Orthosymmetrical monotone functions

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Abstract

A straightforward generalization of the classical inverse of a real function based on reflections leads to several insuperable difficulties. We introduce a new type of inverse w.r.t. monotone bijections ϕ that is determined by the direction of the base vectors of the real Euclidean plane. Inverting a monotone function in the real plane does not necessarily result in a function. Given an increasing real function f, Schweizer and Sklar geometrically construct a set of inverse functions. We will largely extend their construction to our new concept of ϕ-inverses, also incorporating decreasing functions f. Furthermore, the geometrical and algebraical aspects of our approach are elaborated comprehensively. Special attention goes to the symmetry of a monotone function f w.r.t. some monotone bijection ϕ.

1 Introduction

In the real plane $\mathbb{R}^2$, the inverse F^{-1} of a set $F \subseteq \mathbb{R}^2$ is defined as $F^{-1} = \{(x, y) \in \mathbb{R}^2 \mid (y, x) \in F\}$. Geometrically, we obtain F^{-1} by reflecting F about the graph of the first bisector $\mathbf{id} : \mathbb{R} \to \mathbb{R} : x \mapsto x$. For a function f (*i.e.* every element x in the domain of f is mapped to a unique image $f(x)$), its inverse $f^{-1} = \{(x, y) \in \mathbb{R}^2 \mid x = f(y)\}$ is again a function if and only if f is injective. A set F is symmetrical w.r.t. the first bisector if $(x, y) \in F$ whenever $(y, x) \in F$, meaning that the set and its inverse coincide. Analogously, F is symmetrical w.r.t. the second bisector $-\mathbf{id} : \mathbb{R} \to \mathbb{R} : x \mapsto -x$ if it holds that $(x, y) \in F$ whenever $(-y, -x) \in F$. Hence, $F^{-\mathbf{id}} := \{(x, y) \in \mathbb{R}^2 \mid (-y, -x) \in F\}$ can be understood as the inverse of F w.r.t. the second bisector. In particular, $F^{-\mathbf{id}}$ is the reflection of F w.r.t. $-\mathbf{id}$. However, reflections are not always apt to define the inverse of a set w.r.t. a given monotone $\mathbb{R} \to \mathbb{R}$ bijection ϕ. For instance, suppose that ϕ contains part of a circle with center

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(x_0, y_0) belonging to F. There does not exist a unique straight line perpendicular to ϕ that contains (x_0, y_0). This observation forces us to approach the inverse of F in a different way.

2 Inverting monotone functions

2.1 Geometrical construction

Consider a point (x_0, y_0) on F. Due to the strict monotonicity of ϕ, the triplet $((x_0, \phi(x_0)), (x_0, y_0), (\phi^{-1}(y_0), y_0))$ determines a unique rectangle through the point (x_0, y_0), with each side parallel to one of the axes and having at least two vertices on ϕ. The fourth point $(\phi^{-1}(y_0), \phi(x_0))$ of the rectangle belongs to the set

$$F^\phi := \{(x, y) \in \mathbb{R}^2 \mid (\phi^{-1}(y), \phi(x)) \in F\}\,.$$

We call F^ϕ the *ϕ-inverse of F*. It holds that $(x, y) \in F^\phi$ if and only if $(\phi(x), \phi^{-1}(y)) \in F^{-1}$. In case ϕ is the identity function $\mathbf{id}$, $F^{\mathbf{id}} = F^{-1}$ and will still be referred to as the inverse of F. The ϕ-inverse of a function f is again a function if and only if f is injective. Moreover, in this case $f^\phi = \phi \circ f^{-1} \circ \phi$. Note also that $(F^\phi)^\phi = F$.

It is well known that a monotone function $f : [a, b] \to [c, d]$, with $[a, b]$ and $[c, d]$ closed subintervals of $[-\infty, \infty]$ ($a < b$ and $c < d$), has a countable number of discontinuity points. Consider a monotone bijection $\phi : [q, r] \to [s, t]$ such that $[a, b] \subseteq [q, r]$ and $[c, d] \subseteq [s, t]$. If f is not injective or $\operatorname{rng} f \subset [c, d]$, its ϕ-inverse f^ϕ cannot be seen as a $\phi^{-1}([c, d]) \to \phi([a, b])$ function. There are various ways to adjust this ϕ-inverse, ensuring that it becomes a $\phi^{-1}([c, d]) \to \phi([a, b])$ function. Given an increasing function $f : [a, b] \to [c, d]$, Schweizer and Sklar geometrically construct a set of inverse functions [3]. Some additional results for monotone functions are due to Klement *et al.* [1, 2]. We will largely extend these results and associate to each monotone function f a set of ϕ-inverse functions.

Adding vertical segments we complete the graph of f to a continuous line from the point (a, c) to the point (b, d) whenever f is increasing and from the point (a, d) to the point (b, c) whenever f is decreasing. We construct the ϕ-inverse of such a 'completed' curve and delete all but one point from any vertical segment. The set of all $\phi^{-1}([c, d]) \to \phi([a, b])$ functions obtained in this way, is denoted $Q(f, \phi)$. Note that, by definition, for a constant function f the set $Q(f, \phi)$ contains the $\phi^{-1}([c, d]) \to \phi([a, b])$ functions constructed from the increasing completion of f as well as those constructed from the decreasing completion of f. The injectivity and/or surjectivity of f is reflected in the set $Q(f, \phi)$.

Theorem 1. *The following assertions hold:*

(i) f is injective if and only if $|Q(f, \phi)| = 1$.

(ii) f is surjective if and only if $Q(f, \phi)$ contains injective functions only.

(iii) f is bijective if and only if $f^\phi \in Q(f, \phi)$.

For a bijective function f it clearly holds that $Q(f, \phi) = \{f^\phi\}$.

We can introduce an equivalence relation on the class of monotone $[a,b] \to [c,d]$ functions by calling two functions f and h equivalent if their 'completed' curves coincide, or equivalently, if the sets $Q(f,\phi)$ and $Q(h,\phi)$ coincide. The monotone bijection ϕ can be chosen arbitrarily. The equivalence class containing a function f is then given by $Q(g,\phi)$, with $g \in Q(f,\phi)$.

Theorem 2. *The following assertions hold:*

(i) For every $g \in Q(f,\phi)$ it holds that $f \in Q(g,\phi)$.

(ii) For every $g_1, g_2 \in Q(f,\phi)$ it holds that $Q(g_1,\phi) = Q(g_2,\phi)$.

(iii) For every $g \in Q(f,\phi)$ it holds that $h \in Q(g,\phi)$ if and only if $Q(h,\phi) = Q(f,\phi)$.

2.2 The importance of $Q(f, \mathbf{id})$

In order to describe the members of $Q(f,\phi)$ mathematically, we first have to introduce four $\phi^{-1}([c,d]) \to \phi([a,b])$ functions $\overline{f}^{\phi}$, $\overline{f}_{\phi}$, $\underline{f}^{\phi}$ and $\underline{f}_{\phi}$:

$$\begin{aligned}
\overline{f}^{\phi}(x) &= \sup\{t \in \phi([a,b]) \mid f(\phi^{-1}(t)) < \phi(x)\}\\
\overline{f}_{\phi}(x) &= \inf\{t \in \phi([a,b]) \mid f(\phi^{-1}(t)) > \phi(x)\}\\
\underline{f}^{\phi}(x) &= \sup\{t \in \phi([a,b]) \mid f(\phi^{-1}(t)) > \phi(x)\}\\
\underline{f}_{\phi}(x) &= \inf\{t \in \phi([a,b]) \mid f(\phi^{-1}(t)) < \phi(x)\}\,.
\end{aligned}$$

In the following theorem we lay bare the tight connection between the above functions constructed from a monotone bijection ϕ and those constructed from the identity function.

