1 - \alpha_1)}{c} \beta_1 \right\} - \tilde{a}_N^2.$$

(Here we have used the fact that the $\{\tilde{a}_n\}$ and $\{\tilde{b}_n\}$ are periodic.) This gives us (4.13) in the case when E_0 is an interval even if the zeros of $W(z)$ are in E. To prove the general case we note that since $z_i \notin E$, $|T(z_i)| > 1$. Let $q_n^0(x)$ be the orthonormal polynomials associated with $\{\tilde{a}_n^0\}$ and $\{\tilde{b}_n^0\}$. Then it is easy to show [12, (II.22)] that

$$\lim_{n \to \infty} \frac{q_{nk+s}^0(x)}{q_{nk+s+1}^0(x)} = f_s(x), \qquad s = 0, 1, 2, \ldots, k - 1,$$

uniformly on compact sets of $\mathbf{C}\backslash[-1, 1]$. By the definition of the class $M_k(E_0)$ it follows from Poincaré's theorem that

$$\lim_{n \to \infty} \frac{p_{nk+s}(x; \mu_0)}{p_{nk+s+1}(x; \mu_0)} = f_s(x)$$

uniformly on compact sets of $\mathbf{C}\backslash[-1, 1]$. Consequently

$$(4.16) \qquad \lim_{n \to \infty} \frac{p_{nk+s}(T(z_i); \mu_0)}{p_{nk+s+1}(T(z_i); \mu_0)} = f_s(T(z_i)).$$

If we examine the subsequences $\{a_{(nk+s+1)N} : n = 0, 1, 2, \ldots\}$ and $\{p_{N-2}^{((nk+s)N+1)}(x):$ $n = 0, 1, 2\}$ ($s = 0, 1, 2, \ldots, k - 1$), we can use (4.16) to repeat the same reasoning as before and (4.14) follows.

To show (4.15) we note that from (4.5)–(4.7) and (4.10)–(4.12) it is sufficient to show that

$$(4.17) \qquad \sum_{n=1}^{\infty} n^j \left| \frac{p_n(T(z_i); \mu_0)}{p_{n+1}(T(z_i); \mu_0)} - \frac{1}{T(z_i) + \sqrt{T(z_i)^2 - 1}} \right| < \infty, \qquad j = 0, 1.$$

Following [14] we let $\rho(x) = x + \sqrt{x^2 - 1}$ and note that $\rho(x)$ satisfies the following equation:

$$(4.18) \qquad \rho(x) = 2x - 1/\rho(x).$$

From (2.11) we have that for $\mathbf{C}\backslash[-1, 1]$

$$(4.19) \qquad \frac{p_n(x; \mu_0)}{p_{n-1}(x; \mu_0)} = \frac{z - b_{n-1}}{a_n} - \frac{a_{n-1}}{a_n} \frac{p_{n-2}(x; \mu_0)}{p_{n-1}(x; \mu_0)}.$$

Subtracting (4.18) from (4.19) yields

$$
(4.20) \quad \frac{p_n(x;\mu_0)}{p_{n-1}(x;\mu_0)} - \rho(x) = \frac{2z}{a_n}\left(\frac{1}{2} - a_n\right) - \frac{b_{n-1}}{a_n} + \left(1 - \frac{a_{n-1}}{a_n}\right)\frac{1}{\rho(x)}
$$

$$
+ \frac{a_{n-1}}{a_n}\frac{1}{\rho}\frac{p_{n-2}(x;\mu_0)}{p_{n-1}(x;\mu_0)}\left(\frac{p_{n-1}(x;\mu_0)}{p_{n-2}(x;\mu_0)} - \rho(x)\right).
$$

Since $p_{n-2}(x;\mu_0)/p_{n-1}(x;\mu_0) \to 1/\rho(x)$ uniformly on compact subsets of $\mathbf{C}\backslash E_0$, for any given compact set K ($K \cap E_0 = \varnothing$) and ε there exists an N such that

$$
(4.21) \quad \left|\frac{1}{\rho(x)}\frac{p_n(x;\mu_0)}{p_{n-1}(x;\mu_0)}\right| \leq \left(\frac{1}{|\rho(x)| - \varepsilon}\right)^2
$$

for all $n \geq N$ and $x \in K$. Setting

$$
\Delta(n) = |\tfrac{1}{2} - a_n| + |\tfrac{1}{2} - a_{n-1}| + |b_{n-1}|
$$

and

$$
\delta = \frac{1}{(|\rho| - \varepsilon)} < 1,
$$

we find from (4.20) and (4.21) that

$$
\left|\frac{p_n(x;\mu_0)}{p_{n-1}(x;\mu_0)} - \rho(x)\right| \leq \delta^2\left|\frac{p_{n-1}(x;\mu_0)}{p_{n-2}(x;\mu_0)} - \rho(x)\right| + O(\Delta(n)).
$$

