SIAM J. MATH. ANAL.
Vol. 24, No. 2, pp. 499-519, March 1993

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ORTHONORMAL BASES OF COMPACTLY SUPPORTED WAVELETS II. VARIATIONS ON A THEME*

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Abstract. Several variations are given on the construction of orthonormal bases of wavelets with compact support. They have, respectively, more symmetry, more regularity, or more vanishing moments for the scaling function than the examples constructed in Daubechies [*Comm. Pure Appl. Math.*, 41 (1988), pp. 909-996].

Key words. wavelets, orthonormal bases, regularity, symmetry

AMS(MOS) subject classifications. 26A16, 26A18, 26A27, 39B12

1. Introduction. This paper concerns the construction of orthonormal bases of wavelets, i.e., orthonormal bases $\{\psi_{jk};\ j, k \in \mathbb{Z}\}$ for $L^2(\mathbb{R})$, where

$$\psi_{jk}(x) = 2^{-j/2}\psi(2^{-j}x - k) \tag{1.1}$$

for some (very particular!) $\psi \in L^2(\mathbb{R})$. The functions (1.1) are *wavelets* because they are all generated from one single function by dilations and translations. Note that wavelets need not be orthogonal or even linearly independent. In fact, the "first" wavelets were neither [1], [2]. See [3], [4] for discussions of wavelet expansions using nonindependent wavelets, with continuous [3] or discrete [4] dilation and translation labels. Even the special case of orthonormal wavelets need not always be of the form (1.1). Basic dilation factors different from 2 are possible: there exist orthonormal bases in which this factor is any rational $p/q > 1$ [5]; in more than one dimension we may even choose a dilation *matrix* instead of an isotropic dilation factor. In these more general cases, it may be necessary to introduce more than one ψ (but always a finite number). We shall restrict ourselves to one dimension here, and to the dilation factor 2, as in (1.1). Bases with factor 2 are by far the easiest to implement for numerical computations.

All interesting examples of orthonormal wavelet bases can be constructed via *multiresolution analysis.* This is a framework developed by Mallat [6] and Meyer [7], in which the wavelet coefficients $\langle f, \psi_{jk}\rangle$ for fixed j describe the difference between two approximations of f, one with resolution 2^{j-1}, and one with the coarser resolution 2^j. The following succinct review of multiresolution analysis suffices for the understanding of this paper; for more details, examples, and proofs we refer the reader to [6] and [7].

The successive approximation spaces V_j in a multiresolution analysis can be characterized by means of a *scaling function* ϕ. More precisely, we assume that the integer translates of ϕ are an orthonormal basis for the space V_0, which we define to be the approximation space with resolution 1. The approximation spaces V_j with resolution 2^j are then defined as the closed linear spans of the ϕ_{jk} $(k \in \mathbb{Z})$, where

$$\phi_{jk} = 2^{-j/2}\phi(2^{-j}x - k). \tag{1.2}$$

To ensure that projections on the V_j describe successive approximations, we require $V_0 \subset V_{-1}$, which implies

$$\cdots V_2 \subset V_1 \subset V_0 \subset V_{-1} \subset V_{-2} \subset \cdots. \tag{1.3}$$

* Received by the editors May 29, 1990; accepted for publication (in revised form) May 23, 1992.

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This imposes a restriction on ϕ: since $\phi \in V_0 \subset V_{-1} = \overline{\mathrm{Span}\,\{\phi_{-1k};\, k \in \mathbb{Z}\}}$, there must exist c_n such that

$$\phi(x) = \sum_n c_n \phi(2x - n). \tag{1.4}$$

In order to have a complete description of $L^2(\mathbb{R})$, we also impose

$$\bigcap_{j \in \mathbb{Z}} V_j = \{0\}, \qquad \overline{\bigcup_{j \in \mathbb{Z}} V_j} = L^2(\mathbb{R}). \tag{1.5}$$

For every multiresolution analysis as described above, there exists a corresponding orthonormal basis of wavelets defined by

$$\psi(x) = \sum_{n \in \mathbb{Z}} (-1)^n c_{-n+1} \phi(2x - n), \tag{1.6}$$

where c_n are the coefficients in (1.4). We can prove [6], [7] (see also below) that the ψ_{0n} are then an orthonormal basis for the orthogonal complement W_0 of V_0 in V_{-1}. This phenomenon repeats itself at every resolution level j. It follows that, for every j, the $\langle f, \psi_{jk} \rangle$ determine the difference in information between the approximations $P_j f$, $P_{j-1} f$ at resolutions 2^j, 2^{j-1}, respectively:

$$P_{j-1} f = P_j f + \sum_k \langle f, \psi_{jk} \rangle \psi_{jk}.$$

Consequently, by (1.3) and (1.5), the $(\psi_{jk};\, j, k \in \mathbb{Z})$ constitute an orthonormal basis for $L^2(\mathbb{R})$.

One advantage of the "nested" structure of a multiresolution analysis is that it leads to an efficient tree-structured algorithm for the decomposition and reconstruction of functions (given either in continuous or sampled form). Instead of computing all the inner products $\langle f, \psi_{jk} \rangle$ directly, we proceed in a hierarchic way:

—compute $\langle f, \phi_{jk} \rangle$ for the finest resolution level j wanted (if the data are given in a discrete fashion, then these discrete data can just be taken to be $\langle f, \phi_{jk} \rangle$);

—then compute $\langle f, \psi_{j-1k} \rangle$ *and* $\langle f, \phi_{j-1k} \rangle$ at the next finest resolution level by applying (1.4) and (1.7),

$$\langle f, \psi_{j-1k} \rangle = \frac{1}{\sqrt{2}} \sum_n (-1)^n c_{-n+2k+1} \langle f, \phi_{jn} \rangle,$$

$$\langle f, \phi_{j-1k} \rangle = \frac{1}{\sqrt{2}} \sum_n c_{n-2k} \langle f, \phi_{jn} \rangle;$$

—iterate until the coarsest desired resolution level is attained.

The total complexity of this calculation is lower, despite the computation of the seemingly unnecessary $\langle f, \phi_{jk} \rangle$, than if the $\langle f, \psi_{jk} \rangle$ were computed directly.

This brief review shows how to construct an orthonormal basis of wavelets from any "decent" function ϕ satisfying an equation of type (1.4). An example of such a construction is given by the Battle–Lemarié wavelets, consisting of spline functions [8], [9], [10]. In general, constructions starting from a choice of ϕ lead to ϕ, ψ, which are not compactly supported (see, e.g., [15], [25] for a more detailed discussion). The construction can, however, also be viewed differently. The Fourier transform of (1.4) is

$$\hat{\phi}(\xi) = \left[\frac{1}{2} \sum_n c_n\, e^{in\xi/2}\right] \hat{\phi}\left(\frac{\xi}{2}\right),$$

which implies

$$\hat{\phi}(\xi) = \left[\prod_{j=1}^{\infty} m_0(2^{-j}\xi)\right] \hat{\phi}(0), \tag{1.7}$$

with $m_0(\xi)=\frac{1}{2}\sum_n c_n e^{in\xi}$, so that, up to normalization, ϕ is completely determined by the c_n. Fixing the c_n, therefore, also defines a multiresolution analysis. The c_n have to satisfy certain conditions. Combining $\langle\phi_{0k}, \phi_{0l}\rangle=\delta_{kl}$ with (1.4) immediately leads to

$$\sum_n c_n c_{n-2k}=2\delta_{k0}, \tag{1.8}$$

where we have assumed, as we shall do in the sequel, that the c_n are real. In terms of $m_0(\xi)$, (1.8) can be rewritten as

$$|m_0(\xi)|^2+|m_0(\xi+\pi)|^2=1. \tag{1.9}$$

To ensure that ϕ is well defined, the infinite product in (1.7) must converge, which implies $m_0(0)=1$ or

$$\sum_n c_n=2. \tag{1.10}$$

It follows that ϕ is uniquely determined by (1.4), up to normalization, which we fix by requiring $\int dx\,\phi(x)=1$. One can show (see, e.g., [12]) that (1.9) implies that ϕ is in $L^2(\mathbb{R})$, but unfortunately (1.8) is not sufficient to guarantee orthonormality of the ϕ_{0n}. A counterexample is $c_0=c_3=1$, all other $c_n=0$, which leads to $\phi(x)=\frac{1}{3}$ for $0\leqq x<3$, $\phi(x)=0$ otherwise. Such counterexamples are rare, however. If $N_2-N_1=3$, then the example above, $c_0=c_3=1$, is the only one. For a detailed discussion, see [12], [13], [22].