Theorem 3. *1. If ϕ is increasing, then the following identities hold:*

$$\begin{aligned}
\overline{f}^{\phi} &= \overline{f \circ \phi^{-1}}^{\mathbf{id}} \circ \phi = \phi \circ \overline{\phi^{-1} \circ f}^{\mathbf{id}} = \overline{\phi^{-1} \circ f \circ \phi^{-1}}^{\mathbf{id}} = \phi \circ \overline{f}^{\mathbf{id}} \circ \phi\\
\overline{f}_{\phi} &= \overline{f \circ \phi^{-1}}_{\mathbf{id}} \circ \phi = \phi \circ \overline{\phi^{-1} \circ f}_{\mathbf{id}} = \overline{\phi^{-1} \circ f \circ \phi^{-1}}_{\mathbf{id}} = \phi \circ \overline{f}_{\mathbf{id}} \circ \phi\\
\underline{f}^{\phi} &= \underline{f \circ \phi^{-1}}^{\mathbf{id}} \circ \phi = \phi \circ \underline{\phi^{-1} \circ f}^{\mathbf{id}} = \underline{\phi^{-1} \circ f \circ \phi^{-1}}^{\mathbf{id}} = \phi \circ \underline{f}^{\mathbf{id}} \circ \phi\\
\underline{f}_{\phi} &= \underline{f \circ \phi^{-1}}_{\mathbf{id}} \circ \phi = \phi \circ \underline{\phi^{-1} \circ f}_{\mathbf{id}} = \underline{\phi^{-1} \circ f \circ \phi^{-1}}_{\mathbf{id}} = \phi \circ \underline{f}_{\mathbf{id}} \circ \phi\,.
\end{aligned}$$

2. If ϕ is decreasing, then the following identities hold:

$$\begin{aligned}
\overline{f}^{\phi} &= \overline{f \circ \phi^{-1}}^{\mathbf{id}} \circ \phi = \phi \circ \overline{\phi^{-1} \circ f}_{\mathbf{id}} = \underline{\phi^{-1} \circ f \circ \phi^{-1}}^{\mathbf{id}} = \phi \circ \underline{f}_{\mathbf{id}} \circ \phi\\
\overline{f}_{\phi} &= \overline{f \circ \phi^{-1}}_{\mathbf{id}} \circ \phi = \phi \circ \overline{\phi^{-1} \circ f}^{\mathbf{id}} = \underline{\phi^{-1} \circ f \circ \phi^{-1}}_{\mathbf{id}} = \phi \circ \underline{f}^{\mathbf{id}} \circ \phi\\
\underline{f}^{\phi} &= \underline{f \circ \phi^{-1}}^{\mathbf{id}} \circ \phi = \phi \circ \underline{\phi^{-1} \circ f}_{\mathbf{id}} = \overline{\phi^{-1} \circ f \circ \phi^{-1}}^{\mathbf{id}} = \phi \circ \overline{f}_{\mathbf{id}} \circ \phi\\
\underline{f}_{\phi} &= \underline{f \circ \phi^{-1}}_{\mathbf{id}} \circ \phi = \phi \circ \underline{\phi^{-1} \circ f}^{\mathbf{id}} = \overline{\phi^{-1} \circ f \circ \phi^{-1}}_{\mathbf{id}} = \phi \circ \overline{f}^{\mathbf{id}} \circ \phi\,.
\end{aligned}$$

Proof We will prove the theorem for $\underline{f}_\phi$, the other cases being similar. On the one hand, for an increasing bijection ϕ, we obtain that

$$\begin{aligned}\underline{f}_\phi(x) &= \inf\{t \in [\phi(a),\phi(b)] \mid f(\phi^{-1}(t)) < \phi(x)\} = \underline{f\circ\phi^{-1}}_{\mathbf{id}}(\phi(x))\,;\\ &= \phi(\inf\{T\in[a,b] \mid f(T)<\phi(x)\}) = \phi(\underline{f}_{\mathbf{id}}(\phi(x)))\,;\\ &= \inf\{t\in[\phi(a),\phi(b)] \mid \phi^{-1}(f(\phi^{-1}(t))) < x\} = \underline{\phi^{-1}\circ f\circ\phi^{-1}}_{\mathbf{id}}(x)\,;\\ &= \phi(\inf\{T\in[a,b]\mid \phi^{-1}(f(T))<x\}) = \phi(\underline{\phi^{-1}\circ f}_{\mathbf{id}}(x))\,,\end{aligned}$$

for every $x\in\phi^{-1}([c,d])$. On the other hand, for a decreasing bijection ϕ, we obtain that

$$\begin{aligned}\underline{f}_\phi(x) &= \inf\{t\in[\phi(b),\phi(a)] \mid f(\phi^{-1}(t))<\phi(x)\} = \underline{f\circ\phi^{-1}}_{\mathbf{id}}(\phi(x))\,;\\ &=\phi(\sup\{T\in[a,b]\mid f(T)<\phi(x)\}) = \phi(\overline{f}^{\mathbf{id}}(\phi(x)))\,;\\ &=\inf\{t\in[\phi(b),\phi(a)]\mid \phi^{-1}(f(\phi^{-1}(t)))>x\} = \overline{\phi^{-1}\circ f\circ\phi^{-1}}_{\mathbf{id}}(x)\,;\\ &=\phi(\sup\{T\in[a,b]\mid\phi^{-1}(f(T))>x\}) = \phi(\underline{\phi^{-1}\circ f}^{\mathbf{id}}(x))\,,\end{aligned}$$

for every $x\in\phi^{-1}([c,d])$. ■

Thanks to this theorem, properties of $\overline{f}^{\mathbf{id}}$, $\overline{f}_{\mathbf{id}}$, $\underline{f}^{\mathbf{id}}$ and $\underline{f}_{\mathbf{id}}$ are easily translated to properties of $\overline{f}^{\phi}$, $\overline{f}_{\phi}$, $\underline{f}^{\phi}$ and $\underline{f}_{\phi}$. Note also that the theorem even holds for non-monotonic functions f.

Corollary 1. *Both functions $\overline{f}^{\phi}$ and $\overline{f}_{\phi}$ have the same type of monotonicity as ϕ. The monotonicity of the functions $\underline{f}^{\phi}$ and $\underline{f}_{\phi}$ is the opposite of that of ϕ.*

Proof It is easily verified that $\overline{f}^{\mathbf{id}}$ and $\overline{f}_{\mathbf{id}}$ are always increasing and that $\underline{f}^{\mathbf{id}}$ and $\underline{f}_{\mathbf{id}}$ are always decreasing. Taking into account Theorem 3 yields the postulate. ■

As shown in the following theorem, both sets $Q(f,\phi)$ and $Q(f,\mathbf{id})$ are isomorphic.

Theorem 4. *Consider a monotone bijection $\psi:[m,n]\to[u,v]$ such that $[a,b]\subseteq([q,r]\cap[m,n])$ and $[c,d]\subseteq([s,t]\cap[u,v])$. Then $Q(f,\phi)$ and $Q(f,\psi)$ are isomorphic. In particular, for every $g\in Q(f,\phi)$ there exists a unique function $h\in Q(f,\psi)$ such that $\phi^{-1}\circ g\circ\phi^{-1}=\psi^{-1}\circ h\circ\psi^{-1}$.*

Proof It suffices to prove that

$$\mathcal{I}_\phi : Q(f,\phi)\to Q(f,\mathbf{id}) : g\mapsto \mathcal{I}_\phi(g) := \phi^{-1}\circ g\circ\phi^{-1}$$

is an isomorphism for every monotone bijection $\phi:[q,r]\to[s,t]$ such that $[a,b]\subseteq[q,r]$ and $[c,d]\subseteq[s,t]$. Recall the geometrical construction of $Q(f,\phi)$ and $Q(f,\mathbf{id})$. For every $g\in Q(f,\phi)$ we know that $(\phi^{-1}(g(x)),\phi(x))$, with $x\in\phi^{-1}([c,d])$, belongs to the completion of f. Hence, the set

$$\{(\phi(x),\phi^{-1}(g(x)))\mid x\in\phi^{-1}([c,d])\} = \{(X,\mathcal{I}_\phi(g)(X))\mid X\in[c,d]\}$$

indeed defines a $[c,d] \to [a,b]$ function belonging to $Q(f,\mathbf{id})$. Conversely, consider a function $k \in Q(f,\mathbf{id})$, then $\{(k(x),x) \mid x \in [c,d]\}$ is a subset of the completion of f. It is clear that the set

$$\{(\phi^{-1}(x),\phi(k(x))) \mid x \in [c,d]\} = \{(X,\phi(k(\phi(X)))) \mid X \in \phi^{-1}([c,d])\}$$

defines a function belonging to $Q(f,\phi)$ and thus $k = \mathcal{I}_\phi(\phi \circ k \circ \phi)$. We conclude that $\mathcal{I}_\phi$ is surjective. The bijectivity of ϕ ensures that $\mathcal{I}_\phi$ is also injective. Since ϕ is monotone, it holds that $\mathcal{I}_\phi$ is order-preserving and therefore $\mathcal{I}_\phi$ is indeed an order-preserving bijection. ∎