Repeated application of this inequality gives

$$
\left|\frac{p_n(x;\mu_0)}{p_{n-1}(x;\mu_0)} - \rho(x)\right| = O\left[\sum_{k=1}^{n}\delta^{(2n-2k)}\Delta(k)\right],
$$

where we set $a_i = 0$ for $i \leq 0$. If we choose K so that $T(z_i) \in K$, $i = 1, 2, \ldots, N-1$, the assertion will have been proved if we can show that

$$
I = \sum_{n=1}^{\infty} n^j \sum_{k=1}^{n}\delta^{2k-2n}\Delta(k) < \infty, \qquad j = 0, 1.
$$

Interchanging summations in the above equation gives

$$
I = \sum_{k=1}^{\infty}\Delta(k)\delta^{-2k}\sum_{n=k}^{\infty}n^j\delta^{2n}, \qquad j = 0, 1.
$$

If we are in the class M_1^0, then

$$
I = \sum_{k=1}^{\infty}\Delta(k)/(1 - \delta^2) < \infty.
$$

If we are in the class M_1^1, then

$$
I = \frac{1}{(1 - \delta^2)^2}\sum_{k=1}^{\infty}(k - (k-1)\delta^2)\Delta(k)
$$

which gives the result. $\square$

REMARK 5. For other possible rates of convergence of the coefficients see Máté-Nevai-Totik [19].

By using (2.8), Theorem 7 and Theorem 8 one shows that asymptotic formulas generating functions, etc. for the known polynomial system $p_n(x;\mu_0)$ can be carried over to give corresponding formulas for the system $p_n(x;\mu)$. A simple example of the above is the following theorem.

THEOREM 10. *Suppose that (4.14) holds. Then there exists a constant A such that for $j = 0, 1, 2, \ldots, N - 1$*

(4.22) $\qquad \|p_{nN+j-1}^{(1)}(x; \mu)\|_\infty \leq A \max\{\|p_n^{(1)}(x; \mu_0)\|_\infty, \|p_{n-1}^{(1)}(x; \mu_0)\|_\infty\},$

and if $W(z) \neq 0$ for $z \in E$ then there exist a constant B such that for $j = 0, 1, 2, \ldots, N - 1$

(4.23) $\qquad \|p_{nN+j}(x; \mu)\|_\infty \leq B \max\{\|p_n(x; \mu_0)\|_\infty, \|p_{n+1}(x; \mu_0)\|_\infty\}.$

PROOF. Let

$$C_1 = \max_{0 \leq i < N} \max_k \sup_{z \in E} |p_i^{(k)}(z; \mu)|,$$

$$C_2 = \max_n \frac{a_{n+1}^0}{a_{nN+1}}, \quad C_3 = \max_{0 \leq i < N} \max_n \frac{a_{nN+1}}{a_{nN+i+1}}.$$

Then C_1, C_2, C_3 are bounded because of the asymptotic periodicity of the recurrence coefficients $\{a_{n+1}, b_n: n = 0, 1, 2, \ldots\}$. From (4.2) one then easily finds

$$\|p_{nN+j-1}^{(1)}(x; \mu)\|_\infty \leq \frac{a_1}{a_1^0} C_2 \{C_3 C_1 \|p_{n-1}^{(1)}(x; \mu_0)\|_\infty + C_1 \|p_n^{(1)}(x; \mu_0)\|_\infty\}$$

from which (4.22) follows. If $W(z) \neq 0$ for $z \in E$, then

$$C_4 = \inf_{z \in E} |W(z)| > 0$$

and from (4.1) we find

$$\|p_{nN+j}(x; \mu)\|_\infty \leq \frac{C_2}{C_4} \{C_3 C_1 \|p_n(x; \mu_0)\|_\infty + C_1 \|p_{n+1}(x; \mu_0)\|_\infty\},$$

which leads to (4.23). □

V. Potential theory. In this section we will relate some potential-theoretic notions involving the set $E = T^{-1}(E_0)$ to those related to E_0. If E is a compact set and μ is a probability measure on E, then its (logarithmic) energy is

