

ORTHOGONAL POLYNOMIALS WITH WEIGHT  
FUNCTION  $(1-x)^\alpha(1+x)^\beta + M\delta(x+1) + N\delta(x-1)$

BY

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ABSTRACT. We study orthogonal polynomials for which the weight function is a linear combination of the Jacobi weight function and two delta functions at 1 and  $-1$ . These polynomials can be expressed as  ${}_4F_3$  hypergeometric functions and they satisfy second order differential equations. They include Krall's Jacobi type polynomials as special cases. The fourth order differential equation for the latter polynomials is derived in a more simple way.

0. **Introduction.** The nonclassical orthogonal polynomials which are eigenfunctions of a fourth order differential operator were classified by H. L. Krall [6], [7]. These polynomials were described in more details by A. M. Krall [5]. The corresponding weight functions are special cases of the classical weight functions together with a delta function at the end point(s) of the interval of orthogonality. A number of A. M. Krall's results can be obtained in a more satisfactory way:

- (a) Jacobi, Legendre and Laguerre type polynomials are connected with each other by quadratic transformations and a limit formula.
- (b) The power series for the Jacobi type polynomials is of  ${}_3F_2$ -type.
- (c) There is a pair of second order differential operators not depending on  $n$  which connect the Jacobi polynomials  $P_n^{(\alpha,0)}(2x-1)$  and the Jacobi type polynomials  $S_n(x)$ . Combination of these two differentiation formulas yields the fourth order equation for  $S_n(x)$ .

It is the first purpose of the present paper to make these comments to [5]. The second purpose is to describe a more general class of Jacobi type polynomials, with weight function  $(1-x)^\alpha(1+x)^\beta +$  linear combination of  $\delta(x+1)$  and  $\delta(x-1)$ . They can be expressed in terms of Jacobi polynomials as  $((a_n x + b_n)d/dx + c_n)P_n^{(\alpha,\beta)}(x)$  for certain coefficients  $a_n, b_n, c_n$  and their power series in  $\frac{1}{2}(1-x)$  is of  ${}_4F_3$  type. Finally, they satisfy a second order differential equation with polynomial coefficients depending on  $n$ , but of bounded degree, thus generalizing the known result for the Jacobi type polynomials  $S_n(x)$  (cf. Littlejohn & Shore [9]) and providing further examples for the general theory

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of orthogonal polynomials with this property, cf. Atkinson & Everitt [1], Hahn [4].

There are two further motivations for studying this class of orthogonal polynomials. First, as pointed out by Nikishin [11], any new set of orthogonal polynomials for which explicit expressions are available, is welcome because it provides a testing ground for the general theory of orthogonal polynomials. Second, orthogonal polynomials expressible in terms of certain hypergeometric functions may yield possibly new formulas for these hypergeometric functions. I want to acknowledge useful comments by D. Stanton on this topic, which led to section 8 of this paper.

**1. Jacobi polynomials.** We summarize the properties of Jacobi polynomials we need, cf. [3, §10.8].

Let  $\alpha, \beta > -1$ . *Jacobi polynomials*  $P_n^{(\alpha, \beta)}(x)$  are orthogonal polynomials on the interval  $[-1, 1]$  with respect to the weight function  $(1-x)^\alpha(1+x)^\beta$  and with the normalization

$$(1.1) \quad P_n^{(\alpha, \beta)}(1) = (\alpha + 1)_n / n!.$$

Symmetry properties:

$$(1.2) \quad P_n^{(\alpha, \beta)}(-x) = (-1)^n P_n^{(\beta, \alpha)}(x).$$

Differentiation formula:

$$(1.3) \quad \frac{d}{dx} P_n^{(\alpha, \beta)}(x) = \frac{1}{2}(n + \alpha + \beta + 1) P_{n-1}^{(\alpha+1, \beta+1)}(x).$$

Rodrigues formula:

$$(1.4) \quad (-1)^n 2^n n! (1-x)^\alpha (1+x)^\beta P_n^{(\alpha, \beta)}(x) = (d/dx)^n ((1-x)^{n+\alpha} (1+x)^{n+\beta}).$$

Power series expansion:

$$(1.5) \quad P_n^{(\alpha, \beta)}(x) = \frac{(\alpha + 1)_n}{n!} {}_2F_1 \left( -n, n + \alpha + \beta + 1; \alpha + 1; \frac{1-x}{2} \right) \\ = \frac{(\alpha + 1)_n}{n!} \sum_{k=0}^n \frac{(-n)_k (n + \alpha + \beta + 1)_k}{(\alpha + 1)_k k!} \left( \frac{1-x}{2} \right)^k.$$

Laguerre polynomials:

$$(1.6) \quad L_n^\alpha(x) := \lim_{\beta \rightarrow \infty} P_n^{(\alpha, \beta)}(1 - 2\beta^{-1}x),$$

orthogonal on  $[0, \infty)$  with respect to the weight function  $e^{-x}x^\alpha$ .