Corollary 2. *Consider a monotone bijection* $\psi : [m,n] \to [u,v]$ *such that* $[a,b] \subseteq ([q,r] \cap [m,n])$ *and* $[c,d] \subseteq ([s,t] \cap [u,v])$*. Then for every* $g \in Q(f,\phi)$ *there exists a unique function* $h \in Q(f,\psi)$ *such that*

$$\overline{g}^\phi = \overline{h}^\psi,\ \overline{g}_\phi = \overline{h}_\psi,\ \underline{g}^\phi = \underline{h}^\psi \text{ and } \underline{g}_\phi = \underline{h}_\psi\,,$$

whenever ϕ *and* ψ *have the same monotonicity and*

$$\overline{g}^\phi = \underline{h}^\psi,\ \overline{g}_\phi = \underline{h}_\psi,\ \underline{g}^\phi = \overline{h}^\psi \text{ and } \underline{g}_\phi = \overline{h}_\psi\,,$$

whenever ϕ *and* ψ *have opposite types of monotonicity.*

Proof From Theorem 4 we know that, given a function $g \in Q(f,\phi)$, there exists a unique function $h \in Q(f,\psi)$ such that $\phi^{-1} \circ g \circ \phi^{-1} = \psi^{-1} \circ h \circ \psi^{-1}$. The statements then follow immediately from Theorem 3. ∎

3 The set $Q(f,\mathbf{id})$

3.1 Mathematical description

The mathematical description of the set $Q(f,\mathbf{id})$ originates from the following observations dealing with monotone $[a,b] \to [c,d]$ functions f:

(I) if $x \in f([a,b])$, then $f^{-1}(x) = \{y \in [a,b] \mid f(y) = x\}$ is an interval;

(IIa) if f is increasing and $x \in [c,d] \setminus f([a,b])$, then $\overline{f}^{\mathbf{id}}(x) = \overline{f}_{\mathbf{id}}(x)$;

(IIb) if f is decreasing and $x \in [c,d] \setminus f([a,b])$, then $\underline{f}^{\mathbf{id}}(x) = \underline{f}_{\mathbf{id}}(x)$.

In this setting $\sup\emptyset = a$ and $\inf\emptyset = b$. As shown by Schweizer and Sklar [3], the set $Q(f,\mathbf{id})$ can be described as the set of $[c,d] \to [a,b]$ functions g fulfilling

(I)$_{\mathbf{id}}$ $\Big(\forall x \in f([a,b])\Big)\Big(g(x) \in \Big[\inf(f^{-1}(x)), \sup(f^{-1}(x))\Big]\Big)$;

(IIa)$_{\mathbf{id}}$ if f is increasing: $\Big(\forall x \in [c,d] \setminus f([a,b])\Big)\ \Big(g(x) = \overline{f}^{\mathbf{id}}(x) = \overline{f}_{\mathbf{id}}(x)\Big)$;

$\mathbf{(IIb)_{id}}$ if f is decreasing: $\Big(\forall x \in [c,d] \setminus f([a,b])\Big) \Big(g(x) = \underline{f}^{\mathbf{id}}(x) = \underline{f}_{\mathbf{id}}(x)\Big)$.

Special attention is drawn to the constant functions $\boldsymbol{\alpha} : [a,b] \to [c,d] : x \mapsto \alpha$, for some $\alpha \in [c,d]$. These functions are both increasing and decreasing. Therefore, $Q(\boldsymbol{\alpha}, \mathbf{id})$ contains functions fulfilling $\mathbf{(IIa)_{id}}$ as well as functions fulfilling $\mathbf{(IIb)_{id}}$. Whenever $f(a) \neq f(b)$, all elements of $Q(f, \mathbf{id})$ fulfill the same condition: either $\mathbf{(IIa)_{id}}$ or $\mathbf{(IIb)_{id}}$. According to Klement *et al.* [2], in this case we can merge $\mathbf{(IIa)_{id}}$ and $\mathbf{(IIb)_{id}}$ as follows:

$\mathbf{(II)_{id}}$ $\Big(\forall x \in [c,d] \setminus f([a,b])\Big)$

$$\Big(g(x) = \sup\{t \in [a,b] \mid (f(t) - x) \cdot (f(b) - f(a)) < 0\} = \inf\{t \in [a,b] \mid (f(t) - x) \cdot (f(b) - f(a)) > 0\}\Big).$$

In case $f(a) < f(b)$, resp. $f(a) > f(b)$, the function $\overline{f}^{\mathbf{id}}$, resp. $\underline{f}^{\mathbf{id}}$, is known as the *pseudo-inverse* $f^{(-1)}$ of f [2]. For a constant $[a,b] \to [c,d]$ function $\boldsymbol{\alpha}$, Klement *et al.* [2] define the pseudo-inverse as $\boldsymbol{\alpha}^{(-1)} := \mathbf{a}$. This pseudo-inverse does not necessarily coincide with $\overline{\boldsymbol{\alpha}}^{\mathbf{id}}$ or $\underline{\boldsymbol{\alpha}}^{\mathbf{id}}$, which can easily be verified by considering the $[0,1] \to [0,1]$ function $\mathbf{\frac{1}{2}}$. The authors were clearly inspired by the 'supremum expression' in condition $\mathbf{(II)_{id}}$. However, when dealing with constant functions, condition $\mathbf{(II)_{id}}$ can never hold as $\sup \emptyset = a < b = \inf \emptyset$ and the 'supremum expression' in condition $\mathbf{(II)_{id}}$ is neither related to condition $\mathbf{(IIa)_{id}}$ nor to condition $\mathbf{(IIb)_{id}}$. Pseudo-inverses are often used in the construction of triangular norms and conorms (see [1], [2], [4] and [5]). They have been studied extensively in that context. Some of our results concerning the pseudo-inverse of non-constant monotone functions can be (partially) found in [1], [2] or [5]. Our goal was not only to extend the existing knowledge, but also to purify the theorems from superfluous conditions and to rearrange the results in a more insightful way. We also clarified the inversion of constant functions.

We now try to figure out the significance of the four functions $\overline{f}^{\mathbf{id}}$, $\overline{f}_{\mathbf{id}}$, $\underline{f}^{\mathbf{id}}$ and $\underline{f}_{\mathbf{id}}$. In the following theorem we investigate which of these functions belongs to $Q(f, \mathbf{id})$ and can therefore be understood as some kind of inverse of f.

Theorem 5. *The following assertions hold:*

(i) If $f(a) < f(b)$, then a $[c,d] \to [a,b]$ function g belongs to $Q(f, \mathbf{id})$ if and only if $\overline{f}^{\mathbf{id}} \leq g \leq \overline{f}_{\mathbf{id}}$.

(ii) If $f(a) > f(b)$, then a $[c,d] \to [a,b]$ function g belongs to $Q(f, \mathbf{id})$ if and only if $\underline{f}^{\mathbf{id}} \leq g \leq \underline{f}_{\mathbf{id}}$.

(iii) If $f(a) = f(b)$, then a $[c,d] \to [a,b]$ function g belongs to $Q(f, \mathbf{id})$ if and only if $\overline{f}^{\mathbf{id}} \leq g \leq \overline{f}_{\mathbf{id}}$ or $\underline{f}^{\mathbf{id}} \leq g \leq \underline{f}_{\mathbf{id}}$.