$$I(\mu) = \int_E \int_E \log \frac{1}{|x - y|} \, d\mu(x) \, d\mu(y).$$

The (logarithmic) potential of μ is

$$U(z; \mu) = \int_E \log \frac{1}{|z - x|} \, d\mu(x).$$

If Ω_E is the class of all probability measures on E, then the equilibrium energy of E (Robin's constant) is

$$V(E) = \inf_{\mu \in \Omega_E} I(\mu)$$

and the capacity of E is $C(E) = \exp(-V(E))$. A well-known result in potential theory is that for every compact set E with positive capacity there exists a unique measure μ_E such that $V(E) = I(\mu_E)$. This measure is the equilibrium measure (or Frostman measure) for E (Tsuji [**26**]). If the capacity of E is positive, then the Green's function $g_E(z)$ of $(\mathbf{C} \cup \{\infty\})\backslash E$ is the only function that is harmonic in $\mathbf{C}\backslash E$, that behaves at ∞ as

$$g_E(z) = \log |z| + \text{harmonic function}$$

and for which

$$\lim_{\substack{z \to x \\ z \notin E}} g_E(z) = 0$$

for almost every x in E. The Green's function is actually given by

$$g_E(z) = -U(z; \mu_E) - \log C(E)$$

and

$$\lim_{z \to \infty} \{g_E(z) - \log|z|\} = \log \frac{1}{C(E)}.$$

THEOREM 11. *Suppose $C(E_0) > 0$ and let $E = T^{-1}(E_0)$. Then*

(5.1) $$g_E(z) = g_{E_0}(T(z))/N$$

and in particular

(5.2) $$C(E) = \{C(E_0)/c\}^{1/N}.$$

If μ_{E_0} is the equilibrium measure on E_0, then the equilibrium measure μ_E on E can be obtained from μ_{E_0} by taking $W(z) = T'(z)/N$.

PROOF. The function $g_{E_0}(T(z))/N$ is harmonic in $\mathbf{C} \backslash E$, in the neighborhood of infinity it behaves as

$$\frac{1}{N} g_{E_0}(T(z)) = \frac{1}{N}\left\{\log|T(z)| + \log\frac{1}{C(E_0)}\right\} + \text{harmonic function}$$

$$= \log|z| + \frac{1}{N}\log c + \frac{1}{N}\log\frac{1}{C(E_0)} + \text{harmonic function},$$

and

$$\lim_{\substack{z \to E \\ z \notin E}} \frac{1}{N} g_{E_0}(T(z)) = \lim_{\substack{z \to E_0 \\ z \notin E_0}} \frac{1}{N} g_{E_0}(z) = 0$$

almost everywhere. Therefore (5.1) and (5.2) follow immediately. Let $W(z) = T'(z)/N$ and take μ_0 to be the equilibrium measure μ_{E_0} in E_0; then the corresponding measure μ in E satisfies

$$U(z; \mu) = \int_E \log\frac{1}{|z-x|} d\mu(x)$$

$$= \frac{1}{N} \sum_{i=1}^{N} \int_{E_0} \log\frac{1}{|z - T_i^{-1}(x)|} d\mu_0(x).$$

Now

$$\sum_{i=1}^{N} \log|z - T_i^{-1}(x)| = \log\left|\frac{T(z) - x}{c}\right|$$

so that

$$U(z; \mu) = \frac{1}{N} \int_{E_0} \log\frac{1}{|T(z) - x|} d\mu_0(x) + \frac{1}{N}\log c$$

and by (5.1) and (5.2) this shows that μ is the equilibrium measure μ_E on E. □

If $E_0 = [-1, 1]$, then the equilibrium measure μ_{E_0} is the arcsine measure

$$d\mu_{E_0}(x) = \frac{1}{\pi}\frac{1}{\sqrt{1-x^2}}dx, \qquad x \in [-1, 1],$$

and the equilibrium measure on $E = T^{-1}([-1, 1])$ becomes by Theorem 8

$$d\mu_E(x) = \frac{1}{N\pi} \frac{|T'(x)|}{\sqrt{1 - T(x)^2}} \, dx, \qquad x \in E.$$