Differential equation:

$$(1.7) \quad [(1-x^2)d^2/dx^2 + (\beta - \alpha - (\alpha + \beta + 2)x)d/dx] P_n^{(\alpha, \beta)}(x) \\ = -n(n + \alpha + \beta + 1) P_n^{(\alpha, \beta)}(x).$$

2. **Definition.** Fix  $M, N \geq 0$  and  $\alpha, \beta > -1$ . For  $n = 0, 1, 2, \dots$  define

$$(2.1) \quad P_n^{\alpha, \beta, M, N}(x) := ((\alpha + \beta + 1)_n / n!)^2 [(\alpha + \beta + 1)^{-1} (B_n M(1 - x) - A_n N(1 + x)) d/dx + A_n B_n] P_n^{(\alpha, \beta)}(x),$$

where

$$(2.2) \quad A_n := \frac{(\alpha + 1)_n n!}{(\beta + 1)_n (\alpha + \beta + 1)_n} + \frac{n(n + \alpha + \beta + 1)M}{(\beta + 1)(\alpha + \beta + 1)},$$

$$(2.3) \quad B_n := \frac{(\beta + 1)_n n!}{(\alpha + 1)_n (\alpha + \beta + 1)_n} + \frac{n(n + \alpha + \beta + 1)N}{(\alpha + 1)(\alpha + \beta + 1)}.$$

The case  $\alpha + \beta + 1 = 0$  must be understood by continuity in  $\alpha, \beta$ . By using (1.1) and (1.3) we find

$$(2.4) \quad P_n^{\alpha, \beta, M, N}(1) = \frac{(\alpha + 1)_n}{n!} + \frac{(\beta + 1)_n (\alpha + \beta + 2)_n n M}{n! n! (\beta + 1)}.$$

From (1.2) we have the symmetry

$$(2.5) \quad P_n^{\alpha, \beta, M, N}(-x) = (-1)^n P_n^{\beta, \alpha, N, M}(x).$$

3. **Orthogonality.** Define the measure  $\mu$  on  $[-1, 1]$  by

$$(3.1) \quad \int_{-1}^1 f(x) d\mu(x) := \frac{\Gamma(\alpha + \beta + 2)}{2^{\alpha + \beta + 1} \Gamma(\alpha + 1) \Gamma(\beta + 1)} \int_{-1}^1 f(x) (1 - x)^\alpha (1 + x)^\beta dx + Mf(-1) + Nf(1), \quad f \in C([-1, 1]).$$

**THEOREM 3.1.** *The polynomials  $P_n^{\alpha, \beta, M, N}(x)$  are orthogonal polynomials on the interval  $[-1, 1]$  with respect to the measure  $\mu$  and with the normalization (2.4).*

**Proof.** By (2.1) and (2.3),  $P_n^{\alpha, \beta, M, N}(x)$  is a polynomial of degree  $\leq n$ , not identically zero.

In order to prove the orthogonality first assume  $n \geq 2$ . Observe that the polynomials  $(1 + x)^k (1 - x)^{n - k - 1}$  ( $k = 0, 1, \dots, n - 1$ ) form a basis for the space of polynomials of degree  $\leq n - 1$ . If  $1 \leq k \leq n - 2$  then

$$\int_{-1}^1 P_n^{\alpha, \beta, M, N}(x) (1 - x)^{n - k - 1} (1 + x)^k d\mu(x) = 0$$

by integration by parts and the orthogonality property of Jacobi polynomials. Now consider  $k = 0$ :