The structural difference between $\overline{f}^{\mathbf{id}}$, $\overline{f}_{\mathbf{id}}$ and $\underline{f}^{\mathbf{id}}$, $\underline{f}_{\mathbf{id}}$ implies the following corollary:

Corollary 3. *The following assertions hold:*

(i) If $f(a) < f(b)$, then $Q(f, \mathbf{id})$ contains increasing functions only and $\{\underline{f}^{\mathbf{id}}, \underline{f}_{\mathbf{id}}\} \cap Q(f, \mathbf{id}) = \emptyset$.

(ii) If $f(a) > f(b)$, then $Q(f, \mathbf{id})$ contains decreasing functions only and $\{\overline{f}^{\mathbf{id}}, \overline{f}_{\mathbf{id}}\} \cap Q(f, \mathbf{id}) = \emptyset$.

(iii) If $f(a) = f(b)$, then $Q(f, \mathbf{id})$ contains increasing and decreasing functions.

Proof It is easily verified that every function located between $\overline{f}^{\mathbf{id}}$ and $\overline{f}_{\mathbf{id}}$ is increasing and that every function located between $\underline{f}^{\mathbf{id}}$ and $\underline{f}_{\mathbf{id}}$ is decreasing. By definition it holds that $\overline{f}^{\mathbf{id}}(c) = \underline{f}^{\mathbf{id}}(d) = a$ and $\overline{f}_{\mathbf{id}}(d) = \underline{f}_{\mathbf{id}}(c) = b$. Furthermore, $\underline{f}^{\mathbf{id}}(c) = b$ and $\underline{f}_{\mathbf{id}}(d) = a$ whenever $f(a) < f(b)$ and $\overline{f}^{\mathbf{id}}(d) = b$ and $\overline{f}_{\mathbf{id}}(c) = a$ whenever $f(a) > f(b)$. Taking into account the monotonicity of the members of $Q(f, \mathbf{id})$ yields that $\{\underline{f}^{\mathbf{id}}, \underline{f}_{\mathbf{id}}\} \cap Q(f, \mathbf{id}) = \emptyset$ whenever $f(a) < f(b)$ and $\{\overline{f}^{\mathbf{id}}, \overline{f}_{\mathbf{id}}\} \cap Q(f, \mathbf{id}) = \emptyset$ whenever $f(a) > f(b)$. ■

Depending on the monotonicity of f, the functions $\overline{f}^{\mathbf{id}}$, $\overline{f}_{\mathbf{id}}$ or $\underline{f}^{\mathbf{id}}$, $\underline{f}_{\mathbf{id}}$ do not only constitute the boundaries of $Q(f, \mathbf{id})$, they can also be sifted out of $Q(f, \mathbf{id})$ by means of continuity conditions.

Theorem 6. *If $f \notin \{\boldsymbol{c}, \boldsymbol{d}\}$, then the following assertions hold:*

1. *If f is increasing, then*

   (i) $\overline{f}^{\mathbf{id}}$ is the only member of $Q(f, \mathbf{id})$ that is left-continuous and maps c to a;

   (ii) $\overline{f}_{\mathbf{id}}$ is the only member of $Q(f, \mathbf{id})$ that is right-continuous and maps d to b.

2. *If f is decreasing, then*

   (i) $\underline{f}^{\mathbf{id}}$ is the only member of $Q(f, \mathbf{id})$ that is right-continuous and maps d to a;

   (ii) $\underline{f}_{\mathbf{id}}$ is the only member of $Q(f, \mathbf{id})$ that is left-continuous and maps c to b.

The set $Q(\boldsymbol{c}, \mathbf{id})$, resp. $Q(\boldsymbol{d}, \mathbf{id})$, contains exactly two continuous functions: $\underline{\boldsymbol{c}}^{\mathbf{id}} = \boldsymbol{a}$ and $\overline{\boldsymbol{c}}_{\mathbf{id}} = \boldsymbol{b}$, resp. $\overline{\boldsymbol{d}}^{\mathbf{id}} = \boldsymbol{a}$ and $\underline{\boldsymbol{d}}_{\mathbf{id}} = \boldsymbol{b}$. The above theorem has to be adjusted as follows.

Theorem 7. *The following assertions hold:*

1. *(i) $\overline{\boldsymbol{c}}^{\mathbf{id}}$ and $\underline{\boldsymbol{c}}^{\mathbf{id}}$ are the only members of $Q(\boldsymbol{c}, \mathbf{id})$ that are left-continuous and map c to a.*

   (ii) $\overline{\boldsymbol{c}}_{\mathbf{id}}$ is the only member of $Q(\boldsymbol{c}, \mathbf{id})$ that is right-continuous and maps d to b.

   (iii) $\underline{\boldsymbol{c}}^{\mathbf{id}}$ is the only member of $Q(\boldsymbol{c}, \mathbf{id})$ that is right-continuous and maps d to a.

(iv) $\bar{\boldsymbol{c}}_{\mathbf{id}}$ *and* $\underline{\boldsymbol{c}}_{\mathbf{id}}$ *are the only members of* $Q(\boldsymbol{c}, \mathbf{id})$ *that are left-continuous and map* c *to* b.

2. *(i)* $\overline{\boldsymbol{d}}^{\mathbf{id}}$ *is the only member of* $Q(\boldsymbol{d}, \mathbf{id})$ *that is left-continuous and maps* c *to* a.
   (ii) $\overline{\boldsymbol{d}}_{\mathbf{id}}$ *and* $\underline{\boldsymbol{d}}_{\mathbf{id}}$ *are the only members of* $Q(\boldsymbol{d}, \mathbf{id})$ *that are right-continuous and map* d *to* b.
   (iii) $\overline{\boldsymbol{d}}^{\mathbf{id}}$ *and* $\underline{\boldsymbol{d}}^{\mathbf{id}}$ *are the only members of* $Q(\boldsymbol{d}, \mathbf{id})$ *that are right-continuous and map* d *to* a.
   (iv) $\underline{\boldsymbol{d}}_{\mathbf{id}}$ *is the only member of* $Q(\boldsymbol{d}, \mathbf{id})$ *that is left-continuous and maps* c *to* b.

Note that Theorem 6 remains applicable to the other constant functions $\boldsymbol{\alpha}$, with $\alpha \in]c, d[$. The boundary conditions ensure the unicity.

3.2 Properties

In this section we focus on the characteristic properties of the classical inverse and figure out under which conditions these properties are preserved in our new framework. Firstly, we deal with the involutivity of the 'inverse' operator, *i.e.* $(f^{-1})^{-1} = f$. From Theorem 2 we know that $f \in Q(g, \mathbf{id})$, for every $g \in Q(f, \mathbf{id})$. Therefore, interpreting g as some inverse of f and f as some inverse of g, we obtain that in some sense 'inverting' some 'inverse' yields the original function. For monotone bijections f this reasoning is sound as $Q(f, \mathbf{id}) = \{f^{\mathbf{id}}\} = \{f^{-1}\}$ (Theorem 1). Otherwise, whenever f is not bijective, we know that $|Q(f, \mathbf{id})| > 1$ and/or $|Q(g, \mathbf{id})| > 1$, for some $g \in Q(f, \mathbf{id})$ (Theorem 1). We need to find out how the inverse g of f, resp. the inverse of g, should be selected from the set $Q(f, \mathbf{id})$, resp. $Q(g, \mathbf{id})$. Special attention is drawn here to the functions $\overline{f}^{\mathbf{id}}$, $\overline{f}_{\mathbf{id}}$ and $\underline{f}^{\mathbf{id}}$, $\underline{f}_{\mathbf{id}}$.

Theorem 8. *The following assertions hold:*

(i) If there exists a function $g \in Q(f, \mathbf{id})$ *such that* $\overline{g}^{\mathbf{id}} = f$, *then* f *must be increasing, left-continuous and* $f(a) = c$.

(ii) If there exists a function $g \in Q(f, \mathbf{id})$ *such that* $\overline{g}_{\mathbf{id}} = f$, *then* f *must be increasing, right-continuous and* $f(b) = d$.