If we exclude these thin sets of "bad" choices for the c_n (which can be done by various means [6], [7], [12], [13], [15]), then we can build orthonormal bases of wavelets starting from the c_n. Once orthonormality of the ϕ_{0k} is established, all the rest follows easily. Formula (1.6) for ψ leads immediately to orthogonality of the ψ_{0l} and ϕ_{0k},

$$\begin{aligned}\langle\psi_{0l}, \phi_{0k}\rangle &=\frac{1}{2}\sum_{n,m}(-1)^n c_{-n+2l+1}c_{m-2k}\langle\phi_{-1n}, \phi_{-1m}\rangle\\ &=\frac{1}{2}\sum_n(-1)^n c_{-n+2l+1}c_{n-2k}=0.\end{aligned}$$

The last equality follows from the substitution $n=-m+2(k+l)+1$ for the summation index n. Similar manipulations prove

$$\langle\psi_{0l}, \psi_{0k}\rangle=\delta_{kl}$$

and

$$\sum_k[\langle f, \phi_{0k}\rangle\phi_{0k}+\langle f, \psi_{0k}\rangle\psi_{0k}]=\sum_n\langle f, \phi_{-1n}\rangle\phi_{-1n}. \tag{1.11}$$

It follows that both $\{\phi_{-1n};\, n\in\mathbb{Z}\}$ and $\{\phi_{0k}, \psi_{0k};\, k\in\mathbb{Z}\}$ are orthonormal bases for V_{-1}. (In other words, (1.8) ensures that (1.4) and (1.6) describe an orthonormal basis transformation.) It follows that $W_0=\overline{\mathrm{Span}\,(\psi_{0k})}$ is the orthogonal complement of V_0 in V_{-1}, and hence that the $\{\psi_{jk};\, j, k\in\mathbb{Z}\}$ constitute an orthonormal basis for $L^2(\mathbb{R})$.

Constructing ψ from the c_n rather than from ϕ has the advantage of allowing better control over the supports of ϕ and ψ. If $c_n=0$ for $n<N_1$, $n>N_2$, then support $(\phi)\subset[N_1, N_2]$ (see [11a], [14]). In [15] this method was used to construct orthonormal bases of wavelets with compact support, and arbitrarily high preassigned regularity (the size of the support increases linearly with the number of continuous derivatives). These orthonormal basis functions and the associated multiresolution analysis have been tried out for several applications, ranging from image processing to numerical analysis [16]. For some of these applications, variations on the scheme of [15] were requested, emphasizing other properties. The goal of this and the next paper is to present a number of these variations.

The construction in [15] relied on the identity

$$\sum_{j=0}^{N-1}\binom{N-1+j}{j}[(\cos\alpha)^{2N}(\sin\alpha)^{2j}+(\sin\alpha)^{2N}(\cos\alpha)^{2j}]=1. \tag{1.12}$$

Since

$$\left|\frac{1+e^{i\xi}}{2}\right|^2=\left(\cos\frac{\xi}{2}\right)^2,$$

(1.12) suggests the choice

$$m_0(\xi)=\left(\frac{1+e^{i\xi}}{2}\right)^N Q(e^{i\xi}), \tag{1.13}$$

where Q is a trigonometric polynomial with real coefficients such that

$$|Q(e^{i\xi})|^2=\sum_{j=0}^{N-1}\binom{N-1+j}{j}\left(\frac{1-\cos\xi}{2}\right)^j. \tag{1.14}$$

By (1.12), any such m_0 will satisfy (1.9). To determine Q, we have to extract the "square root" of the right-hand side of (1.5). This can be done by using a lemma of Riesz [17]. Denote the right-hand side of (1.14) by $P_N(e^{i\xi})$, and extend P_N to all of $\mathbb{C}$. We have $\overline{P_N(z)}=P_N(\bar{z})$ and $P_N(z^{-1})=P_N(z)$. Consequently, the zeros of P_N come either in real duplets, r_k and r_k^{-1}, or in complex quadruplets, z_l, $\bar{z}_l$, z_l^{-1}, and $\bar{z}_l^{-1}$,

$$\begin{aligned}P_N(z)&=4^{-N+1}\binom{2N-2}{N-1}z^{-N+1}\prod_k(z-r_k)(z-r_k^{-1})\\&\quad\cdot\prod_l(z-z_l)(z-\bar{z}_l)(z-z_l^{-1})(z-\bar{z}_l^{-1})\\&=4^{-N+1}\binom{2N-2}{N-1}\prod_k\frac{(z-r_k)(r_k-z^{-1})}{r_k^2}\\&\quad\cdot\prod_l\frac{(z-z_l)(z-\bar{z}_l)(z_l-z^{-1})(\bar{z}_l-z^{-1})}{|z_l|^2z^2}.\end{aligned}$$

It follows that $P_N(e^{i\xi})=|Q_N(e^{i\xi})|^2$, with

$$Q_N(z)=2^{-N+1}\binom{2N-2}{N-1}^{1/2}\prod_k\frac{(z-r_k)}{\sqrt{|r_k|}}\prod_l\frac{(z^2+|z_l|^2-2z\,\mathrm{Re}\,z_l)}{|z_l|}. \tag{1.15}$$

This gives a recipe for the construction of m_0:

(1) For given N, determine the zeros of P_N;

(2) Choose one zero out of every pair of real zeros r_l, r_l^{-1} of P_N, and one conjugated pair out of every quadruplet z_k, z_k^{-1}, $\bar{z}_k$, $\bar{z}_k^{-1}$.

(3) Compute the product Q_N, and substitute into (1.12).

The result is a polynomial in $e^{i\xi}$ of degree $2N-1$, corresponding to an orthonormal basis of wavelets in which the basic wavelet ψ_N has support width $2N-1$. Since (1.6) can be rewritten as

$$\hat{\psi}(\xi)=e^{i((\xi/2)+\pi)}\overline{m_0\left(\frac{\xi}{2}+\pi\right)}\hat{\phi}\left(\frac{\xi}{2}\right),$$

and since (1.13) has a zero of order N at π, it follows that ψ_N has N vanishing moments,

$$\int dx\, x^l \psi_N(x) = 0, \qquad l = 0, 1, \ldots, N-1,$$

which is useful for quantum field theory [18] and numerical analysis applications [19]. The regularity of the ψ_N constructed in [15] increases linearly with their support width, $\psi_N \in C^{\alpha(N)}$, with $\lim_{N\to\infty} N^{-1}\alpha(N) = .2075$ [23], [24], [25]. Plots of ϕ and ψ for various values of N can be found in [15], [25].

Depending on the application they had in mind, several scientists (mathematicians or engineers) have requested possible variations on the construction in [15]. The following are the most recurrent wish items.

(1) More symmetry: the functions ϕ, ψ in [15] are very asymmetric. Complete symmetry is incompatible with the orthonormal basis condition (see [15, p. 971], or § 2 below), but is less asymmetry possible?

(2) Better frequency resolution: orthonormal bases with basic multiplication factor 2 correspond to frequency intervals of 1 octave. Is better possible (e.g., $\frac{1}{2}$ octave), without giving up compact support?

(3) More regularity: is better regularity than in [15] achievable for the same support width?

(4) More vanishing moments: for a fixed support width $2N-1$, the ψ_N of [15] have the maximum number of vanishing moments. The functions ϕ_N do not satisfy any moment condition, except $\int dx\, \phi_N(x) = 1$. For numerical analysis applications, it may be useful to give up some zero moments of ψ in order to obtain zero moments for ϕ, i.e., to have

$$\begin{aligned} &\int dx\, \phi(x) = 1, \\ &\int dx\, x^l \phi(x) = 0, \qquad l = 1, \ldots, L, \\ &\int dx\, x^l \psi(x) = 0, \qquad l = 0, \ldots, L. \end{aligned} \tag{1.16}$$

How can such ϕ, ψ be constructed? They would have the advantage that inner products with smooth functions are particularly appealing:

$$\begin{aligned} \int dx\, \phi_{-jk}(x) f(x) &= 2^{j/2} \int dx\, \phi(2^j(x - 2^{-j}k)) f(x) \\ &= 2^{-j/2} f(2^{-j}k) + \text{correction terms in } f^{(L+1)} \end{aligned}$$

(use the Taylor expansion of f around $2^{-j}k$; the second through $(L+1)$th terms vanish because of (1.16)). Moreover, if the $(L+1)$th derivative of f is uniformly bounded, then the correction terms in this formula are of order $2^{-(L+1/2)j}$.

The purpose of this and the next paper is to show how such variations can be constructed. In § 2 we handle symmetry, in § 3 regularity, and in § 4 vanishing moments for ϕ. The next paper shows how to obtain better frequency localization.

2. More symmetry. If we restrict our attention to orthonormal bases of *compactly supported wavelets only*, then it is impossible to obtain ψ which is either symmetric or antisymmetric, except for the trivial Haar case ($c_0 = 1$, $c_1 = -1$, all other $c_n = 0$). This is the content of the following theorem.

THEOREM 2.1. *Let ψ, ϕ be defined as in* § 1, *from a finite set of coefficients* c_n *satisfying* (1.9) *and* (1.11), *with orthonormal* ϕ_{0n}. *If* ψ *is either symmetric or antisymmetric around some axis, then* ψ *is the Haar function.*

A proof can be found in [25, Chap. 8].

It is thus a fact of life that symmetric or antisymmetric ψ, however desirable they might be in applications, are just not possible within a framework of orthonormal bases of continuous, compactly supported wavelets. On the other hand, ϕ and ψ do not really need to be quite as asymmetric as in [15], where the extreme asymmetry of ψ, ϕ proceeds from choices made in their construction. In practice, the $2(N-1)$ zeros of P_N consist of one real pair r, r^{-1} and n_0^{-1} quadruplets of complex zeros z_l, $\bar{z}_l$, z_l^{-1}, $\bar{z}_l^{-1}$ if $N=2n_0$ is even, and of n_0 quadruplets if $N=2n_0+1$ is odd. To construct Q_N, we need to select one of the two real zeros, and one pair z_l, $\bar{z}_l$ out of every quadruplet. The choice made in [15] is the so-called *extremal phase* choice: we chose systematically all zeros with modulus smaller than one. Other choices may lead to less asymmetric ϕ. The following argument shows why.

