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Orthogonal wavelets with compact support on locally compact abelian groups

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Abstract. We extend and improve the results of W. Lang (1998) on the wavelet analysis on the Cantor dyadic group $\mathcal{C}$. Our construction is realized on a locally compact abelian group G which is defined for an integer $p \geqslant 2$ and coincides with $\mathcal{C}$ when $p = 2$. For any integers $p, n \geqslant 2$ we determine a function φ in $L^2(G)$ which

1) is the sum of a lacunary series by generalized Walsh functions,
2) has orthonormal "integer" shifts in $L^2(G)$,
3) satisfies "the scaling equation" with p^n numerical coefficients,
4) has compact support whose Haar measure is proportional to p^n,
5) generates a multiresolution analysis in $L^2(G)$.

Orthogonal wavelets ψ with compact supports on G are defined by such functions φ. The family of these functions φ is in many respects analogous to the well-known family of Daubechies' scaling functions. We give a method for estimating the moduli of continuity of the functions φ, which leads to sharp estimates for small p and n. We also show that the notion of adapted multiresolution analysis recently suggested by Sendov is applicable in this situation.

§ 1. Introduction

Multiresolution analysis in L^2-spaces on locally compact abelian groups is one of the fundamental concepts of wavelet theory (see, for example, [1], [2]) and can be defined in the following way.

Let G be a locally compact abelian group and let $L^2(G)$ be the corresponding Lebesgue space (see [3]). Suppose that H is a discrete subgroup of G such that the quotient group G/H is compact. Let A be an automorphism of G such that $A(H)$ is a proper subgroup of H. A collection of closed subspaces $V_j \subset L^2(G)$, $j \in \mathbb{Z}$, is called a *multiresolution analysis* in $L^2(G)$ associated with H and A if it satisfies the following conditions:

(i) $V_j \subset V_{j+1}$ for $j \in \mathbb{Z}$,

(ii) $\overline{\bigcup V_j} = L^2(G)$ and $\bigcap V_j = \{0\}$,

(iii) $f(\,\cdot\,) \in V_j \iff f(A\,\cdot\,) \in V_{j+1}$ for $j \in \mathbb{Z}$,

(iv) $f(\,\cdot\,) \in V_0 \implies f(\,\cdot - h) \in V_0$ for $h \in H$,

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(v) there is a function $\varphi \in L^2(G)$ such that the system $\{\varphi(\cdot - h) \mid h \in H\}$ is an orthonormal basis of V_0.

The function φ in condition (v) is called a *scaling function* in $L^2(G)$.

By conditions (iii) and (v), the system $\{\varphi(A \cdot -h) \mid h \in H\}$ is an orthonormal basis of V_1. Since $\varphi \in V_0 \subset V_1$, we have an expansion

$$\varphi(x) = \sum_{h \in H} c(h)\varphi(Ax - h).$$

A function ψ is called an *orthogonal wavelet* in $L^2(G)$ if the system of functions $\{\psi(A^j \cdot -h) \mid j \in \mathbb{Z}, h \in H\}$ is an orthogonal basis of $L^2(G)$. If the quotient group $H/A(H)$ consists of two cosets (that is, the index of the subgroup $A(H)$ in H is 2), then the coefficients $\{c(h)\}$ determine coefficients $\{d(h)\}$ such that the formula

$$\psi(x) = \sum_{h \in H} d(h)\varphi(Ax - h)$$

defines an orthogonal wavelet in $L^2(G)$. If $\operatorname{card} H/A(H) = i > 2$ and ϕ is a given scaling function, then one can define *wavelets* $\psi_1, \dots, \psi_{i-1}$ in such a way that the system of functions

$$\{\psi_l(A^j \cdot -h) \mid 1 \leqslant l \leqslant i-1,\ j \in \mathbb{Z},\ h \in H\}$$

is an orthogonal bases of $L^2(G)$.

Classical wavelet analysis on the line corresponds to the case when $G = \mathbb{R}$, $H = \mathbb{Z}$ and $Ax = 2x$ for $x \in \mathbb{R}$ (see, for example, [1]). If $G = \mathbb{R}^d$, then the subgroup H is a lattice in $\mathbb{R}^d$ and the automorphism A can be given by a non-singular $d \times d$-matrix (see [1], § 10.3, and [4]–[6]). For some groups G different from $\mathbb{R}^d$, multiresolution analysis was studied in [7]–[11]. In particular, the group analogues of the B-spline wavelet bases in $L^2(\mathbb{R})$ are defined in [7] and [8]. The book [12] describes structural features of locally compact abelian groups G for which there exist subgroups H and automorphisms A with the properties mentioned above ($A(H) \subset H$, $A(H) \neq H$, H is discrete and the quotient group G/H is compact).

In this paper we construct a set of scaling functions on a group G which is determined by an integer $p \geqslant 2$ and consists of all sequences of the form

$$x = (x_j) = (\dots, 0, 0, x_k, x_{k+1}, x_{k+2}, \dots),$$

where $x_j \in \{0, 1, \dots, p-1\}$ for $j \in \mathbb{Z}$ and $x_j = 0$ for $j < k = k(x)$. The group operation on G is denoted by $\oplus$ and is defined as termwise addition modulo p:

$$(z_j) = (x_j) \oplus (y_j) \iff z_j = x_j + y_j \pmod p \qquad \text{for} \quad j \in \mathbb{Z}.$$

A topology on G is introduced via the following complete system of neighbourhoods of zero:

$$U_l = \{(x_j) \in G \mid x_j = 0 \text{ for } j \leqslant l\}, \qquad l \in \mathbb{Z}.$$

The inverse operation of $\oplus$ is denoted by $\ominus$. (We have $x \ominus x = \theta$, where θ is the zero sequence.) It is clear that each neighbourhood U_l is a subgroup of G, $U_{l+1} \subset U_l$ for $l \in \mathbb{Z}$, and $\bigcap U_l = \{\theta\}$.

Put $U = U_0$. When $p = 2$, the subgroup U is isomorphic to the *Cantor dyadic group*, which is the topological Cartesian product of countably many cyclic groups of order 2 with the discrete topology. It is well known that U is a perfect nowhere-dense totally disconnected metrizable space and, therefore, U is homeomorphic to the Cantor ternary set (see [13], Russian p. 138). Some authors identify the Cantor dyadic group with the whole group G when $p = 2$ (see, for example, [10]).

Elements of harmonic analysis on the group G and some of their applications to problems in coding and digital signal processing are given in [14] and [15]. See also [16], Chpts. 14, 15 for applications of the Cantor dyadic group to the theory of Fourier series.

We define the Lebesgue spaces $L^q(G)$, $1 \leqslant q \leqslant \infty$, with respect to the Haar measure μ on Borel subsets of G normalized by $\mu(U) = 1$ (see, for example, [3]). We denote the inner product and the norm in $L^2(G)$ by $(\,\cdot\,,\,\cdot\,)$ and $\|\cdot\|$ respectively.

The group dual to G is denoted by G^* and consists of all sequences of the form

$$\omega = (\omega_j) = (\dots, 0, 0, \omega_k, \omega_{k+1}, \omega_{k+2}, \dots),$$

where $\omega_j \in \{0, 1, \dots, p-1\}$ for $j \in \mathbb{Z}$ and $\omega_j = 0$ for $j < k = k(\omega)$. We introduce the operations of addition and subtraction, the neighbourhoods $\{U_l^*\}$ of zero and the Haar measure μ^* for G^* as above for G. The value of a character $\omega \in G^*$ on an element $x \in G$ is given by

$$\chi(x, \omega) = \exp\biggl(\frac{2\pi i}{p} \sum_{j=1}^{\infty} x_j \omega_{1-j}\biggr),$$

and the Fourier transform of a function $f \in L^1(G)$ is defined by

$$\widehat{f}(\omega) = \int_G f(x) \overline{\chi(x, \omega)}\, d\mu(x).$$

(See [3], Ch. 8 for basic properties of the Fourier transform on $L^2(G)$.) By the Plancherel formula,

$$(f, g) = (\widehat{f}, \widehat{g}), \qquad f, g \in L^2(G).$$

Take a discrete subgroup $H = \{(x_j) \in G \mid x_j = 0, j > 0\}$ of G and define an automorphism $A \in \operatorname{Aut} G$ by $(Ax)_j = x_{j+1}$. It is easily seen that the quotient group $H/A(H)$ contains p elements and the annihilator $H^\perp$ of the subgroup H consists of all sequences $(\omega_j) \in G^*$ which satisfy $\omega_j = 0$ for $j > 0$.

For any function $\varphi \in L^2(G)$, we set

$$\begin{aligned} \varphi_{j,h}(x) &= p^{j/2} \varphi(A^j x \ominus h), \qquad j \in \mathbb{Z}, \quad h \in H, \\ V_j &= \operatorname{clos}_{L^2(G)} \operatorname{span}\{\varphi_{j,h} \mid h \in H\}, \qquad j \in \mathbb{Z}. \end{aligned} \tag{1.1}$$

If the system $\{\varphi(\,\cdot\, - h) \mid h \in H\}$ is orthonormal and the family of subspaces (1.1) is a multiresolution analysis in $L^2(G)$, then φ is a scaling function. Moreover, the system $\{\varphi_{j,h} \mid h \in H\}$ is an orthonormal basis of V_j for every $j \in \mathbb{Z}$, and the standard method (see, for example, [4], [8]) enables us to define wavelets $\psi_1, \dots, \psi_{p-1}$ such that the system of functions

$$\psi_{l,j,h}(x) = p^{j/2}\psi_l(A^j x \ominus h), \qquad 1 \leqslant l \leqslant p-1, \quad j \in \mathbb{Z}, \quad h \in H,$$

is an orthonormal basis of $L^2(G)$. When $p = 2$, only one wavelet ψ is obtained and the system $\{2^{j/2}\psi(A^j \cdot \ominus h) \mid j \in \mathbb{Z}, h \in H\}$ is an orthonormal basis of $L^2(G)$ (see [10], § 3).

