Orthonormal mode sets for the two-dimensional fractional Fourier transformation

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A family of orthonormal mode sets arises when Hermite–Gauss modes propagate through lossless first-order optical systems. It is shown that the modes at the output of the system are eigenfunctions for the symmetric fractional Fourier transformation if and only if the system is described by an orthosymplectic ray transformation matrix. Essentially new orthonormal mode sets can be obtained by letting helical Laguerre–Gauss modes propagate through an antisymmetric fractional Fourier transformer. The properties of these modes and their representation on the orbital Poincaré sphere are studied. © 2007 Optical Society of America

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Probably the two best-known complete orthonormal sets of modes used in optics are (i) the Hermite–Gauss (HG) modes $\mathcal{H}_{m,n}(\mathbf{r})=\mathcal{H}_m(x)\mathcal{H}_n(y)$ with

$$\mathcal{H}_n(x) = 2^{1/4}(2^n n!)^{-1/2} H_n(\sqrt{2\pi}x)\exp(-\pi x^2), \tag{1}$$

where $H_n(\cdot)$ denotes the Hermite polynomials, and (ii) the helical Laguerre–Gauss (LG) modes[1]

$$\mathcal{L}_{m,n}(\mathbf{r}) = 2^{1/2}\left[\frac{(\min\{m,n\})!}{(\max\{m,n\})!}\right]^{1/2}(\sqrt{2\pi}r)^{|m-n|} \times\exp[i(m-n)\phi]L_{\min\{m,n\}}^{(|m-n|)}(2\pi r^2)\exp(-\pi r^2), \tag{2}$$

where $L_n^{(\alpha)}(\cdot)$ denotes the generalized Laguerre polynomials; as usual, spatial coordinates are represented by the two-dimensional column vector $\mathbf{r}=(x,y)^t=(r\cos\phi, r\sin\phi)^t$, where the superscript t denotes transposition. The existence of other sets of orthogonal modes, obtained from HG or LG modes by linear canonical transformations (LCTs), has been reported.[2,3]

LCTs describe the evolution of the complex optical field amplitude $f(\mathbf{r})$ when it propagates through a lossless first-order optical system: $f_i(\mathbf{r}_i)\rightarrow f_o(\mathbf{r}_o)$, with

$$f_o(\mathbf{r}_o) = \mathcal{R}^{\mathbf{T}}[f_i(\mathbf{r}_i)](\mathbf{r}_o) = \int K^{\mathbf{T}}(\mathbf{r}_o,\mathbf{r}_i) f_i(\mathbf{r}_i)\mathrm{d}\mathbf{r}_i. \tag{3}$$

The kernel $K^{\mathbf{T}}(\mathbf{r}_o,\mathbf{r}_i)$ of a LCT is parameterized by the system's real symplectic ray transformation matrix[4] $\mathbf{T}$, which relates the properly normalized dimensionless position $\mathbf{r}_i$ and direction $\mathbf{q}_i$ of an incoming ray to the position $\mathbf{r}_o$ and direction $\mathbf{q}_o$ of the corresponding outgoing ray through the matrix-vector relation

$$\begin{bmatrix}\mathbf{r}_o\\ \mathbf{q}_o\end{bmatrix} = \mathbf{T}\begin{bmatrix}\mathbf{r}_i\\ \mathbf{q}_i\end{bmatrix} \equiv \begin{bmatrix}\mathbf{A} & \mathbf{B}\\ \mathbf{C} & \mathbf{D}\end{bmatrix}\begin{bmatrix}\mathbf{r}_i\\ \mathbf{q}_i\end{bmatrix}, \tag{4}$$

where $\mathbf{q}=(q_x,q_y)^t$.

LCTs include several easily interpretable transformations in phase space, like scaling, image rotation, rotations in various position-direction (i.e., $\mathbf{rq}$) planes, shearing operations, etc. The important subset of so-called *orthosymplectic* systems arises when the ray transformation matrix is not only symplectic but also orthogonal; an orthosymplectic system can be described elegantly by the unitary matrix[5] $\mathbf{U}=\mathbf{A}+i\mathbf{B}=\mathbf{D}-i\mathbf{C}$. Three basic examples of orthosymplectic systems are the separable fractional Fourier transformer (FT), the rotator, and the gyrator,[5,6] with unitary representations

