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ORTHOGONAL RATIONAL FUNCTIONS AND INTERPOLATORY PRODUCT RULES ON THE UNIT CIRCLE.
II: QUADRATURE AND CONVERGENCE*

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Abstract: Let $\mathcal{R}$ be the space of rational functions with poles among $\{\alpha_k, 1/\overline{\alpha}_k\}_{k=0}^{\infty}$ with $\alpha_0 = 0$ and $|\alpha_k| < 1$, $k \geq 1$. We consider a sequence $\{\mathcal{R}_n\}_{n=0}^{\infty}$ of nested subspaces with $\cup_{n=0}^{\infty}\mathcal{R}_n = \mathcal{R}$. First we recall from part I how to find orthogonal bases for $\mathcal{R}$ for a positive measure on the unit circle. These are used in the construction of interpolatory quadrature rules for integrals with respect to a complex measure on the unit circle. Integration for the $(2n+1)$-point rule is exact for all $f \in \mathcal{R}_n$. Also their convergence is discussed as $n \to \infty$. Finally we discuss the convergence of multipoint rational approximants to the Riesz-Herglotz transform associated with such a complex measure.

Keywords: orthogonal rational functions, multipoint Padé approximants, numerical quadrature.

AMS Classification: 41A55, 33C45

1 Introduction

Throughout the paper we use the following notations for the unit circle and the open unit disk: $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$, $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$; where $\mathbb{C}$ is the set of complex numbers and $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. The space of polynomials of degree at most n is denoted as Π_n and Π is the space of all polynomials. For any pair of nonnegative integers (m, n), we denote by $\Lambda_{m,n}$ the linear space of all Laurent polynomials of the form $L(z) = \sum_{j=-m}^{n} c_j z^j$, $c_j \in \mathbb{C}$, and Λ is the space of all Laurent polynomials.

Let $\alpha = \{\alpha_n : n = 0, 1, \ldots\}$ $(\alpha_0 = 0)$ be an arbitrary sequence in $\mathbb{D}$. We denote the Blaschke factor $\zeta_k(z)$ as

$$\zeta_k(z) = \frac{\overline{\alpha}_k}{|\alpha_k|}\frac{\alpha_k - z}{1 - \overline{\alpha}_k z}, \quad k = 1, 2, \ldots$$

if $\alpha_k \neq 0$ and we set $\zeta_k(z) = z$ if $\alpha_k = 0$. The Blaschke products are given by $B_0 = 1$,

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$B_k(z) = \zeta_1(z)\cdots\zeta_k(z)$. These generate the space $\mathcal{L}_n = \text{span}\{B_0, B_1, \ldots, B_n\}$ and $\mathcal{L} = \cup_{n=0}^{\infty}\mathcal{L}_n$. Setting $\pi_0 = 1$ and $\pi_n(z) = \prod_{k=1}^{n}(1-\overline{\alpha}_k z)$ for $n > 0$, we can represent $R \in \mathcal{L}_n$ as $R = q/\pi_n$ with $q \in \Pi_n$. By introducing the substar transformation, i.e. $f_* = \overline{f(1/\overline{z})}$, we can also define $\mathcal{L}_{n*} = \{f : f_* \in \mathcal{L}_n\} = \text{span}\{1, B_{1*}, \ldots, B_{n*}\}$ and $\mathcal{L}_* = \cup_{n=0}^{\infty}\mathcal{L}_{n*}$. If $R \in \mathcal{L}_{n*}$, then $R = q/\omega_n$ with $q \in \Pi_n$ and where $\omega_0 = 1$ and $\omega_n(z) = \prod_{j=1}^{n}(z-\alpha_j)$ for $n > 0$. Furthermore, let $\mathcal{R} = \mathcal{L} + \mathcal{L}_*$ and for m and n nonnegative integers, denote

$$\mathcal{R}_{m,n} = \mathcal{L}_{m*} + \mathcal{L}_n = \left\{\frac{p}{\omega_m \pi_n} : p \in \Pi_{m+n}\right\}.$$

Note that $B_{n*} = 1/B_n$ and hence $\mathcal{R}_{m,n} = \text{span}\{1/B_m, \ldots, 1/B_1, 1, B_1, \ldots, B_n\}$ and $\mathcal{R}_{0,n} = \mathcal{L}_n$. When all $\alpha_k = 0$, then $\mathcal{R}_{m,n} = \Lambda_{m,n}$ and $\mathcal{L}_n = \Pi_n$.

Given a positive measure on $\mathbb{T}$, we first recall from part I of this paper how to construct orthogonal bases for $\mathcal{R}$. This is done in Section 2. This generalizes an earlier paper by Thron [12] concerning Laurent polynomials (i.e., $\alpha_k = 0$ for all k) to the rational situation ($\{\alpha_k\}$ an arbitrary sequence in $\mathbb{D}$). Such an orthogonal basis will allow us to characterize the nodes $\{x_{j,N}\}_{j=1}^{N}$ which are all on $\mathbb{T}$ and then we can define also unique coefficients $\{A_{j,N}\}_{j=1}^{N}$ in certain quadrature formulas of the form

$$I_N\{f\} = \sum_{j=1}^{N} A_{j,N} f(x_{j,N}), \quad x_{j,N} \in \mathbb{T}. \tag{1.1}$$

These are introduced in Section 3. These formulas are constructed to approximate integrals on the unit circle which are of the form

$$I_\rho\{f\} = \int_{-\pi}^{\pi} f(e^{i\theta})\rho(\theta)d\theta, \tag{1.2}$$

where ρ is a complex function defined on $[-\pi, \pi]$ such that $\int_{-\pi}^{\pi} |\rho(\theta)|d\theta < \infty$. The quadrature formula (1.1) is constructed such that it is exact for any function $f \in \mathcal{R}_{p,q}$, for appropriate subspaces $\mathcal{R}_{p,q}$ (p and q depending on N). Under appropriate conditions for ρ and α, we proved in [5] that the quadrature formulas converge to the exact integral for integrands in the class of Lipschitz continuous functions. In this paper, we make use of a paper by Sloan and Smith [10] to construct quadrature formulas for which this convergence can be extended to the class of functions bounded and integrable on $\mathbb{T}$. At the same time it gives an adaptation of an earlier result concerning interpolatory product integration rules on intervals of the real line [10] to interpolatory product integration rules for the unit circle.

REMARK 1 To obtain quadrature formulas with a maximal domain of validity, i.e. which are exact in $\mathcal{R}_{p,q}$ with the dimension of $\mathcal{R}_{p,q}$ as large as possible, we should take the $x_{j,N} \in \mathbb{T}$ to be the zeros of so called para-orthogonal rational functions on $\mathbb{T}$ with respect to a positive measure. Since we consider here integrals with a complex measure $d\mu(\theta) = \rho(\theta)d\theta$, we need to introduce some auxiliary positive measure $d\psi$ on $\mathbb{T}$. The orthogonal rational functions we shall need here are the ones studied in Part I of this paper. We first recall the necessary results in the next section.