A sequence of real numbers $(\alpha_n)_{n\in\mathbb{Z}}$ is said to define a *linear phase filter* if the phase of the function $\alpha(\xi)=\sum_n \alpha_n e^{in\xi}$ is a linear function of ξ, i.e., if, for some $l\in\mathbb{Z}/2$,

$$\alpha(\xi)=e^{il\xi}|\alpha(\xi)|.$$

This means that the α_n are symmetric around l, $\alpha_n=\alpha_{2l-n}$. If the sequence does not define a linear phase filter, then the deviation from linearity of the phase of $\alpha(\xi)$ reflects the asymmetry of the α_n. The Fourier transform of ϕ is given by the infinite product (1.7). If c_n were symmetric around l, then we would have $m_0(\xi)=e^{il\xi}|m_0(\xi)|$, hence

$$\hat{\phi}(\xi)=\exp\left[il\sum_{j=1}^{\infty}2^{-j}\xi\right]\prod_{j=1}^{\infty}|m_0(2^{-j}\xi)||\hat{\phi}(0)|=e^{il\xi}|\hat{\phi}(\xi)|,$$

so that ϕ would be symmetric around l as well. As explained above, this is impossible for c_n satisfying (1.8). The closer the phase of m_0 is to linear phase, the closer the phase of $\hat{\phi}$ will be to linear phase, and the less asymmetric ϕ will be. In our case, m_0 is a product of factors of type

$$\begin{aligned}(z-z_l)(z-\bar{z}_l)&=e^{i\xi}(e^{i\xi}-R_l e^{i\alpha_l})(1-e^{-i\xi}R_l e^{-i\alpha_l})\\&=e^{i\xi}[e^{i\xi}-2R_l\cos\alpha_l+R_l^2 e^{-i\xi}],\end{aligned}\tag{2.1}$$

with possibly an extra factor

$$(z-r)=e^{i\xi/2}[e^{i\xi/2}-r\,e^{i\xi/2}].\tag{2.2}$$

The total phase of m_0 is a sum of the phase contributions of each factor. Apart from linear phase terms, the phase contributions of (2.1) and (2.2) are, respectively,

$$\Phi_l(\xi)=\operatorname{arctg}\left(\frac{(1-R_l^2)\sin\xi}{(1+R_l^2)\cos\xi-2R_l\cos\alpha_l}\right),\tag{2.3}$$

$$=\operatorname{arctg}\left(\frac{1+r}{1-r}\,tg\frac{\xi}{2}\right).\tag{2.4}$$

The valuation of arctg should be chosen so that Φ_l is continuous in $[0,2\pi]$, and $\Phi_l(0)=0$. Since the denominator in (2.3) has two zeros, namely,

$$\xi=\xi_l=\operatorname{Arc}\cos\left(\frac{2R_l}{1+R_l^2}\cos\alpha_l\right)$$

and $\xi = 2\pi - \xi_l$, $\Phi_l(2\pi) = \Phi_l(0) + \varepsilon_l 2\pi$, with $\varepsilon_l = \pm 1$. Something similar happens in the $(z-r)$ case. In order to extract only the nonlinear part of Φ_l, we define, therefore,

$$\Psi_l(\xi) = \operatorname{arctg}\left(\frac{(1-R_l^2)\sin\xi}{(1+R_l^2)\cos\xi - 2R_l\cos\alpha_l}\right) - \frac{\xi}{2\pi}\Phi_l(2\pi)$$

or

$$= \operatorname{arctg}\left(\frac{1+r}{1-r}\, tg\frac{\xi}{2}\right) - \frac{\xi}{2\pi}\Phi_l(2\pi).$$

In order to obtain m_0 as close to linear phase as possible, we have to choose the zeros to retain from every quadruplet or duplet in such a way that $\Psi_{tot}(\xi) = \sum_l \Psi_l(\xi)$ is as close to zero as possible. In practice, we have $2^{\lfloor N/2 \rfloor}$ choices (and not 2^{N-1}, as was mistakenly stated in [15]). This number can be reduced by another factor of 2: for every choice, the complementary choice (choosing all the other zeros) leads to the complex conjugate m_0 (up to a phase shift), and, therefore, to the mirror image of ϕ. For $N=2$ or 3, there is, therefore, effectively only one pair ϕ_N, ψ_N. For $N \geqq 4$, we can compare the $2^{\lfloor N/2 \rfloor - 1}$ different choices for Ψ_{tot} in order to find the closest to linear phase. It turns our that the net effect of a change of choice from z_l, $\bar{z}_l$ to z_l^{-1}, $\bar{z}_l^{-1}$ is most significant if R_l is close to 1, and if α_l is close to either zero or π. In Fig. 1 we show the graphs for $\Psi_{tot}(\xi)$ for $N=4$ and 10, both for the original construction in [15] and for the case with flattest Ψ_{tot}. The "least asymmetric" ϕ and ψ, associated with the flattest possible Ψ_{tot}, are plotted in Fig. 2 for $N=4$ and 10. A table for the corresponding c_n can be found in [25, p. 198], as well as figures for $N=6, 8$.

Remarks.

(1) In this discussion we have restricted ourselves to the case where m_0 and $|Q|^2$ are given by (1.13) and (1.14), respectively. This means that the ϕ in Fig. 2 are the least asymmetric possible, given that N moments of ψ are zero, and that ϕ has support width $2N-1$. (This is the minimum width for N vanishing moments.) If ϕ may have larger support width, then it can be made even more symmetric. These wider solutions correspond to a variation on (1.14), i.e., to

$$|Q(e^{i\xi})|^2 = \sum_{j=0}^{N-1} \binom{N-1+j}{j} \left(\frac{1-\cos\xi}{2}\right)^j + \left(\frac{1-\cos\xi}{2}\right)^N R(\cos\xi), \tag{2.5}$$

where R is any odd polynomial such that the right-hand side of (2.5) is positive for

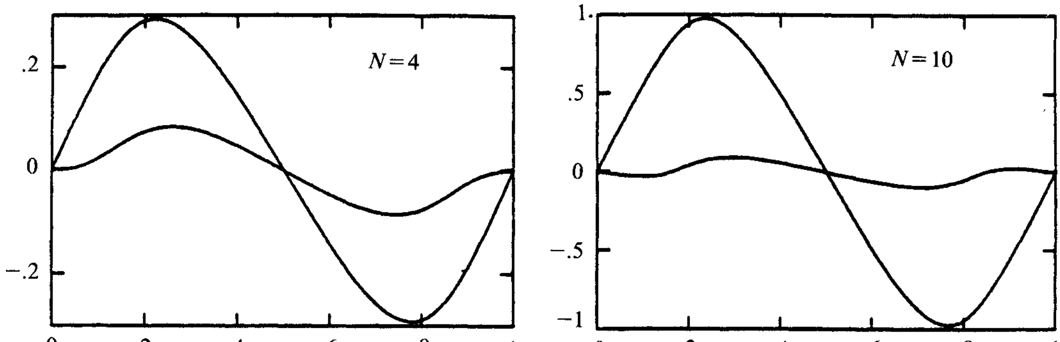

FIG. 1. *Plots of $\Psi_{tot}(\xi)$ for the cases $N=4$ and 10. In both cases we plot Ψ_{tot} for the construction in* [15], *and the much flatter Ψ_{tot} corresponding to the closest to linear phase choice. The horizontal axis gives $\xi/2\pi$, the vertical axis Ψ_{tot}/π.*

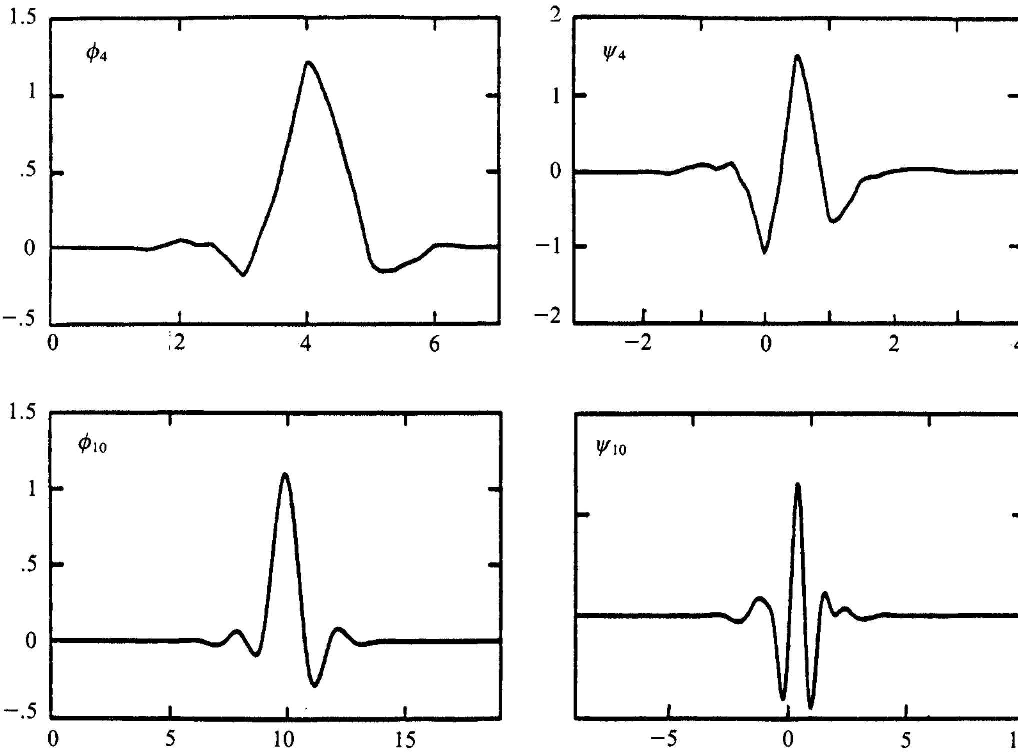

FIG. 2. *Plots of* ϕ_N, ψ_N *closest to linear phase, for the cases* $N=4$ *and* 10. *In every case, support* $(\phi_N)=[0, 2N-1]$, *support* $(\psi_N)=[-N+1, N]$.

all ξ. The functions ϕ constructed in § 4, for instance, are more symmetric than those in Fig. 2, but they have large support width.