Let $\mathbb{R}_+ = [0, +\infty)$. We define a map $\lambda\colon G \to \mathbb{R}_+$ by

$$\lambda(x) = \sum_{j\in\mathbb{Z}} x_j p^{-j}, \qquad x = (x_j) \in G.$$

Note that λ takes the subgroup U onto the interval $[0,1]$ and defines an isomorphism of the measure spaces (G,μ) and $(\mathbb{R}_+,\nu)$, where ν is the Lebesgue measure on $\mathbb{R}_+$ (compare [16], Ch. 14, Exercise 14.16). The image of H under λ is the set of non-negative integers: $\lambda(H) = \mathbb{Z}_+$. For every $\alpha \in \mathbb{N}$, let $h_{[\alpha]}$ and $h^-_{[\alpha]}$ denote the elements of G such that

$$\lambda(h_{[\alpha]}) = \lambda(h^-_{[\alpha]}) = \alpha,$$

where the terms of the sequence $h_{[\alpha]}$ (resp. $h^-_{[\alpha]}$) are eventually equal to 0 (resp. to $p-1$). We also set $h_{[\alpha]} = \theta$ for $\alpha = 0$. Hence $h_{[\alpha]} \in H$ for all $\alpha \in \mathbb{Z}_+$. For G^*, we define the map $\lambda^*\colon G^* \to \mathbb{R}_+$, the automorphism $B \in \operatorname{Aut}G^*$, the subgroup U^* and the elements $\omega_{[\alpha]}$, $\omega^-_{[\alpha]}$ of $H^\perp$ simlarly to λ, A, U and $h_{[\alpha]}$, $h^-_{[\alpha]}$ respectively. We note that

$$\chi(Ax,\omega) = \chi(x,B\omega), \qquad x \in G, \quad \omega \in G^*.$$

The *generalized Walsh functions* $\{W_\alpha\}$ for the group G can be defined by

$$W_\alpha(x) = \chi(x,\omega_{[\alpha]}), \qquad \alpha \in \mathbb{Z}_+, \quad x \in G.$$

(The family $\{W_\alpha\}$ is sometimes called the *Price system*, see [14], Russian p. 30.) These functions are continuous on G and satisfy the orthogonality relations

$$\int_U W_\alpha(x)\overline{W_\beta(x)}\,d\mu(x) = \delta_{\alpha,\beta} \qquad \alpha,\beta \in \mathbb{Z}_+,$$

where $\delta_{\alpha,\beta}$ is the Kronecker delta. It is also known that the system $\{W_\alpha\}$ is complete in $L^2(U)$. The corresponding system for the group G^* is defined by

$$W^*_\alpha(\omega) = \chi(h_{[\alpha]},\omega), \qquad \alpha \in \mathbb{Z}_+, \quad \omega \in G^*.$$

The system $\{W^*_\alpha\}$ is an orthonormal basis of $L^2(U^*)$.

For $s \in \mathbb{Z}_+$ put
$$U^*_{n,s} = B^{-n}(\omega_{[s]}) \oplus B^{-n}(U^*),$$
so that $\lambda^*(U^*_{n,s}) = [sp^{-n}, (s+1)p^{-n}]$. The sets $U^*_{n,s}$, $0 \leqslant s \leqslant p^n - 1$, are cosets of the subgroup $B^{-n}(U^*)$ in the group U^*.

In this paper, given any $p, n \in \mathbb{N}$ with $p \geqslant 2$, we define coefficients a_α such that the functional equation

$$\varphi(x) = p \sum_{\alpha=0}^{p^n-1} a_\alpha \varphi(Ax \ominus h_{[\alpha]}) \tag{1.2}$$

has a solution $\varphi \in L^2(G)$ with the following properties:

1) the system $\{\varphi(\,\cdot\, \ominus h) \mid h \in H\}$ is orthonormal in $L^2(G)$,

2) $\operatorname{supp}\varphi \subset U_{1-n}$,

3) φ generates a multiresolution analysis in $L^2(G)$ by (1.1).

For example, if $n = 1$, then all $a_\alpha = 1/p$, and a solution of (1.2) is the Haar function $\varphi = \mathbf{1}_U$ (where $\mathbf{1}_E$ is the characteristic function of a subset $E \subset G$). In the general case, φ is given by a generalized Walsh series expansion and the coefficients a_α are found from a system of linear algebraic equations using the discrete Vilenkin–Chrestenson transform. The author [17] reported on an analogous construction of scaling functions in the L^2-space on the positive half-line $\mathbb{R}_+$.

To formulate a theorem, we introduce some notation. Let $\mathbb{N}_0(p,n)$ be the set of all positive integers $m \geqslant p^{n-1}$ whose p-ary expansion

$$m = \sum_{j=0}^{k} \mu_j p^j, \qquad \mu_j \in \{0, 1, \dots, p-1\}, \quad \mu_k \neq 0, \quad k = k(m) \in \mathbb{Z}_+ \tag{1.3}$$

contains no n-tuple $(\mu_j, \mu_{j+1}, \dots, \mu_{j+n-1})$ coinciding with any of the n-tuples

$$(0,0,\dots,0,1), (0,0,\dots,0,2), \dots, (0,0,\dots,0,p-1).$$

Put $\mathbb{N}(p,n) = \{1, 2, \dots, p^{n-1} - 1\} \bigcup \mathbb{N}_0(p,n)$. In particular, we have

$$\mathbb{N}(2,2) = \{2^{j+1} - 1 \mid j \in \mathbb{Z}_+\} = \{1, 3, 7, 15, 31, \dots\},$$
$$\mathbb{N}(p,2) = \biggl\{ \sum_{j=0}^{k} m_j p^j \mid m_j \in \{1, 2, \dots, p-1\},\ k \in \mathbb{Z}_+ \biggr\}, \qquad p \geqslant 3.$$

For every $m \in \mathbb{N}(p,n)$, $1 \leqslant m \leqslant p^n - 1$, we choose a (real or complex) number b_m in such a way that the following conditions hold for all $j \in \{1, 2, \dots, p^{n-1} - 1\}$:

$$b_j \neq 0 \quad \text{and} \quad |b_j|^2 + |b_{p^{n-1}+j}|^2 + |b_{2p^{n-1}+j}|^2 + \dots + |b_{(p-1)p^{n-1}+j}|^2 = 1. \tag{1.4}$$

In particular, when $p = n = 2$ we obtain only one equality: $|b_1|^2 + |b_3|^2 = 1$, and when $p = 3$, $n = 2$ we have

$$|b_1|^2 + |b_4|^2 + |b_7|^2 = |b_2|^2 + |b_5|^2 + |b_8|^2 = 1.$$

The conditions (1.4) will be used to prove the orthonormality of the system $\{\varphi(\cdot \ominus h) \mid h \in H\}$ in the theorem below. (In connection with the inequality $b_j \neq 0$, see Remark 1.1 and (3.4).)

Furthermore, for $m \in \mathbb{N}(p,n)$ and $1 \leqslant m \leqslant p^n - 1$, we set

$$\gamma(i_1, i_2, \dots, i_n) = b_m \quad \text{if} \quad m = i_1 p^0 + i_2 p^1 + \dots + i_n p^{n-1}, \qquad i_j \in \{0, 1, \dots, p-1\}.$$

Then we represent every $m \in \mathbb{N}(p,n)$ by the p-ary expansion (1.3) and define the coefficients $\{c(m)\}$ by

$$\begin{gathered} c(m) = \gamma(\mu_0, 0, 0, \dots, 0, 0) \quad \text{if} \quad k(m) = 0; \\ c(m) = \gamma(\mu_1, 0, 0, \dots, 0, 0)\gamma(\mu_0, \mu_1, 0, \dots, 0, 0) \quad \text{if} \quad k(m) = 1; \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ c(m) = \gamma(\mu_k, 0, 0, \dots, 0, 0)\gamma(\mu_{k-1}, \mu_k, 0, \dots, 0, 0) \dots \gamma(\mu_0, \mu_1, \mu_2, \dots, \mu_{n-2}, \mu_{n-1}) \end{gathered}$$

if $k = k(m) \geqslant n - 1$. Note that the subscripts of each factor (from the second onwards) in the last product are obtained by "shifting" the subscripts of the previous factor one position to the right and filling the vacant first position with a new digit from the p-ary expansion (1.3) of the number m. (Hence the factor $\gamma(\mu_{k-l-1}, \mu_{k-l}, \dots, \mu_{k-1}, \mu_k, 0 \dots, 0)$ follows $\gamma(\mu_{k-l}, \mu_{k-l+1}, \dots, \mu_k, 0, 0 \dots, 0)$.)

For $s \in \{0, 1, \dots, p^n - 1\}$ put

$$d_s^{(n)} = \begin{cases} 1 & \text{if } s = 0, \\ b_s & \text{if } s = j + lp^{n-1}, \ 1 \leqslant j \leqslant p^{n-1} - 1, \ 0 \leqslant l \leqslant p - 1, \\ 0 & \text{if } s = p^n - lp^{n-1}, \ 1 \leqslant l \leqslant p - 1. \end{cases} \tag{1.5}$$

Theorem. *Let φ be given by the expansion*

$$\varphi(x) = p^{1-n} \mathbf{1}_U(A^{1-n}x) \Bigg(1 + \sum_{m \in \mathbb{N}(p,n)} c(m) W_m(A^{1-n}x) \Bigg), \qquad x \in G, \tag{1.6}$$

and let coefficients a_α be defined by

$$a_\alpha = \frac{1}{p^n} \sum_{s=0}^{p^n - 1} d_s^{(n)} W_\alpha^*(B^{-n}\omega_{[s]}), \quad 0 \leqslant \alpha \leqslant p^n - 1. \tag{1.7}$$

Then

(a) *the function φ satisfies* (1.2),

(b) *the system $\{\varphi(\cdot \ominus h) \mid h \in H\}$ is orthonormal in $L^2(G)$,*

(c) *the family $\{V_j\}$ given by* (1.1) *is a multiresolution analysis in $L^2(G)$.*

Remark 1.1. In terms of the Fourier transform, (1.2) may be rewritten as

$$\widehat{\varphi}(\omega) = m_0(B^{-1}\omega)\, \widehat{\varphi}(B^{-1}\omega), \tag{1.8}$$

where

$$m_0(\omega) = \sum_{\alpha=0}^{p^n-1} a_\alpha \overline{W_\alpha^*(\omega)}.$$

Using the discrete Vilenkin–Chrestenson transform (see, for example, [18]), we obtain that the coefficients a_α defined by (1.7) satisfy the system of equations

$$\sum_{\alpha=0}^{p^n-1} a_\alpha \overline{W_\alpha^*(B^{-n}\omega_{[s]})} = d_s^{(n)}, \qquad 0 \leqslant s \leqslant p^n - 1,$$

and, conversely, (1.7) follows from this system. Since the polynomial $m_0(\omega)$ is constant on $U_{n,s}^*$, equations (1.7) are equivalent to

$$m_0(\omega) = d_s^{(n)} \quad \text{for} \quad \omega \in U_{n,s}^*, \qquad 0 \leqslant s \leqslant p^n - 1. \tag{1.9}$$

Since $d_0^{(n)} = 1$, we see from (1.8) that

$$\widehat{\varphi}(\omega) = \prod_{j=1}^{\infty} m_0(B^{-j}\omega),$$

where, for each $\omega \in G^*$, all but finitely many of the factors are equal to 1. Thus $\widehat{\varphi}$ is continuous on G^*. Since

$$\bigcup_{s=0}^{p^{n-1}-1} U_{n,s}^* = B^{-1}(U^*),$$

we obtain from (1.4), (1.5) and (1.9) that

$$m_0(B^{-1}\omega) \neq 0, \qquad \omega \in U^*.$$

Moreover, (1.6) implies that $\operatorname{supp}\varphi \subset U_{1-n}$.