(iii) If there exists a function $g \in Q(f, \mathbf{id})$ *such that* $\underline{g}^{\mathbf{id}} = f$, *then* f *must be decreasing, right-continuous and* $f(b) = c$.

(iv) If there exists a function $g \in Q(f, \mathbf{id})$ *such that* $\underline{g}_{\mathbf{id}} = f$, *then* f *must be decreasing, left-continuous and* $f(a) = d$.

Proof Consider a monotone function f and suppose that there exists a function $g \in Q(f, \mathbf{id})$ such that $\underline{g}_{\mathbf{id}} = f$. In particular, it then holds that $f(a) = \underline{g}_{\mathbf{id}}(a) = \inf\{t \in [c, d] \mid g(t) < a\} = d$. The decreasingness of f is an immediate consequence of Corollary 1. From Corollary 3 and the definition of $\underline{g}_{\mathbf{id}}$ we can derive that g must be decreasing. Theorems 6 and 7 then ensure that $f = \underline{g}_{\mathbf{id}}$ is always left-continuous. The other cases are proven in the same way. ∎

Note that neither $\underline{g}^{\mathbf{id}} = f$ nor $\underline{g}_{\mathbf{id}} = f$ can hold if $f(a) < f(b)$ and $g \in Q(f, \mathbf{id})$. Indeed, in contrast to f, both functions $\underline{g}^{\mathbf{id}}$ and $\underline{g}_{\mathbf{id}}$ are decreasing (Corollary 1). Similarly, if $f(a) > f(b)$, there does not exist a function $g \in Q(f, \mathbf{id})$ such that $\overline{g}^{\mathbf{id}} = f$ or $\overline{g}_{\mathbf{id}} = f$.

Theorem 9. *Let f be non-constant.*

1. *For an increasing function f it holds that:*
   - *(i) If f is left-continuous and $f(a) = c$, then $\overline{g}^{\mathbf{id}} = f$ for every $g \in Q(f, \mathbf{id})$.*
   - *(ii) If f is right-continuous and $f(b) = d$, then $\overline{g}_{\mathbf{id}} = f$ for every $g \in Q(f, \mathbf{id})$.*
2. *For a decreasing function f it holds that:*
   - *(i) If f is right-continuous and $f(b) = c$, then $\underline{g}^{\mathbf{id}} = f$ for every $g \in Q(f, \mathbf{id})$.*
   - *(ii) If f is left-continuous and $f(a) = d$, then $\underline{g}_{\mathbf{id}} = f$ for every $g \in Q(f, \mathbf{id})$.*

Proof Consider a left-continuous decreasing function f for which $f(a) = d$ and take $g \in Q(f, \mathbf{id})$. Theorem 5 and the left-continuity of f ensure that

$$g(f(x) + \epsilon) \leq \underline{f}_{\mathbf{id}}(f(x) + \epsilon) = \inf\{t \in [a, b] \mid f(t) < f(x) + \epsilon\} < x\,, \tag{1}$$

for every $x \in [a, b]$ such that $f(x) < d$ and with $\epsilon \in\,]0, d - f(x)]$. Moreover, it holds that

$$g(f(x) - \epsilon) \geq \underline{f}^{\mathbf{id}}(f(x) - \epsilon) = \sup\{t \in [a, b] \mid f(t) > f(x) - \epsilon\} \geq x\,, \tag{2}$$

for every $x \in [a, b]$ such that $c < f(x)$ and with $\epsilon \in\,]0, f(x) - c]$. Consider an arbitrary $x \in [a, b]$ such that $f(x) \in\,]c, d[$ and let $\epsilon \in\,]0, \min(d - f(x), f(x) - c)]$. As g is decreasing, combining Eqs. (1) and (2) leads to

$$\underline{g}_{\mathbf{id}}(x) = \inf\{t \in [c, d] \mid g(t) < x\} = f(x)\,. \tag{3}$$

In case $f(x) = c$, then Eq. (1), with arbitrary $\epsilon \in\,]0, d - c]$, also implies Eq. (3). In a similar way, Eq. (2) implies Eq. (3) whenever $f(x) = d$. We conclude that $\underline{g}_{\mathbf{id}} = f$. The other cases are proven in a similar way. ∎

From the proof of the previous theorem one can easily derive the following result concerning the constant functions $\boldsymbol{c}$ and $\boldsymbol{d}$.

Theorem 10. *For every $[c, d] \to [a, b]$ function g the following assertions hold:*

(i) $\overline{g}^{\mathbf{id}} = \boldsymbol{c}$ *if and only if* $\overline{\boldsymbol{c}}^{\mathbf{id}} \leq g \leq \overline{\boldsymbol{c}}_{\mathbf{id}}$.

(ii) $\overline{g}_{\mathbf{id}} = \boldsymbol{d}$ *if and only if* $\overline{\boldsymbol{d}}^{\mathbf{id}} \leq g \leq \overline{\boldsymbol{d}}_{\mathbf{id}}$.

(iii) $\underline{g}^{\mathbf{id}} = \boldsymbol{c}$ *if and only if* $\underline{\boldsymbol{c}}^{\mathbf{id}} \leq g \leq \underline{\boldsymbol{c}}_{\mathbf{id}}$.

(iv) $\underline{g}_{\mathbf{id}} = \boldsymbol{d}$ *if and only if* $\underline{\boldsymbol{d}}^{\mathbf{id}} \leq g \leq \underline{\boldsymbol{d}}_{\mathbf{id}}$.

Proof The sufficient conditions immediately follow from the proof of Theorem 9. Suppose now that $\underline{g}_{\mathbf{id}} = \boldsymbol{d}$, then necessarily $g(x) = b$ whenever $x \in [c, d[$ and hence $\underline{\boldsymbol{d}}^{\mathbf{id}} \leq g \leq \underline{\boldsymbol{d}}_{\mathbf{id}}$. The other cases are proven in the same way. ■

Note that if, for example, $\underline{g}_{\mathbf{id}}$ equals a non-constant left-continuous function f fulfilling $f(a) = d$, then it does not necessarily hold that $g \in Q(f, \mathbf{id})$ (consider for example $g = \mathbf{id}_{[0,1]}$ and let $[a, b] = [c, d] = [0, 1]$). This prevents us from further generalizing Theorem 9.

In classical analysis it holds that $f^{-1} \circ f = \mathbf{id}_{[a,b]}$ (*i.e.* $\{y \in [a, b] \mid f(x) = f(y)\} = \{x\}$ for every $x \in [a, b]$) if and only if f is injective. It is easily verified that $\overline{f}^{\mathbf{id}} \circ f \leq \mathbf{id}_{[a,b]} \leq \overline{f}_{\mathbf{id}} \circ f$ whenever f is increasing and $\underline{f}^{\mathbf{id}} \circ f \leq \mathbf{id}_{[a,b]} \leq \underline{f}_{\mathbf{id}} \circ f$ whenever f is decreasing.

Theorem 11. *There exists a function $g \in Q(f, \mathbf{id})$ such that $g \circ f = \mathbf{id}_{[a,b]}$ if and only if f is injective.*

Proof We present the proof for a decreasing function f. If $g \circ f = \mathbf{id}_{[a,b]}$ holds for some $g \in Q(f, \mathbf{id})$, then g must be surjective. From Theorem 1 it then follows that $Q(g, \mathbf{id})$ contains only injective functions. Since $f \in Q(g, \mathbf{id})$ (Theorem 2), this means that f must be injective. Conversely, assume that f is an injective decreasing $[a, b] \to [c, d]$ function. Expressing the injectivity of f

$$(\forall x \in [a, b[)(\forall \epsilon \in]0, b - x])(f(x + \epsilon) < f(x))\,,$$

is equivalent with

$$\underline{f}_{\mathbf{id}}(f(x)) = \inf\{t \in [a, b] \mid f(t) < f(x)\} = x\,,$$

for every $x \in [a, b]$. Recall from Theorems 1 and 5 that $Q(f, \mathbf{id}) = \{\underline{f}_{\mathbf{id}}\}$. Hence, $g \circ f = \mathbf{id}_{[a,b]}$, with $g \in Q(f, \mathbf{id})$. ■

For monotone $[a, b] \to [c, d]$ functions f, it holds that $f \circ f^{-1} = \mathbf{id}_{[c,d]}$ (*i.e.* $f(\{y \mid x = f(y)\}) = \{x\}$ for every $x \in [c, d]$) if and only if f is bijective. The injectivity of f ensures that f^{-1} is a function. Since $Q(f, \mathbf{id})$ only contains functions, the injectivity of f will become superfluous when replacing f^{-1} by some $g \in Q(f, \mathbf{id})$.

