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# Orthonormal Wavelet Expansion and Its Application to Turbulence

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Orthonormal wavelet expansion is applied to experimental data of turbulence. A direct relation is found between the wavelet spectrum and the Fourier spectrum. The orthonormal wavelet analysis with conditional sampling is applied to data of wind turbulence, yielding Kolmogorov's spectrum and the dissipation correlation with the intermittency exponent $\mu \simeq 0.2$.

Fourier transform method is a fundamental and indispensable tool in data analysis since it enables us to decompose data into components with different scales. Many fundamental properties of physical systems have been described in terms of Fourier spectrum, that is, the amplitude of Fourier coefficients. However, since Fourier spectrum totally ignores the phase of each Fourier coefficient, it lacks information about positions of local events which underlie the characteristics of the spectrum. The Fourier spectrum analysis therefore encounters difficulty in analyzing data in temporal or spatial intervals which include different kinds of local events.

The method of scale analysis applicable even to such complicated situations should enable us to identify the origin of characteristics of the spectrum with local events occurring in physical space. This requirement would be satisfied at least partially by an expansion in terms of basis functions which are local both in physical and Fourier space, although the locality is limited in its extent by the uncertainty principle. In this paper, we adopt an expansion method in terms of orthonormal wavelets (discrete wavelet analysis) as one of such types of expansion method.

The orthonormal wavelet expansion is a discrete version of continuous wavelet analysis.[1)] The latter is an integral transform method with kernel functions obtained by translating and dilating a localized function (analyzing wavelet). The continuous wavelet transform of a square integrable function is an isometric transform between a Hilbert space ($L^2$ space on $R^n$) and $L^2$ space on a locally compact topological group (a group of translation and dilation) with its Haar measure.[2)~4)] The continuous wavelet is a useful tool especially for studying a singularity or a fractal structure of a given function.[5)~9)] In particular, the energy cascade process in fully-developed turbulence has been captured remarkably in such an analysis.[7)] However, it is not very advantageous if one is interested in the energetic aspect because the kernel functions are not mutually orthogonal and no physically immediate meaning can be associated with the expansion coefficients.[10)]

Recently, mathematicians have succeeded in constructing a wavelet expansion in terms of orthonormal wavelets, which allows a clear and conventional physical interpretation of expansion coefficients from the energetic point of view. The orthonormal wavelets are obtained from a single function, the analyzing wavelet $\psi(t)$. In

Meyer's procedure,[10),11)] the analyzing wavelet is constructed from a real, even, non-negative and infinitely differentiable function $\phi(\omega)$ satisfying the conditions that (1) $\phi(\omega)$ is monotonically decreasing for $\omega \geqq 0$, (2) $\phi(\omega)=1$ ($|\omega| \leqq 2\pi/3$), $0(|\omega| \geqq 4\pi/3)$, (3) $\phi(\omega)^2+\phi(\omega-2\pi)^2=1$ ($2\pi/3 \leq |\omega| \leq 4\pi/3$). These conditions do not determine $\phi(\omega)$ uniquely, and here we take $\phi(\omega)=\sqrt{(h(\omega)h(-\omega))}$, $h(\omega)=f(4\pi/3-\omega)/(f(\omega-2\pi/3)+f(4\pi/3-\omega))$, $f(\omega)=\exp(-1/\omega^2)(\omega>0)$, $0(\omega \leqq 0)$. The analyzing wavelet $\psi(t)$ is given by $\psi(t)=(1/2\pi)\int\exp(i\omega t)\hat{\psi}(\omega)d\omega$, where $\hat{\psi}(\omega)=\exp(-i\omega/2)\sqrt{\phi(\omega/2)^2-\phi(\omega)^2}$. This analyzing wavelet is a real function so localized that it decays to zero faster than any negative power of $|x|$ as $|x| \to \infty$. The orthonormal wavelets are obtained from the analyzing wavelet $\psi(t)$ through discrete dilation and translation as $\psi_{j,k}(t)=2^{j/2}\psi(2^j t-k)$ ($j$, $k$: integers). The integers $j$ and $k$ specify respectively the spatial scale and the position of the wavelet $\psi_{j,k}(t)$ in the form of $k/2^j$. These orthonormal wavelets have been proved to form a complete orthonormal basis of $L^2(R)$ as

$$f(t)=\sum_{j=-\infty}^{\infty}\sum_{k=-\infty}^{\infty}\alpha_{j,k}\psi_{j,k}(t)\,, \qquad \alpha_{j,k}=\int_{-\infty}^{\infty}\psi_{j,k}(t)^* f(t)dt\,, \tag{1}$$

where $*$ denotes complex conjugate.