Remark 1.2. The generalized Walsh functions $\{w_m\}$ on the positive half-line $\mathbb{R}_+$ are defined by

$$w_0(t) \equiv 1, \quad w_m(t) = \prod_{j=0}^{k} (r(p^j t))^{\mu_j}, \qquad m \in \mathbb{N}, \quad t \in \mathbb{R}_+.$$

Here the μ_j are taken from the p-ary expansion (1.3) of m, and the function r is defined on $[0,1)$ by

$$r(t) = \begin{cases} 1 & \text{if } t \in [0, 1/p), \\ \exp(2\pi i l) & \text{if } t \in [lp^{-1}, (l+1)p^{-1}), \ \ l = 1, 2, \dots, p-1, \end{cases}$$

and is extended to $\mathbb{R}_+$ by putting $r(t+1) = r(t)$ for all $t \in \mathbb{R}_+$. Orthogonal series by the systems $\{W_m\}$ and $\{w_m\}$ are studied simultaneously (see, for example, [14] and [15]) if one replaces the Fourier transform in $L^2(G)$ by the Walsh–Fourier transform (for $p = 2$) or by the corresponding multiplicative transform in $L^2(\mathbb{R}_+)$

(for $p > 2$), see [14], Russian p. 127. Therefore, along with scaling functions φ of type (1.6), it is natural to study their analogues Φ on $\mathbb{R}_+$. These are defined by

$$\Phi(t) = p^{1-n}\mathbf{1}_{[0,1)}(p^{1-n}t)\Big(1 + \sum_{m\in\mathbb{N}(p,n)} c(m)w_m(p^{1-n}t)\Big), \qquad t \in \mathbb{R}_+,$$

where $\mathbb{N}(p, n)$ and $c(m)$ are the same as in (1.6). Every such function Φ satisfies an equation of the type

$$\Phi(t) = p\sum_{\alpha=0}^{p^n-1} a_\alpha \Phi(pt \ominus_p \alpha), \qquad t \in \mathbb{R}_+,$$

and generates a multiresolution analysis in the L^2-space on $(\mathbb{R}_+, \oplus_p)$. (We denote by $\oplus_p$ and $\ominus_p$ the operations of "addition" and "subtraction" modulo p defined by the p-ary expansions of elements of $\mathbb{R}_+$, see [14], [15].)

Remark 1.3. Using the equality

$$\sum_{m=0}^{p^{n-1}-1} \chi(y, m) = \begin{cases} p^{n-1} & \text{if } y \in U_{n-1}, \\ 0 & \text{if } y \in U \setminus U_{n-1}, \end{cases}$$

we easily verify that (1.6) determines the function $\varphi = \mathbf{1}_{U_{1-n}}$ if $b_1 = b_2 = \cdots = b_{p^{n-1}-1} = 1$. Hence the theorem also holds for $n = 1$ (the Haar case) if we put $\mathbb{N}(p, 1) = \varnothing$.

Example 1.4. If we take $p = n = 2$ and put $b_1 = a$, $b_3 = b$, then (1.6) takes the form

$$\varphi(x) = \frac{1}{2}\mathbf{1}_U(A^{-1}x)\Big(1 + a\sum_{j=0}^{\infty} b^j W_{2^{j+1}-1}(A^{-1}x)\Big), \qquad x \in G, \tag{1.10}$$

where $a \neq 0$, $|a|^2 + |b|^2 = 1$. In this case, the coefficients of (1.2) are defined by

$$a_0 = \frac{1+a+b}{4}, \qquad a_1 = \frac{1+a-b}{4}, \qquad a_2 = \frac{1-a-b}{4}, \qquad a_3 = \frac{1-a+b}{4}.$$

The formula (1.10) was found by Lang (see [10]).

Example 1.5. For $p = 2$ and $n = 3$, the conditions (1.4) take the form

$$|b_1|^2 + |b_5|^2 = |b_2|^2 + |b_6|^2 = |b_3|^2 + |b_7|^2 = 1.$$

Putting $b_1 = a,\ b_2 = b,\ b_3 = c,\ b_5 = \alpha,\ b_6 = \beta$ and $b_7 = \gamma$, we see from (1.7) that

$$\begin{aligned}
a_0 &= \frac{1}{8}(1 + a + b + c + \alpha + \beta + \gamma),\\
a_1 &= \frac{1}{8}(1 + a + b + c - \alpha - \beta - \gamma),\\
a_2 &= \frac{1}{8}(1 + a - b - c + \alpha - \beta - \gamma),\\
a_3 &= \frac{1}{8}(1 + a - b - c - \alpha + \beta + \gamma),\\
a_4 &= \frac{1}{8}(1 - a + b - c - \alpha + \beta - \gamma),\\
a_5 &= \frac{1}{8}(1 - a + b - c + \alpha - \beta + \gamma),\\
a_6 &= \frac{1}{8}(1 - a - b + c - \alpha - \beta + \gamma),\\
a_7 &= \frac{1}{8}(1 - a - b + c + \alpha + \beta - \gamma).
\end{aligned}$$

Moreover, we have

$$\begin{array}{lll}
\gamma(1,0,0) = a, & \gamma(0,1,0) = b, & \gamma(1,1,0) = c,\\
\gamma(1,0,1) = \alpha, & \gamma(0,1,1) = \beta, & \gamma(1,1,1) = \gamma
\end{array}$$

and

$$c(m) = \gamma(\mu_k, 0, 0)\gamma(\mu_{k-1}, \mu_k, 0)\ldots\gamma(\mu_0, \mu_1, \mu_2),$$

where

$$m = \sum_{i=0}^{k} \mu_i 2^i, \qquad \mu_k \neq 0, \quad \mu_i \in \{0, 1\}.$$

Observing that

$$\mathbb{N}(2,3) = \{1, 2, 3, 5, 6, 7, 10, 11, 13, 14, 15, 21, 22, 23, 26, 27, 29, 30, 31, 42, \ldots\},$$

and defining $y = A^{-2}x$, we see from (1.6) that

$$\begin{aligned}
\varphi(x) &= \frac{1}{4}\,\mathbf{1}_U(y)\Big(1 + \sum_{m\in\mathbb{N}(2,3)} c(m)W_m(y)\Big)\\
&= \frac{1}{4}\,\mathbf{1}_U(y)\big(1 + aW_1(y) + abW_2(y) + acW_3(y) + ab\alpha W_5(y)\\
&\qquad + ac\beta W_6(y) + ac\gamma W_7(y) + ab^2\alpha W_{10}(y) + abc\alpha W_{11}(y)\\
&\qquad + ac\alpha\beta W_{13}(y) + ac\beta\gamma W_{14}(y) + ac\gamma^2 W_{15}(y) + ab^2\alpha^2 W_{21}(y)\\
&\qquad + abc\alpha\beta W_{22}(y) + ab\alpha\beta\gamma W_{23}(y) + abc\alpha\beta W_{26}(y)\\
&\qquad + ac^2\alpha\beta W_{27}(y) + ac\alpha\beta\gamma W_{29}(y) + ac\beta\gamma^2 W_{30}(y) + \ldots\big).
\end{aligned} \tag{1.11}$$

Remark 1.6. The expansion (1.11) was found in [10] in the following cases:

1) $a = 1$, $b = 0$, $|c| < 1$ (and $\alpha = 0$, $\beta = 1$, $|\gamma| < 1$);

2) $|a| < 1$, $b = 1$, $c = 0$ (and $|\alpha| < 1$, $\beta = 0$, $\gamma = 1$).

Besides these two cases, the formula (1.6) was known only for $p = n = 2$ (Example 1.4). The above (easily computerizable) rules for calculating the coefficients a_α, $c(m)$ and the elements of the set $\mathbb{N}(p, n)$ are new. We recommend using (1.7) and the fast Vilenkin–Chrestenson transform (see, for example, [18], § 4) to compute the a_α.

In § 2 we prove the theorem stated above. In § 3 we show that the construction of an adapted multiresolution analysis suggested in [19] may be modified to solve the problem on the "optimal" sampling of the parameters b_m for the approximation of a "signal" f by its projections to subspaces of $L^2(G)$. Finally, in § 4 we discuss the smoothness of the scaling functions constructed above. For example, if $p = n = 2$, then the smoothness of φ is characterized by the sequence

$$\Omega_j(\varphi) := \sup\{|\varphi(x) - \varphi(y)| \mid x, y \in U_{-1},\ x \ominus y \in U_j\}, \qquad j \in \mathbb{N}.$$

In particular, we shall establish the following estimate (which is sharp) for functions φ given by (1.10): $\Omega_j(\varphi) \leqslant C2^{-\mu j}$ with $\mu = \log_2(1/|b|)$ (compare [1], Russian page 319 and [10], p. 541).

§ 2. Proof of the theorem

Let $l \in \{0, 1, \dots, p-1\}$. We denote by δ_l the sequence $\omega = (\omega_j)$ such that $\omega_1 = l$ and $\omega_j = 0$ for $j \neq 1$ (in particular, $\delta_0 = \theta$). It is easily seen that

$$\{\omega \in H^* \mid \chi(x, \omega) = 1,\ x \in A(H)\} = \{\delta_0, \delta_1, \dots, \delta_{p-1}\}. \tag{2.1}$$

Hence the set $\{\delta_l\}$ is the annihilator of the subgroup $A(H)$ in H.

Lemma 2.1. *Let*

$$m_0(\omega) = \sum_{\alpha=0}^{p^n-1} a_\alpha \overline{W_\alpha^*(\omega)} \tag{2.2}$$

be a polynomial satisfying the following conditions:

(a) $m_0(\theta) = 1$,

(b) $m_0(B^{-1}\omega) \neq 0$ *for* $\omega \in U^*$,

(c) $\sum_{l=0}^{p-1} |m_0(B^{-n}\omega_{[s]} \oplus \delta_l)|^2 = 1$ *for* $s \in \{0, 1, \dots, p^{n-1} - 1\}$.