$$\mathbf{U}_f(\gamma_x,\gamma_y) = \begin{bmatrix}\exp(i\gamma_x) & 0\\ 0 & \exp(i\gamma_y)\end{bmatrix}, \tag{5}$$

$$\mathbf{U}_r(\alpha) = \begin{bmatrix}\cos\alpha & \sin\alpha\\ -\sin\alpha & \cos\alpha\end{bmatrix}, \tag{6}$$

$$\mathbf{U}_g(\alpha) = \begin{bmatrix}\cos\alpha & i\sin\alpha\\ i\sin\alpha & \cos\alpha\end{bmatrix}, \tag{7}$$

respectively. All three systems have a rotating character: the rotator performs a rotation in the (x,y) and (q_x,q_y) planes, the separable fractional FT in the (x,q_x) and (y,q_y) planes (possibly with different angles), and the gyrator in the (x,q_y) and (y,q_y) planes. As mentioned in Ref. 7, the rotator and the gyrator are similar to an antisymmetric fractional FT (i.e., $\gamma_x=-\gamma_y$) in the sense of similarity of their respective unitary matrices:

$$\mathbf{U}_r(\alpha) = \mathbf{U}_g(-\pi/4)\mathbf{U}_f(-\alpha,\alpha)\mathbf{U}_g(\pi/4), \tag{8}$$

$$\mathbf{U}_g(\alpha) = \mathbf{U}_r(-\pi/4)\mathbf{U}_f(\alpha,-\alpha)\mathbf{U}_r(\pi/4). \tag{9}$$

The subscripts f, r, and g for the unitary matrices $\mathbf{U}$ (and their corresponding ray transformation matrices $\mathbf{T}$) will be used throughout to indicate the corresponding transformations.

HG and LG modes are eigenfunctions for the symmetric fractional FT (i.e., $\gamma_x=\gamma_y$).[6,8] From the similarity between the Fresnel transformation,[9] which describes in the paraxial scalar approximation the propagation of light in free space, and the symmetric fractional Fourier transformation, it follows that the modes for the symmetric fractional FT are stable: they do not change their form during propagation in free space, except for some scaling and phase modulation. HG modes are also eigenfunctions for the *nonsymmetric* fractional FT (i.e., $\gamma_x \neq \gamma_y$). The LG modes are eigenfunctions for the rotation operation [see Eq. (6)] and can be obtained[5] by letting the HG modes propagate through a gyrator for which the gyrating angle is an odd multiple of $\pi/4$; we have, for example, $\mathcal{R}^{\mathbf{T}_g(-\pi/4)}[\mathcal{H}_{m,n}(\cdot)](\mathbf{r})=\mathcal{L}_{m,n}(\mathbf{r})(-i)^{m-|m-n|}$. Note that a gyrator with gyrating angle $\pm\pi/2$ transforms a positive-vortex LG mode into a negative-vortex one, and vice versa.

In this Letter, we find that among all orthonormal sets of modes generated from the HG (or LG) set by LCTs, only the modes generated by an *orthosymplectic* system are eigenfunctions for the symmetric fractional FT. The expression for these modes, their properties, and their location on the Poincairé sphere[10] are discussed.

It has been shown[3] that the orthonormal modes $\mathcal{H}^{\mathbf{M}}_{m,n}(\mathbf{r})=\mathcal{R}^{\mathbf{M}}[\mathcal{H}_{m,n}(\cdot)](\mathbf{r})$, which are obtained after propagation of the HG modes $\mathcal{H}_{m,n}(\mathbf{r})$ through a first-order optical system described by the symplectic matrix $\mathbf{M}$, and which we will call *symplectic* HG modes, are eigenfunctions of the transformation associated with the matrix $\mathbf{M}\mathbf{T}_f(\gamma_x,\gamma_y)\mathbf{M}^{-1}$. Since the angles γ_x and γ_y can be chosen arbitrarily, the symplectic HG modes $\mathcal{H}^{\mathbf{M}}_{m,n}(\mathbf{r})$ are eigenfunctions of a family of transformations. If the matrices $\mathbf{M}$ and $\mathbf{T}_f(\gamma_x,\gamma_y)$ commute, we have $\mathbf{M}\mathbf{T}_f(\gamma_x,\gamma_y)\mathbf{M}^{-1}=\mathbf{T}_f(\gamma_x,\gamma_y)\mathbf{M}\mathbf{M}^{-1}=\mathbf{T}_f(\gamma_x,\gamma_y)$, and we conclude that in that case the symplectic HG modes $\mathcal{H}^{\mathbf{M}}_{m,n}(\mathbf{r})$ are also eigenfunctions of the separable fractional FT $\mathbf{T}_f(\gamma_x,\gamma_y)$.