(2) We can achieve even more symmetry by going a little beyond the multiresolution scheme explained in § 1, and by "mirroring" the filters at every odd step. For more details, see [25, p. 256].

(3) In [21] the construction of orthonormal bases of wavelets is generalized to "biorthogonal bases," i.e., to two dual unconditional bases $\{\psi_{jk};\ j, k \in \mathbb{Z}\}$ and $\{\tilde{\psi}_{jk};\ j, k \in \mathbb{Z}\}$. The construction in [21] corresponds to a decomposition + reconstruction scheme in which the reconstruction filters differ from the decomposition filters. In this more general framework, complete symmetry *can* be achieved. Orthonormality is then lost, however, which is less desirable for some applications.

3. More regularity. The regularity of the wavelets ψ_n constructed in [15] increases linearly with their support width, $\psi_N \in C^{\alpha(N)}$, $\lim N^{-1}\alpha(N) = .2075$. The technique used in [15] to control the regularity of ϕ_N, ψ_N involved constructing $m_0(\xi)$ so that it contained the factor $\frac{1}{2}(1+e^{i\xi})$ with as high multiplicity as possible,

$$m_0(\xi) = \left(\frac{1+e^{i\xi}}{2}\right)^N Q_N(\xi), \tag{3.1}$$

where Q_N is a polynomial in $e^{i\xi}$ of order $N-1$ (see § 1). Since $\prod_{j=0}^{\infty}(1+\exp(i2^{-j}\xi))/2 = e^{i\xi}(\sin \xi/\xi)$, we find (use (1.7))

$$\hat{\phi}_N(\xi) = e^{iN\xi/2}\left[\frac{\sin \xi/2}{\xi/2}\right]^N \prod_{j=1}^{\infty} Q_N(2^{-j}\xi).$$

Together with control on the infinite product of Q_N (see [15]), this leads to decay for $\hat{\phi}_N$ as $|\xi| \to \infty$, hence to regularity for ϕ_N, ψ_N.

In this argument, imposing high order divisibility of m_0 by $\frac{1}{2}(1+e^{i\xi})$ is used as a technical tool to obtain regularity. On the other hand, regularity for ϕ *implies* that m_0 is of type (3.1). More precisely, if ϕ is compactly supported and $\phi \in C^L$, then m_0 must be divisible by $[\frac{1}{2}(1+e^{i\xi})]^L$; see [22], [21]. Since $\phi_N \in C^{\mu N}$ for large N, with $\mu \simeq .2$, this means that at least $\frac{1}{5}$ of the factors $(1+e^{i\xi})$ in $m_{0,N}$ are necessary. Can the others be dispensed with, allowing even shorter support for the same regularity, or higher regularity for the same support width? The answer is yes.

In [11b], an alternative way was used to determine the regularity of functions ϕ satisfying an equation of type (1.4). Unlike the methods in [15], the method of [11b] does not use the Fourier transform. Instead, two N-dimensional matrices T_0, T_1 are defined, $(T_0)_{i,j} = c_{2i-j-1}$, $(T_1)_{i,j} = c_{2i-j}$, $1 \leqq i, j \leqq N$, where we assume $c_n = 0$ for $n < 0$ or $n > N$. Divisibility of m_0 by $(1+e^{i\xi})$ with multiplicity L is equivalent to

$$\sum_{n=0}^{N} c_n (-1)^n n^l = 0, \qquad l = 0, \ldots, L-1. \tag{3.2}$$

In terms of the matrices T_0, T_1, this implies that there exists a flag of subspaces $U_1 \subset \cdots \subset U_L$ of $\mathbb{R}^N$, with $\dim U_j = j$, such that

- U_j is left-invariant under both T_0, T_1.
- The left restrictions of T_0, T_1 to U_j have the j eigenvalues $1, \frac{1}{2}, \ldots, 2^{-j+1}$.

Let V_L be the subspace for $\mathbb{R}^N$ orthogonal to U_L; V_L is right invariant for T_0, T_1. If, for some $\lambda < 1$, $C > 0$, and for all $m \in \mathbb{N}$,

$$\| T_{d_1} \cdots T_{d_m} |_{V_L} \| \leqq C \lambda^m 2^{-m(L-1)} \qquad (d_j = 1 \text{ or } 0), \tag{3.3}$$

then (3.2) implies that $\phi \in C^L$, and that its Lth derivative $\phi^{(L)}$ is Hölder continuous with exponent $|\log_2 \lambda|$; if λ is best possible in (3.3), then $|\log_2 \lambda|$ is the best possible Hölder exponent for $\phi^{(L)}$. In principle (3.3) involves infinitely many inequalities; in practice we substitute finitely many conditions sufficient to ensure that (3.3) holds for all m [11b, Prop. 3.11]. The value of ϕ and its derivatives at any point x in support (ϕ) is governed by the behavior of the infinite product $T_{d_1(x)} T_{d_2(x)} \cdots T_{d_m(x)} \cdots$, where $d_j(x)$ are the digits in the binary expansion of x, $x = \lfloor x \rfloor + \sum_{j=1}^{\infty} d_j(x) 2^{-j}$. Special, "local" inequalities of type (3.3), valid only for certain sequences $(d_n)_{n \in \mathbb{N}}$, can, therefore, be translated into local regularity estimates, leading, in many examples, to a hierarchy of fractal sets corresponding to different local Hölder exponents. For more details, see [11b].

This approach can be used to study the regularity of compactly supported basis wavelets, which all correspond to an equation of type (1.4) with finitely many coefficients. For the examples of [15], this analysis was carried out in [11b] for $N = 2, 3, 4$ (for higher N, checking (3.3) becomes very complicated). In these three cases, the best possible Hölder exponent for the highest order well-defined derivative of ϕ_N was determined; these results were significantly better than what had been obtained in [15] via Fourier analysis. Table 1 compares the regularity results of [15] and [11b].

The optimal estimates obtained in [11] illustrate again that some of the factors $(1+e^{i\xi})$ of m_0, or, equivalently, some of the sum rules (3.2), which we impose in order to obtain regularity, are "wasted" in the final construction. N sum rules can deliver up to $N-1$ continuous derivatives if everything else cooperates; because of the other constraints on the c_n (i.e., (1.8)), wavelets do not achieve this optimal number. We can, therefore, drop some of the sum rules, and use the additional degrees of freedom

TABLE 1

The regularity of the wavelets of [15], *as obtained via Fourier methods (middle column) or via the matrix method of* [11b] *(right column). The integer part of the entry is the number of times* ϕ *is continuously differentiable; the decimal part is the Hölder exponent of the highest order well-defined derivative.*

N	Best estimate in [15]	Optimal result, obtained in [11b]
2	$.5-\varepsilon$	.5500...
3	.915	1.0878...
4	1.275	1.5179...

to obtain better λ in (3.3), i.e., better regularity than in Table 1. We present here the cases of wavelets with support width 3 and 5.

For $|\text{support } \phi| = 3$, there is a one-parameter family of choices c_n satisfying (1.8) and (1.10) (see [15, p. 946]), namely,

$$c_0 = \frac{\nu(\nu-1)}{(\nu^2+1)}, \quad c_1 = \frac{(1-\nu)}{(\nu^2+1)}, \quad c_2 = \frac{(\nu+1)}{(\nu^2+1)}, \quad c_3 = \frac{\nu(\nu+1)}{(\nu^2+1)}.$$

These c_n satisfy $\sum c_n(-1)^n = 0$; imposing a second sum rule leads to $\nu = \pm 3^{-1/2}$, which corresponds to the "standard" case $N=2$. The matrices T_0 and T_1 are 3×3-matrices; since we have one sum rule, we can restrict our attention to the reduced matrices $T_0|_{V_1}$, $T_1|_{V_1}$,

$$T_0|_{V_1} = \frac{1}{\nu^2+1}\begin{pmatrix} \nu(\nu-1) & 0 \\ 1-\nu^2 & \nu(\nu+1) \end{pmatrix}, \qquad T_1|_{V_1} = \frac{1}{\nu^2+1}\begin{pmatrix} \nu(\nu+1) & \nu(\nu-1) \\ 0 & 1-\nu^2 \end{pmatrix}.$$