Then the function

$$g(\omega) = \prod_{j=1}^{\infty} m_0(B^{-j}\omega) \tag{2.3}$$

belongs to $L^2(G^*)$. *Moreover, if* φ *is obtained from* g *by the inverse Fourier transform* (*so that* $\varphi \in L^2(G)$ *and* $\widehat{\varphi} = g$), *then* $\{\varphi(\cdot \ominus h) \mid h \in H\}$ *is an orthonormal system in* $L^2(G)$ *and* φ *generates a multiresolution analysis in* $L^2(G)$ *by* (1.1).

Proof. Let g be defined by (2.3), where $m_0(\omega)$ has the form (2.2) and satisfies conditions (a)–(c). Since $m_0(\omega \oplus h^*) = m_0(\omega)$ for $h^* \in H^\perp$ and the polynomial $m_0(\omega)$ is constant on each $U^*_{n,s}$, $s \in \{0, 1, \dots, p^n - 1\}$, assertion (c) is equivalent to

$$\sum_{l=0}^{p-1} |m_0(\omega \oplus \delta_l)|^2 = 1, \qquad \omega \in G^*. \tag{2.4}$$

Step 1. We shall check that

$$\int_{G^*} |g(\omega)|^2 \, d\mu^*(\omega) \leqslant 1. \tag{2.5}$$

For every positive integer k, we put

$$\mu^{[k]}(\omega) = \prod_{j=1}^{k} m_0(B^{-j}\omega) \mathbf{1}_{U^*}(B^{-k}\omega), \qquad \omega \in G^*.$$

By condition (a), $m_0(\omega) = 1$ for $\omega \in U^*_{n,0}$. Thus it follows from (2.3) that

$$\lim_{k \to \infty} \mu^{[k]}(\omega) = g(\omega), \qquad \omega \in G^*. \tag{2.6}$$

Observing that $B^{-n}\omega \in U^*_{n,0}$ for $\omega \in U^*$, we have

$$g(\omega) = \prod_{j=1}^{n-1} m_0(B^{-j}\omega), \qquad \omega \in U^*.$$

By condition (b), there is a constant $c_1 > 0$ such that

$$|m_0(B^{-j}\omega)| \geqslant c_1, \qquad j \in \mathbb{N}, \quad \omega \in U^*,$$

and so $c_1^{1-n} |g(\omega)| \geqslant \mathbf{1}_{U^*}(\omega)$ for $\omega \in G^*$. Therefore

$$|\mu^{[k]}(\omega)| = \prod_{j=1}^{k} |m_0(B^{-j}\omega)| \mathbf{1}_{U^*}(B^{-k}\omega) \leqslant c_1^{1-n} \prod_{j=1}^{k} |m_0(B^{-j}\omega)| |g(B^{-k}\omega)|,$$

which by (2.3) yields

$$|\mu^{[k]}(\omega)| \leqslant c_1^{1-n} |g(\omega)|, \qquad k \in \mathbb{N}, \quad \omega \in G^*. \tag{2.7}$$

Now, let

$$I_k(s) := \int_{G^*} |\mu^{[k]}(\omega)|^2 \, \overline{W^*_s(\omega)} \, d\mu^*(\omega), \qquad k \in \mathbb{N}, \quad s \in \mathbb{Z}_+.$$

Setting $\zeta = B^{-k}\omega$, we find that

$$I_k(s) = \int_{U^*_{k,0}} \prod_{j=1}^{k} |m_0(B^{-j}\omega)|^2 \overline{W_s^*(\omega)}\, d\mu^*(\omega)$$
$$= p^k \int_{U^*} |m_0(\zeta)|^2 \prod_{j=1}^{k-1} |m_0(B^j\zeta)|^2 \overline{W_s^*(B^k\zeta)}\, d\mu^*(\zeta). \tag{2.8}$$

Since $U^* = \bigcup_{l=0}^{p-1} \big(B^{-1}(U^*) \oplus \delta_l\big)$, where the δ_l are defined by (2.1), we have

$$I_k(s) = p^{k-1} \int_{U^*} \sum_{l=0}^{p-1} |m_0(B^{-1}\omega \oplus \delta_l)|^2 \prod_{j=1}^{k-1} |m_0(B^{j-1}\omega)|^2 \overline{W_s^*(B^{k-1}\omega)}\, d\mu^*(\omega),$$

and, in view of (2.4),

$$I_k(s) = p^{k-1} \int_{U^*} \prod_{j=0}^{k-2} |m_0(B^j\omega)|^2 \overline{W_s^*(B^{k-1}\omega)}\, d\mu^*(\omega).$$

Hence, by (2.8),

$$I_k(s) = I_{k-1}(s).$$

When $k = 1$, we similarly have

$$I_1(s) = p \int_{U^*} |m_0(\omega)|^2\, \overline{W_s^*(B\omega)}\, d\mu^*(\omega) = \int_{U^*} \overline{W_s^*(\omega)}\, d\mu^*(\omega),$$

where the last integral is equal to $\delta_{0,s}$ since the system $\{W_\alpha^*\}$ is orthonormal in $L^2(U^*)$. Hence,

$$I_k(s) = \delta_{0,s}, \qquad k \in \mathbb{N}, \quad s \in \mathbb{Z}_+. \tag{2.9}$$

In particular, for all $k \in \mathbb{N}$,

$$I_k(0) = \int_{G^*} |\mu^{[k]}(\omega)|^2\, d\mu^*(\omega) = 1.$$

Using (2.6) and Fatou's lemma, we obtain (2.5).

Step 2. Let us define a function φ in $L^2(G)$ by the condition $\widehat{\varphi} = g$, where g is given by (2.3). Using Lebesgue's dominated convergence theorem, we see from (2.6), (2.7) and (2.9) that

$$\int_{G^*} |\widehat{\varphi}(\omega)|^2 \overline{W_s^*(\omega)}\, d\mu^*(\omega) = \lim_{k\to\infty} I_k(s) = \delta_{0,s}.$$

By the Plancherel formula, it follows that the system $\{\varphi(\cdot \ominus h) \mid h \in H\}$ is orthonormal in $L^2(G)$.

Step 3. We note that φ satisfies (1.2) if the coefficients a_α in (1.2) are the same as in the polynomial (2.2). Indeed, (2.3) implies that

$$\widehat{\varphi}(\omega) = m_0(B^{-1}\omega)\,\widehat{\varphi}(B^{-1}\omega). \tag{2.10}$$

Therefore (1.2) follows from (2.10) by the Fourier inversion formula. Hence the condition $V_j \subset V_{j+1}$ holds for the family $\{V_j\}$ given by (1.1) with our φ. Condition (v) of the definition of a multiresolution analysis follows from the results of Step 2. Conditions (iii) and (iv) follow immediately from (1.1).

Step 4. We shall prove that $\bigcap V_j = \{0\}$. Condition (v) and equation (1.1) imply that $\{\varphi_{j,h} \mid h \in H\}$ is an orthonormal basis of V_j for every $j \in \mathbb{Z}$. Therefore the orthogonal projection $P_j\colon L^2(G) \to V_j$ acts by the formula

$$P_j f = \sum_{h\in H} (f, \varphi_{j,h})\varphi_{j,h}, \qquad f \in L^2(G). \tag{2.11}$$

Suppose that $f \in \bigcap V_j$ and fix $\varepsilon > 0$. The set $C_0(G)$ of compactly supported continuous functions on G is dense in $L^2(G)$ (see [3], (12.10)). Choose $f_0 \in C_0(G)$ such that $\|f - f_0\| < \varepsilon$. Then

$$\|f - P_j f_0\| \leqslant \|P_j(f - f_0)\| \leqslant \|f - f_0\| < \varepsilon,$$

and so

$$\|f\| \leqslant \|P_j f_0\| + \varepsilon \tag{2.12}$$

for all $j \in \mathbb{Z}$. If $\operatorname{supp} f_0 \subset U_l$, then

$$(P_j f_0, \varphi_{j,h}) = (f_0, \varphi_{j,h}) = p^{j/2} \int_{U_l} f_0(x)\overline{\varphi(A^j x \ominus h)}\, d\mu(x),$$

where the number l depends on f_0. Using the Cauchy–Schwarz inequality, we get

$$\|P_j f_0\|^2 = \sum_{h\in H} |(P_j f_0, \varphi_{j,h})|^2 \leqslant \|f_0\|^2 \sum_{h\in H} p^j \int_{U_l} |\varphi(A^j x \ominus h)|^2\, d\mu(x).$$

For $j < l$ we have

$$\sum_{h\in H} p^j \int_{U_l} |\varphi(A^j x \ominus h)|^2\, d\mu(x) = \int_{S_{l,j}} |\varphi(x)|^2\, d\mu(x),$$

where

$$S_{l,j} := \bigcup_{h\in H} \{y \ominus h \mid y \in U_{l-j}\}.$$

Hence,

$$\|P_j f_0\|^2 \leqslant \|f_0\|^2 \int_G \mathbf{1}_{S_{l,j}}(x)|\varphi(x)|^2\, d\mu(x). \tag{2.13}$$

It is easy to check that $\lim_{j\to-\infty}\mathbf{1}_{S_{l,j}}(x)=0$ for all $x\notin H$. By Lebesgue's theorem, we see from (2.13) that

$$\lim_{j\to-\infty}\|P_ju\|=0.$$

Applying (2.12), we conclude that $\|f\|\leqslant\varepsilon$ and hence $\bigcap V_j=\{0\}$.