$$I := \int_{-1}^1 P_n^{\alpha, \beta, M, N}(x) (1 - x)^{n - 1} d\mu(x).$$

The continuous part of  $\mu$  yields a contribution

$$I_1 := \frac{\Gamma(\alpha + \beta + 1)(n + \alpha + \beta + 1)B_n M((\alpha + \beta + 1)_n)^2}{2^{\alpha + \beta + 3 - n} \Gamma(\alpha + 1) \Gamma(\beta + 1) (n!)^2} \times \int_{-1}^1 P_{n-1}^{(\alpha+1, \beta+1)}(x) (1-x)^{\alpha+1} (1+x)^\beta dx,$$

where we used (1.3) and the orthogonality property of Jacobi polynomials. Now substitute (1.4), integrate by parts and evaluate the resulting beta integral:

$$I_1 = (-1)^{n-1} 2^{n-1} (\alpha + 1)_n B_n M(\alpha + \beta + 1)_n / (n!)^2.$$

The discrete part of  $\mu$  yields a contribution  $-I_1$  to  $I$  (use (1.5), (1.2) and (1.1)) so  $I = 0$ . The case  $k = n - 1$  follows from the case  $k = 0$  by (2.5).

Finally consider the case  $n = 1$ . By (1.5) we have

$$P_1^{(\alpha, \beta)}(x) = (\alpha + 1) - \frac{1}{2}(\alpha + \beta + 2)(1 - x),$$

so

$$P_1^{\alpha, \beta, M, N}(x) = -\frac{1}{2}(\alpha + 1)(\alpha + \beta + 1)B_1(1 - x) + \frac{1}{2}(\beta + 1)(\alpha + \beta + 1)A_1(1 + x).$$

Hence

$$\int_{-1}^1 P_1^{\alpha, \beta, M, N}(x) d\mu(x) = 0$$

by evaluating the beta integrals.  $\square$

**4. Special cases.** Of course:

$$(4.1) \quad P_n^{\alpha, \beta, 0, 0}(x) = P_n^{(\alpha, \beta)}(x).$$

Next we have

$$(4.2) \quad P_n^{\alpha, \beta, M, 0}(x) = \left[ 1 + \frac{M(\beta + 1)_n (\alpha + \beta + 1)_n}{(\alpha + 1)_n n! (\alpha + \beta + 1)} \left( (1-x) \frac{d}{dx} + \frac{n(n + \alpha + \beta + 1)}{\beta + 1} \right) \right] P_n^{(\alpha, \beta)}(x),$$

$$(4.3) \quad S_n(x) = M P_n^{\alpha, 0, (\alpha+1)/M, 0}(2x - 1) = ((1-x)d/dx + n(n + \alpha + 1) + M) P_n^{(\alpha, 0)}(2x - 1),$$

where  $S_n(x)$  are Krall's [5, §16,17] Jacobi type polynomials, orthogonal with respect to the measure  $((1-x)^\alpha + M^{-1}\delta(x)) dx$  on  $[0, 1]$ .

Furthermore,

$$(4.4) \quad P_n^{\alpha, \alpha, M, M}(x) = \left( 1 + \frac{M(2\alpha + 2)_n n}{(\alpha + 1)n!} \right) \cdot \left[ 1 + \frac{M(2\alpha + 1)_n}{n! (2\alpha + 1)} \left( -2x \frac{d}{dx} + \frac{n(n + 2\alpha + 1)}{\alpha + 1} \right) \right] P_n^{(\alpha, \alpha)}(x),$$

$$(4.5) \quad P_n^{(\alpha)}(x) = \frac{\alpha^2}{\alpha + \frac{1}{2}n(n + 1)} P_n^{0, 0, 1/(2\alpha), 1/(2\alpha)}(x) = (-x d/dx + \alpha + \frac{1}{2}n(n + 1)) P_n(x),$$

where  $P_n^{(\alpha)}(x)$  are Krall's [5, §4.5] Legendre type polynomials, orthogonal with respect to the measure  $\frac{1}{2}(\alpha + \delta(x - 1) + \delta(x + 1)) dx$  on  $[-1, 1]$ .