2 Orthogonal rational functions on the unit circle

Suppose ψ is a positive Borel measure on $[-\pi, \pi]$ and define an inner product for functions defined on $\mathbb{T}$ by

$$\langle f, g \rangle = \int_{-\pi}^{\pi} f(e^{i\theta})\overline{g(e^{i\theta})}d\psi(\theta). \tag{2.1}$$

In this section, we recall the construction of sequences $\{\sigma_n\}_0^\infty$ and $\{\tau_n\}_0^\infty$ of functions in $\mathcal{R}$ which are orthogonal with respect to this inner product, and which span the space $\mathcal{R}$. The sequence σ_n is related to the nesting

$$\mathcal{R}_{0,0} \subset \mathcal{R}_{0,1} \subset \mathcal{R}_{1,1} \subset \mathcal{R}_{1,2} \subset \mathcal{R}_{2,2} \subset \cdots \tag{2.2}$$

which means that $\sigma_{2n} \in \mathcal{R}_{n,n} - \mathcal{R}_{n-1,n}$ and $\sigma_{2n+1} \in \mathcal{R}_{n,n+1} - \mathcal{R}_{n,n}$ for $n = 0, 1, \ldots$. Similarly, the sequence τ_n is related to the nesting

$$\mathcal{R}_{0,0} \subset \mathcal{R}_{1,0} \subset \mathcal{R}_{1,1} \subset \mathcal{R}_{2,1} \subset \cdots \tag{2.3}$$

thus $\tau_{2n} \in \mathcal{R}_{n,n} - \mathcal{R}_{n,n-1}$ and $\tau_{2n+1} \in \mathcal{R}_{n+1,n} - \mathcal{R}_{n,n}$ for $n = 0, 1, \ldots$.

Let us consider the table of points

$$\tilde{\alpha} = \{0, \tilde{\alpha}_1, \tilde{\alpha}_2, \tilde{\alpha}_3, \tilde{\alpha}_4, \ldots, \tilde{\alpha}_{2m-1}, \tilde{\alpha}_{2m}, \ldots\} = \{0, \alpha_1, \alpha_1, \alpha_2, \alpha_2, \ldots, \alpha_m, \alpha_m, \ldots\}.$$

Let $\tilde{\zeta}_n(z)$ be the Blaschke factors for the table $\tilde{\alpha}$ and $\tilde{B}_n(z)$ the corresponding Blaschke products. Set $\tilde{\mathcal{L}}_n = \text{span}\{1, \tilde{B}_1, \ldots, \tilde{B}_n\}$ and $\tilde{\mathcal{L}} = \cup_{n=0}^{\infty} \tilde{\mathcal{L}}_n$.

Let $\{\tilde{\phi}_n\}_{n=0}^\infty$ be the orthogonal functions in $\tilde{\mathcal{L}}$ obtained by orthogonalizing the basis $\{1, \tilde{B}_1, \tilde{B}_2, \ldots\}$ corresponding to the table $\tilde{\alpha}$ with respect to the inner product (2.1). Note that

$$\tilde{B}_{2m} = B_m^2 \quad \text{and} \quad \tilde{B}_{2m+1} = B_{m+1}B_m, \quad m \geq 0.$$

Introduce the superstar conjugate as

$$\tilde{\phi}_n^*(z) = \tilde{B}_n(z)\tilde{\phi}_{n*}(z).$$

and the functions

$$\sigma_{2n} = B_{n*}\tilde{\phi}_{2n}^* \quad \text{and} \quad \sigma_{2n+1} = B_{n*}\tilde{\phi}_{2n+1}, \tag{2.4}$$

and

$$\tau_{2n} = \tilde{\phi}_{2n}B_{n*} \quad \text{and} \quad \tau_{2n+1} = \tilde{\phi}_{2n+1}^* B_{(n+1)*}. \tag{2.5}$$

In part I [6] we have proved the following

<u>THEOREM</u> 2.1 *The functions $\{\sigma_n\}_0^\infty$ form a basis for $\mathcal{R}$ which is orthogonal with respect to the inner product (2.1) and it respects the ordering (2.2). Thus $\mathcal{R}_{n-1,n} \perp \sigma_{2n} \in \mathcal{R}_{n,n}$ and $\mathcal{R}_{n,n} \perp \sigma_{2n+1} \in \mathcal{R}_{n,n+1}$.*

The functions $\{\tau_n\}_0^\infty$ form a basis for $\mathcal{R}$ which is orthogonal with respect to the inner product (2.1) and it respects the ordering (2.3). Thus that $\mathcal{R}_{n,n-1} \perp \tau_{2n} \in \mathcal{R}_{n,n}$ and $\mathcal{R}_{n,n} \perp \tau_{2n+1} \in \mathcal{R}_{n+1,n}$.

3 Interpolatory quadrature rules

In the two remaining sections we are concerned with the evaluation of the integral (1.2). Recall that we assume that ρ is a measurable function on $[-\pi,\pi]$ such that

$$\int_{-\pi}^{\pi} |\rho(\theta)| d\theta < \infty. \tag{3.1}$$

For any function σ defined on $[-\pi,\pi]$, we use the following notation

$$I_\sigma\{f\} = \int_{-\pi}^{\pi} f(e^{i\theta})\sigma(\theta)d\theta.$$

Note that f is a function defined on $\mathbb{T}$, but with a slight abuse of notation, we shall occasionally also write $I_\sigma\{f\}$ when f is a function defined on $[-\pi,\pi]$, and we then mean of course $I_\sigma\{f\} = \int_{-\pi}^{\pi} f(\theta)\sigma(\theta)d\theta$.

Now we want to arrive at the construction of quadrature formulas to approximate the integral $I_\rho\{f\}$. Let $X = \{x_{j,n} : j = 1,\ldots,n; n = 1,2,\ldots\}$ be a triangular array of distinct points on $\mathbb{T}$. For a fixed natural number n set $N = 2n+1$ and consider the quadrature formulas of the form

$$I_N^\rho\{f\} = \sum_{j=1}^{N} A_{j,N} f(x_{j,N}). \tag{3.2}$$

The following facts about interpolatory quadrature formulas can be found in [4]. Given the nodes $\{x_{j,N}\}_1^N$, it is always possible to find uniquely defined coefficients $\{A_{j,N}\}_1^N$ (depending on ρ) such that $I_N^\rho\{f\} = I_\rho\{f\}$ for all $f \in \mathcal{R}_{n,n}$. Furthermore, $I_N^\rho\{f\} = I_\rho\{L_N^f\}$ where L_N^f is the unique function in $\mathcal{R}_{n,n}$ satisfying the interpolation conditions

$$L_N^f(x_{j,N}) = f(x_{j,N}), \quad j = 1,2,\ldots,N.$$

In this section we propose a choice for the table X which gives exact integrals in a much larger space and in the next section, it is proved that for this choice, the sequence $I_N^\rho\{f\}$ converges to $I_\rho\{f\}$ for any function bounded on $\mathbb{T}$ and integrable (with respect to ρ).