Step 5. We shall now prove that

$$\overline{\bigcup V_j}=L^2(G). \tag{2.14}$$

Let $f\in(\bigcup V_j)^{\perp}$ and $\varepsilon>0$. We choose $u\in L^2(G)$ such that $\widehat{u}\in C_0(G^*)$ and $\|\widehat{f}-\widehat{u}\|<\varepsilon$. For every $j\in\mathbb{Z}_+$, we see from (2.11) that

$$\|P_jf\|^2=(P_jf,P_jf)=(f,P_jf)=0$$

and

$$\|P_ju\|=\|P_j(f-u)\|\leqslant\|f-u\|=\|\widehat{f}-\widehat{u}\|<\varepsilon. \tag{2.15}$$

Fix a number $j\in\mathbb{N}$ such that $\operatorname{supp}\widehat{u}\subset U^*_{-j}$ and $B^{-j}\omega\in U^*_{n-1}$ for all $\omega\in\operatorname{supp}\widehat{u}$. Since the system $\{p^{-j/2}W^*_{\alpha}(B^{-j}\,\cdot)\}$ is orthonormal and complete in $L^2(U^*_{-j})$, we see that $\Gamma(\omega)=\widehat{u}(\omega)\,\widehat{\varphi}(B^{-j}\omega)$ satisfies

$$p^{-j}\int_{U^*_{-j}}|\Gamma(\omega)|^2\,d\mu^*(\omega)=\sum_{\alpha\in\mathbb{Z}_+}|c_{\alpha}(\Gamma)|^2, \tag{2.16}$$

where

$$c_{\alpha}(\Gamma)=p^{-j/2}\int_{U^*_{-j}}\Gamma(\omega)\overline{W^*_{\alpha}(B^{-j}\omega)}\,d\mu^*(\omega).$$

Since

$$\int_G\varphi(A^jx\ominus h_{[\alpha]})\overline{\chi(x,\omega)}\,d\mu(x)=p^{-j}\widehat{\varphi}(B^{-j}\omega)\overline{W^*_{\alpha}(B^{-j}\omega)},$$

we get

$$p^{-j/2}(u,\varphi_{j,h})=p^{-j}\int_{U^*_{-j}}\Gamma(\omega)\overline{W^*_{\alpha}(B^{-j}\omega)}\,d\mu^*(\omega).$$

Therefore, by (2.16) we have

$$\|P_ju\|^2=\sum_{h\in H}|(u,\varphi_{j,h})|^2=\int_{U^*_{-j}}|\widehat{u}(\omega)|^2|\widehat{\varphi}(B^{-j}\omega)|^2\,d\mu^*(\omega). \tag{2.17}$$

As $m_0(\omega)=1$ on $U^*_{n,0}$ and $B^{-j}\omega\in U^*_{n+1}$ for $\omega\in\operatorname{supp}\widehat{u}$, it follows from (2.3) that $\widehat{\varphi}(B^{-j}\omega)=1$ for all $\omega\in\operatorname{supp}\widehat{u}$. (We recall that $\widehat{\varphi}=g$.) Since $\operatorname{supp}\widehat{u}\subset U^*_{-j}$, we see from (2.15) and (2.17) that

$$\varepsilon>\|P_ju\|=\|\widehat{u}\|=\|u\|.$$

Consequently, $\|f\| < \varepsilon + \|u\| < 2\varepsilon$. Thus $(\bigcup V_j)^\perp = \{0\}$ and hence (2.14) holds. The lemma is proved.

Remark 2.2. In connection with condition (b) of Lemma 2.1, we note that in general there are points $\omega \in U^*$ at which $m_0(\omega) = 0$ (see Examples 1.4, 1.5 and Proposition 3.3).

Proof of the theorem. Put $X_{n-1} = \mathbf{1}_{U^*_{n-1}}$. For every $x \in G,\ m \in \mathbb{N}$ we have

$$\begin{aligned}\int_G X_{n-1}(\omega \ominus B^{1-n}\omega_{[m]})\chi(x,\omega)\,d\mu^*(\omega) &= \chi(x, B^{1-n}\omega_{[m]})\int_{U_{n-1}} \chi(x,\omega)\,d\mu^*(\omega)\\ &= p^{1-n}\mathbf{1}_U(A^{1-n}x)\chi(A^{1-n}x,\omega_{[m]}) = p^{1-n}\mathbf{1}_U(A^{1-n}x)W_m(A^{1-n}x).\end{aligned}$$

Using the Fourier transform, we see from this and (1.6) that

$$\widehat{\varphi}(\omega) = X_{n-1}(\omega) + \sum_{m\in\mathbb{N}(p,n)} c(m)X_{n-1}(\omega \ominus B^{1-n}\omega_{[m]}).$$

Hence, for $m \in \mathbb{N}(p,n)$

$$\widehat{\varphi}(\omega) = \begin{cases} 1 & \text{if } \omega \in U^*_{n-1},\\ c(m) & \text{if } \omega \in U^*_{n-1,m},\\ 0 & \text{otherwise.}\end{cases} \tag{2.18}$$

We now suppose that the coefficients a_α are given by (1.7). According to (1.9), the polynomial

$$m_0(\omega) = \sum_{\alpha=0}^{p^n-1} a_\alpha \overline{W^*_\alpha(\omega)}$$

satisfies

$$m_0(B^{-n}(\omega_{[s]})) = d_s^{(n)}, \qquad 0 \leqslant s \leqslant p^n - 1.$$

Hence, by (2.18) and the definition of $c(m)$, we have

$$\widehat{\varphi}(\omega) = \prod_{j=1}^{\infty} m_0(B^{-j}\omega)$$

and so

$$\widehat{\varphi}(\omega) = m_0(B^{-1}\omega)\,\widehat{\varphi}(B^{-1}\omega).$$

This proves assertion (a). Using Lemma 2.1, we conclude that assertions (b) and (c) also hold. The theorem is proved.

§ 3. On orthonormality conditions in $L^2(G)$ and optimization of parameters

The following assertion can be deduced from the generalized Poisson summation formula (see, for example, [2], p. 377). For completeness, we give an elementary proof here.

Proposition 3.1. *Let $\varphi \in L^2(G)$. The system $\{\varphi(\,\cdot \ominus h) \mid h \in H\}$ is orthonormal in $L^2(G)$ if and only if*

$$\sum_{h^* \in H^\perp} |\widehat{\varphi}(\omega \ominus h^*)|^2 = 1 \qquad \text{for a. e.} \quad \omega \in G^*. \tag{3.1}$$

Proof. The function

$$\Phi(\omega) = \sum_{h^* \in H^\perp} |\widehat{\varphi}(\omega \ominus h^*)|^2$$

is $H^\perp$-periodic: $\Phi(\omega \oplus h^*) = \Phi(\omega)$ for all $h^* \in H^\perp$. Furthermore, it has a finite L^1-norm on U^* because

$$\begin{aligned} 0 \leqslant \int_{U^*} \Phi(\omega)\, d\mu^*(\omega) &= \int_{U^*} \sum_{h^* \in H^\perp} |\widehat{\varphi}(\omega \ominus h^*)|^2\, d\mu^*(\omega) \\ &= \sum_{h^* \in H^\perp} \int_{U^* \oplus h^*} |\widehat{\varphi}(\omega)|^2\, d\mu^*(\omega) = \int_{G^*} |\widehat{\varphi}(\omega)|^2\, d\mu^*(\omega) < +\infty. \end{aligned}$$

Let $\{\widehat{\Phi}(h)\}$ be the Fourier coefficients of Φ with respect to the system $\{\chi(h,\,\cdot\,)\}$. For any $h \in H$, we get the following equations by the change of variables $\eta = \omega \ominus h^*$:

$$\begin{aligned} \widehat{\Phi}(h) &= \int_{U^*} \Phi(\omega)\overline{\chi(h,\omega)}\, d\mu^*(\omega) = \int_{U^*} \overline{\chi(h,\omega)} \sum_{h^* \in H^\perp} |\widehat{\varphi}(\omega \ominus h^*)|^2\, d\mu^*(\omega) \\ &= \sum_{h^* \in H^\perp} \int_{U^* \oplus h^*} \overline{\chi(h,\eta)} |\widehat{\varphi}(\eta)|^2\, d\mu^*(\eta) = \int_{G^*} |\widehat{\varphi}(\omega)|^2 \overline{\chi(h,\omega)}\, d\mu^*(\omega). \end{aligned}$$

Applying the Plancherel equality, we get

$$\int_G \varphi(x \ominus h)\overline{\varphi(x)}\, d\mu(x) = \widehat{\Phi}(h), \qquad h \in H.$$

To complete the proof, we note that (3.1) is equivalent to $\widehat{\Phi}(h) = \delta_{\theta,h}$ for $h \in H$.

Proposition 3.2. *If a function $\varphi \in L^2(G)$ satisfies* (1.2) *and the system $\{\varphi(\,\cdot \ominus h) \mid h \in H\}$ is orthonormal in $L^2(G)$, then*

$$\sum_{l=0}^{p-1} |m_0(\omega \oplus \delta_l)|^2 = 1 \quad \text{for all} \quad \omega \in G^*, \tag{3.2}$$

where

$$m_0(\omega) = \sum_{\alpha=0}^{p^n-1} a_\alpha \overline{W_\alpha^*(\omega)}.$$

Proof. For $l \in \{0, 1, \dots, p-1\}$ we put

$$H_l^\perp = \{h^* \in H^\perp \mid B^{-1}h^* \ominus \delta_l \in H^\perp\}$$

(note that $h_0^* = l$ for $h^* = (h_j^*) \in H_l^\perp$). Since $\widehat{\varphi}$ is continuous on G^* (see Remark 1.1), we see from (3.1) that

$$\sum_{h^* \in H_l^\perp} |\widehat{\varphi}(B^{-1}\omega \oplus B^{-1}h^*)|^2 = 1, \qquad l \in \{0, 1, \dots, p-1\}$$

for any $\omega \in G^*$. It follows from this and (1.8) that

$$\begin{aligned} 1 &= \sum_{h^* \in H^\perp} \big|m_0\big(B^{-1}(\omega \oplus h^*)\big)\big|^2 \big|\widehat{\varphi}\big(B^{-1}(\omega \oplus h^*)\big)\big|^2 \\ &= \sum_{l=0}^{p-1} |m_0(B^{-1}\omega \oplus \delta_l)|^2 \sum_{h^* \in H_l^\perp} |\widehat{\varphi}(B^{-1}\omega \oplus B^{-1}h^*)|^2 = \sum_{l=0}^{p-1} |m_0(B^{-1}\omega \oplus \delta_l)|^2. \end{aligned}$$

Thus (3.2) holds. The proof is complete.

A subset E of G^* is said to be *congruent to U^* modulo $H^\perp$* if $\mu^*(E) = 1$ and, for each $\omega \in E$, there is an element $h^* \in H^\perp$ such that $\omega \oplus h^* \in U^*$. We have the following analogue of Cohen's theorem (see [20]).

Proposition 3.3. *Let m_0 be a polynomial of the form*

$$m_0(\omega) = \sum_{h \in H} a(h)\overline{\chi(h, \omega)},$$

where $a(\,\cdot\,)$ is a finitary function on H. Then the following conditions are equivalent.