By using Theorem 3.1 we obtain the quadratic transformations

$$(4.6) \quad \frac{P_{2n}^{\alpha, \alpha, M, M}(x)}{P_{2n}^{\alpha, \alpha, M, M}(1)} = \frac{P_n^{\alpha, -1/2, 0, 2M}(2x^2 - 1)}{P_n^{\alpha, -1/2, 0, 2M}(1)},$$

$$(4.7) \quad \frac{P_{2n+1}^{\alpha, \alpha, M, M}(x)}{P_{2n+1}^{\alpha, \alpha, M, M}(1)} = \frac{xP_n^{\alpha, 1/2, 0, (4\alpha+6)M}(2x^2 - 1)}{P_n^{\alpha, 1/2, 0, (4\alpha+6)M}(1)}.$$

In particular, these formulas connect Krall's Legendre and Jacobi type polynomials with each other.

$$(4.8) \quad L_n^{\alpha, N}(x) := \lim_{\beta \rightarrow \infty} P_n^{\alpha, \beta, 0, N}(1 - 2\beta^{-1}x) = \left[ 1 + \frac{N(\alpha + 1)_n}{n!} \left( \frac{d}{dx} + \frac{n}{\alpha + 1} \right) \right] L_n(x),$$

orthogonal polynomials on the interval  $[0, \infty)$  with respect to the measure  $((\Gamma(\alpha + 1))^{-1}e^{-x}x^\alpha + N\delta(x)) dx$  on the interval  $[0, \infty)$  and with the normalization  $L_n^{\alpha, N}(0) = (\alpha + 1)_n/n!$  (cf. (1.6), (2.5), (4.2) and Theorem 3.1).

$$(4.9) \quad R_n(x) = RL_n^{0, R^{-1}}(x),$$

where  $R_n(x)$  are Krall's [5, §10,11] Laguerre type polynomials, orthogonal with respect to the measure  $(e^{-x} + R^{-1}\delta(x)) dx$  on  $[0, \infty)$ .

**5. Expression as hypergeometric series.** By (1.5) and (2.1) we have

$$\begin{aligned} & \frac{n! n! n!}{(\alpha + 1)_n(\alpha + \beta + 1)_n(\alpha + \beta + 1)_n} P_n^{\alpha, \beta, M, N}(1 - 2x) \\ &= [(\alpha + \beta + 1)^{-1}(-B_n Mx + A_n N(1 - x))d/dx + A_n B_n] \\ & \cdot \left( \sum_{k=0}^n \frac{(-n)_k (n + \alpha + \beta + 1)_k}{(\alpha + 1)_k k!} x^k \right). \end{aligned}$$

By straightforward calculations we obtain

$$(5.1) \quad \begin{aligned} \frac{P_n^{\alpha, \beta, M, N}(1 - 2x)}{P_n^{\alpha, \beta, M, N}(1)} &= \frac{(\alpha + 1)_n(\alpha + \beta + 1)_n}{(\alpha + 1)(\beta + 1)_n n! A_n} \\ & \cdot \sum_{k=0}^n \frac{(-n)_k (n + \alpha + \beta + 1)_k}{(\alpha + 2)_k k!} \left[ -MB_n(\alpha + \beta + 1)^{-1}k^2 \right. \\ & + (NA_n(\alpha + \beta + 1)^{-1}\beta - MB_n(\alpha + \beta + 1)^{-1}(\alpha + 1) + A_n B_n)k \\ & \left. + \frac{(\alpha + 1)(\beta + 1)_n n!}{(\alpha + 1)_n(\alpha + \beta + 1)_n} A_n \right] x^k. \end{aligned}$$

For  $M, N > 0$  this becomes

$$(5.2) \quad \frac{P_n^{\alpha, \beta, M, N}(1 - 2x)}{P_n^{\alpha, \beta, M, N}(1)} = {}_4F_3 \left( \begin{matrix} -n, n + \alpha + \beta + 1, -a_n + 1, b_n + 1 \\ \alpha + 2, -a_n, b_n \end{matrix} \middle| x \right),$$

where  $a_n > n, b_n > 0$  and

$$a_n b_n = \frac{(\alpha + 1)(\alpha + \beta + 1)(\beta + 1)_n n! A_n}{(\alpha + 1)_n (\alpha + \beta + 1)_n M B_n},$$

$$a_n - b_n = \beta N M^{-1} A_n B_n^{-1} + (\alpha + \beta + 1) M^{-1} A_n - \alpha - 1.$$