Theorem 9 shows that the coefficients for this system are in the class $M_N^1(E)$. However in this case $p_n(x; \mu_0) = T_n(x) = \frac{1}{2}(\rho(x)^n + \rho(x)^{-n})$ are the Chebyshev polynomials of the first kind. Consequently

$$\frac{T_n(T(z_i))}{T_{n+1}(T(z_i))} = \begin{cases} 1 & \text{if } T(z_i) = 1, \\ -1 & \text{if } T(z_i) = -1, \end{cases}$$

and if $|T(z_i)| > 1$ then

$$\frac{T_n(T(z_i))}{T_{n+1}(T(z_i))} = \frac{1}{\rho(T(z_i))}[1 + O(\rho^{-n}(T(z_i)))].$$

Thus (4.5)–(4.7) and (4.10)–(4.12) imply the following:

LEMMA 3. *Let $p_n(x; \mu_E)$ be the orthonormal polynomials associated with μ_E and let $\{a_n\}$ and $\{b_n\}$ be the coefficients associated with these polynomials. Finally, let $\{\tilde{a}_i\}$ and $\{\tilde{b}_{i-1}\}$ $(i = 1, 2, \ldots, N-1)$ be the limits of a_{nN+i} and b_{nN+i} respectively. Then*

$$|a_{nN+i} - \tilde{a}_i| = O(\sigma^{-n}) \quad and \quad |b_{nN+i} - \tilde{b}_i| = O(\sigma^{-n}),$$

where $\sigma = \min_{z_i \notin E_0}\{\rho(T(z_i))\}$. Here z_i, $i = 1, 2, \ldots, N-1$, are the zeros of $T'(z)$. □

REMARK 6. The Szegő class on a set E consists of all measures $\mu(x)$ with absolutely continuous part $\sigma(x)$ having the property that $\ln \sigma(x) \in L_1(\mu_E)$. Denote by S_{E_0} and S_E the Szegő classes on E_0 and E respectively. Then it follows from (2.6) that

$$\frac{1}{N\pi} \int_E |\ln \sigma(x)| \frac{|T'(x)|}{\sqrt{1 - T(x)^2}} \, dx$$

$$= \frac{1}{N\pi} \int_E \{|\ln |W(x)|| + |\ln \sigma_0(Tx)|\} \frac{|T'(x)|}{\sqrt{1 - T(x)^2}} \, dx,$$

which implies that $\phi(T, W): S_{E_0} \to S_E$.

REMARK 7. If we choose μ_0 such that it is absolutely continuous on $[-1, 1]$ with weight $\sigma_0(x) = \frac{1}{\pi}(1 - x^2)^{-1/2}/\rho(x)$, where ρ is a positive polynomial of degree l on $[-1, 1]$, then the $\{p_n(x; \mu_0)\}$ are Bernstein-Szegő polynomials on $[-1, 1]$ (Szegő [25, §2.6]). The measure μ on $E = T^{-1}([-1, 1])$ is also absolutely continuous on E with weight

$$\sigma(x) = \frac{1}{\pi} \frac{|W(x)|}{\sqrt{1 - T(x)^2}} \frac{1}{\rho(T(x))}.$$

If $W(x)$ has all its zeros at the endpoints of the intervals of E, then this measure gives Bernstein-Szegő polynomials on E (Akhiezer [1]).

VI. Sieved orthogonal polynomials. If we take

$$T(x) = c\hat{T}_N(x) = c2^{-N+1}T_N(x),$$

where $T_N(x)$ is the Chebyshev polynomial of the first kind of degree N and $W = T'(x)/N = c\hat{U}_{N-1}(x) = c2^{-N+1}U_{N-1}(x)$, where U_{N-1} is the Chebyshev polynomial of the second kind of degree $N - 1$, then T and W satisfy all the conditions of Lemma 1 if $c \geq 2^{N-1}$. This special case gives rise to so-called sieved orthogonal polynomials, studied by Al-Salam, Allaway, and Askey [2], Charris and Ismail [9] and Ismail [15–17]. The use of the polynomial mapping T gives a new approach to studying sieved orthogonal polynomials and enables one to rediscover many results found by the previously mentioned authors. If we start with a weight function $\sigma_0(x)$ on $[-1, 1]$, then from (2.6) we see that the orthogonal polynomials on $T^{-1}[-1, 1]$ have the weight function

$$\sigma(x) = c|\hat{U}_{N-1}(x)|\sigma_0(c\hat{T}_N(x)).$$

Alternatively letting $\sigma_0^{(1)}$ be the weight function for the associated polynomials we find from (2.17) and (2.14) that for appropriate $\sigma_0^{(1)}(x)$,