As we discussed above, the interpolatory quadrature formulas with $N = 2n+1$ nodes are exact in the space $\mathcal{R}_{n,n}$ of dimension $2n+1$. However by a special choice for these nodes, one can make the formulas exact in spaces of dimension $2N-1$. Therefore, the nodes should be chosen as the zeros of so called para-orthogonal functions. The quadrature formulas have then a maximal domain of validity and they are called rational Szegő formulas. For more details about this kind of formulas, see [3, 4]. Thus we need para-orthogonal functions, hence orthogonal functions, hence an inner product, hence some positive measure. If the function ρ in $I_\rho\{f\}$ is a positive weight function, then we can take of course $d\mu(\theta) = \rho(\theta)d\theta$ to define a positive definite inner product. Since however we assumed ρ to be complex, we shall need an auxiliary positive measure ψ. For our purpose we shall take ψ to be absolutely continuous, i.e., it is of the form $d\psi(\theta) = \omega(\theta)d\theta$ with ω a positive weight function. The theory of para-orthogonal functions that we shall apply of course covers this situation. Thus our inner product of Section 2 will be replaced by an inner product of the form

$$\langle f,g\rangle = \int_{-\pi}^{\pi} f(e^{i\theta})\overline{g(e^{i\theta})}\omega(\theta)d\theta \tag{3.3}$$

where $\omega(\theta)$ is a weight function defined on $[-\pi, \pi]$, i.e., $\omega(\theta) > 0$ a.e. on $[-\pi, \pi]$ and $\int_{-\pi}^{\pi} \omega(\theta) d\theta < \infty$. In the remainder of this paper, unless stated otherwise, all inner products, orthogonality relations and L_2-norms will refer to the weight ω i.e., to the inner product (3.3).

Let $\delta_N \in \mathbb{T}$ be an arbitrary unimodular complex number and let $\{x_{j,N}\}_{j=1}^{N}$ be the zeros of the so called para-orthogonal rational function (see [3, 4])

$$\tilde{\phi}_N(z) + \delta_N \tilde{\phi}_N^*(z), \quad \tilde{\phi}_N \in \tilde{\mathcal{L}}_N - \tilde{\mathcal{L}}_{N-1}, \quad \tilde{\phi}_N \perp \tilde{\mathcal{L}}_{N-1}.$$

These zeros $x_{j,N}$ are all simple and on $\mathbb{T}$. Let $Q_N\{f\}$ be the Nth rational Szegő quadrature formula with respect to the weight function $\omega(\theta)$ and table $\tilde{\alpha}$, i.e.,

$$Q_N\{f\} = \sum_{j=1}^{N} w_{j,N} f(x_{j,N}). \tag{3.4}$$

The coefficients $w_{j,N} > 0$ for $j = 1, \ldots, N$ are uniquely defined by the condition

$$Q_N\{f\} = I_\omega\{f\}, \quad \forall f \in \tilde{\mathcal{R}}_{N-1,N-1}.$$

Note that since $F \in \tilde{\mathcal{R}}_{N-1,N-1}$ iff $F = f g_*$ with $f, g \in \mathcal{R}_{n,n}$, it follows that the previous condition is equivalent with

$$Q_N\{fg\} = I_\omega\{fg\} = \langle f, g_* \rangle, \quad \forall f, g \in \mathcal{R}_{n,n}.$$

If $g \in L^2_\omega(\mathbb{T})$, i.e., $\langle g, g \rangle = \|g\|^2_{\omega,2} < \infty$, then its orthogonal projection onto $\mathcal{R}_{n,n}$ can be expressed with the help of an orthonormal basis for $\mathcal{R}_{n,n}$. Such orthogonal bases were described in Section 2, namely $\{\sigma_n\}_0^{2n}$ and $\{\tau_n\}_0^{2n}$ which we now assume to be normalized, thus $\langle \sigma_k, \sigma_l \rangle = \delta_{kl}$ and $\langle \tau_k, \tau_l \rangle = \delta_{kl}$. The orthogonal projection is then given by

$$\begin{aligned} P^g_{2n} &= \sum_{j=0}^{2n} c_j \sigma_j, \quad c_j = \langle g, \sigma_j \rangle \\ &= \sum_{j=0}^{2n} d_j \tau_j, \quad d_j = \langle g, \tau_j \rangle . \end{aligned}$$

LEMMA 3.1 *Given $g \in L^2_\omega(\mathbb{T})$, let P^g_{2n} be its orthogonal projection onto $\mathcal{R}_{n,n}$. Then we have*

$$\langle P^g_{2n}, f \rangle = \langle g, f \rangle, \quad \textit{and} \quad I_\omega\{P^g_{2n} f\} = I_\omega\{gf\}, \quad \forall f \in \mathcal{R}_{n,n}.$$

PROOF. For the first identity, note that since $\{\sigma_k\}_0^{2n}$ is an orthonormal basis for $\mathcal{R}_{n,n}$ it suffices to prove that $\langle P^g_{2n}, \sigma_k \rangle = \langle g, \sigma_k \rangle$ for $k = 0, 1, \ldots, 2n$. This follows from

$$\langle P^g_{2n}, \sigma_k \rangle = \left\langle \sum_{j=0}^{2n} c_j \sigma_j, \sigma_k \right\rangle = \sum_{j=0}^{2n} c_j \langle \sigma_j, \sigma_k \rangle = c_k = \langle g, \sigma_k \rangle$$

so that the first formula is proved.

For the second formula, we note that if $f \in \mathcal{R}_{n,n}$, then f_* is also in $\mathcal{R}_{n,n}$ so that we also have

$$I_\omega\{P^g_{2n} f\} = \langle P^g_{2n}, f_* \rangle = \langle g, f_* \rangle = I_\omega\{fg\}, \quad \forall f \in \mathcal{R}_{n,n}.$$

□

Now setting $g(e^{i\theta}) = \rho(\theta)/\omega(\theta)$ and assuming $g \in L^2_\omega(\mathbb{T})$, i.e.,

$$g(e^{i\theta}) = \frac{\rho(\theta)}{\omega(\theta)} \in L^2_\omega(\mathbb{T}), \tag{3.5}$$

we have

THEOREM 3.2 *Let $\tilde{\phi}_n$ denote the orthogonal functions with respect to the weight ω and the table $\tilde{\alpha}$ as given before. Set $N = 2n+1$ and let $\{x_{j,N}\}_{j=1}^N$ be the zeros of the para-orthogonal function $\tilde{\phi}_N + \delta_N \tilde{\phi}_N^*$, ($\delta_N \in \mathbb{T}$) and $\{w_{j,N}\}_{j=1}^N$ the corresponding weights for the Nth rational Szegő formula (3.4). Suppose the quadrature formula (3.2) is exact in $\mathcal{R}_{n,n}$, i.e.,*

$$I_N^\rho\{f\} \equiv \sum_{j=1}^N A_{j,N} f(x_{j,N}) = I_\rho\{f\} \equiv \int_{-\pi}^{\pi} f(e^{i\theta})\omega(\theta)d\theta, \quad \forall f \in \mathcal{R}_{n,n}. \tag{3.6}$$

Then

$$A_{j,N} = w_{j,N} P_{N-1}^g(x_{j,N}), \quad j = 1, \ldots, N$$

where $P_{N-1}^g = P_{2n}^g$ is the orthogonal projection in $L^2_\omega(\mathbb{T})$ of g onto $\mathcal{R}_{n,n}$ and g is as in (3.5).