We restrict our attention to $\nu \geqq 0$. (A change of sign $\nu \to -\nu$ corresponds to $c_n \to c_{3-n}$, i.e., to mirroring ϕ with respect to $\frac{3}{2}$.) Since (3.3) has to hold, in particular when all the d_j are identical, $d_j \equiv 0$ or $d_j \equiv 1$, the constant λ is bounded below by the spectral radii $\rho(T_j|_{V_1})$ of $T_j|_{V_1}$, $j=0$ or 1. It follows that (3.3) can only be satisfied if $\nu < 1$. For $\nu \geqq 1/\sqrt{3}$, we can find M so that both $MT_j|_{V_1}M^{-1}$, $j=0,1$, are symmetric; consequently,

$$\begin{aligned} \lambda &\leqq \max\left(\|MT_j|_{V_1}M^{-1}\|;\ j=0,1\right) = \max\left(\rho(MT_j|_{V_1}M^{-1});\ j=0,1\right) \\ &= \max\left(\rho(T_j|_{V_1});\ j=0,1\right) = \frac{\nu(\nu+1)}{1+\nu^2}. \end{aligned}$$

This is, moreover, the best possible λ. If $\nu < 1/\sqrt{3}$, then $T_0|_{V_1}$ and $T_1|_{V_1}$ are not simultaneously "symmetrizable," and we have to do some more work. In every case

$$\lambda \geqq \max\left(\frac{\nu(\nu+1)}{1+\nu^2}, \frac{1-\nu^2}{1+\nu^2}\right). \tag{3.4}$$

For $\nu = .25$, e.g., the tricks of Proposition 3.11 in [11b] suffice to show that equality holds in (3.4), and

$$\lambda = \frac{1-1/16}{1+1/16} = .88235\ldots.$$

The lowest value for the right-hand side of (3.4), and, therefore, the best candidate for the "most regular possible" ϕ, occurs for $\nu = .5$. In this case, $T_1|_{V_1}$ has only one (degenerate) eigenvalue, .6, and the matrix is not diagonalizable. Since (3.3) has to hold for $d_j \equiv 1$, it follows that we can at best hope to establish $\lambda = .6(1+\varepsilon)$.

In fact, we cannot achieve even this much. It turns out that $[\rho(T_0, T^{12})]^{1/13} \simeq .659676 \cdots > .6$, meaning that we can certainly not hope for a smaller λ than $.659 \cdots$. Using all the tricks in Proposition 3.11 in [11b], and checking a collection of building blocks with up to 17 factors, we find $\lambda \leqq .666$. More work leads to smaller upper bounds for λ; presumably the best value is the .659 obtained above.

Figure 3 shows the function ϕ for a few choices of ν ($\nu = .75$, .5 and .25). In each case ϕ is continuous, and we can compute its Hölder exponent from our estimate for λ. Even with our less than optimal estimate $\lambda \leqq .666$, the case $\nu = .5$ leads to a better Hölder exponent than the "standard" example $\nu = 1/\sqrt{3}$. This might be surprising: the graph of ϕ for $\nu = .5$ seems more jagged than for $\nu = 1/\sqrt{3}$. However, the peaks in the $\nu = .5$ example are "less sharp": the steepest slope of the peak around $x = 1$, e.g., is

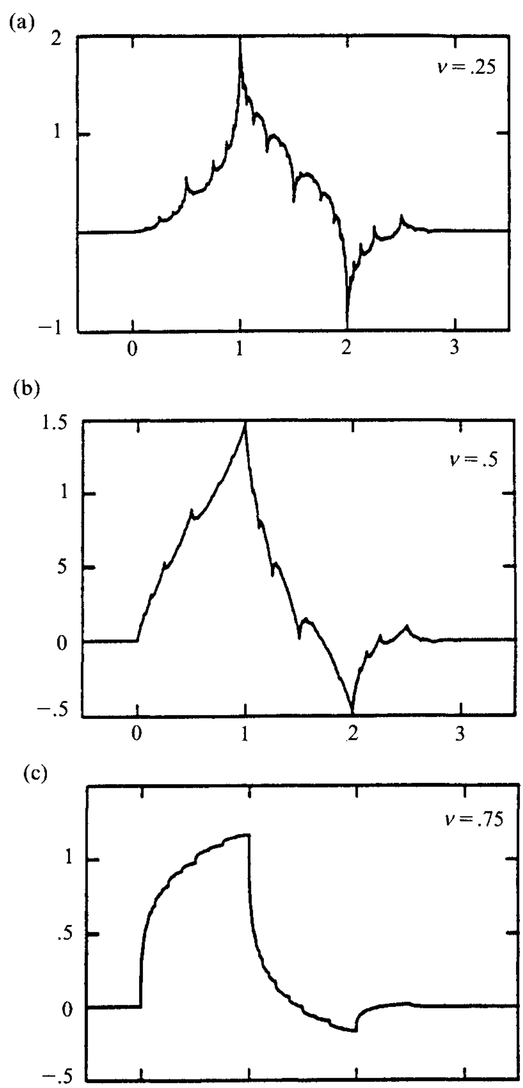

FIG. 3. *The functions ϕ defined by $c_0 = \nu(\nu+1)/(1+\nu^2)$, $c_1 = (\nu+1)/(\nu^2+1)$, $c_2 = (1-\nu)/(\nu^2+1)$, $c_3 = \nu(\nu-1)/(\nu^2+1)$, for different values of ν. As outlined in the text we can prove that the Hölder exponents of these functions are at least* (a) .180572, (b) .5864, (c) .251539. *For* a, c *these numbers are sharp*: *for* b *the true Hölder exponent is conjectured to be* .60017.

less steep than its counterpart for $\nu = 1/\sqrt{3}$, and this steepness is what is really expressed by a low Hölder exponent.

For $|\text{support }\phi| = 5$ we have no analytical expression for all the possible choices of the c_n. Since the "standard" example, with its 3 sum rules, achieves C^1-regularity (see Table 1), for which at least 2 sum rules are necessary, we can drop at most one sum rule. We explore what this extra degree of freedom can give us by perturbing around the standard example. More precisely, we have

$$m_0(\xi) = \left(\frac{1+e^{i\xi}}{2}\right)^2 Q(\xi), \tag{3.5}$$

with

$$|Q(\xi)|^2 = P(\cos \xi), \tag{3.6}$$

$$P(x) = 2 - x + \frac{a}{4}(1-x)^2, \tag{3.7}$$

where a can be chosen freely, subject to the constraint that the right-hand side of (3.6) is nonnegative for all ξ. The example of [15] with support width 5 corresponds to m_0 with a zero of order 3 at $\xi = \pi$, hence to P with a zero at $x = -1$, which gives $a = 3$. If we impose that P has a zero close to $x = -1$, e.g., at $x = -1 - \delta$ (where $\delta \geqq 0$, since otherwise the positivity constraint would be violated), then $a = 4(\delta+3)/(\delta+1)(\delta+2)^2$, and $P(x) = (x+1+\delta)/(\delta+1)(\delta+2)^2[x^2(\delta+3) - x(\delta+3)^2 + 2(\delta+2)^2]$. The other two roots of P are, therefore, given by $x_\pm = \frac{1}{2}(\delta+3) \pm \frac{1}{2}[(\delta+3)^2 - 8(\delta+2)^2/(\delta+3)]^{1/2}$. Each of the three roots of $P(x)$, namely, $x_0 = -1-\delta$, and $x_\pm$, corresponds to two roots in $z = e^{i\xi}$ of $P(\cos \xi)$ (use $\frac{1}{2}(z+z^{-1}) = x \Rightarrow z = x \pm \sqrt{x^2-1}$). This leads to the candidates $Q_\varepsilon(\xi) = N(e^{i\xi} + \delta + 1 + \varepsilon\sqrt{\delta(\delta+2)})\ (e^{i\xi} - z_+(\delta))\ (e^{i\xi} - z_-(\delta))$, where $z_\pm(\delta) = x_\pm(\delta) - \sqrt{x_\pm(\delta)^2 - 1}$ and $\varepsilon = \pm 1$. The choice $\varepsilon = +1$ corresponds to choosing all the zeros of Q inside the unit circle; the choice $\varepsilon = -1$ gives one (real) zero outside, and two complex conjugate zeros inside the unit circle. For $\varepsilon = +1$, the choice $\delta = 0$ (i.e., the example of [15]) minimizes $\max(\rho(T_0|_{V_2}), \rho(T_1|_{V_2}))$ (where ρ denotes the spectral radius), so that $\delta = 0$ leads to the most regular ϕ. For $\varepsilon = -1$, the situation is different. We find a minimum for $\max(\rho(T_0|_{V_2}), (\rho(T_1|_{V_2}))$ at $\delta = .07645485\ldots$ (value determined numerically). As in the case where $|\text{support }\phi| = 2$, this minimum for the spectral