Practically, one of the most interesting properties of these orthonormal wavelets is the compactness of their support in Fourier space; $\hat{\psi}(\omega) \neq 0$ only for $2\pi/3<|\omega|<8\pi/3$. This property suggests through (1) that the wavelet spectrum $E_j=\sum_{k=-\infty}^{\infty}|\alpha_{j,k}|^2$ reflects the energy spectrum integrated from $|\omega|=2^j 2\pi/3$ to $2^j 8\pi/3$ in Fourier space. In other words, if the Fourier spectrum $E(\omega)$ has a power law as $\omega^{-p}$ then $E_j$ is expected to behave as $2^{-(p-1)j}$, and vice versa. This direct relation between the wavelet spectrum and the Fourier spectrum, together with spatial locality of wavelets, will open a variety of applications of the orthonormal wavelet expansion, including a local spectrum of a temporal or spatial structure.

We tested its usefulness by applying it to a shock solution of Burgers equation.[12)] Noticing that the r.h.s. of (1) is essentially a convolution, we can compute $\alpha_{j,k}$ numerically by using FFTs. The wavelet spectrum was found to have a scaling form $E_j \sim 2^{-j}$ in agreement with Fourier spectrum of $k^{-2}$ (replace $t$ and $\omega$ in the above by $x$ and $k$, respectively). For each $j$, we can find the spatial position of a local event dominant for the wavelet spectrum $E_j$, by identifying the maximum value of $|\alpha_{j,k}|^2$ over $k$. The spatial distribution of $|\alpha_{j,k}|^2$ for each $j$ has two sharp peaks which surround a particular spatial position, that is, the position of the shock (graphs omitted). This confirms a well-known fact that the Fourier spectrum $k^{-2}$ comes from a shock structure and thereby demonstrates sensitivity of wavelets to local events.

Now we apply the orthonormal wavelet expansion to data of atmospheric turbulence, obtained by a single hot-wire anemometer at a sampling rate of 100 Hz for about 8 minutes. The wind was weak and the mean velocity is about 0.5 m/s. In the following we implicitly employ Taylor's hypothesis assuming that it may be valid at least for sufficiently small scale motion. In Fig. 1 we show the (averaged) Fourier spectrum of the signal, and each was transformed with $10^{12}$ points. We can see a power-law behavior in higher wavenumber part. The slope is nearly equal to $-5/3$ of the inertial subrange spectrum of turbulence, but it cannot be distinguished from,

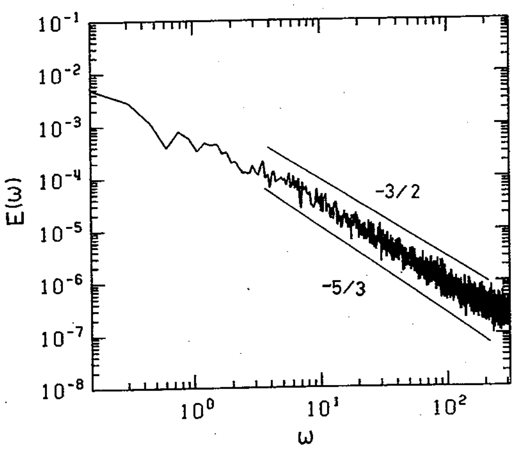


Fig. 1. Fourier spectrum of wind turbulence. The straight lines show slopes of $-3/2$ and $-5/3$. The angular frequency $\omega$ denotes $2\pi$ times frequency (Hz).