We first derive which symplectic matrices $\mathbf{M}$—with submatrices $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, and $\mathbf{D}$ [see Eq. (4)]—commute with the ray transformation matrix $\mathbf{T}_f(\gamma,\gamma)$ of a symmetric fractional FT. For commutation we require $\mathbf{M}\mathbf{T}_f(\gamma,\gamma)=\mathbf{T}_f(\gamma,\gamma)\mathbf{M}$, which leads to the two conditions $\mathbf{C}=-\mathbf{B}$ and $\mathbf{D}=\mathbf{A}$. The matrix $\mathbf{M}$ therefore needs to be orthosymplectic. We thus conclude that only if $\mathbf{M}$ is orthosymplectic, $\mathbf{M}=\mathbf{M}_o$, the orthonormal sets of modes $\mathcal{H}^{\mathbf{M}_o}_{m,n}(\mathbf{r})$ are eigenfunctions for the symmetric fractional FT $\mathbf{T}_f(\gamma,\gamma)$, with eigenvalues $\exp[-i(m+n)\gamma]$. We will call such modes *orthosymplectic* HG modes. Note that in optics we often deal with a fractional FT that is defined with an additional phase factor $\exp(-i\gamma)$; in that case the eigenvalues read $\exp[-i(m+n+1)\gamma]$.

The explicit form for orthosymplectic HG modes can be obtained from the general expression[3]

$$\mathcal{H}^{\mathbf{M}_o}_{m,n}(x,y) = \frac{2^{1/2}(-1)^{m+n}\exp[\pi(x^2+y^2)]}{2^{m+n}\sqrt{\pi^{m+n}m!n!\det\mathbf{U}}} \times\left(U^*_{11}\frac{\partial}{\partial x}+U^*_{21}\frac{\partial}{\partial y}\right)^m\left(U^*_{12}\frac{\partial}{\partial x}+U^*_{22}\frac{\partial}{\partial y}\right)^n \exp[-2\pi(x^2+y^2)], \tag{10}$$

where U_{11}, U_{12}, U_{21}, and U_{22} are the entries of the unitary matrix $\mathbf{U}$. Note in particular the symmetry relation $\mathcal{H}^{\mathbf{M}_o}_{m,n}(-\mathbf{r})=(-1)^{m+n}\mathcal{H}^{\mathbf{M}_o}_{m,n}(\mathbf{r})$ and the property $[\mathcal{H}^{\mathbf{M}_o}_{m,n}(\mathbf{r})]^*=\mathcal{H}^{\mathbf{M}_o^{-1}}_{m,n}(\mathbf{r})$. Using the derivative relations[3]

$$\left[\frac{\partial}{\partial x},\frac{\partial}{\partial y}\right]^t f_{m,n} = 2\sqrt{\pi}\mathbf{U}^*[\sqrt{m}f_{m-1,n},\sqrt{n}f_{m,n-1}]^t - 2\pi f_{m,n}[x,y]^t \tag{11}$$

and the recurrence relations[3]

$$2\sqrt{\pi}[x,y]^t f_{m,n} = \mathbf{U}[\sqrt{m+1}f_{m+1,n},\sqrt{n+1}f_{m,n+1}]^t + \mathbf{U}^*[\sqrt{m}f_{m-1,n},\sqrt{n}f_{m,n-1}]^t \tag{12}$$

for the orthonormal modes $\mathcal{H}^{\mathbf{M}_o}_{m,n}(x,y)=f_{m,n}(x,y)$, we can derive that the z component of the orbital angular momentum[2,10] of these modes takes the form

$$L_z^{m,n} = \int_{-\infty}^{\infty}\mathrm{Im}\left\{f^*_{m,n}(x,y)\left[x\frac{\partial f_{m,n}}{\partial y}-y\frac{\partial f_{m,n}}{\partial x}\right]\right\}\mathrm{d}x\mathrm{d}y = 2\,\mathrm{Im}\{mU_{11}U^*_{21}-nU_{22}U^*_{12}\}. \tag{13}$$