PROOF. Set $W_{j,N} = w_{j,N} P_{N-1}^g(x_{j,N})$, $j = 1, \ldots, N$ and define the quadrature formula

$$S_N\{f\} = \sum_{j=1}^N W_{j,N} f(x_{j,N}).$$

Then, for any $f \in \mathcal{R}_{n,n}$, we have by the previous lemma and because the rational Szegő quadrature $Q_N\{F\}$ is exact for all $F \in \tilde{\mathcal{R}}_{N-1,N-1} = \tilde{\mathcal{R}}_{2n,2n}$, hence for all $F = gf$ with $f, g \in \mathcal{R}_{n,n}$,

$$\begin{aligned} S_N\{f\} &= \sum_{j=1}^N w_{j,N} P_{N-1}^g(x_{j,N}) f(x_{j,N}) = Q_N\{P_{N-1}^g f\} = I_\omega\{P_{N-1}^g f\} \\ &= \int_{-\pi}^{\pi} P_{N-1}^g(e^{i\theta}) f(e^{i\theta})\omega(\theta)d\theta = \int_{-\pi}^{\pi} g(e^{i\theta}) f(e^{i\theta})\omega(\theta)d\theta \\ &= \int_{-\pi}^{\pi} f(e^{i\theta})\frac{\rho(\theta)}{\omega(\theta)}\omega(\theta)d\theta = \int_{-\pi}^{\pi} f(e^{i\theta})\rho(\theta)d\theta = I_\rho\{f\}. \end{aligned}$$

Thus the quadrature rule $S_N\{f\}$ is exact for all $f \in \mathcal{R}_{n,n}$. The coefficients of such a quadrature formula are uniquely defined, whence $W_{j,N} = A_{j,N}$ for $j = 1, \ldots, N$ and the theorem is proved. □

Note that if $\rho(\theta) > 0$ a.e. on $[-\pi, \pi]$, then one can take $\omega = \rho$, so that $g \equiv 1$, hence $A_{j,N} = w_{j,N}$ for $j = 1, \ldots, N$ and then the quadrature formulas I_N^ρ, S_N and Q_N coincide. Thus, in that case the interpolatory quadrature formula I_N^ρ is the same as the rational Szegő formula Q_N.

4 Convergence

In this section we prove the convergence of the quadrature formulas $I_N^\rho\{f\}$ as given in Theorem 3.2 for any function f integrable and bounded on $\mathbb{T}$. The interpolatory quadrature rule $I_N^\rho\{f\}$ can be written as the integral $I_\rho\{L_N^f\}$ of the uniquely defined function $L_N^f \in \mathcal{R}_{n,n}$ which interpolates f in the nodes $x_{j,N}$. Thus it will not be a surprise that the convergence of $I_N^\rho\{f\}$ to $I_\rho\{f\}$ will follow from the convergence of L_N^f to f. The proof that the interpolating rational function does converge will be our first objective.

We first prove a density result.

LEMMA 4.1 *Suppose ω is a weight function on $[-\pi,\pi]$ with $\int_{-\pi}^{\pi}\omega(\theta)d\theta < \infty$. Then $\mathcal{R}$ is dense in $L_\omega^2(\mathbb{T})$ if $\sum_{j=1}^{\infty}(1-|\alpha_j|)=\infty$.*

PROOF. Let $C(\mathbb{T})$ be the class of continuous functions on $\mathbb{T}$. As a direct consequence of the closure criterion discussed in [1, p. 244], it can be seen that if $\sum(1-|\alpha_j|)=\infty$, then $\mathcal{R}$ is dense in $C(\mathbb{T})$ with respect to the uniform norm. On the other hand, $C(\mathbb{T})$ is also dense in $L_\omega^2(\mathbb{T})$ (see e.g. [8]). Thus for $f \in L_\omega^2(\mathbb{T})$ and any $\epsilon > 0$, there exists a function $h \in C(\mathbb{T})$ such that

$$\|f-h\|_{\omega,2} = \left[\int_{-\pi}^{\pi} |f(e^{i\theta}) - h(e^{i\theta})|^2\omega(\theta)d\theta\right]^{1/2} < \epsilon.$$

Furthermore there exists a function $R \in \mathcal{R}$ so that

$$\|h-R\|_\infty = \max_{x\in\mathbb{T}} |h(x)-R(x)| < \epsilon.$$

Hence, setting $\int_{-\pi}^{\pi}\omega(\theta)d\theta = K^2$, with $0 < K < \infty$,

$$\|h-R\|_{\omega,2}^2 = \int_{-\pi}^{\pi} |h(e^{i\theta}) - R(e^{i\theta})|^2\omega(\theta)d\theta < \epsilon^2\int_{-\pi}^{\pi}\omega(\theta)d\theta = \epsilon^2K^2.$$

Thus

$$\|f-R\|_{\omega,2} = \|f-h+h-R\|_{\omega,2} \le \|f-h\|_{\omega,2} + \|h-R\|_{\omega,2} < \epsilon + \epsilon K$$

and this proves the lemma. □

Although this is not used in the remainder of this paper, we point out that Lemma 4.1 implies L_2-convergence of the rational trigonometric functions just like for trigonometric polynomials. In fact they are a special case (all $\alpha_k = 0$) in the following

COROLLARY 4.2 *Let P_n^f denote the nth partial sum of the Fourier expansion of a function $f \in L_\omega^2(\mathbb{T})$ with respect to any orthonormal basis of $\mathcal{R}$ and let $\sum_{j=1}^{\infty}(1-|\alpha_j|)=\infty$. Then $\lim_{n\to\infty}\|f-P_n^f\|_{\omega,2} = 0$.*

PROOF. Let $\{\gamma_n\}_0^\infty$ be the orthonormal basis and $\mathcal{R}_n = \mathrm{span}\{\gamma_0,\ldots,\gamma_n\}$. Then P_n^f is the orthogonal projection of f onto $\mathcal{R}_n$. For any $\epsilon > 0$, there exists an $R \in \mathcal{R}$ such that by the previous lemma $\|f-R\|_{\omega,2} < \epsilon$. Assume $R \in \mathcal{R}_m$ for some finite m. Because P_n^f is an optimal approximant from $\mathcal{R}_n$ [8], it follows that for any $n > m$: $P_n^f = R$, thus $\|f-P_n^f\|_{\omega,2} = \|f-R\|_{\omega,2} < \epsilon$, from which the corollary follows. □