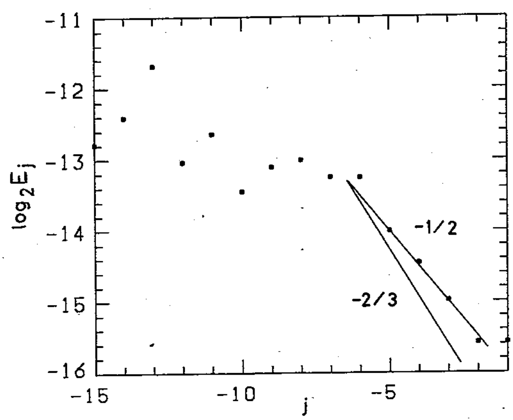


Fig. 2. Wavelet spectrum of wind turbulence. The straight lines show slope of $-1/2$ and $-2/3$.

for example, $-3/2$. We show in Fig. 2 the wavelet spectrum obtained from $2^{15}$ points of data ($-15 \leqq j \leqq -1$), in which the logarithm of $E_j$ to the base 2 is plotted against $j$ in order to see the power-law dependence directly (the Nyquist frequency corresponds to $j=-1$). For $j$ larger than $-7$, we can see a power-law behavior with exponent closer to $-1/2$ than to $-2/3$. Note that the wavelet spectrum gives a more sensitive check of a power-law, because it corresponds to the Fourier spectrum multiplied by wavenumber and because fluctuations are smoothed out by an averaging effect in the wavelet spectrum.

In order to examine the discrepancy in the slope from $-2/3$ (Kolmogorov), we adopt a working hypothesis that external disturbances contaminate the inertial subrange of fully-developed turbulence intermittently. Such events must give rise to relatively large values of the spectrum in the inertial subrange. Therefore we classify the wavelet coefficients into disturbed and undisturbed ones; disturbed coefficients are defined by $|\alpha_{j,k}|^2 > F\langle|\alpha_{j,k}|^2\rangle_k$, while undisturbed one by $|\alpha_{j,k}|^2 \leqq F\langle|\alpha_{j,k}|^2\rangle_k$, where $\langle\rangle_k$ denotes the average over all $k$ for each value of $j$, and $F$ is an arbitrarily chosen threshold value. Then we introduce conditional wavelet spectra, that is, the wavelet spectra in disturbed periods and undisturbed periods as

$$E^d{}_j = \{2^j/(\text{number of disturbed coefficients})\} \sum_{\text{disturbed coefficients}} |\alpha_{j,k}|^2 ,$$

$$E^u{}_j = \{2^j/(\text{number of undisturbed coefficients})\} \sum_{\text{undisturbed coefficients}} |\alpha_{j,k}|^2 .$$

Each of these represents the wavelet spectrum which we would have if the whole time interval would consist of the disturbed or undisturbed periods. We have no apriori reason to choose a particular value of $F$ and tried a number of values of $F$.

Figure 3 shows an example of the conditional spectra with $F=5$. The slope of the disturbed spectra is close to $-1/2$, while that of the undisturbed spectrum to $-2/3$, the value expected in the inertial subrange. Actually, the same results were

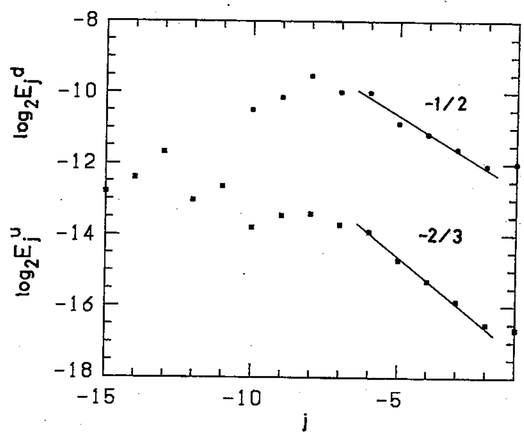


Fig. 3. Conditional wavelet spectra; the disturbed spectrum (circles) and the undisturbed one (squares). The straight lines show slopes of $-1/2$ and $-2/3$.

obtained for $0.01 \leqq F \leqq 15$, and thus such a feature is robust to the change of the value of $F$. For $F \geqq 20$, both the spectra were found to take a power-law with exponent $-1/2$. These results support the working hypothesis and also show the usefulness of the method of conditional sampling. The origin of strong turbulence is supposed to be a violent fluid motion in an atmospheric boundary layer, associated with the experimental environment.