Theorem 12. *There exists a function $g \in Q(f, \mathbf{id})$ such that $f \circ g = \mathbf{id}_{[c,d]}$ if and only if f is surjective. Moreover, the surjectivity of f implies that $f \circ g = \mathbf{id}_{[c,d]}$ for every $g \in Q(f, \mathbf{id})$.*

Proof We present the proof for a decreasing function f. Clearly, $f \circ g = \mathbf{id}_{[c,d]}$, for some $g \in Q(f, \mathbf{id})$, implies the surjectivity of f. Conversely, suppose that f is surjective, then f is continuous, $f(a) = d$ and $f(b) = c$. By definition it then holds that

$$\begin{aligned} f(\underline{f}^{\mathbf{id}}(x)) &= f(\sup\{t \in [a, b] \mid f(t) > x\}) \\ &= x = f(\inf\{t \in [a, b] \mid f(t) < x\}) = f(\underline{f}_{\mathbf{id}}(x))\,, \end{aligned}$$

for every $x \in [c, d]$. Taking into account that $\underline{f}^{\mathbf{id}} \leq g \leq \underline{f}_{\mathbf{id}}$, for every $g \in Q(f, \mathbf{id})$ (Theorem 5), this leads to $f \circ \underline{f}^{\mathbf{id}} = f \circ g = f \circ \underline{f}_{\mathbf{id}} = \mathbf{id}_{[c,d]}$. ■

Combining Theorems 11 and 12, we obtain the following corollary.

Corollary 4. *There exists a function $g \in Q(f, \mathbf{id})$ such that $g \circ f = \mathbf{id}_{[a,b]}$ and $f \circ g = \mathbf{id}_{[c,d]}$ if and only if f is bijective.*

4 The set $Q(f, \phi)$

4.1 Increasing bijections

In this section we will generalize our previous results concerning the set $Q(f, \mathbf{id})$, to properties of the set $Q(f, \phi)$ where f is a monotone $[a, b] \to [c, d]$ function and ϕ is an increasing $[q, r] \to [s, t]$ bijection fulfilling $[a, b] \subseteq [q, r]$ and $[c, d] \subseteq [s, t]$. The isomorphy between $Q(f, \mathbf{id})$ and $Q(f, \phi)$ (see Theorem 4) allows a straightforward conversion of the properties of $Q(f, \mathbf{id})$ to those of $Q(f, \phi)$: for every $g \in Q(f, \phi)$, we know that $\phi^{-1} \circ g \circ \phi^{-1}$ belongs to $Q(f, \mathbf{id})$. Throughout this translation process we make extensively use of Theorem 3 and Corollary 2, where $\psi = \mathbf{id}$. The proofs are elementary and therefore left out.

The set $Q(f, \phi)$ can be described as the set of all $[\phi^{-1}(c), \phi^{-1}(d)] \to [\phi(a), \phi(b)]$ functions g fulfilling

($\overline{\mathrm{I}}$) $\Big(\forall\, x \in \phi^{-1}(f([a, b]))\Big)\Big(g(x) \in \phi\Big(\Big[\inf(f^{-1}(\phi(x))), \sup(f^{-1}(\phi(x)))\Big]\Big)\Big)$;

($\overline{\mathrm{IIa}}$) if f is increasing: $\Big(\forall\, x \in \phi^{-1}\left([c, d] \setminus f([a, b])\right)\Big)\ \Big(g(x) = \overline{f}^{\phi}(x) = \overline{f}_{\phi}(x)\Big)$;

($\overline{\mathrm{IIb}}$) if f is decreasing: $\Big(\forall\, x \in \phi^{-1}\left([c, d] \setminus f([a, b])\right)\Big)\ \Big(g(x) = \underline{f}^{\phi}(x) = \underline{f}_{\phi}(x)\Big)$.

For a constant function $\boldsymbol{\alpha}$, with $\alpha \in [c, d]$, the set $Q(\boldsymbol{\alpha}, \phi)$ contains functions fulfilling **($\overline{\mathrm{IIa}}$)** as well as functions fulfilling **($\overline{\mathrm{IIb}}$)**. The following theorems point out the significance and importance of the functions $\overline{f}^{\phi}, \overline{f}_{\phi}, \underline{f}^{\phi}$ and $\underline{f}_{\phi}$.

Theorem 13. *The following assertions hold:*

(i) If $f(a) < f(b)$, then a $\phi^{-1}([c, d]) \to \phi([a, b])$ function g belongs to $Q(f, \phi)$ if and only if $\overline{f}^{\phi} \le g \le \overline{f}_{\phi}$.

(ii) If $f(a) > f(b)$, then a $\phi^{-1}([c, d]) \to \phi([a, b])$ function g belongs to $Q(f, \phi)$ if and only if $\underline{f}^{\phi} \le g \le \underline{f}_{\phi}$.

(iii) If $f(a) = f(b)$, then a $\phi^{-1}([c, d]) \to \phi([a, b])$ function g belongs to $Q(f, \phi)$ if and only if $\overline{f}^{\phi} \le g \le \overline{f}_{\phi}$ or $\underline{f}^{\phi} \le g \le \underline{f}_{\phi}$.

It is clear that $Q(f, \phi)$ only contains increasing, resp. decreasing, functions provided that $f(a) < f(b)$, resp. $f(a) > f(b)$. Depending on the monotonicity of f, the functions $\overline{f}^{\phi}$, $\overline{f}_{\phi}$ and $\underline{f}^{\phi}$, $\underline{f}_{\phi}$ can also be characterized by means of some continuity conditions.

Theorem 14. *If $f \notin \{\boldsymbol{c}, \boldsymbol{d}\}$, then the following assertions hold:*

1. *If f is increasing, then*

   (i) *$\overline{f}^{\phi}$ is the only member of $Q(f,\phi)$ that is left-continuous and maps $\phi^{-1}(c)$ to $\phi(a)$;*

   (ii) *$\overline{f}_{\phi}$ is the only member of $Q(f,\phi)$ that is right-continuous and maps $\phi^{-1}(d)$ to $\phi(b)$.*

2. *If f is decreasing, then*

   (i) *$\underline{f}^{\phi}$ is the only member of $Q(f,\phi)$ that is right-continuous and maps $\phi^{-1}(d)$ to $\phi(a)$;*

   (ii) *$\underline{f}_{\phi}$ is the only member of $Q(f,\phi)$ that is left-continuous and maps $\phi^{-1}(c)$ to $\phi(b)$.*

Dealing with the constant functions $\boldsymbol{c}$ and $\boldsymbol{d}$, we have to reformulate Theorem 7 in a similar way. This adjustment has been omitted since it is straightforward yet lengthy. Next, we wonder which properties of f^{ϕ} remain preserved for $\overline{f}^{\phi}$, $\overline{f}_{\phi}$ and $\underline{f}^{\phi}$, $\underline{f}_{\phi}$.

Theorem 15. *The following assertions hold:*

(i) *If there exists a function $g \in Q(f,\phi)$ such that $\overline{g}^{\phi} = f$, then f must be increasing, left-continuous and $f(a) = c$.*

(ii) *If there exists a function $g \in Q(f,\phi)$ such that $\overline{g}_{\phi} = f$, then f must be increasing, right-continuous and $f(b) = d$.*

(iii) *If there exists a function $g \in Q(f,\phi)$ such that $\underline{g}^{\phi} = f$, then f must be decreasing, right-continuous and $f(b) = c$.*

(iv) *If there exists a function $g \in Q(f,\phi)$ such that $\underline{g}_{\phi} = f$, then f must be decreasing, left-continuous and $f(a) = d$.*

Also the converse property holds.