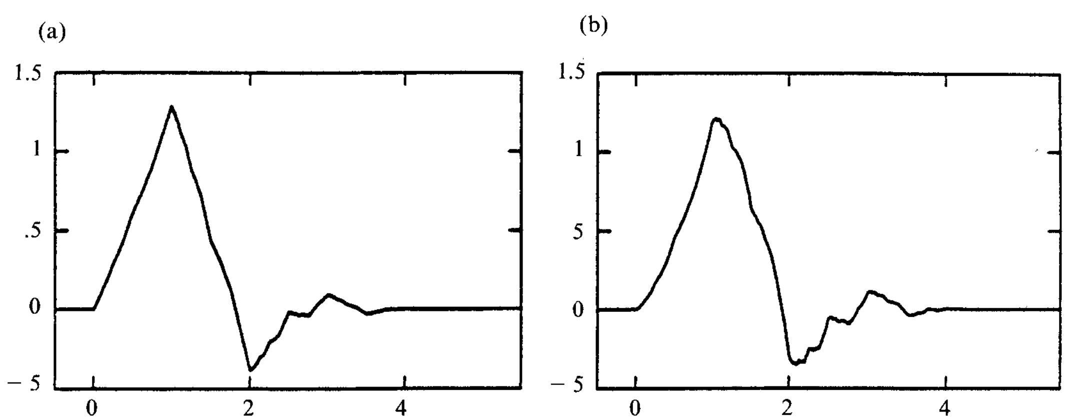

FIG. 4. *Two examples of ϕ with $|\text{support }\phi| = 5$.* (a) *Corresponds to the construction in* [15], (b) *is the "most regular" ϕ constructed here. In both cases $\phi \in C^1$; the Hölder exponent of ϕ' is* .0878 *for* (a), *and at least* .40198 *for* (b) (*it is conjectured to be* .41762 *for* (b)).

radii $\rho(T_j|_{V_2})$ corresponds to a degenerate largest eigenvalue of $T_1|_{V_2}$, and we find again that $T_1|_{V_2}$ is not diagonalizable. Consequently, we can only hope to establish

$$\lambda \leqq 2(1+\varepsilon)\max\left(\rho(T_0|_{V_2}), \rho(T_1|_{V_2})\right) = (1+\varepsilon).74865\cdots.$$

In order to obtain $\varepsilon < .01$, we already have to consider a large number of building blocks $T_{d_1}\cdots T_{d_m}$, the longest of which has $d_j = 1$ for $j = 1, \ldots, m$, and $m \geqq 700$! It seems likely that arbitrarily small ε can be attained by more work. Figure 4 shows both the standard example of [15] and the most regular ϕ obtained here for $|\text{support } \phi| = 5$. It is apparent that the present example is much more regular; both functions are C^1 (even though the function of [15] seems to have peaks, these peaks are not really sharp—see [11b]), but the Hölder exponent of ϕ' is significantly better in the example constructed here.

4. Vanishing moments for ϕ. In this subsection we want to construct ϕ, ψ with compact support,

$$|\text{supp } \phi| = |\text{supp } \psi| = 2M - 1$$

and such that

$$\int dx\, \phi(x) = 1,$$

$$\int dx\, x^l \phi(x) = 0 \quad \text{for } l = 1, \ldots, L-1, \tag{4.1}$$

$$\int dx\, x^l \psi(x) = 0 \quad \text{for } l = 0, \ldots, L-1.$$

The need for orthonormal bases with this property first came up in the application of wavelet bases to numerical analysis in the work of Beylkin, Coifman, and Rokhlin [19]. The desirability of vanishing moments for ϕ is explained in the introduction: if (4.1) is satisfied, then the inner product of ϕ_{jk} with a smooth function f only depends on $f(2^j k)$ and derivatives of f of order $\geqq L$. (In a later version of their work, Beylkin, Coifman, and Rokhlin did not require (4.1), however.) Imposing such vanishing moments on ϕ also increases its symmetry. Because these orthonormal wavelet bases with vanishing moments for both ϕ and ψ were requested by Coifman, I have named these wavelets *coiflets*. Condition (4.1) corresponds to a *coiflet of order L*.

The Fourier transforms of ϕ, ψ are given by $\hat\phi(\xi) = \prod_{j=1}^{\infty} m_0(2^{-j}\xi)$ $\hat\psi(\xi) = m_1(\xi/2)\hat\phi(\xi/2)$, with

$$m_0(\xi) = \sum_{n=N_1}^{N_2} c_n e^{in\xi}, \qquad m_1(\xi) = \sum_n (-1)^n c_{-n+1} e^{in\xi} = -e^{i\xi}\overline{m_0(\xi+\pi)}.$$

Note that the lower limit N_1 in the sum over n will in general not be zero in this subsection: we have lost our freedom to translate by integers because (4.1) is not invariant under such translations (the conditions on ψ are translation-invariant, but the conditions on ϕ are not). The conditions (4.1) are equivalent to

$$\hat\phi(0) = 1, \quad \left(\frac{d^l}{d\xi^l}\hat\phi\right)(0) = 0 \quad \text{for } l = 1, \ldots, L-1,$$

$$\left(\frac{d^l}{d\xi^l}\hat\psi\right)(0) = 0 \quad \text{for } l = 0, \ldots, L-1.$$

In terms of m_0, these become

$$m_0^{(l)}(\xi+\pi)=0 \quad \text{for } l=0,\dots,L-1, \tag{4.2}$$

$$m_0(0)=1, \quad m_0^{(l)}(0)=0 \quad \text{for } l=1,\dots,L-1. \tag{4.3}$$

By (4.2), m_0 has a zero of order L in $\xi=\pi$. Consequently, m_0 has to be of the form

$$m_0(\xi)=\left(\frac{1+e^{i\xi}}{2}\right)^L Q(e^{i\xi}), \tag{4.4}$$

where

$$|Q(e^{i\xi})|^2=\sum_{j=0}^{L-1}\binom{L-1+j}{j}\left(\frac{1-\cos\xi}{2}\right)^j+\left(\frac{1-\cos\xi}{2}\right)^L R(\cos\xi), \tag{4.5}$$

and R is an odd polynomial [15]. On the other hand (4.3) implies

$$m_0(\xi)=1+(1-e^{i\xi})^L S(e^{i\xi}). \tag{4.6}$$

Together, (4.4) and (4.6) lead to L independent linear constraints on the coefficients of S. Imposing that Q be of the form (4.5), with R an odd polynomial, leads to further quadratic constraints. For small values of L, the whole collection of constraint equations can be solved more or less by hand; for values of L larger than 6, the situation becomes untractable. We propose, therefore, an approach which from the start satisfies (4.2) and (4.3) (the linear constraints on S are built in), and we tackle (4.5) afterwards.

For the sake of convenience, we restrict ourselves to L even, $L=2K$. A similar analysis can be carried out for L odd. We impose that m_0 be of the form

$$m_0(\xi)=\left(\cos^2\frac{\xi}{2}\right)^K\left[\sum_{k=0}^{K-1}\binom{K-1+k}{k}\left(\sin^2\frac{\xi}{2}\right)^k+\left(\sin^2\frac{\xi}{2}\right)^K f(\xi)\right]. \tag{4.7}$$

Since $\cos^2\xi/2=\frac{1}{4}e^{-i\xi}(1+e^{i\xi})^2$, this clearly has a zero of order $2K$ at $\xi=\pi$. On the other hand, (4.7) can be rewritten as (use (1.13))

$$m_0(\xi)=1+\left(\sin^2\frac{\xi}{2}\right)^K\left[-\sum_{k=0}^{K-1}\binom{K-1+k}{k}\left(\cos^2\frac{\xi}{2}\right)^k+\left(\cos^2\frac{\xi}{2}\right)^K f(\xi)\right].$$

This clearly satisfies (4.3). It remains, therefore, to tailor f so that m_0 satisfies (1.10).

For the sake of convenience we shall use f such that

$$f(\xi)=\sum_{n=0}^{K'} f_n e^{in\xi}, \tag{4.8}$$

i.e., $f_n=0$ for all $n<0$. This is by no means the only choice possible; we could also decide to distribute the f_n as symmetrically around zero as possible, so that the support of ϕ would be more symmetrical around $x=0$. It turns out, however, that this symmetrical choice can lead to larger support widths for ϕ than (4.8) (this happens, e.g., for $K=3$). From (4.5) we obtain

$$\begin{aligned}&\left|\sum_{k=1}^{K-1}\binom{K-1+k}{k}\left(\sin^2\frac{\xi}{2}\right)^k+\left(\sin^2\frac{\xi}{2}\right)^K f(\xi)\right|^2\\ &\quad=\sum_{j=0}^{2K-1}\binom{2K-1+j}{j}\left(\sin^2\frac{\xi}{2}\right)^j+\left(\sin^2\frac{\xi}{2}\right)^{2K} R(\cos\xi),\end{aligned} \tag{4.9}$$

where R is an odd polynomial. Rewriting (4.9) leads to

$$\left[\sum_{k=0}^{K-1}\binom{K-1+k}{k}s_2^k\right]^2+\sum_{k=0}^{K-1}\binom{K-1+k}{k}s_2^{k+K}[f(\xi)+\overline{f(\xi)}]$$
$$+s_2^{2K}|f(\xi)|^2=\sum_{j=0}^{2K-1}\binom{2K-1+j}{j}s_2^j+s_2^{2K}R(\cos\xi), \tag{4.10}$$

where s_2 denotes $\sin^2(\xi/2)$. We shall determine the f_n by identifying coefficients of s_2^j.

Both $f(\xi)+\overline{f(\xi)}$ and $|f(\xi)|^2$ can be written as polynomials in $\cos\xi$, hence in s_2. It follows that only the first term in the left-hand side of (4.10), which is independent of f, contains terms in s_2^j with $j\leqq K-1$. Fortunately, these terms cancel the corresponding terms in s_2^j in the right-hand side of (4.10) because of the identity

$$\sum_{k=0}^{K}\binom{N_1-1+k}{k}\binom{N_2-1+K-k}{K-k}=\binom{N_1+N_2-1+K}{K}. \tag{4.11}$$

(See [26, (5.27)].)