$$d\mu^{(1)}(x) = \left(\frac{a_1^0}{a_1}\right)^2 \frac{1}{c|\hat{U}_{N-1}(x)|}\sigma_0^{(1)}(c\hat{T}_N(x))\, dx + \sum_{i=1}^{N-1} \rho_i\delta(x - x_i)\, dx,$$

where x_i, $i = 1, 2, \ldots, N - 1$, are the zeros of $U_{N-1}(x)$ and

$$\rho_i = \frac{(a_1^0)^2 S(c\hat{T}_N(x_i); \mu_0) - c\hat{T}_N(x_i) + b_0^0}{a_1^2 c\hat{U}_{N-1}^1(x_i)}.$$

Specific choices of σ_0 or appropriate $\sigma_0^{(1)}$ (see Remark 3) (Ultraspherical weights, Pollaczek weights) then give the formula for the weight function obtained in [2, 15–16]. If the spectral measure μ_0 on $[-1, 1]$ is symmetric, then one can explicitly find the recurrence coefficients for the new orthogonal polynomials in terms of the recurrence coefficients of the polynomials on $[-1, 1]$.

THEOREM 12. *Suppose* $T(x) = c\hat{T}_N(x)$ *and* $W(x) = c\hat{U}_{N-1}(x)$ *with* $c \geq 2^{N-1}$ *and let* $b_n^0 = 0$ $(n = 0, 1, 2, \ldots)$. *Then the recurrence coefficients* $\{a_{n+1}, b_n: n = 0, 1, 2, \ldots\}$ *of the new orthogonal system are given by*

(6.1) $$b_n = 0,$$

(6.2) $$a_{nN+j} = \frac{1}{2}, \qquad j = 2, \ldots, N - 1,$$

(6.3) $$a_{nN}^2 = \frac{a_n^0}{c}2^{N-2}\frac{p_{n-1}(c2^{-N+1}; \mu_0)}{p_n(c2^{-N+1}; \mu_0)},$$

(6.4) $$a_{nN+1}^2 = \frac{1}{2} - \frac{a_n^0}{c}2^{N-2}\frac{p_{n-1}(c2^{-N+1}; \mu_0)}{p_n(c2^{-N+1}; \mu_0)}.$$

PROOF. First consider the rational function

$$\frac{\hat{p}_{N-1}^{(1)}(x; \mu)}{\hat{p}_N(x; \mu)} = \frac{U_{N-1}(x)}{T_N(x)},$$

where the equality follows from Theorems 1 and 2. The coefficients of the (Jacobi) continued fraction are $\{b_0; a_n^2, b_n : n = 1, \ldots, N-1\}$ and because we are dealing with Chebyshev polynomials we have

$$a_1 = 1/\sqrt{2}; \qquad b_0 = b_1 = 0,$$

$$a_j = \frac{1}{2}; \qquad b_j = 0, \qquad j = 2, 3, \ldots, N-1.$$

Suppose next that we have proved

$$a_{nN+j} = \frac{1}{2}, \quad b_{nN+j} = 0 \qquad (j = 2, 3, \ldots, N-1); \quad b_{nN+1} = 0$$

for every $n = 0, 1, \ldots, k-1$. Formula (3.10) becomes for $n = k$

$$a_{kN}^2 \hat{p}_{N-2}^{(kN+1)}(z; \mu) + a_{kN+1}^2 \hat{p}_{N-2}^{(kN+2)}(z; \mu) = 2^{-N+1} U_{N-2}(z).$$

Comparing the leading coefficients gives

(6.5) $$a_{kN}^2 + a_{kN+1}^2 = \tfrac{1}{2}$$

and by hypothesis $\hat{p}_{N-2}^{(lN-N+1)}(z; \mu) = 2^{-N+2} U_{N-2}(x)$; therefore

$$a_{kN+1}^2 \hat{p}_{N-2}^{(kN+2)}(z; \mu) = 2^{-N+2} U_{N-2}(x)\{\tfrac{1}{2} - a_{kN}^2\}$$

from which we find $\hat{p}_{N-2}^{(kN+2)}(z; \mu) = 2^{-N+2} U_{N-2}(x)$. The coefficients in the (Jacobi) continued fraction for