Theorem 16. *Let f be non-constant.*

1. *For an increasing function f it holds that:*

   (i) *If f is left-continuous and $f(a) = c$, then $\overline{g}^{\phi} = f$ for every $g \in Q(f,\phi)$.*

   (ii) *If f is right-continuous and $f(b) = d$, then $\overline{g}_{\phi} = f$ for every $g \in Q(f,\phi)$.*

2. *For a decreasing function f it holds that:*

   (i) *If f is right-continuous and $f(b) = c$, then $\underline{g}^{\phi} = f$ for every $g \in Q(f,\phi)$.*

   (ii) *If f is left-continuous and $f(a) = d$, then $\underline{g}_{\phi} = f$ for every $g \in Q(f,\phi)$.*

Although Theorems 11, 12 and Corollary 4 can be easily transformed to properties on the set $Q(f,\phi)$, it still remains unclear what the meaning is of $g \circ f$ and $f \circ g$ with $g \in Q(f,\phi)$. Also, $f^{\phi} \circ f$ and $f \circ f^{\phi}$ have no straightforward interpretation.

4.2 Decreasing bijections

Let f be a monotone $[a,b] \to [c,d]$ function and ϕ a decreasing $[q,r] \to [s,t]$ bijection fulfilling $[a,b] \subseteq [q,r]$ and $[c,d] \subseteq [s,t]$. As in the previous section, Theorem 3 and Corollary 2 will be used to convert the properties of $Q(f,\mathbf{id})$ into properties of $Q(f,\phi)$. The set $Q(f,\phi)$ can be described as the set of all $[\phi^{-1}(d),\phi^{-1}(c)] \to [\phi(b),\phi(a)]$ functions g fulfilling

(<u>I</u>) $\Big(\forall\, x \in \phi^{-1}(f([a,b]))\Big)\Big(g(x) \in \phi\Big(\Big[\inf(f^{-1}(\phi(x))), \sup(f^{-1}(\phi(x)))\Big]\Big)\Big)$;

(<u>IIa</u>) if f is increasing: $\Big(\forall\, x \in \phi^{-1}\left([c,d] \setminus f([a,b])\right)\Big)\,\Big(g(x) = \underline{f}^{\phi}(x) = \underline{f}_{\phi}(x)\Big)$;

(<u>IIb</u>) if f is decreasing: $\Big(\forall\, x \in \phi^{-1}\left([c,d] \setminus f([a,b])\right)\Big)\,\Big(g(x) = \overline{f}^{\phi}(x) = \overline{f}_{\phi}(x)\Big)$.

Working with decreasing bijections instead of increasing bijections reverses the role of the functions $\overline{f}^{\phi}, \overline{f}_{\phi}$ and $\underline{f}^{\phi}, \underline{f}_{\phi}$ in the description of the set $Q(f,\phi)$.

Theorem 17. *The following assertions hold:*

(i) If $f(a) < f(b)$, then a $\phi^{-1}([c,d]) \to \phi([a,b])$ function g belongs to $Q(f,\phi)$ if and only if $\underline{f}^{\phi} \leq g \leq \underline{f}_{\phi}$.

(ii) If $f(a) > f(b)$, then a $\phi^{-1}([c,d]) \to \phi([a,b])$ function g belongs to $Q(f,\phi)$ if and only if $\overline{f}^{\phi} \leq g \leq \overline{f}_{\phi}$.

(iii) If $f(a) = f(b)$, then a $\phi^{-1}([c,d]) \to \phi([a,b])$ function g belongs to $Q(f,\phi)$ if and only if $\overline{f}^{\phi} \leq g \leq \overline{f}_{\phi}$ or $\underline{f}^{\phi} \leq g \leq \underline{f}_{\phi}$.

Every function located between $\underline{f}^{\phi}$ and $\underline{f}_{\phi}$ is increasing and every function located between $\overline{f}^{\phi}$ and $\overline{f}_{\phi}$ is decreasing. In the following theorem we try to pinpoint the functions $\overline{f}^{\phi}$, $\overline{f}_{\phi}$ and $\underline{f}^{\phi}$, $\underline{f}_{\phi}$ by means of their continuity.

Theorem 18. *If $f \notin \{\boldsymbol{c},\boldsymbol{d}\}$, then the following assertions hold:*

1. *If f is increasing, then*
   - *(i) $\underline{f}^{\phi}$ is the only member of $Q(f,\phi)$ that is left-continuous and maps $\phi^{-1}(d)$ to $\phi(b)$;*
   - *(ii) $\underline{f}_{\phi}$ is the only member of $Q(f,\phi)$ that is right-continuous and maps $\phi^{-1}(c)$ to $\phi(a)$.*
2. *If f is decreasing, then*
   - *(i) $\overline{f}^{\phi}$ is the only member of $Q(f,\phi)$ that is right-continuous and maps $\phi^{-1}(c)$ to $\phi(b)$;*
   - *(ii) $\overline{f}_{\phi}$ is the only member of $Q(f,\phi)$ that is left-continuous and maps $\phi^{-1}(d)$ to $\phi(a)$.*

Thanks to the following two theorems we can derive under which conditions it holds that

$$\overline{\overline{f}^{\phi}}^{\phi} = f,\ \overline{\overline{f}_{\phi}}_{\phi} = f,\ \underline{\underline{f}^{\phi}}^{\phi} = f \text{ or } \underline{\underline{f}_{\phi}}_{\phi} = f\,.$$

Theorem 19. *The following assertions hold:*

(i) If there exists a function $g \in Q(f,\phi)$ such that $\overline{g}^{\phi} = f$, then f must be decreasing, right-continuous and $f(b) = c$.

(ii) If there exists a function $g \in Q(f,\phi)$ such that $\overline{g}_{\phi} = f$, then f must be decreasing, left-continuous and $f(a) = d$.

(iii) If there exists a function $g \in Q(f,\phi)$ such that $\underline{g}^{\phi} = f$, then f must be increasing, left-continuous and $f(a) = c$.

(iv) If there exists a function $g \in Q(f,\phi)$ such that $\underline{g}_{\phi} = f$, then f must be increasing, right-continuous and $f(b) = d$.

Theorem 20. *Let f be non-constant.*

1. *For a decreasing function f it holds that:*

   (i) If f is right-continuous and $f(b) = c$, then $\overline{g}^{\phi} = f$ for every $g \in Q(f,\phi)$.

   (ii) If f is left-continuous and $f(a) = d$, then $\overline{g}_{\phi} = f$ for every $g \in Q(f,\phi)$.

2. *For an increasing function f it holds that:*

   (i) If f is left-continuous and $f(a) = c$, then $\underline{g}^{\phi} = f$ for every $g \in Q(f,\phi)$.

   (ii) If f is right-continuous and $f(b) = d$, then $\underline{g}_{\phi} = f$ for every $g \in Q(f,\phi)$.

5 Symmetrical monotone functions

5.1 Orthosymmetry

Generalizing the classical notion of symmetry, we call a set $F \subseteq \mathbb{R}^2$ *ϕ-symmetrical* if it coincides with its ϕ-inverse, *i.e.* $(\phi^{-1}(y), \phi(x)) \in F \Leftrightarrow (x,y) \in F$. Unfortunately, when dealing with monotone $[a,b] \to [c,d]$ functions f only bijections can coincide with their ϕ-inverse. Indeed, if f has discontinuity points, its ϕ-inverse f^{ϕ} will not be defined on a closed interval. Otherwise, if f is not injective, its ϕ-inverse will not be a function on $\phi^{-1}([c,d])$. To overcome these problems we will generalize the classical concept of symmetry by means of the set $Q(f,\phi)$, containing the inverse functions associated with f. We call a monotone $[a,b] \to [c,d]$ function f *ϕ-orthosymmetrical* if $f \in Q(f,\phi)$. The prefix 'ortho' refers to the rectangle-based construction of $Q(f,\phi)$ (see Section 2.1). Since every element of $Q(f,\phi)$ is a $\phi^{-1}([c,d]) \to \phi([a,b])$ function, it must hold that $\phi^{-1}([c,d]) = [a,b]$. Hence, it suffices to consider monotone $[a,b] \to [c,d]$ bijections ϕ only. From now on f denotes a monotone $[a,b] \to [c,d]$ function and ϕ denotes an arbitrary monotone $[a,b] \to [c,d]$ bijection.