Moreover, the kernel $K^{\mathbf{T}_f(\gamma,\gamma)}(\mathbf{r}_o,\mathbf{r_i})$ of the symmetric fractional FT can be represented as a series of products of the orthosymplectic HG modes:

$$K^{\mathbf{T}_f(\gamma,\gamma)}(\mathbf{r}_o,\mathbf{r}_i) = \sum_{m,n=0}^{\infty}\exp[-i(m+n)\gamma]\,\mathcal{H}^{\mathbf{M}_o}_{m,n}(\mathbf{r}_o) \times[\mathcal{H}^{\mathbf{M}_o}_{m,n}(\mathbf{r}_i)]^*. \tag{14}$$

For $\mathbf{M}_o=\mathbf{I}$ we get the normal expansion in the common HG modes $\mathcal{H}_{m,n}(\mathbf{r})$.

We might wonder whether the commutation of an orthosymplectic system with the symmetric fractional FT, $\mathbf{U}\mathbf{U}_f(\gamma,\gamma)=\mathbf{U}_f(\gamma,\gamma)\mathbf{U}$, can be extended to the separable *nonsymmetric* case. From the requirement $\mathbf{U}\mathbf{U}_f(\gamma_x,\gamma_y)=\mathbf{U}_f(\gamma_x,\gamma_y)\mathbf{U}$ and realizing that $\mathbf{U}_f(\gamma_x,\gamma_y)$ is a diagonal matrix, we derive that either $\mathbf{U}$ should also be a diagonal matrix (corresponding to a separable fractional FT) or $\gamma_x=\gamma_y$ (which implies that we are dealing with a symmetric fractional FT that we considered before). We thus conclude that the only complete orthonormal set of modes that are eigenfunctions for the separable fractional Fourier transformation for *all* possible angles γ_x and γ_y is the set of *common* HG modes $\mathcal{H}_{m,n}(\mathbf{r})$. The *symmetric* case $\gamma_x=\gamma_y$ leads to many possible sets $\mathcal{H}^{\mathbf{M}_o}_{m,n}(\mathbf{r})$, i.e., the *orthosymplectic* HG modes as described above. The construction of eigenfunctions for a particular pair of the angles γ_x and γ_y has been considered in Ref. 11, along with a derivation of some properties of these eigenfunctions.

Let us now find all essentially different orthosymplectic HG mode sets. Any orthosymplectic system with unitary representation $\mathbf{U}$ can be decomposed[12] as an antisymmetric fractional FT $\mathbf{U}_f(\gamma,-\gamma)$ embedded in between two rotators $\mathbf{U}_r(\alpha)$ and $\mathbf{U}_r(\beta)$, in cascade with a symmetric fractional FT $\mathbf{U}_f(\varphi,\varphi)=\exp(i\varphi)\mathbf{I}$:

$$\mathbf{U}=\mathbf{U}_r(\beta)\mathbf{U}_f(\gamma,-\gamma)\mathbf{U}_r(\alpha)\exp(i\varphi). \tag{15}$$

When this system operates on an orthosymplectic HG mode $\mathcal{H}_{m,n}^{\mathbf{M}_o}(\mathbf{r})$, the symmetric fractional FT $\mathbf{U}_f(\varphi,\varphi)$ yields only a constant-phase multiplication $\exp[-i(m+n)\varphi]$, while the rotator $\mathbf{U}_r(\beta)$ yields a mere rotation of the resulting output mode. Modes that result after such a constant-phase multiplication or a mere rotation will not be considered as essentially new modes. The generation of new modes is thus reduced to the transformation associated with $\mathbf{U}_f(\gamma,-\gamma)\mathbf{U}_r(\alpha)$. Since we are free to choose any orthosymplectic HG mode set as the starting set for the generation of new modes, let us choose the LG modes: LG modes are eigenfunctions for rotation, and therefore the action of the rotator $\mathbf{U}_r(\alpha)$ can be omitted. So, all different sets of orthosymplectic modes can be obtained from the LG set by an antisymmetric fractional FT described by the matrix $\mathbf{U}_f(\gamma,-\gamma)$. In particular, for $\gamma=\pi/4$ we obtain the common HG set rotated at $\pi/4$ with respect to the Ox axis.