For  $M = 0, N \neq 0$ :

$$(5.3) \quad \frac{P_n^{\alpha, \beta, 0, N}(1 - 2x)}{P_n^{\alpha, \beta, 0, N}(1)} = {}_3F_2 \left( \begin{matrix} -n, n + \alpha + \beta + 1, c_n + 1 \\ \alpha + 2, c_n \end{matrix} \middle| x \right),$$

where

$$c_n = \frac{(\alpha + 1)(\beta + 1)_n n!}{(N(\alpha + \beta + 1)^{-1} \beta + B_n)(\alpha + 1)_n (\alpha + \beta + 1)_n}.$$

For  $N = 0, M \neq 0$ :

$$(5.4) \quad \frac{P_n^{\alpha, \beta, M, 0}(1 - 2x)}{P_n^{\alpha, \beta, M, 0}(1)} = {}_3F_2 \left( \begin{matrix} -n, n + \alpha + \beta + 1, -(\alpha + \beta + 1)M^{-1}A_n + 1 \\ \alpha + 1, -(\alpha + \beta + 1)M^{-1}A_n \end{matrix} \middle| x \right).$$

Combination of (4.3), (2.5) and (5.3) yields Krall’s power series expansion [5, §16]. Combination of (4.6), (4.7), (2.5) and (5.4) yields power series expansion in  $x$  for  $P_n^{\alpha, \beta, M, M}(x)$ , cf. [5, §4].

**6. Second order differential equations.** In view of the observations in Atkinson & Everitt [1, §6] and the definition (3.1) of our orthogonality measure it is no surprise that the polynomials  $P_n^{\alpha, \beta, M, N}$  will satisfy a linear second order ordinary differential equation with polynomial coefficients,  $n$ -dependent but of bounded degree. Hahn [4, §6] points out that, if  $\{u_n\}$  and  $\{y_n\}$  are systems of orthogonal polynomials and  $u_n = y_n + q_n y'_n$  for certain first degree polynomials  $q_n$ , then the  $y_n$ ’s satisfy second order o.d.e.’s of the above type. Our relation (2.1) has this form, but here Hahn’s observation yields nothing new, since the second order o.d.e. for the  $P_n^{(\alpha, \beta)}$ ’s is already well-known (cf. (1.7)). However, we can prove:

PROPOSITION 6.1 *Let  $\{u_n\}$  and  $\{y_n\}$  be systems of orthogonal polynomials such that*

$$(6.1) \quad u_n = p_n y_n + q_n y'_n,$$

$$(6.2) \quad y''_n + \alpha_n y'_n + \beta_n y_n = 0,$$

where  $p_n, q_n, \alpha_n, \beta_n$  are rational functions which are quotients of polynomials of bounded degree. Then

$$(6.3) \quad y_n = r_n u_n + s_n u'_n,$$

$$(6.4) \quad u''_n + \gamma_n u'_n + \delta_n u_n = 0,$$

for certain rational functions  $r_n, s_n, \gamma_n, \delta_n$  which are quotients of polynomials of bounded degree.

**Proof.** Clearly, we only need to prove the proposition for sufficiently large  $n$  and under the assumption that  $q_n$  is not identically zero. Eliminate  $y'_n$  and  $y''_n$  from (6.1), (6.2) and the equation obtained by differentiating (6.1) once. Then we obtain

$$(q_n(p'_n - \beta_n q_n) - p_n(p_n + q'_n - \alpha_n q_n))y_n = (-p_n + \alpha_n q_n - q'_n)u_n + q_n u'_n.$$

Here the coefficient of  $y_n$  is not identically zero for sufficiently large  $n$ , because, otherwise, not all zeros of  $u_n$  would be simple, in contradiction to Szegő [12, Theorem 3.3.1]. This proves (6.3). Next eliminate  $y_n$  and  $y'_n$  from (6.1), (6.3) and the first derivative of (6.3). Then we obtain

$$s_n u''_n + (p_n s_n + q_n(r_n + s'_n))u'_n + (p_n r_n + q_n r'_n - 1)u_n = 0.$$

Since we assumed  $q_n \neq 0$ , we have  $s_n \neq 0$ . This proves (6.4).  $\square$