$$\frac{\hat{p}_{N-2}^{(kN+2)}(x; \mu)}{\hat{p}_{N-1}^{(kN+1)}(x; \mu)} = \frac{1}{2} \frac{U_{N-2}(x)}{U_{N-1}(x)}$$

are $\{b_{kN+1}; a_{kN+j}^2, b_{kN+j} : j = 2, \ldots, N-1\}$ and since we are dealing with Chebyshev polynomials of the second kind we have

$$b_{kN+j} = 0, \qquad j = 1, \ldots, N-1,$$

$$a_{kN+j} = \tfrac{1}{2}, \qquad j = 2, \ldots, N-1.$$

Also $b_{kN} = 0$ by (3.7), so that (6.1) and (6.2) are true for every n. Because of (6.1) and (6.2) we have $\hat{p}_{N-2}^{(nN+1)}(z; \mu) = 2^{-N+2} U_{N-2}(z)$ so that (4.7) becomes

$$a_{nN}^2 = -\frac{a_n^0}{c} \frac{2^{N-2}}{U_{N-2}(z_i)} \frac{p_{n-1}(T(z_i); \mu_0)}{p_n(T(z_i); \mu_0)},$$

where z_i is a zero of U_{N-1}. Now $U_{N-2}(z_i) = -T_N(z_i) = \pm 1$ from which (6.3) follows. The result in (6.4) is a combination of (6.3) and (6.5). $\square$

The formulas for the intermediate polynomials $p_{nN+j}(x; \mu)$ and $p_{nN+j-1}^{(1)}(x; \mu)$ in Theorem 7 become quite simple. Indeed, because of (6.1) and (6.2) we have for $j = 1, \ldots, N-2$

$$2a_{(n+1)N} p_{N-j-1}^{(nN+j+1)}(x; \mu) = U_{N-j-1}(x), \qquad p_{j-1}^{(nN+1)}(x; \mu) = U_{j-1}(x)$$

so that under the conditions of Theorem 12 we have for $1 \le j < N$

$$p_{nN+j}(x; \mu) = \frac{2^{N-1}}{c} \frac{a_{n+1}^0}{a_{nN+1}} \left\{ \frac{a_{nN+1}}{a_{nN+n}} U_{N-j-1}(x) p_n(T(x); \mu_0) \right.$$

$$\left. + U_{j-1}(x) p_{n+1}(T(x); \mu_0) \right\} / U_{N-1}(x)$$

and

$$p_{nN+j-1}^{(1)}(x;\mu) = \frac{1}{\sqrt{2}a_1^0}\frac{a_{nN+1}^0}{a_{nN+1}}\left\{\frac{a_{nN+1}}{a_{nN+n}}U_{N-j-1}(x)p_{n-1}^{(1)}(T(x);\mu_0)\right.$$
$$\left. + U_{j-1}(x)p_n^{(1)}(T(x);\mu_0)\right\}.$$

The special case $c = 2^{N-1}$ gives a mapping such that $T^{-1}([-1,1]) = [-1,1]$. The weight function σ may vanish at the zeros of the Chebyshev polynomial U_{N-1}. As for the associated polynomials we see that for appropriate weight functions $\sigma_0^{(1)}$ we may find mass points of $\mu^{(1)}$ in $[-1,1]$. For example if one chooses

$$\sigma_0^{(1)} = \frac{\Gamma(\alpha+\frac{3}{2})(1-x^2)^\alpha}{\sqrt{\pi}\Gamma(\alpha+1)}, \qquad \alpha > 0,$$

then

$$S(\pm 1, \mu_0^{(1)}) = \pm\frac{(\alpha+\frac{1}{2})}{\alpha}$$

[**25**, p. 77] and we find that

$$d\mu^{(1)}(x) = \frac{2(a_1^0)\Gamma(\alpha+\frac{3}{2})}{\sqrt{\pi}\Gamma(\alpha+1)}\frac{(1-T_N^2(x))^\alpha}{|U_{N-1}(x)|}\,dx$$
$$+ \sum_{i=1}^{N-1}\frac{2}{\alpha}(\alpha - (\alpha+2)(a_1^0)^2)\sin^2\left(\frac{i\pi}{N}\right)\delta\left(x - \cos\frac{i\pi}{N}\right)\,dx,$$

where one must choose $(a_1^0)^2 \le \alpha/(\alpha+\frac{1}{2})$ (see Remark 3).

We note that the above calculations easily carry over to the case when $\mu_0^{(1)}$ is a Jacobi weight with $\alpha, \beta > 0$.

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