If we want to define these modes on the orbital Poincaré sphere,[10] similar to the one used for presentation of polarized light, we have to embed the antisymmetric fractional FT into a rotator and its inverse:

$$\mathbf{U}_g^{(\beta)}(\alpha)=\mathbf{U}_r(-\pi/4+\beta)\mathbf{U}_f(\alpha,-\alpha)\mathbf{U}_r(\pi/4-\beta). \tag{16}$$

We will call the corresponding system, which reduces to a gyrator [see Eq. (9)] for $\beta=0$, a *generalized* gyrator. Starting from the LG mode $\mathcal{L}_{m,n}(\mathbf{r})$, living on one of the poles of the (m,n)-Poincaré sphere, and applying the generalized gyrator $\mathbf{U}_g^{(\varphi/2)}(\theta/2)$ to this mode, the entire sphere can be populated by the modes $\mathcal{L}_{m,n}^{(\theta,\varphi)}(\mathbf{r})=\mathcal{R}^{\mathbf{T}_g^{\varphi/2}(\theta/2)}[\mathcal{L}_{m,n}(\cdot)](\mathbf{r})$, where the parameters $\theta\in[0,\pi]$ and $\varphi\in[-\pi,\pi]$ indicate the colatitude of a parallel and the longitude of a meridian on the sphere, respectively. For a picture of the Poincaré sphere, in which the same parameters θ and φ are used, we refer to Ref. 10. Since the determinant of the matrix $\mathbf{U}_g^{(\beta)}(\alpha)$ equals 1, there is no separable fractional FT (with determinant $\exp[i(\gamma_x+\gamma_y)]$) on the (m,n)-Poincaré sphere, except the antisymmetric one $(\gamma_x+\gamma_y=0)$; this is in accordance with the fact that all modes on the sphere are eigenfunctions of a symmetric fractional FT and that the symmetric part of a fractional FT is thus not visible.

All essentially new modes can be found on the main meridian $(\varphi=0)$ of the (m,n)-Poincaré sphere, with the positive- and negative-vortex LG modes on the two poles $(\theta=0$ or $\theta=\pi)$. The common HG mode lives on the equator $(\theta=\pi/2)$. For $\varphi=0$ and a certain $\theta\in[0,\pi]$, we obtain a *gyrating* mode $\mathcal{L}_{m,n}^{(\theta,0)}(\mathbf{r})$, which is transformed into the mode $\mathcal{L}_{m,n}^{(\theta,\varphi)}(\mathbf{r})$ by moving along the parallel of colatitude θ by an appropriate rotation and phase factor multiplication.

To derive a closed-form expression for the gyrating modes, we can use Eq. (10), and realize that the gyrating modes that live on the northern hemisphere, say, can be obtained by a gyrator $\mathbf{U}_g(-\pi/4+\theta/2)$ operating on the common HG mode $\mathcal{H}_{m,n}(\mathbf{r})$. We then find that the z component of the orbital angular momentum of the gyrating mode $\mathcal{L}_{m,n}^{(\theta,0)}(\mathbf{r})=\mathcal{H}_{m,n}^{\mathbf{T}_g(-\pi/4+\theta/2)}(\mathbf{r})$ equals $L_z^{m,n}=(m-n)\cos\theta$, which corresponds to the result in Ref. 10. Note that $L_z^{m,n}$ does not change by moving along a parallel of the (m,n)-Poincaré sphere.

Just as LG modes $\mathcal{L}_{m,n}(\mathbf{r})$ are eigenfunctions for the rotator $\mathbf{U}_r(\alpha)$, the *gyrating* modes $\mathcal{L}_{m,n}^{(\theta,0)}(\mathbf{r})$ are eigenfunctions with eigenvalues $\exp[i(m-n)\alpha]$ for the *generalized* rotator $\mathbf{U}_r^{(\theta/2)}(\alpha)$, defined by analogy with the generalized gyrator [see Eqs. (8) and (16)] as

$$\mathbf{U}_r^{(\beta)}(\alpha)=\mathbf{U}_g(-\pi/4+\beta)\mathbf{U}_f(-\alpha,\alpha)\mathbf{U}_g(\pi/4-\beta). \tag{17}$$

The normal rotator $\mathbf{U}_r(\cdot)$ arises on the poles $(\theta=0$ or $\theta=\pi)$, in which case we are indeed dealing with the rotationally symmetric LG modes (with positive or negative vortex). On the equator $(\theta=\pi/2)$, the generalized rotator reduces to the antisymmetric fractional FT $\mathbf{U}_r^{(\pi/4)}(\alpha)=\mathbf{U}_f(-\alpha,\alpha)$, with the common HG modes as its eigenfunctions.

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