Now apply Prop. 6.1 to the case  $y_n = P_n^{\alpha, \beta}, u_n = P_n^{\alpha, \beta, M, N}$ . It follows from (2.1) and (1.7) that

$$(6.5) \quad (a_n(x)d/dx + b_n(x))P_n^{\alpha, \beta, M, N}(x) = ((\alpha + \beta + 1)/n!)^2 c_n(x)P_n^{\alpha, \beta}(x),$$

where

$$a_n(x) := (B_n M - A_n N - (B_n M + A_n N)x)(1 - x^2),$$

$$b_n(x) := (\alpha + \beta + 1)(B_n M + A_n N + A_n B_n)x^2 + 2((\alpha + 1)A_n N - (\beta + 1)B_n M)x + (\beta - \alpha + 1)B_n M + (\alpha - \beta + 1)A_n N - A_n B_n(\alpha + \beta + 1),$$

$$c_n(x) := A_n B_n b_n(x) - n(n + \alpha + \beta + 1) \times (\alpha + \beta + 1)^{-1}(B_n M - A_n N - (B_n M + A_n N)x)^2.$$

From (6.5) one can calculate the second order o.d.e. for  $P_n^{\alpha, \beta, M, N}$ . Littlejohn & Shore [9] derive special cases of this o.d.e. for the polynomials (4.3), (4.5), (4.9) in a different, more complicated way.

**7. Fourth order differential equation for Krall's Jacobi type polynomials.** Fix  $\alpha > -1$  and  $M > 0$ . Let  $S_n(x)$  be defined by (4.3). Combination of (4.3) and (1.7) yields

$$(7.1) \quad S_n(x) = [x(x - 1)d^2/dx^2 + (\alpha + 1)xd/dx + M]P_n^{(\alpha, 0)}(2x - 1).$$

Observe that, for arbitrary polynomials  $f, g$  we have

$$(7.2) \quad \int_0^1 g(x)[x(x - 1)d^2/dx^2 + (\alpha + 1)xd/dx + M]f(x)((1 - x)^\alpha + M^{-1}\delta(x)) dx = \int_0^1 f(x)[x(x - 1)d^2/dx^2 + ((\alpha + 3)x - 2)d/dx + M + \alpha + 1]g(x)(1 - x)^\alpha dx.$$

Formulas (7.1), (7.2) and the orthogonality properties of  $S_n(x)$  and  $P_n^{(\alpha,0)}(2x-1)$  imply:

$$(7.3) \quad \begin{aligned} & ((n + \alpha + 1)(n + 1) + M)(n(n + \alpha) + M)P_n^{(\alpha,0)}(2x - 1) \\ &= \left[ x(x - 1) \frac{d^2}{dx^2} + ((\alpha + 3)x - 2) \frac{d}{dx} + M + \alpha + 1 \right] S_n(x), \end{aligned}$$

where the coefficient of  $P_n^{(\alpha,0)}(2x-1)$  is obtained by comparing the coefficients of  $x^n$  at both sides of (7.3). Combination of (7.1) and (7.3) yields.

**THEOREM 7.1.** *The polynomials  $S_n(x)$  are eigenfunctions of a fourth order differential operator with polynomial coefficients not depending on  $n$ .*

A calculation leads to the explicit form of Krall’s [5, §14] differential equation.

Recently Littlejohn [8], proved that the polynomials  $P_n^{0,0,M,N}(x)$  (notation of the present paper) are eigenfunctions of a sixth order differential operator. The above techniques also apply to this case and would lead to an eighth order differential operator.

**8. Quadratic transformations for hypergeometric functions.** Many of the formulas for terminating  ${}_2F_1$  hypergeometric functions can be derived from similar formulas for Jacobi polynomials (use (1.5)) obtained by properties of orthogonal polynomials. Similarly, results for the polynomials  $P_n^{\alpha,\beta,M,N}$  obtained here can be translated in terms of hypergeometric functions of the form

$${}_4F_3 \left( \begin{matrix} -n, b, \theta_1 + 1, \theta_2 + 1 \\ c, \theta_1, \theta_2 \end{matrix} \middle| x \right), \quad n = 0, 1, 2, \dots$$

(use (5.2)). In particular, (4.6) and (4.7) will imply quadratic transformations for such  ${}_4F_3$ -functions. In this section we will give an independent derivation of these quadratic transformations, also in the nonterminating case.

