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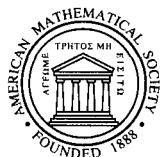
Colloquium Publications

Volume 54, Part 2

# Orthogonal Polynomials on the Unit Circle

## Part 2: Spectral Theory

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