1) *The polynomial m_0 satisfies*

$$m_0(\theta) = 1, \quad \sum_{l=0}^{p-1} |m_0(\omega \oplus \delta_l)|^2 = 1, \qquad \omega \in G^*, \tag{3.3}$$

and there is a compact set E congruent to U^* modulo $H^\perp$ and containing a neighbourhood of zero such that

$$\inf_{j \in \mathbb{N}} \inf_{\omega \in E} |m_0(B^{-j}\omega)| > 0. \tag{3.4}$$

2) *There is a function $\varphi \in L^2(G)$ whose Fourier transform can be written as*

$$\widehat{\varphi}(\omega) = \prod_{j=1}^{\infty} m_0(B^{-j}\omega)$$

and the system $\{\varphi(\,\cdot\, \ominus h) \mid h \in H\}$ is orthonormal in $L^2(G)$.

We note that since E is compact, there is a number j_0 such that $m_0(B^{-j}\omega) = 1$ for all $j > j_0$, $\omega \in E$. Therefore (3.4) holds if the polynomial $m_0(\omega)$ does not vanish on the sets $B^{-1}(E), \dots, B^{-j_0}(E)$.

Remark 3.4. If φ satisfies the conditions of Proposition 3.3, then the family $\{V_j\}$ given by (1.1) is a multiresolution analysis in $L^2(G)$ (compare [1], Russian p. 248).

Let φ be given by (1.6) and let the a_α be determined from (1.7). (Then (3.4) holds for $E = U^*$ by Remark 1.1.) As above, for each $j \in \mathbb{Z}$, let P_j be the orthogonal projection of $L^2(G)$ onto the subspace

$$V_j = \mathrm{clos}_{L^2(G)} \mathrm{span}\{\varphi_{j,h} \mid h \in H\}.$$

Given f in $L^2(G)$, it is natural to choose the parameters b_m in (1.4) such that the approximations $f \approx P_j f$ be optimal. If we know that f belongs to some class $\mathcal{M}$ in $L^2(G)$, then we can use the methods of approximation theory (see, for example, [14], Ch. 10 and [21], Ch. 2) to seek the parameters b_m that minimize the quantity

$$\sup\{\|f - P_j f\| \mid f \in \mathcal{M}\}$$

for a fixed j and to study the behaviour of this quantity as $j \to +\infty$. As in the recent paper [19], we shall discuss a different approach to the problem of optimal approximations $f \approx P_j f$ of a given function f.

For every $j \in \mathbb{Z}$, let W_j be the orthogonal complement of V_j in V_{j+1} and let Q_j be the orthogonal projection of $L^2(G)$ to W_j. Since $\{V_j\}$ is a multiresolution analysis, for any $f \in L^2(G)$ we have

$$f = \sum_j Q_j f = P_0 f + \sum_{j \geqslant 0} Q_j f$$

and

$$\lim_{j \to +\infty} \|f - P_j f\| = 0, \qquad \lim_{j \to -\infty} \|P_j f\| = 0.$$

Also, it is easy to see that

$$P_j f = Q_{j-1} f + Q_{j-2} f + \dots + Q_{j-s} f + P_{j-s} f, \qquad j \in \mathbb{Z}, \quad s \in \mathbb{N}.$$

The equality $V_j = V_{j-1} \oplus W_{j-1}$ means that W_{j-1} contains the "details" needed for passing from the $(j-1)$th level of approximation to the more exact jth level. Since $\|P_j f\|^2 = \|P_{j-1} f\|^2 + \|Q_{j-1} f\|^2$, it is natural to choose the parameters b_m that maximize $\|P_{j-1} f\|$ (or, equivalently, minimize $\|Q_{j-1} f\|$).

We write (1.2) as

$$\varphi(x) = \sqrt{p} \sum_{\alpha=0}^{p^n-1} \widetilde{a}_\alpha \varphi\bigl(Ax \ominus h_{[\alpha]}\bigr),$$

where

$$\widetilde{a}_\alpha = \sqrt{p}\, a_\alpha = \sqrt{p}\, \bigl(\varphi(\,\cdot\,), \varphi\bigl(A \cdot \ominus h_{[\alpha]}\bigr)\bigr).$$

Put $\varphi_j(x) = p^{j/2}\varphi(A^j x)$. Then

$$\varphi_{j-1}(x) = \sum_{\alpha=0}^{p^n-1} \widetilde{a}_\alpha \varphi_j\big(x \ominus A^{-j} h_{[\alpha]}\big) \tag{3.5}$$

and

$$\varphi_j(x \ominus A^{-j}h) = \varphi_{j,h}(x) = p^{j/2}\varphi_j(A^j x \ominus h), \qquad j \in \mathbb{Z}, \quad h \in H.$$

Given $f \in L^2(G)$, we set

$$f(j,h) := (f, \varphi_{j,h}) = \int_G f(x)\overline{\varphi_j(x \ominus A^{-j}h)}\, d\mu(x), \qquad j \in \mathbb{Z}, \quad h \in H.$$

Using the relations (3.5), we obtain

$$\begin{aligned} f(j-1,h) &= \int_G f(x)\overline{\varphi_{j-1}(x \ominus A^{-j+1}h)}\, d\mu(x) \\ &= \sum_{\alpha=0}^{p^n-1} \overline{\widetilde{a}}_\alpha \int_G f(x)\overline{\varphi_j\big(x \ominus A^{-j}\big(Ah \oplus h_{[\alpha]}\big)\big)}\, d\mu(x) \end{aligned}$$

and hence

$$f(j-1,h) = \sum_{\alpha=0}^{p^n-1} \overline{\widetilde{a}}_\alpha f\big(j, Ah \oplus h_{[\alpha]}\big). \tag{3.6}$$

Since

$$P_j f = \sum_{h \in H} f(j,h)\varphi_{j,h}, \qquad j \in \mathbb{Z},$$

we see from (3.6) that

$$\begin{aligned} \|P_{j-1}f\|^2 &= \sum_{h\in H} |f(j-1,h)|^2 = \sum_{h \in H}\left|\sum_{\alpha=0}^{p^n-1} \overline{\widetilde{a}}_\alpha f\big(j, Ah \oplus h_{[\alpha]}\big)\right|^2 \\ &= \sum_{h\in H}\left(\sum_{\alpha,\beta=0}^{p^n-1} \overline{\widetilde{a}}_\alpha \widetilde{a}_\beta f\big(j, Ah \oplus h_{[\alpha]}\big)\overline{f\big(j, Ah \oplus h_{[\beta]}\big)}\right). \end{aligned} \tag{3.7}$$

For $0 \leqslant \alpha,\ \ \beta \leqslant p^n - 1$ we put

$$F_{\alpha,\beta}(j) := \sum_{h\in H} f\big(j, Ah \oplus h_{[\alpha]}\big)\overline{f\big(j, Ah \oplus h_{[\beta]}\big)}.$$

Then $F_{\beta,\alpha}(j) = \overline{F_{\alpha,\beta}(j)}$ and (3.7) implies that

$$\|P_{j-1}f\|^2 = \sum_{\alpha,\beta=0}^{p^n-1} F_{\alpha,\beta}(j)\overline{\widetilde{a}}_\alpha \widetilde{a}_\beta. \tag{3.8}$$

We denote by $\mathcal{U}(p,n)$ the set of vectors $u=(u_0,u_1,\dots,u_{p^n-1})$ such that

$$u_0=1,\quad u_j=0,\qquad j\in\{p^{n-1},2p^{n-1},\dots,(p-1)p^{n-1}\},$$

and

$$u_j\neq 0,\quad \sum_{l=0}^{p-1}|u_{lp^{n-1}+j}|^2=1,\qquad j\in\{1,2,\dots,p^{n-1}-1\}.$$

For every $u=(u_0,u_1,\dots,u_{p^n-1})$ in $\mathcal{U}(p,n)$, we define the coefficients $a_\alpha(u)$ by the system

$$\sum_{\alpha=0}^{p^n-1}a_\alpha(u)\overline{W_\alpha^*\bigl(B^{-n}\omega_{[s]}\bigr)}=u_s,\qquad 0\leqslant s\leqslant p^n-1.$$

Fix a positive integer j_0. If a vector u^* is a solution of the extremal problem

$$\sum_{\alpha,\beta=0}^{p^n-1}F_{\alpha,\beta}(j_0)\overline{a_\alpha(u)}a_\beta(u)\to\max,\qquad u\in\mathcal{U}(p,n),\tag{3.9}$$

then $\varphi^*_{j_0-1}$ is defined by

$$\varphi^*_{j_0-1}(x)=\sum_{\alpha=0}^{p^n-1}a_\alpha(u^*)\varphi_{j_0}\bigl(x\ominus A^{-j_0}h_{[\alpha]}\bigr).$$

We see from (3.8) and (3.9) that $\|P_j^*f\|\geqslant\|P_jf\|$ for $j=j_0-1$. We similarly construct $\varphi^*_{j_0-2}$ on the base of $\varphi^*_{j_0-1}$ and so on. Then we fix a number s and replace the orthogonal projections P_jf ($j\in\{j_0-1,\dots,j_0-s\}$) by the orthogonal projections P_j^*f of f to the subspaces V_j^* given by

$$V_j^*=\operatorname{clos}_{L^2(G)}\operatorname{span}\{\varphi_j^*(\,\cdot\,-A^{-j}h)\mid h\in H\}.$$

The effectiveness of this method of adaptation can be demonstrated by numerical examples using the entropy criterion. (Similar examples for the Daubechies' multiresolution analysis and for wavelet-packets are given in [19] and [22].)

§ 4. On the smoothness of scaling functions

Let φ be a scaling function in $L^2(G)$ defined by (1.6) (so that $\operatorname{supp}\varphi\subset U_{1-n}$). We recall that for $l\in\mathbb{Z}$,

$$U_l=\{(x_j)\in G\mid x_j=0\text{ for }j\leqslant l\}\qquad\text{and}\qquad U_l\supset U_{l+1}.$$

The *modulus of continuity* of a continuous complex-valued function f on U_{1-n} is the following sequence $\{\Omega_j(f)\}$, $j\geqslant 1-n$:

$$\Omega_j(f)=\sup\{|f(x)-f(y)|\mid x,y\in U_{1-n},\ x\ominus y\in U_j\}.$$

Given any non-increasing sequence with zero limit,

$$\varepsilon_{1-n} \geqslant \varepsilon_{2-n} \geqslant \varepsilon_{3-n} \geqslant \dots, \qquad \lim_{j\to\infty} \varepsilon_j = 0,$$

one can find a function $f \in C(U_{1-n})$ such that $\Omega_j(f) = \varepsilon_j$ for all $j \geqslant 1-n$ (see [23]). A method for estimating the smoothness of scaling functions was suggested in [24]. Concentrating on the case $p = 2$ for brevity, we shall show how to apply this method to estimate the modulus of continuity of φ. We shall assume that the coefficients a_α are real, although all arguments can easily be extended to the complex case.