Our starting point is

$$(8.1) \quad \begin{aligned} & (1-x)^{-a} {}_4F_3 \left( \begin{matrix} \frac{1}{2}a, \frac{1}{2}a + \frac{1}{2}, b, d + 1 \\ c, b + a + 2 - c, d \end{matrix} \middle| -\frac{4x}{(1-x)^2} \right) \\ &= {}_5F_4 \left( \begin{matrix} a, 1 + a - c, c - 1 - b, \theta_1 + 1, \theta_2 + 1 \\ c, a + b + 2 - c, \theta_1, \theta_2 \end{matrix} \middle| x \right), \end{aligned}$$

where

$$\theta_1 + \theta_2 = a, \quad \theta_1 \theta_2 = \frac{d(1 + a - c)(c - 1 - b)}{d - b}$$

(formula due to D. Stanton, private communication). For the proof expand the



left hand side as a power series. On letting  $b \rightarrow \infty$  in (8.1) we obtain

$$(8.2) \quad (1-x)^{-a} {}_3F_2\left(\begin{matrix} \frac{1}{2}a, \frac{1}{2}a + \frac{1}{2}, d+1 \\ c, d \end{matrix} \middle| -\frac{4x}{(1-x)^2}\right) \\ = {}_4F_3\left(\begin{matrix} a, 1+a-c, \theta_1+1, \theta_2+1 \\ c, \theta_1, \theta_2 \end{matrix} \middle| -x\right),$$

where  $\theta_1 + \theta_2 = a$ ,  $\theta_1\theta_2 = d(1+a-c)$ . Observe that (8.1) tends to formula (22) in Niblett [10] as  $b \rightarrow \infty$  and to formula 4.5 (1) in [2] as  $d \rightarrow b$ .

A linear transformation formula is given by

$$(8.3) \quad (1-x)^{-a} {}_4F_3\left(\begin{matrix} a, b, d+1, e+1 \\ c, d, e \end{matrix} \middle| \frac{x}{x-1}\right) = {}_4F_3\left(\begin{matrix} a, c-b-2, \theta_1+1, \theta_2+1 \\ c, \theta_1, \theta_2 \end{matrix} \middle| x\right),$$

where

$$\theta_1 + \theta_2 = \frac{(d+e)(b^2 - bc + 2b) + de(-2b + 2c - 3) + (1-c)b}{(d-b)(e-b)}, \\ \theta_1\theta_2 = \frac{de(c-b-1)(c-b-2)}{(d-b)(e-b)}.$$

For the proof again expand the left hand side. A limit case of (8.3) is

$$(8.4) \quad (1-x)^{-a} {}_3F_2\left(\begin{matrix} a, b, d+1 \\ c, d \end{matrix} \middle| \frac{x}{x-1}\right) \\ = {}_3F_2\left(\begin{matrix} a, c-b-1, d(c-b-1)(d-b)^{-1} + 1 \\ c, d(c-b-1)(d-b)^{-1} \end{matrix} \middle| x\right).$$

Substitution of (8.3) and (8.4) into (8.2) yields the two formulas

$$(8.5) \quad {}_3F_2\left(\begin{matrix} a, b, d+1 \\ a+b+\frac{3}{2}, d \end{matrix} \middle| 4x(1-x)\right) = {}_4F_3\left(\begin{matrix} 2a, 2b, \theta_1+1, \theta_2+1 \\ a+b+\frac{3}{2}, \theta_1, \theta_2 \end{matrix} \middle| x\right),$$

where

$$\theta_1 + \theta_2 = \frac{4ab + a + b + d + \frac{1}{2}}{a + b - d + \frac{1}{2}}, \\ \theta_1\theta_2 = \frac{4(a + \frac{1}{2})(b + \frac{1}{2})d}{a + b - d + \frac{1}{2}},$$

$$(8.6) \quad (1-2x) {}_3F_2\left(\begin{matrix} a, b, d+1 \\ d+b+\frac{1}{2}, d \end{matrix} \middle| 4x(1-x)\right) = {}_4F_3\left(\begin{matrix} 2a-1, 2b-1, \theta_1+1, \theta_2+1 \\ a+b+\frac{1}{2}, \theta_1, \theta_2 \end{matrix} \middle| x\right),$$

where

$$\theta_1 + \theta_2 = \frac{4ab - a - b - d + \frac{1}{2}}{a + b - d - \frac{1}{2}}$$

$$\theta_1 \theta_2 = \frac{4(a - \frac{1}{2})(b - \frac{1}{2})d}{a + b - d - \frac{1}{2}}.$$

Formulas (8.5) and (8.6) imply (4.6) and (4.7), respectively.

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