In Szegő's book [11, Thm. 1.5.4] we find

LEMMA 4.3 *Let ω be a finite positive weight function on $[-\pi,\pi]$ and f a real-valued and bounded function on $[-\pi,\pi]$ such that the Riemann integral $\int_{-\pi}^{\pi} f(\theta)\omega(\theta)d\theta$ exists. Then for any $\epsilon > 0$, there exist trigonometric polynomials p and P such that*

$$\int_{-\pi}^{\pi}[P(\theta)-p(\theta)]\omega(\theta)d\theta < \epsilon$$

and

$$-M-\epsilon \le p(\theta) \le f(\theta) \le P(\theta) \le M+\epsilon, \quad \forall\theta\in[-\pi,\pi]$$

with

$$M = \max\left\{\left|\inf_{\theta\in[-\pi,\pi]} f(\theta)\right|, \left|\sup_{\theta\in[-\pi,\pi]} f(\theta)\right|\right\}.$$

Suppose we denote the set of trigonometric polynomials of degree n at most by $\mathcal{T}_n$ and set $\mathcal{T} = \cup_{n=0}^{\infty}\mathcal{T}_n$. Let us now associate with the table α the classes $\mathcal{S}_n$ of real rational trigonometric functions which are of the form

$$R\in\mathcal{S}_n \Leftrightarrow R(\theta) = \frac{T(\theta)}{|\omega_n(e^{i\theta})|^2} \quad \text{with } T\in\mathcal{T}_n$$

and where as before $\omega_n(z) = \prod_{j=1}^{n}(z-\alpha_j)$. Furthermore set $\mathcal{S} = \cup_{n=0}^{\infty}\mathcal{S}_n$. We can now give a rational variant of the previous lemma which will be required in Theorem 4.5.

LEMMA 4.4 *Let ω be a given weight function on $[-\pi,\pi]$ such that $\int_{-\pi}^{\pi}\omega(\theta)d\theta < \infty$ and let f be a bounded function on $[-\pi,\pi]$ such that the integral*

$$\int_{-\pi}^{\pi} f(\theta)\omega(\theta)d\theta$$

exists. Assume also that $\sum(1-|\alpha_k|) = \infty$. Then, for any $\epsilon > 0$ there exist rational trigonometric functions $S, T\in\mathcal{S}$ which satisfy

$$\int_{-\pi}^{\pi}[T(\theta)-S(\theta)]\omega(\theta)d\theta < \epsilon$$

and

$$-M-\epsilon \le S(\theta) \le f(\theta) \le T(\theta) \le M+\epsilon, \quad \forall\theta\in[-\pi,\pi]$$

with

$$M = \max\left\{\left|\inf_{\theta\in[-\pi,\pi]} f(\theta)\right|, \left|\sup_{\theta\in[-\pi,\pi]} f(\theta)\right|\right\}.$$

PROOF. Let ϵ' be an arbitrary positive number. Then by Lemma 4.3 there exist trigonometric polynomials $p, P\in\mathcal{T}$ such that

$$-M-\epsilon' \le p(\theta) \le f(\theta) \le P(\theta) \le M+\epsilon', \quad \forall\theta\in[-\pi,\pi]$$

and

$$\int_{-\pi}^{\pi}[P(\theta)-p(\theta)]\omega(\theta)d\theta < \epsilon'.$$

It is well known (see [1, p. 244]) that $\mathcal{S}$ is dense in the space of continuous functions iff $\sum(1-|\alpha_j|)=\infty$. Thus since p and P are continuous functions in $[-\pi,\pi]$, there exist sequences $\{r_n\}$ and $\{R_n\}$ in $\mathcal{S}$ such that

$$\lim_{n\to\infty} r_n(\theta)=p(\theta), \quad \text{and} \quad \lim_{n\to\infty} R_n(\theta)=P(\theta), \quad \text{uniformly in } [-\pi,\pi].$$

Thus, for any $\epsilon''>0$, we can find an integer n_0 such that for all $n\geq n_0$

$$r_n(\theta)-\epsilon''<p(\theta)<r_n(\theta)+\epsilon'', \quad \forall\theta\in[-\pi,\pi]$$

and

$$R_n(\theta)-\epsilon''<P(\theta)<R_n(\theta)+\epsilon'', \quad \forall\theta\in[-\pi,\pi].$$

Because the convergence $r_n\to p$ and $R_n\to P$ is uniform in $[-\pi,\pi]$, we also have

$$\lim_{n\to\infty} I_\omega\{r_n\}=I_\omega\{p\} \quad \text{and} \quad \lim_{n\to\infty} I_\omega\{R_n\}=I_\omega\{P\}.$$

Thus, for any $\epsilon'''>0$, there exists an integer n_1 such that for all $n\geq n_1$

$$-\epsilon'''+I_\omega\{p\}<I_\omega\{r_n\}<\epsilon'''+I_\omega\{p\}$$

and

$$-\epsilon'''+I_\omega\{P\}<I_\omega\{R_n\}<\epsilon'''+I_\omega\{P\}$$

Taking $n>\max\{n_0,n_1\}$ and defining

$$S(\theta)=r_n(\theta)-\epsilon'' \quad \text{and} \quad T(\theta)=R_n(\theta)+\epsilon''$$

we see that S and T are both in $\mathcal{S}$ and

$$S(\theta)=r_n(\theta)-\epsilon''<p(\theta)\leq f(\theta)\leq P(\theta)<R_n(\theta)+\epsilon''=T(\theta).$$

Moreover

$$T(\theta)<P(\theta)+2\epsilon''\leq M+\epsilon'+2\epsilon''$$

and

$$S(\theta)>p(\theta)-2\epsilon''\geq -M-\epsilon'-2\epsilon''.$$

Furthermore (we assume without loss of generality that $\int_{-\pi}^{\pi}\omega(\theta)d\theta=1$)

$$\begin{aligned}
I_\omega\{T-S\} &= I_\omega\{R_n+\epsilon''-r_n+\epsilon''\}\\
&= I_\omega\{R_n\}-I_\omega\{r_n\}+2\epsilon''\\
&< I_\omega\{P\}-I_\omega\{p\}+2(\epsilon''+\epsilon''')\\
&= I_\omega\{P-p\}+2(\epsilon''+\epsilon''')<\epsilon'+2(\epsilon''+\epsilon''').
\end{aligned}$$

Now by taking $\epsilon'<\epsilon/3$, $\epsilon''<\epsilon/6$ and $\epsilon'''<\epsilon/6$, the result follows. □

Given $N=2n+1$ distinct points $\{x_{j,N}\}_{j=1}^{N}$ on $\mathbb{T}$ and a function f defined on $\mathbb{T}$, we can consider the uniquely defined function from $\mathcal{R}_{n,n}$ which interpolates f in these points. This function will be denoted as L_{2n+1}^{f}. We are now able to prove that L_{2n+1}^{f} converges to f in $L_\omega^2(\mathbb{T})$ if the interpolation points are chosen to be the zeros of the para-orthogonal rational functions as in Theorem 3.2, with ω and α as in Lemma 4.4.