We next concern ourselves with the terms in s_2^j, $j=K,\dots,2K-1$. Only the first two terms in the left-hand side of (4.10) contribute, leading to linear constraints in the f_n. Define g_n by

$$f(\xi)+\overline{f(\xi)}=\sum_{n=0}^{K'}g_n s_2^n. \tag{4.12}$$

Using $s_2=-\frac{1}{4}e^{-i\xi}(1-e^{i\xi})^2$, we find that the f_n and g_n are related through

$$f_0=\frac{1}{2}\sum_{n=0}^{K'}\binom{2n}{n}4^{-n}g_n,$$
$$f_k=(-1)^k\sum_{n=k}^{K'}\binom{2n}{n-k}4^{-n}g_n\quad\text{for } k\neq 0. \tag{4.13}$$

In practice we will determine the g_n and then calculate the f_n and f via (4.13).

Identification of the terms in s_2^j, $j=K,\dots,2K-1$ on both sides of (4.10) gives

$$\sum_{k=j-K+1}^{K-1}\binom{K-1+k}{k}\binom{K-1+j-k}{j-k}$$
$$+\sum_{k=0}^{\min(K',j-K)}\binom{j-1-k}{j-K-k}g_k=\binom{2K-1+j}{j}.$$

Using (4.11) again, and substituting $j=K+l$, $l=0,\dots,K-1$, we can reduce this to

$$\sum_{m=\max(0,l-K')}^{l}\binom{K-1+m}{m}g_{l-m}=2\sum_{k=0}^{l}\binom{K-1+k}{k}\binom{2K-1+l-k}{K+l-k}. \tag{4.14}$$

This is a system of K linear equations in $\min(K,K'+1)$ unknowns. It has no solutions if $K'+1<K$. If $K'\geqq K-1$, then the invertibility of the triangular matrix

$$M_{ij}=\binom{K-1+i-j}{i-j},\qquad K-1\geqq i\geqq j\geqq 0$$

immediately leads to

$$g_k=2\binom{2K-1+k}{K+k},\qquad k=0,\dots,K-1.$$

It remains to determine the $g_K, \dots, g_{K'}$. They are given by the constraint that

$$\sum_{k=0}^{K-1} \sum_{l=0}^{K'-K} \binom{K-1+k}{k} g_{K+l} s_2^{k+l} + |f(\xi)|^2 \tag{4.15}$$

should be an odd polynomial in $\cos \xi$. Since (4.15) can be rewritten as a polynomial of degree K' in $\cos \xi$, this results in $\lfloor (K'+1)/2 \rfloor$ equations for $K'-K+1$ unknowns. It follows that $K' \geqq 2K-1$ (no miraculous cancellations occur). In the examples worked out here, $K'=2K-1$. In these examples a solution has to be found for a system of K quadratic equations in K unknowns; every such solution corresponds to a coiflet of order $2K$, with support width $3K-1$.

The system of K equations to be solved can be written out a little more explicitly. Writing x_m, $m=0,\dots,K-1$ for the K unknown g_{K+m}, we have

$$|f(\xi)|^2 = \sum_{l=-(2K-1)}^{2K-1} e^{il\xi} \sum_{k=\max(0,-l)}^{\min(2K-1,2K-1-l)} f_k \overline{f_{l+k}}$$

with

$$\begin{aligned} f_k = (1-\tfrac{1}{2}\delta_{k0})(-1)^k \Bigg[2 \sum_{n=k}^{K-1} \binom{2n}{n-k} 4^{-n} \binom{2K-1+n}{K+n} \\ + \sum_{m=0}^{K-1} \binom{2m+2K}{m+K-k} 4^{-m-K} x_m \Bigg], \qquad 0 \leqq k \leqq K-1 \end{aligned} \tag{4.16}$$

$$f_k = (-1)^k \sum_{m=k-K}^{K-1} \binom{2m+2K}{m+K-k} 4^{-m-K} x_m, \qquad K \leqq k \leqq 2K-1. \tag{4.17}$$

On the other hand, the first term in (4.15) can be rewritten as

$$\begin{aligned} &\sum_{j=0}^{2K-2} s_2^j \sum_{m=\max(0,j-K+1)}^{\min(j,K-1)} \binom{K-1+j-m}{j-m} x_m \\ &= \sum_{l=-(2K-2)}^{2K-2} e^{il\xi} (-1)^l \sum_{j=|l|}^{2K-2} 4^{-j} \binom{2j}{j+l} \sum_{m=\max(0,j-K+1)}^{\min(j,K-1)} \binom{K-1+j-m}{j-m} x_m. \end{aligned}$$

The K equations in the unknowns $x_0, \dots, x_{K-1}$ are, therefore,

$$\sum_{k=0}^{2(K-r)-1} f_k \overline{f_{2r+k}} + \sum_{j=2r}^{2K-2} 4^{-j} \binom{2j}{j+2r} \sum_{m=\max(0,j-K+1)}^{\min(j,K-1)} \binom{K-1+j-m}{j-m} x_m = 0, \tag{4.18}$$

where $r=0,\dots,K-1$, and where (4.16), (4.17) have to be substituted for the f_k.

As a quadratic system (4.18) can have many solutions or no solutions at all. The following heuristic argument suggests that (4.18) will have solutions for sufficiently large K. We can rewrite (4.7) as

$$\begin{aligned} m_0(\xi) = \frac{1}{2} + 2^{-4K+1} K \binom{2K}{K} \sum_{k=0}^{K-1} \frac{(-1)^k}{2k+1} \binom{2K-1}{K+k} (e^{i(2k+1)\xi} + e^{-i(2k+1)\xi}) \\ + \left(\cos^2 \frac{\xi}{2}\right)^K \left(\sin^2 \frac{\xi}{2}\right)^K f(\xi). \end{aligned} \tag{4.19}$$

Let us concentrate on the first two terms in (4.19). For large K, the coefficient of $e^{i(2k+1)\xi}$ tends to

$$2^{-4K+1} K \binom{2K}{K} \frac{(-1)^k}{2k+1} \binom{2K-1}{K+k} \underset{K\to\infty}{\sim} \frac{(-1)^k}{\pi(2k+1)},$$

which is exactly the Fourier coefficient of the characteristic function $\chi(\xi)=1$ for $|\xi|\leqq \pi/2$, 0 for $|\xi|\geqq \pi/2$,

$$\chi(\xi)=\frac{1}{2}+\sum_{k=0}^{\infty}(-1)^k\frac{1}{\pi(2k+1)}(e^{i(2k+1)\xi}+e^{-i(2k+1)\xi}).$$

This is, in fact, a perfectly legitimate choice for m_0: $m_0=\chi$ leads to $\hat{\phi}(\xi)=1$ for $|\xi|\leqq \pi, 0$ otherwise, or $\phi(x)=\sin \pi x/\pi x$. The corresponding wavelet basis is C^∞, satisfies (4.1) for arbitrarily large L, but has rather slow decay at ∞. Our ansatz (4.7) or (4.19) for m_0 can, therefore, be viewed as a truncation to finite length of χ, consistent with the restrictions (4.2), (4.3), and where an additional f has to be introduced to fit (1.9). Since for $K\to\infty$, χ itself already satisfies all the conditions (1.9), (4.2), (4.3), it seems reasonable to hope that for large K, a slight perturbation of χ might satisfy (1.9), (4.2), (4.3).

Based on this perturbation argument, we can look for a solution to (4.18) "close to" $x_m\equiv 0$. For $K=1,2,3,4$, and 5 we have (1°) determined the system (4.18) with the symbolic manipulation package MACSYMA, (2°) found a solution by Newton's method, starting from the initial point $x_m\equiv 0$, $m=0,\dots,K-1$. The resulting m_0 are tabulated in Table 2. For $K=5$ the coefficients are given with less precision than for $K\leqq 4$ because the roundoff error, even with double precision, was sufficient to perturb decimals beyond the 10th decimal. Note that Table 2 corrects a mistake in the first entry in the corresponding Table 8.1 in [25]. Graphs for the corresponding ϕ, ψ can be found in [25, Fig. 8.3].

Remarks.

(1) The functions ϕ and ψ corresponding to Table 2 are almost symmetric. For some of these examples, there exists a pair of biorthogonal bases very close to the orthonormal basis (their graphs are almost indistinguishable), which have, moreover, the advantage of corresponding to rational c_n (see [21]).

(2) The approach given above has the merit of giving a method for the construction of coiflets of any order L (modulo the solution of a system of $L/2$ quadratic equations in $L/2$ variables). It does not necessarily give the smoothest coiflet of order L, however! For small L, everything can be worked out more or less by hand, and we find some solutions different from the coiflets given above.

For $L=2$, the smoothest coiflet is found by substituting

$$f(\xi)=a\,e^{i\xi}+b\,e^{2i\xi},$$

rather than (4.8) into (4.7), leading to a less symmetric coiflet with support width 5; in this case support $\phi=[-1,4]$. The system of quadratic equations reduces to a single equation, so that everything can be solved explicitly. We find

$$a=(s-1)/2,\quad b=(-s+3)/2,\quad \text{with } s=\pm\sqrt{15}.$$

The choice $s=-\sqrt{15}$ gives the most regular coiflet of order 2. The corresponding ϕ is plotted in Fig. 5. This ϕ is continuously differentiable; using the methods of [11] we find that its derivative has Hölder exponent .191814....