The undisturbed wavelet coefficients, which gives the Kolmogorov's spectrum, are considered to represent fully-developed turbulence. We can check this view by examining the third-order moment $\langle \delta v(l)^3 \rangle$, whose $l$-dependence in the inertial subrange is rigorously derived from the Navier-Stokes equation as $\langle \delta v(l)^3 \rangle \sim l$.[13] Here $\delta v(l)$ denotes the magnitude of velocity characteristic to eddies of size $l$, and $\langle\ \rangle$ denotes an average. In terms of the wavelet coefficients, we have $\langle \delta v(l)^q \rangle \sim \{2^{jq/2}/(\text{number of undisturbed coefficients})\}\sum_k |a_{j,k}|^q$, where the sum is taken only over undisturbed coefficients and $l \sim 2^{-j}$. The third-order moment, denoted by $T_j$, with $F=5$ is shown in Fig. 4, which confirms that the undisturbed coefficients represent fully-developed turbulence. We remark that large values of undisturbed coefficients appear to locate where velocity gradients are large, while it is difficult to identify them with specific spatial structures.

Finally, we evaluate the intermittency exponent $\mu$ of fully-developed turbulence, which is not due to external disturbances but to internal dynamics. The exponent $\mu$ has different definitions according to the phenomenological theory employed.[14)~16)] Here we adopt the definitions in $\beta$-model and log-normal model, because these are

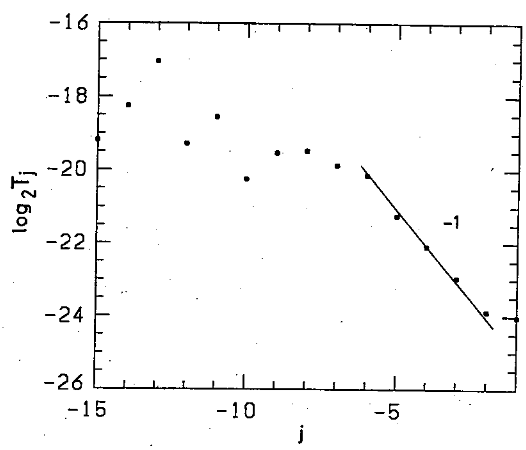


Fig. 4. The third-order moment in the undisturbed periods. The straight line shows slope of $-1$.

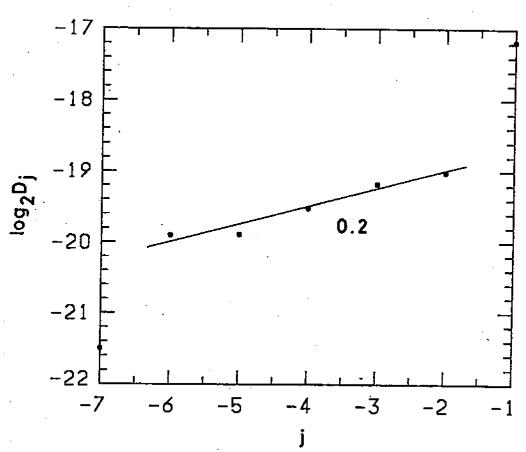


Fig. 5. The correlation of energy dissipation. The straight line shows slope of 0.2.

most frequently used and give the same correlation of energy dissipation $\varepsilon(x)$ as $\langle\varepsilon(x)\varepsilon(x+l)\rangle \sim \langle\delta v(l)^6/l^2\rangle \sim l^{-\mu}$. The dissipation correlation $\langle\delta v(l)^6/l^2\rangle$ in terms of the undisturbed wavelet coefficients, denoted by $D_j$, is shown in Fig. 5 ($F=5$).*) The result in Fig. 5 is consistent with $\mu \sim 0.2$ obtained by Anselmet and his coworkers[17] in a more sophisticated experiment. The consistency of $\mu$ in spite of the rather limited data demonstrates efficiency of the orthonormal wavelet expansion method in data analysis.

In this paper we focused our attention to the statistical properties of fully-developed turbulence. The application of the wavelet analysis to spatially local strucure of fully-developed turbulence is of particular interest. Also, the wavelet analysis can be applied to other aspects of fluid turbulence, such as transport phenomena and coherent structures in 3D and 2D turbulences. Studies on these issues are now in progress and will be reported elsewhere.

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*) We note that smaller values of F might distort the intermittent character of undisturbed parts more seriously.