THEOREM 4.5 *Let $N = 2n+1$ and $\{x_{j,N}\}_{j=1}^N$ be the zeros of the para-orthogonal function $\tilde{\phi}_N + \delta_N \tilde{\phi}_N^*$, ($\delta_N \in \mathbb{T}$) associated with the weight ω and the table $\tilde{\alpha}$ as in Theorem 3.2 and assume $\sum(1 - |\alpha_j|) = \infty$. Then the sequence $\{L_{2n+1}^f\}$ with L_{2n+1}^f the function in $\mathcal{R}_{n,n}$ interpolating f in the points $\{x_{j,N}\}_{j=1}^N$, converges to f in $L_\omega^2(\mathbb{T})$ norm, i.e.,*

$$\lim_{n\to\infty} \|L_{2n+1}^f - f\|_{\omega,2}^2 = \lim_{n\to\infty} \int_{-\pi}^{\pi} |L_{2n+1}^f(e^{i\theta}) - f(e^{i\theta})|^2 \omega(\theta) d\theta = 0$$

for any function bounded on $\mathbb{T}$ for which the Riemann integral $I_\omega\{f\}$ exists.

PROOF. If f is a complex-valued function on $\mathbb{T}$, then we define the real-valued functions f_1, f_2 on $\mathbb{T}$ as $f_1(\theta) = \text{Re } f(e^{i\theta})$ and $f_2(\theta) = \text{Im } f(e^{i\theta})$ so that $f(e^{i\theta}) = f_1(\theta) + i f_2(\theta)$. It is obvious that if $L_{2n+1}^f = R \in \mathcal{R}_{n,n}$ is the function interpolating f in the points $x_{j,N}$, then $R_1(e^{i\theta}) = \text{Re } R(e^{i\theta})$ and $R_2(^{i\theta}) = \text{Im } R(e^{i\theta})$ interpolate f_1, respectively f_2 in the points $x_{j,N}$. Hence, the proof for complex functions will follow if we can prove the theorem for real valued functions. Note that if f is real valued, then also L_{2n+1}^f will be real valued. To prove the convergence for real-valued funcions, note that

$$\|L_{2n+1}^f - f\|_{\omega,2}^2 = \|f\|_{\omega,2}^2 + \|L_{2n+1}^f\|_{\omega,2}^2 - 2I_\omega\{fL_{2n+1}^f\}.$$

Now $L_{2n+1}^f \in \mathcal{R}_{n,n}$ implies that $[L_{2n+1}^f]^2 \in \tilde{\mathcal{R}}_{2n,2n} = \tilde{\mathcal{R}}_{N-1,N-1}$. Hence

$$I_\omega\{[L_{2n+1}^f]^2\} = Q_N\{[L_{2n+1}^f]^2\} = \sum_{j=1}^N w_{j,N}[L_{2n+1}^f(x_{j,N})]^2 = Q_N\{f^2\}.$$

Furthermore, by the Cauchy-Schwartz inequality

$$|I_\omega\{fL_{2n+1}^f\}|^2 \leq \|f\|_{\omega,2}^2 \|L_{2n+1}^f\|_{\omega,2}^2 = I_\omega\{f^2\} Q_n\{f^2\}.$$

Since $\lim_{n\to\infty} Q_N\{f^2\} = I_\omega\{f^2\}$, it follows that

$$\limsup_{n\to\infty} \|f - L_{2n+1}^f\|_{\omega,2}^2 \leq 4\|f\|_{\omega,2}^2.$$

Given $\epsilon > 0$, it can be deduced from Lemma 4.4 that there exists a real-valued $S \in \mathcal{R}$ such that

$$-M - \epsilon \leq f(t) \leq S(t) \leq M + \epsilon, \quad \forall t \in \mathbb{T}$$

and

$$\int_{-\pi}^{\pi} |f(e^{i\theta}) - S(e^{i\theta})| \omega(\theta) d\theta < \epsilon.$$

Hence

$$\|f - S\|_{\omega,2}^2 = \int_{-\pi}^{\pi} |f(e^{i\theta}) - S(e^{i\theta})||f(e^{i\theta}) - S(e^{i\theta})| \omega(\theta) d\theta \leq 2(M + \epsilon)\epsilon.$$

Since for sufficiently large n, we have $L_{2n+1}^S = S$, we get

$$f - L_{2n+1}^f = f - S + S - L_{2n+1}^f = f - S - L_{2n+1}^{f-S}.$$

Therefore

$$\limsup_{n\to\infty} \|f - L_{2n+1}^f\|_{\omega,2}^2 = \limsup_{n\to\infty} \|(f - S) - L_{2n+1}^{f-S}\|_{\omega,2}^2 \leq 4\|f - S\|_{\omega,2}^2 \leq 8\epsilon(M + \epsilon)$$

from which it follows that

$$\limsup_{n\to\infty} \|f - L_{2n+1}^f\|_{\omega,2}^2 = 0.$$

This proves the lemma. □

Our main result concerning the convergence of the interpolatory quadrature formulas $I_N^\rho\{f\}$ now follows easily.

THEOREM 4.6 *Suppose ρ is a complex function and ω a weight function such that*

$$\int_{-\pi}^{\pi} \frac{|\rho(\theta)|^2}{\omega(\theta)} d\theta < \infty \tag{4.1}$$

and let $\sum(1-|\alpha_j|) = \infty$. Then, with $N = 2n+1$, it holds for the quadrature formulas $I_N^\rho\{f\}$ of Theorem 3.2 that

$$\lim_{n\to\infty} I_N^\rho\{f\} = \lim_{n\to\infty} \sum_{j=1}^N A_{j,N} f(x_{j,N}) = \int_{-\pi}^{\pi} f(e^{i\theta})\rho(\theta)d\theta = I_\rho\{f\}$$

for all functions f bounded on $\mathbb{T}$ for which $I_\rho\{f\}$ exists as a Riemann integral.

PROOF. If L_{2n+1}^f is as in the previous theorem, then we know that

$$I_N^\rho\{f\} = I_\rho\{L_{2n+1}^f\},$$

Setting $\int_{-\pi}^{\pi} \frac{|\rho(\theta)|^2}{\omega(\theta)} d\theta = K^2$ with $0 < K < \infty$, we get by the Cauchy-Schwartz inequality, that

$$|I_\rho\{f\} - I_N^\rho\{f\}| = |I_\rho\{f - L_{2n+1}^f\}| \le \|f - L_{2n+1}^f\|_{\omega,2} \cdot K.$$

By Theorem 4.5, the proof follows. □

So we have proved the convergence of the sequence $I_N^\rho\{f\}$ of quadrature formulas (3.6) for all integrable functions f, thus also for all functions $f \in C(\mathbb{T})$. By Polya's theorem [11, Thm. 15.2.1], there exists an absolute constant B such that

$$S_N = \sum_{j=1}^N |A_{j,N}| \le B. \tag{4.2}$$

The next theorem implies as a special case (take $f \equiv 1$) that the sequence S_N converges, namely

$$\lim_{n\to\infty} S_n = \int |\rho(\theta)| d\theta.$$

THEOREM 4.7 *Suppose ρ is a complex function satisfying (4.1). Assume $\sum(1-|\alpha_n|) = \infty$, set $N = 2n+1$ and let $\{x_{j,N}\}_{j=1}^N$ and $\{A_{j,N}\}_{j=1}^N$ be as defined in Theorem 3.2. Then*

$$\lim_{n\to\infty} \sum_{j=1}^N |A_{j,N}| f(x_{j,N}) = \int_{-\pi}^{\pi} f(e^{i\theta})|\rho(\theta)| d\theta \tag{4.3}$$

for all bounded Riemann integrable functions f.