For $L=4$, we find, unlike the $L=2$ case, that the best regularity for ϕ is achieved by distributing its support as symmetrically as possible. This corresponds to choosing

$$f(\xi)=a\,e^{-i\xi}+b+c\,e^{i\xi}+d\,e^{2i\xi}.$$

The resulting set of equations reduces to two linear and two quadratic equations. All this can be reduced to one equation for a of degree 4, which has 2 real and 2 complex solutions. One of the real roots leads to a twice continuously differentiable ϕ,

TABLE 2
The coefficients for coiflets of order 2K, K = 1 or 5. Note: In this table, the coefficients are normalized so that their sum is 1.

	n	$\frac{1}{2}c_n$		n	$\frac{1}{2}c_n$
$K=1$	−2	−.051429484095	$K=4$	0	.553126452562
	−1	.238929728471		1	.307157326198
	0	.602859456942		2	−.047112738865
	1	.272140543058		3	−.068038127051
	2	−.051429972847		4	.027813640153
	3	−.011070271529		5	.017735837438
				6	−.010756318517
$K=2$	−4	.011587596739		7	−.004001012886
	−3	−.029320137980		8	.002652665946
	−2	−.047639590310		9	.000895594529
	−1	.273021046535		10	−.000416500571
	0	.574682393857		11	−.000183829769
	1	.294867193696		12	.000044080354
	2	−.054085607092		13	.000022082857
	3	−.042026480461		14	−.000002304942
	4	.016744410163		15	−.000001262175
	5	.003967883613			
	6	−.001289203356	$K=5$	−10	−.0001499638
	7	−.000509505399		−9	.0002535612
				−8	.0015402457
$K=3$	−6	−.002682418671		−7	−.0029411108
	−5	.005503126709		−6	−.0071637819
	−4	.016583560479		−5	.0165520664
	−3	−.046507764479		−4	.0199178043
	−2	−.043220763560		−3	−.0649972628
	−1	.286503335274		−2	−.0368000736
	0	.561285256870		−1	.2980923235
	1	.302983571773		0	.5475054294
	2	−.050770140755		1	.3097068490
	3	−.058196250762		2	−.0438660508
	4	.024434094321		3	−.0746522389
	5	.011229240962		4	.0291958795
	6	−.006369601011		5	.0231107770
	7	−.001820458916		6	−.0139736879
	8	.000790205101		7	−.0064800900
	9	.000329665174		8	.0047830014
	10	−.000050192775		9	.0017206547
	11	−.000024465734		10	−.0011758222
				11	−.0004512270
$K=4$	−8	.000630961046		12	.0002137298
	−7	−.001152224852		13	.0000993776
	−6	−.005194524026		14	−.0000292321
	−5	.011362459244		15	−.0000150720
	−4	.018867235378		16	.0000026408
	−3	−.057464234429		17	.0000014593
	−2	−.039652648517		18	−.0000001184
	−1	.293667390895		19	−.0000000673

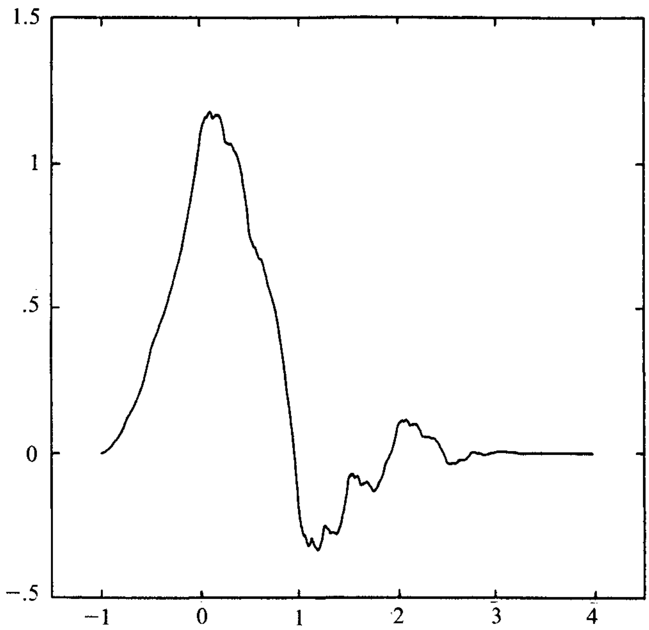

FIG. 5. *Plot of ϕ for the coiflet of order 2 with the highest regularity.*

corresponding to

$$
\begin{aligned}
c_{-5} &= -.008089728693, & c_1 &= .503931298301,\\
c_{-4} &= -.001473073456, & c_2 &= .443259223184,\\
c_{-3} &= .027620978693, & c_3 &= .010862015621,\\
c_{-2} &= .000661782050, & c_4 &= -.136801026363,\\
c_{-1} &= -.029586627843, & c_5 &= -.004737936078,\\
c_0 &= .168333606358, & c_6 &= .026019488227.
\end{aligned}
$$

As in Table 2, these c_n are normalized so that their sum equals 1. We have plotted the corresponding ϕ in Fig. 6.

For $L=6$ explicit computation of all the solutions is more complicated, but still feasible. There exists no solution so that support $(\phi)=[-8,9]$. For the ansatz

$$f(\xi)=a\,e^{-i\xi}+b+c\,e^{i\xi}+d\,e^{2i\xi}+e\,e^{3i\xi}+f\,e^{4i\xi},$$

corresponding to support $(\phi)=[-7,10]$, there are two solutions. The most regular of these solutions is twice differentiable; it is given by

$$
\begin{aligned}
c_{-7} &= -.000152916987, & c_2 &= .269094527854,\\
c_{-6} &= .000315249697, & c_3 &= .558133106629,\\
c_{-5} &= .001443474332, & c_4 &= .322997271647,\\
c_{-4} &= -.001358589300, & c_5 &= -.040303265359,\\
c_{-3} &= -.007915890196, & c_6 &= -.069655118535,\\
c_{-2} &= .006194347829, & c_7 &= .015323777973,\\
c_{-1} &= .025745731466, & c_8 &= .013570199856,\\
c_0 &= -.039961569717, & c_9 &= -.002466300927,\\
c_1 &= -.049807716931, & c_{10} &= -.001196319329.
\end{aligned}
$$

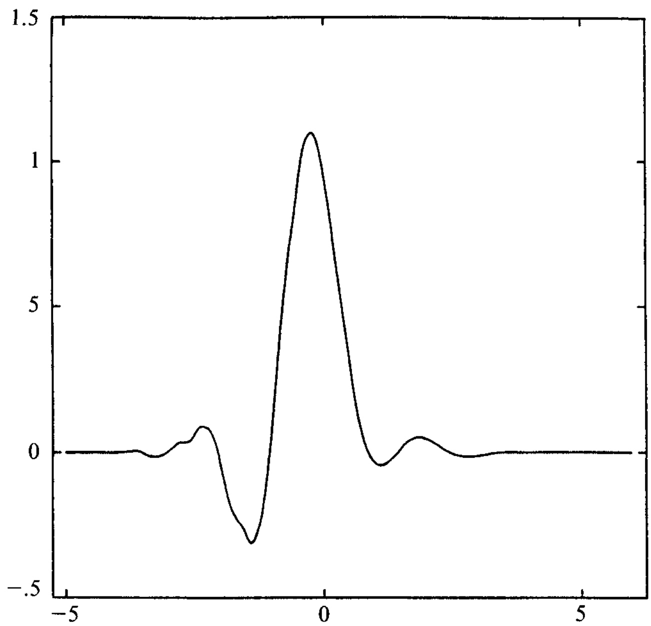

FIG. 6. *Plot of ϕ for the most regular coiflet of order* 4.

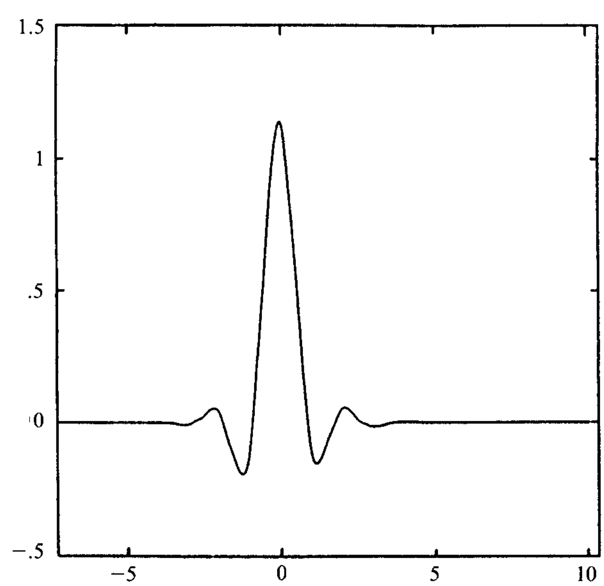

FIG. 7. *Plot of ϕ for the most regular coiflet of order* 6, *with support* $\phi = [-7, 11]$.

The function ϕ is plotted in Fig. 7. The coiflets used in [19] for $L = 2, 4, 6$ correspond to the scaling functions ϕ in Figs. 5, 6, and 7.

Acknowledgments. I would like to thank everyone who suggested the usefulness of the variations constructed in this paper, in particular, M. Basseville, G. Beylkin, A. Cohen, R. Coifman, and Y. Meyer.

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