So we take $p = 2$ and $c_\alpha = 2a_\alpha$ for $\alpha \in \{0, 1, \dots, 2^n - 1\}$, where the a_α are real and defined by (1.7). Then

$$\varphi(x) = \sum_{\alpha=0}^{2^n-1} c_\alpha \varphi\bigl(Ax \ominus h_{[\alpha]}\bigr), \qquad x \in G, \tag{4.1}$$

$$\sum_{\alpha=0}^{2^n-1} c_\alpha = 2, \qquad \sum_{\alpha=0}^{2^{n-1}-1} c_{2\alpha} = \sum_{\alpha=0}^{2^{n-1}-1} c_{2\alpha+1} = 1. \tag{4.2}$$

Put $N = 2^{n-1}$. Using (1.6) and the easily verified equalities

$$\sum_{\alpha=0}^{N-1} W_m\bigl(A^{1-n}h_{[\alpha]}\bigr) = \sum_{\alpha=0}^{N-1} W_m\bigl(A^{1-n}h^-_{[\alpha+1]}\bigr) = 0, \qquad m \in \mathbb{N}(2, n),$$

we get

$$\sum_{\alpha=0}^{N-1} \varphi\bigl(h_{[\alpha]}\bigr) = \sum_{\alpha=0}^{N-1} \varphi\bigl(h^-_{[\alpha+1]}\bigr) = 1. \tag{4.3}$$

By definition, the equality $k = i \oplus_2 j$ for $i, j, k \in \{1, 2, \dots, 2N-1\}$ means that

$$k_s = i_s + j_s \pmod 2, \qquad s \in \{0, 1, \dots, 2N-1\},$$

where i_s, j_s, k_s are digits of the binary expansions

$$i = \sum_{s=0}^{2N-1} i_s 2^s, \qquad j = \sum_{s=0}^{2N-1} j_s 2^s, \qquad k = \sum_{s=0}^{2N-1} k_s 2^s.$$

We define $N \times N$ matrices T_0 and T_1 by

$$(T_0)_{i,j} = c_{2(i-1)\oplus_2(j-1)}, \qquad (T_1)_{i,j} = c_{(2i-1)\oplus_2(j-1)}, \tag{4.4}$$

where $i, j \in \{1, 2, \dots, N\}$. In particular, for $n = 2$

$$T_0 = \begin{pmatrix} c_0 & c_1 \\ c_2 & c_3 \end{pmatrix}, \qquad T_1 = \begin{pmatrix} c_1 & c_0 \\ c_3 & c_2 \end{pmatrix},$$

and for $n = 3$

$$T_0 = \begin{pmatrix} c_0 & c_1 & c_2 & c_3 \\ c_2 & c_3 & c_0 & c_1 \\ c_4 & c_5 & c_6 & c_7 \\ c_6 & c_7 & c_4 & c_5 \end{pmatrix}, \qquad T_1 = \begin{pmatrix} c_1 & c_0 & c_3 & c_2 \\ c_3 & c_2 & c_1 & c_0 \\ c_5 & c_4 & c_7 & c_6 \\ c_7 & c_6 & c_5 & c_4 \end{pmatrix}.$$

Let $e_1 = (1, 1, \dots, 1)$ be the N-dimensional row vector all of whose components are equal to 1. Then by (4.2),

$$e_1 T_0 = e_1 T_1 = e_1. \tag{4.5}$$

For any N-dimensional vectors $v = (v_1, \dots, v_N)$ and $w = (w_1, \dots, w_N)$ we set

$$v \cdot w^{\mathrm{T}} := \sum_{j=1}^{N} v_j w_j, \qquad \|v\| := \sqrt{v \cdot v^{\mathrm{T}}}\,,$$

where the column vectors v^{T}, w^{T} are obtained from the row vectors v, w by transposition.

We define E_1 to be the subspace of $\mathbb{R}^N$ orthogonal to e_1:

$$E_1 := \{u = (u_1, \dots, u_N)^{\mathrm{T}} \mid e_1 \cdot u = 0\}.$$

For any real $N \times N$ matrix M we put

$$\|M\| := \sup\{\|Mu\|/\|u\| \mid u \in \mathbb{R}^N,\ u \neq 0\},$$
$$\|M|_{E_1}\| := \sup\{\|Mu\|/\|u\| \mid u \in E_1,\ u \neq 0\}.$$

It is well known that $\|M\|$ coincides with the square-root of the largest eigenvalue of the matrix $M^{\mathrm{T}}M$.

We have the following analogue of Theorem 2.2 in [24].

Proposition 4.1. *Let φ be a function given by* (1.6) *and let the coefficients a_α be defined by* (1.7) *for $p = 2$. Assume that the elements of the $N \times N$ matrices T_0, T_1 are given by* (4.4), *where $N = 2^{n-1}$ and $c_\alpha = 2a_\alpha$. If, for all $m \in \mathbb{N}$, we have*

$$\max\bigl\{\|T_{d_1} T_{d_2} \dots T_{d_m}|_{E_1}\| \mid d_j \in \{0, 1\},\ 1 \leqslant j \leqslant m\bigr\} \leqslant Cq^m, \tag{4.6}$$

where $0 < q < 1$ and $C > 0$, then φ is continuous on U_{1-n} and the following inequality holds for any integer $j \geqslant n - 1$:

$$\Omega_j(\varphi) \leqslant Cq^j. \tag{4.7}$$

Proof. For any $x \in U$ and $\alpha \in \{0, 1, \dots, N-1\}$ we put

$$\varphi_0\bigl(x \oplus h_{[\alpha]}\bigr) := \bigl(1 - \lambda(x)\bigr)\, \varphi\bigl(h_{[\alpha]}\bigr) + \lambda(x)\, \varphi\bigl(h^-_{[\alpha+1]}\bigr). \tag{4.8}$$

The group U has two cosets with respect to the subgroup $A^{-1}(U)$:

$$U_{1,0} := A^{-1}(U), \qquad U_{1,1} := A^{-1}\big(h_{[1]}\big) \oplus A^{-1}(U).$$

We define a sequence $v_j(x)$ of vector-valued functions for $j \in \mathbb{Z}_+$ and $x \in U$ by

$$v_0(x) := \big(\varphi_0(x), \varphi_0(x \oplus h_{[1]}), \dots, \varphi_0(x \oplus h_{[N-1]})\big)^{\mathrm{T}},$$
$$v_{j+1}(x) := \begin{cases} T_0 v_j(Ax) & \text{if} \quad x \in U_{1,0}, \\ T_1 v_j(Ax \ominus h_{[1]}) & \text{if} \quad x \in U_{1,1}. \end{cases}$$

For each $x = (x_j)$ in U, we put $d_j(x) = x_j$ and

$$\tau(x) := \begin{cases} Ax & \text{if} \quad x \in U_{1,0}, \\ Ax \ominus h_{[1]} & \text{if} \quad x \in U_{1,1}. \end{cases}$$

Then

$$v_{j+1}(x) = T_{d_1(x)} v_j(\tau x)$$

and hence

$$v_j(x) = T_{d_1(x)} T_{d_2(x)} \dots T_{d_j(x)} v_0(\tau^j x). \tag{4.9}$$

We see from (4.3) and (4.8) that

$$\begin{aligned} e_1 \cdot v_0(x) &= \sum_{\alpha=0}^{N-1} \varphi_0\big(x \oplus h_{[\alpha]}\big) \\ &= \big(1 - \lambda(x)\big) \sum_{\alpha=0}^{N-1} \varphi\big(h_{[\alpha]}\big) + \lambda(x) \sum_{\alpha=0}^{N-1} \varphi\big(h^-_{[\alpha+1]}\big) = 1. \end{aligned}$$

Then (4.6), (4.9) imply that

$$e_1 \cdot v_j(x) = e_1 T_{d_1(x)} T_{d_2(x)} \dots T_{d_j(x)} v_0(\tau^j x) = e_1 \cdot v_0(\tau^j x) = 1.$$

Hence, for each $l \in \mathbb{Z}_+$ we have

$$e_1 \cdot \big(v_{j+l}(x) - v_l(x)\big) = 0.$$

Thus the following formulae hold for all $x \in U$ and $j, l \in \mathbb{Z}_+$:

$$e_1 \cdot v_j(x) = 1, \qquad v_{j+l}(x) - v_l(x) \in E_1. \tag{4.10}$$

Using (4.9), we have

$$v_{j+l}(x) - v_j(x) = T_{d_1(x)} T_{d_2(x)} \dots T_{d_j(x)} [v_l(\tau^j x) - v_0(\tau^j x)]. \tag{4.11}$$

For $l = 1$ we see from (4.6), (4.10) and (4.11) that

$$\|v_{j+1}(x) - v_j(x)\| \leqslant C q^j \sup_{y \in U} \|v_1(y) - v_0(y)\|.$$

Therefore,

$$\begin{aligned}\|v_j(x)\| &\leqslant \|v_0(x)\| + \sum_{l=1}^{j} \|v_l(x) - v_{l-1}(x)\| \\ &\leqslant \sup_{y\in U} \|v_0(y)\| + C(1-q)^{-1} \sup_{y\in U} \|v_1(y) - v_0(y)\|.\end{aligned}$$

Thus the sequence $\{v_j(\,\cdot\,)\}$ is uniformly bounded on U:

$$\sup\{\|v_j(x)\| \mid x \in U,\ j \in \mathbb{Z}_+\} < \infty. \tag{4.12}$$

As above, we see from (4.6), (4.10) and (4.11) that

$$\|v_{j+l}(x) - v_j(x)\| \leqslant Cq^j \sup_{y\in U} \|v_l(y) - v_0(y)\|$$

for every $x \in U$ and $l \in \mathbb{N}$. Combining this with (4.12), we get

$$\sup_{x\in U} \|v_{j+l}(x) - v_j(x)\| \leqslant Cq^j, \tag{4.13}$$

where C is a constant independent of l. Hence $\{v_j(\,\cdot\,)\}$ is a Cauchy sequence in the space $[C(U)]^N = C(U) \times \cdots \times C(U)$.