PROOF. In order to display that the coefficients $A_{j,N}$ depend on ρ, we write $A_{j,N} = A_{j,N}(\rho)$. For ease of notation, we also write

$$J_N^\rho\{f\} = \sum_{j=1}^N |A_{j,N}| f(x_{j,N}) \quad \text{and} \quad J_\rho\{f\} = \int_{-\pi}^{\pi} f(e^{i\theta})|\rho(\theta)|d\theta.$$

We will repeatedly make use of the elementary inequality $||a| - |b|| \leq |a - b|$.

Choose a function $\hat{\rho} \in L_\omega^2(\mathbb{T})$. By the triangle inequality and because the coefficients $A_{j,N}(\rho)$ depend linearly on ρ, it follows that

$$\begin{aligned} T &= |J_N^\rho\{f\} - J_\rho\{f\}| \\ &= |J_N^\rho\{f\} - J_N^{\hat{\rho}}\{f\} + J_{\hat{\rho}}\{f\} - J_\rho\{f\} + J_N^{\hat{\rho}}\{f\} - J_{\hat{\rho}}\{f\}| \\ &\leq J_N^{\rho-\hat{\rho}}\{|f|\} + J_{\rho-\hat{\rho}}\{|f|\} + |J_N^{\hat{\rho}}\{f\} - J_{\hat{\rho}}\{f\}| \end{aligned}$$

Since f is bounded, there exists a number $L(f)$ such that $|f(e^{i\theta})| \leq L(f) < \infty$. Hence

$$T \leq L(f) \int_{-\pi}^{\pi} |\rho(\theta) - \hat{\rho}(\theta)|d\theta + L(f) \sum_{j=1}^N |A_{j,N}(\rho - \hat{\rho})| + |I_N^{\hat{\rho}}\{f\} - I_{\hat{\rho}}\{f\}|.$$

Recall that we have set $g(e^{i\theta}) = \rho(\theta)/\omega(\theta)$ and that $A_{j,N}(\rho) = w_{j,N} P_{N-1}^g(x_{j,N})$. Choose now $\hat{g} \in \mathcal{R}_{m,m}$ arbitrary. Then from the definition of $P_{N-1}^{\hat{g}}$ (the projection of $\hat{g}$ onto $\mathcal{R}_{n,n}$) it follows that for $n \geq m$ one has $P_{N-1}^{\hat{g}}(z) = \hat{g}(z)$ for $z \in \mathbb{T}$. Define $\hat{\rho}$ by $\hat{\rho}(\theta) = \hat{g}(e^{i\theta})\omega(\theta)$. From Theorem 3.2, we have

$$A_{j,N}(\hat{\rho}) = w_{j,N} P_{N-1}^{\hat{g}}(x_{j,N}) = w_{j,N} \hat{g}(x_{j,N}).$$

Thus $T \leq T_1 + T_2 + T_3$ with

$$\begin{aligned} T_1 &= L(f) \int_{-\pi}^{\pi} |g(e^{i\theta}) - \hat{g}(e^{i\theta})|\omega(\theta)d\theta \\ T_2 &= L(f) \sum_{j=1}^N w_{j,N} |P_{N-1}^{g-\hat{g}}(x_{j,N})| \\ T_3 &= \left| \sum_{j=1}^N w_{j,N} |\hat{g}(x_{j,N})| f(x_{j,N}) - \int_{-\pi}^{\pi} f(e^{i\theta})|\hat{g}(e^{i\theta})|\omega(\theta)d\theta \right|. \end{aligned}$$

To estimate T_1, we use the Cauchy-Schwarz inequality to get

$$T_1 \leq L(f) \left[\int_{-\pi}^{\pi} |g(e^{i\theta}) - \hat{g}(e^{i\theta})|^2 \omega(\theta)d\theta\right]^{1/2} \left[\int_{-\pi}^{\pi} \omega(\theta)d\theta\right]^{1/2} = L(f)\|g - \hat{g}\|_{\omega,2} K(\omega)$$

with $K(\omega) = \|1\|_{\omega,2} < \infty$.

For T_2 we note that for a given function $S \in \mathcal{R}_{n,n}$ we have $SS_* \in \tilde{\mathcal{R}}_{2n,2n} = \tilde{\mathcal{R}}_{N-1,N-1}$. Since the Nth rational Szegő formula is exact in $\tilde{R}_{N-1,N-1}$ and using the Cauchy-Schwarz inequality for summation we get

$$\begin{aligned} \sum_{j=1}^N w_{j,N} |S(x_{j,N})| &\leq \left[\sum_{j=1}^N w_{j,N}\right]^{1/2} \left[\sum_{j=1}^N w_{j,N} |S(x_{j,N})|^2\right]^{1/2} \\ &= \left[\sum_{j=1}^N w_{j,N}\right]^{1/2} \left[\int_{-\pi}^{\pi} |S(x_{j,N})|^2 \omega(\theta)d\theta\right]^{1/2} = K(\omega)\|S\|_{\omega,2}. \end{aligned}$$

Hence, setting $S(e^{i\theta}) = P_{N-1}^{g-\hat{g}}(e^{i\theta})$, one gets

$$\sum_{j=1}^{N} w_{j,N} |P_{N-1}^{g-\hat{g}}(x_{j,N})| \leq K(\omega)\|g - \hat{g}\|_{\omega,2}.$$

Thus we see that $T_1 + T_2 \leq 2L(f)K(\omega)\|g - \hat{g}\|_{\omega,2}$.

By (4.1) we know that $\|g\|_{\omega,2} < \infty$, i.e., $g \in L^2_\omega(\mathbb{T})$. Therefore, by Lemma 4.1, there exists a $\hat{g} \in \mathcal{R}_{n_0,n_0}$ for some n_0, such that

$$\|g - \hat{g}\|_{\omega,2} < \frac{\epsilon}{4L(f)K(\omega)}.$$

Thus $T_1 + T_2 < \epsilon/2$ if $n > n_0$.

The term T_3 represents the error of the Nth rational Szegő formula for the weight ω and the points $\tilde{\alpha}$. By the assumptions of the theorem, it follows that $|\hat{g}|f$ will be integrable with respect to ω whenever $\|g - \hat{g}\|_{\omega,2}$ is finite. Because the rational Szegő quadrature formula converges for all integrable functions (see [4, Corollary 6]), there exists an integer n_1 such that for all $n > n_1$ and for any positive ϵ we have $T_3 < \epsilon/2$. It thus follows that for $n > \max\{n_0, n_1\}$, we can always find a $\hat{g}$ such that $T < \epsilon$ with ϵ arbitrary small and thus the theorem follows. □

5 Multipoint Padé approximation

To conclude, we show how the quadrature formulas $I_N^\rho\{f\}$ as defined in Theorem 3.2 allow us to construct a rational approximant to the Riesz-Herglotz transform of the complex measure μ (recall $d\mu(\theta) = \rho(\theta)d\theta$ with $\int_{-\pi}^{\pi} |\rho(\theta)|d\theta < \infty$). It will be shown that this approximant converges uniformly on compact subsets of $\hat{\mathbb{C}} - \mathbb{T}$.

The Riesz-Herglotz transform of $d\mu(\theta) = \rho(\theta)d\theta$ is defined as [2]

$$F_\rho(z) = \int_{-\pi}^{\pi} D(t,z)\rho(\theta)d\theta \quad \text{with} \quad D(t,z) = \frac{t+z}{t-z} \quad \text{and} \quad t = e^{i\theta}.$$

In [5] and [7] we proved the convergence for certain quadrature formulas which were exact in certain subspaces of $\mathcal{R}$. This convergence was obtained from convergence properties of associated multipoint rational approximants to F_ρ which have poles on $\mathbb{T}$. Here we follow the opposite approach and prove the convergence of multipoint rational approximants to F_ρ, associated with the quadrature formulas considered in this paper, from the convergence of the quadrature formulas.

Let p, q and n be nonnegative integers and set $N = 2n + 1$. A rational function F_N of type (N, N), i.e., $F_N = P_N/Q_N$ with $P_N, Q_N \in \Pi_N$ is said to be a multipoint rational approximant (MRA) to F_ρ of order $(p+1, q+1)$ (in the strong sense) if

(5.1) $$\frac{F_\rho - F_N}{\omega_p \pi_q} \quad \text{with } \omega_n \text{ and } \pi_n \text{ as defined in Section 1,}$$

is analytic in $\hat{\mathbb{C}} - \mathbb{T}$. For further details concerning this definition see [7]. Let us assume that $p = q = n$ (so that $p + q = N - 1$), then for a given polynomial $Q_N \in \Pi_N$, there exists a unique polynomial $P_N \in \Pi_n$ such that (5.1) is satisfied. In [7] such a MRA is called a multipoint Padé-type approximant (MPTA). In Walsh [13, Chap. 8] these are called rational approximants with preassigned poles. These MPTAs are the ones we shall be concerned with.

THEOREM 5.1 *Suppose $\rho(\theta)$ is a complex function and ω a positive function satisfying (4.1) and let $\sum(1 - |\alpha_j|) = \infty$. For $n = 1, 2, \dots$ set $N = 2n + 1$ and consider F_N defined by*

$$F_N(z) = I_n^\rho\{D(\cdot, z)\}.$$

Then

1. *F_N is a MPTA, i.e., a MRA of order $(n + 1, n + 1)$ to F_ρ*
2. *$\lim_{n\to\infty} F_N(z) = F_\rho(z)$ uniformly on compact subsets of $\hat{\mathbb{C}} - \mathbb{T}$.*

PROOF.

1. From its definition, it follows that

$$F_N(z) = I_N^\rho\{D(\cdot, z)\} = \sum_{j=1}^{N} A_{j,N} \frac{x_{j,N} + z}{x_{j,N} - z}$$

so that F_N is a rational function of type (N, N) whose denominator has poles $x_{j,N} \in \mathbb{T}$, $j = 1, \dots, N$. In [5, § 2] it is proved that it is also an MRA of order $(p, q) = (n+1, n+1)$ to F_ρ, hence a MPTA to F_ρ.

2. Since $\{I_N^\rho\{f\}\}$, $N = 2n+1$, $n = 1, 2, \dots$ converges to $I_\rho\{f\}$ for any bounded integrable function f on $\mathbb{T}$ (Theorem 4.6), it follows that $F_N(z) = I_N^\rho\{D(\cdot, z)\}$ converges pointwise to $F_\rho(z) = I_\rho\{\{D(\cdot, z)\}$. Since by Theorem 4.7, $\sum_{j=1}^{N} |A_{j,N}| \leq B < \infty$ with B independent of N, we conclude that F_N is a normal family in compact subsets of $\hat{\mathbb{C}} - \mathbb{T}$, so that by the Stieltjes-Vitali theorem [9], we have uniform convergence in compact subsets of $\hat{\mathbb{C}} - \mathbb{T}$.

□

REMARK 2 We emphasise that our only assumption concerning the table $\alpha \subset \mathbb{D}$ is that $\sum(1 - |\alpha_j|) = \infty$. In our earlier paper [5] we also obtained convergence results (local uniform and geometric) for sequences of MPTAs whose poles are zeros of para-orthogonal polynomials. However, the proof required the rather restrictive condition $\lim_{n\to\infty} \alpha_n = a \in \mathbb{D}$ [5, § 3]. Under the same restriction we obtained the rate of convergence for the corresponding quadrature formulas within the class of functions analytic in a neighbourhood of $\mathbb{T}$. In § 4 of the same paper, convergence of the quadrature formulas was obtained when the set α

consists of points contained in a compact subset of $\mathbb{D}$ (hence $\sum(1 - |\alpha_n|) = \infty$) and for the class of functions continuous on $\mathbb{T}$ and satisfying a certain Lipschitz condition (hence integrable). The main advantage of choosing as nodes for the quadrature formulas the zeros of para-orthogonal rational functions which are not determined by the table α (as in [5]) but by the auxiliary table $\tilde{\alpha}$ (as in this paper), is that it is now possible to deduce (4.2) which was needed to prove the convergence in Theorem 5.1.

REMARK 3 In this paper μ was supposed to be a complex-valued measure (here we assumed $d\mu(\theta) = \rho(\theta)d\theta$ with ρ a complex-valued function). So we had to introduce another measure ψ, finite and positive, (here we chose $d\psi(\theta) = \omega(\theta)d\theta$ with ω a positive weight function) to obtain an inner product, hence (para-)orthogonal rational functions, and hence simple zeros on $\mathbb{T}$ which could be used as nodes for the quadrature formulas. However, if μ itself is a positive measure, then we can take $\psi = \mu$ and the optimal quadrature formulas (namely the rational Szegő formulas with a maximal domain of validity [4]) are obtained using zeros of para-orthogonal rational functions associated with the table α. Then of course, the auxiliary table $\tilde{\alpha}$ will not be of any use. Convergence results for these rational Szegő quadrature formulas (for any integrable function) and for the associated multipoint Padé approximants (locally uniformly in $\hat{\mathbb{C}} - \mathbb{T}$) were obtained in [4, 7].

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