The limit $\widetilde{v}(\,\cdot\,)$ of the sequence $\{v_j(\,\cdot\,)\}$ is continuous on U. Hence we see from (4.13) that

$$\sup_{x\in U} \|\widetilde{v}(x) - v_j(x)\| \leqslant Cq^j. \tag{4.14}$$

Let

$$v(x) := \bigl(\varphi(x), \varphi\bigl(x \oplus h_{[1]}\bigr), \dots, \varphi\bigl(x \oplus h_{[N-1]}\bigr)\bigr)^{\mathrm{T}}.$$

Then

$$v(x) = T_{d_1(x)} v(\tau x)$$

for all $x \in U$. Letting $j \to \infty$ in the equality $v_{j+1}(x) = T_{d_1(x)} v_j(\tau x)$, we conclude that $\widetilde{v}(x) = v(x)$. Combining this with (4.14), we have

$$\sup_{x\in U_{1-n}} \|\varphi(x) - \varphi_j(x)\| \leqslant Cq^j \tag{4.15}$$

for all $\in \mathbb{Z}_+$.

We fix an integer $j \geqslant n$ and choose elements

$$\begin{aligned} x &= (\dots, 0, 0, x_{2-n}, x_{3-n}, \dots, x_0, x_1, \dots), \\ y &= (\dots, 0, 0, y_{2-n}, y_{3-n}, \dots, y_0, y_1, \dots) \end{aligned}$$

of U_{1-n} such that $y \ominus x \in U_j \setminus U_{j+1}$. Then $x_i = y_i$ for $2-n \leqslant i \leqslant j$ and $x_{j+1} \neq y_{j+1}$. Hence x and y belong to the class $A^{-j}\bigl(h_{[m]} \oplus U\bigr)$, where

$$m = \sum_{i=2-n}^{j} x_i 2^{j-i} = \sum_{i=2-n}^{j} y_i 2^{j-i}.$$

According to (4.15), we have

$$\begin{aligned}|\varphi(x)-\varphi(y)| &\leqslant |\varphi(x)-\varphi_j(x)| + \bigl|\varphi_j(x)-\varphi_j\bigl(A^{-j}h_{[m]}\bigr)\bigr| \\ &\quad + \bigl|\varphi_j\bigl(A^{-j}h_{[m]}\bigr)-\varphi_j(y)\bigr| + |\varphi_j(y)-\varphi(y)| \\ &\leqslant 2Cq^j + \bigl|\varphi_j(x)-\varphi_j\bigl(A^{-j}h_{[m]}\bigr)\bigr| + \bigl|\varphi_j(y)-\varphi_j\bigl(A^{-j}h_{[m]}\bigr)\bigr|. \end{aligned} \tag{4.16}$$

Suppose that $x\in U$, that is, $x_0=x_{-1}=\dots=x_{2-n}=0$. Then

$$m = x_1 2^{j-1} + x_2 2^{j-2} + \dots + x_j 2^0$$

and, similarly to (4.11),

$$v_j(x) - v_j\bigl(A^{-j}h_{[m]}\bigr) = T_{d_1(x)}T_{d_2(x)}\dots T_{d_j(x)}\bigl[v_0(\tau^j x) - v_0\bigl(\tau^j A^{-j}h_{[m]}\bigr)\bigr]. \tag{4.17}$$

Taking into account that the $v_0(\tau^j\,\cdot)$ are uniformly bounded on U and that

$$\begin{gathered}\bigl|\varphi_j(x)-\varphi_j\bigl(A^{-j}h_{[m]}\bigr)\bigr| \leqslant \bigl\|v_j(x)-v_j\bigl(A^{-j}h_{[m]}\bigr)\bigr\|, \\ \|T_{d_1(x)}T_{d_2(x)}\dots T_{d_j(x)}\| \leqslant Cq^j,\end{gathered}$$

we obtain from (4.17) that

$$\bigl|\varphi_j(x)-\varphi_j\bigl(A^{-j}h_{[m]}\bigr)\bigr| \leqslant Cq^j. \tag{4.18}$$

Now suppose that $x\in U_{n-1}\setminus U$. Put $x' = x\ominus h_{[k]}$ and $m'=m-k$, where

$$k = x_0 2^j + x_{-1}2^{j+1} + \dots + x_{2-n}2^{j+n-2}.$$

Then $x'\in U$ and

$$v_j(x') - v_j\bigl(A^{-j}h_{[m']}\bigr) = T_{d_1(x')}T_{d_2(x')}\dots T_{d_j(x')}\bigl[v_0(\tau^j x') - v_0\bigl(\tau^j A^{-j}h_{[m']}\bigr)\bigr].$$

Since

$$\bigl|\varphi_j(x)-\varphi_j\bigl(A^{-j}h_{[m]}\bigr)\bigr| \leqslant \bigl\|v_j(x')-v_j\bigl(A^{-j}h_{[m']}\bigr)\bigr\|,$$

we get (4.18) again. The last term in (4.16) is estimated similarly. Thus the inequality (4.7) holds. This proves the proposition.

Remark 4.2. The following formula is useful for computing values of φ:

$$v(x) = \begin{cases} T_0 v(Ax) & \text{if} \quad x\in U_{1,0}, \\ T_1 v\bigl(Ax\ominus h_{[1]}\bigr) & \text{if} \quad x\in U_{1,1}. \end{cases} \tag{4.19}$$

In particular, setting $x=\theta$ in (4.19), we see that the vector

$$v(\theta) = \bigl(\varphi\bigl(h_{[0]}\bigr), \varphi\bigl(h_{[1]}\bigr), \dots, \varphi\bigl(h_{[N-1]}\bigr)\bigr)^{\mathrm{T}}$$

is an eigenvector of the matrix T_0 with eigenvalue 1. We note that the number 1 belongs to the spectrum of T_0 by (4.2). When $p = n = 2$ we see from (4.3) and (4.19) that

$$\varphi(h_{[0]}) = \frac{1+a-b}{2(1-b)}, \qquad \varphi(h^{-}_{[1]}) = \frac{1+a+b}{2(1+b)},$$
$$\varphi(h_{[1]}) = \frac{1-a-b}{2(1-b)}, \qquad \varphi(h^{-}_{[2]}) = \frac{1-a+b}{2(1+b)},$$

where the parameters a, b are the same as in (1.10). If $p = 2$ and $n = 3$ while the function φ is defined by (1.11) with $a = 1$ and $|\gamma| < 1$, then

$$\begin{aligned}
\varphi(h_{[0]}) &= \frac{1+c-\gamma}{2(1-\gamma)} - \frac{c(1-\beta)}{4(1-\gamma)} + \frac{b}{4}, \\
\varphi(h_{[1]}) &= \frac{1-c-\gamma}{2(1-\gamma)} + \frac{c(1-\beta)}{4(1-\gamma)} - \frac{b}{4}, \\
\varphi(h_{[2]}) &= \frac{b}{2} - \frac{c(1-\beta)}{2(1-\gamma)}, \\
\varphi(h_{[3]}) &= -\frac{b}{2} + \frac{c(1-\beta)}{2(1-\gamma)},
\end{aligned}$$

where $\varphi(h_{[2]}) = \varphi(h_{[3]}) = 0$ if $b = 0$ ($\beta = 1$) or $b = c$ ($\beta = \gamma$).

Example 4.3. If the function φ is given by (1.10), then

$$\Omega_j(\varphi) \leqslant C|b|^j, \qquad j \in \mathbb{N}. \tag{4.20}$$

Indeed, for $n = 2$ we have

$$E_1 = \{v \in \mathbb{R}^2 \mid v_1 + v_2 = 0\} = \{te_1^0 \mid t \in \mathbb{R}\},$$
$$T_0 e_1^0 = be_1^0, \qquad T_1 e_1^0 = -be_1^0,$$

where $e_1^0 = (-1,1)^{\mathrm{T}}$. Therefore the inequality (4.20) follows from Proposition 4.1. Since

$$\varphi(h_{[0]}) - \varphi(A^{-j}h_{[1]}) = \frac{ab^j}{1-b},$$

we see that the estimate (4.20) is sharp. Since $|b| < 1$, the continuity of φ follows from (4.20).

Example 4.4. If the function φ is given by (1.11) with $a = 1$ and $0 < |\gamma| < 1$, then

$$\Omega_j(\varphi) \leqslant C|\gamma|^j, \qquad j \in \mathbb{N}. \tag{4.21}$$

Indeed, the vectors

$$e_1^0 = \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}, \qquad e_2^0 = \begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \end{pmatrix}, \qquad e_3^0 = \begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix}$$

form a basis of the space E_1 when $n=3$. Since $a=1$, we have

$$
\begin{gathered}
T_0 e_1^0 = T_1 e_1^0 = 0,\\
T_0 e_2^0 = -T_1 e_2^0 = \beta e_1^0 + \gamma e_2^0,\\
T_0 e_3^0 = T_1 e_3^0 = b e_1^0 + c e_2^0.
\end{gathered}
$$

Using these equalities, we get the following formulae for any vector $w = \nu_1 e_1^0 + \nu_2 e_2^0 + \nu_3 e_3^0$ in the space E_1:

$$
T_0^2 w = -T_1 T_0 w = (\nu_2\gamma + \nu_3 c) T_0 e_2^0, \qquad T_1^2 w = -T_0 T_1 w = (\nu_2 \gamma - \nu_3 c) T_0 e_2^0. \tag{4.22}
$$

In particular,

$$
T_0^2 e_2^0 = T_1^2 e_2^0 = -T_0 T_1 e_2^0 = -T_1 T_0 e_2^0 = \gamma T_0 e_2^0.
$$

Thus the following equation holds for all $d_1, d_2, \dots, d_m \in \{0,1\}$:

$$
T_{d_1} T_{d_2} \dots T_{d_m} e_2^0 = \pm \gamma^{m-1} T_0 e_2^0.
$$

Hence we see from (4.22) that

$$
\|T_{d_1} T_{d_2} \dots T_{d_m}|_{E_1}\| \leqslant C|\gamma|^m,
$$

and the estimate (4.21) follows from Proposition 4.1. If $a=1$, $b=0$, and $|\gamma|<1$, then (1.11) yields the following equations for all $j \in \mathbb{N}$:

$$
\varphi\bigl(h_{[0]}\bigr) - \varphi\bigl(A^{-j} h_{[1]}\bigr) = \frac{c\gamma^{j+3}}{4(1-\gamma)}\,.
$$

Thus the estimate (4.21) is also sharp.

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