Theorem 21. *If f is ϕ-orthosymmetrical, then every member of $Q(f,\phi)$ is ϕ-orthosymmetrical.*

Proof If f is ϕ-orthosymmetrical, then there exists a function $g \in Q(f,\phi)$ such that $f = g$. Based on Theorem 2, we then know that $h \in Q(h,\phi)$, for every $h \in Q(f,\phi)$. ∎

From Theorem 1 we know that $Q(f,\phi) = \{f^\phi\}$ whenever f is bijective. Hence, ϕ-symmetry can be related to ϕ-orthosymmetry.

Theorem 22. *A monotone $[a,b] \to [c,d]$ bijection is ϕ-symmetrical if and only if it is ϕ-orthosymmetrical. Monotone $[a,b] \to [c,d]$ bijections are the only monotone $[a,b] \to [c,d]$ functions that can be both ϕ-symmetrical and ϕ-orthosymmetrical.*

As ϕ itself is ϕ-symmetrical this leads to the following:

Corollary 5. *Every monotone $[a,b] \to [c,d]$ bijection ϕ is ϕ-orthosymmetrical.*

The following theorem yields necessary and sufficient conditions for ϕ-orthosymmetry.

Theorem 23. *A non-constant function is ϕ-orthosymmetrical if and only if*

1. $\overline{f}^\phi \le f \le \overline{f}_\phi$ *if f and ϕ have the same type of monotonicity;*
2. $\underline{f}^\phi \le f \le \underline{f}_\phi$ *if f and ϕ have opposite types of monotonicity.*

The only ϕ-orthosymmetrical constant functions are $\boldsymbol{c}$ and $\boldsymbol{d}$.

Proof The first part is an immediate consequence of Theorems 13 and 17. Consider now a constant function $\boldsymbol{\alpha}$ and suppose that $\alpha \in \,]c,d[$. By definition, it holds that $\overline{\boldsymbol{\alpha}}^\phi(a) = \overline{\boldsymbol{\alpha}}_\phi(a) \in \{c,d\}$ and $\underline{\boldsymbol{\alpha}}^\phi(a) = \underline{\boldsymbol{\alpha}}_\phi(a) \in \{c,d\}$. However, we know from Theorems 13 and 17 that $\overline{\boldsymbol{\alpha}}^\phi(a) \le \boldsymbol{\alpha}(a) \le \overline{\boldsymbol{\alpha}}_\phi(a)$ or $\underline{\boldsymbol{\alpha}}^\phi(a) \le \boldsymbol{\alpha}(a) \le \underline{\boldsymbol{\alpha}}_\phi(a)$, which leads to a contradiction. We conclude that $\alpha \in \{c,d\}$. Because $\boldsymbol{c} = \underline{\boldsymbol{c}}^\phi \in Q(\boldsymbol{c},\phi)$ and $\boldsymbol{d} = \underline{\boldsymbol{d}}_\phi \in Q(\boldsymbol{d},\phi)$, the constant functions $\boldsymbol{c}$ and $\boldsymbol{d}$ are indeed ϕ-orthosymmetrical. ∎

For a non-constant function f it is impossible that $\underline{f}^\phi \le f \le \underline{f}_\phi$ if f and ϕ have the same type of monotonicity. Similarly, $\overline{f}^\phi \le f \le \overline{f}_\phi$ cannot occur if f and ϕ have opposite types of monotonicity. This is easily illustrated by evaluating the functions in $x = a$. It enables us to simplify the previous theorem.

Corollary 6. *f is ϕ-orthosymmetrical if and only if $\overline{f}^\phi \le f \le \overline{f}_\phi$ or $\underline{f}^\phi \le f \le \underline{f}_\phi$.*

Based on Theorems 14 and 18 we can provide simple methods to verify whether non-constant, left- or right-continuous, monotone $[a,b] \to [c,d]$ functions are ϕ-orthosymmetrical or not. Depending on the continuity of f, the monotonicity of f and ϕ, and given some additional boundary conditions, we have to verify whether $f = \overline{f}^\phi$, $f = \overline{f}_\phi$, $f = \underline{f}^\phi$ or $f = \underline{f}_\phi$ holds. Moreover, given the bijection ϕ, these equalities fix the monotonicity and continuity of f, and imply its ϕ-orthosymmetry. The explicit formulation of these results has been omitted as they are straightforwardly obtained by combining Corollary 1, Theorems 23, 13 and 15 and by combining Corollary 1, Theorems 23, 17 and 19.

5.2 Properties

In general, a monotone function f can only be ϕ-symmetrical if it coincides with the bijection ϕ or if it has the opposite type of monotonicity. Similar results hold when considering ϕ-orthosymmetry.

Theorem 24. *If f is non-constant and ϕ-orthosymmetrical, then one of the following assertions holds:*

1. *$f = \phi$;*
2. *f and ϕ have opposite types of monotonicity.*

Proof It suffices to prove that $f = \phi$ whenever f and ϕ have the same type of monotonicity. From Theorem 23 we know that

$$\sup\{t \in [c,d] \mid f(\phi^{-1}(t)) < \phi(x)\} \leq f(x) \leq \inf\{t \in [c,d] \mid f(\phi^{-1}(t)) > \phi(x)\}\,,$$

for every $x \in [a,b]$. In particular this means that $\phi(x) \leq f(\phi^{-1}(t))$ whenever $t \in \,]f(x), d]$ and that $\phi(x) \geq f(\phi^{-1}(t))$ whenever $t \in [c, f(x)[$. Suppose that there exists $x \in [a,b]$ such that $f(x) < \phi(x)$. If we choose arbitrary $t \in \,]f(x), \phi(x)[$, then the increasingness of $f \circ \phi^{-1}$ implies that $f(\phi^{-1}(t)) \leq f(x) < \phi(x)$, which contradicts $\phi(x) \leq f(\phi^{-1}(t))$. Similarly, suppose that there exists $x \in [a,b]$ such that $\phi(x) < f(x)$, then for every $t \in \,]\phi(x), f(x)[$, we get that $\phi(x) < f(x) \leq f(\phi^{-1}(t))$, a contradiction. We conclude that $f = \phi$. ∎

Dealing with monotone $[a,b] \to [c,d]$ bijections ψ we know that it suffices to investigate their ϕ-symmetry only (see Theorem 22). Explicitly, ψ is ϕ-symmetrical if and only if $\psi = \psi^\phi = \phi \circ \psi^{-1} \circ \phi$ or equivalently $\phi = \psi \circ \phi^{-1} \circ \psi$, which expresses the ψ-symmetry of ϕ. We say that ϕ and ψ form a *symmetrical pair*. The question arises now how to construct a symmetrical pair (ϕ, ψ), given one of its components. The following theorem tackles this problem.

Theorem 25. *Consider a monotone $[a,b] \to [c,d]$ bijection ϕ. Then a monotone $[a,b] \to [c,d]$ bijection ψ is ϕ-symmetrical if and only if $\psi = \phi$ or there exists a number $\alpha \in \,]a,b[$ and a monotone $[a,\alpha] \to \phi([\alpha,b])$ bijection γ with the opposite type of monotonicity as ϕ such that*

$$\psi(x) = \begin{cases} \gamma(x) & , \text{ if } x \in [a,\alpha]\,, \\ \phi(\gamma^{-1}(\phi(x))) & , \text{ if } x \in [\alpha,b]\,. \end{cases} \tag{4}$$

Proof In case $\psi = \phi$ or Eq. (4) holds, we immediately obtain that $\psi = \phi \circ \psi^{-1} \circ \phi$. The latter expresses the ϕ-symmetry of ψ. Conversely, if $\psi \neq \phi$ is ϕ-symmetrical, then $\psi = \phi \circ \psi^{-1} \circ \phi$. Since ψ and ϕ have opposite types of monotonicity (see Theorem 24), it holds that $\psi(a) = \phi(b)$ and $\psi(b) = \phi(a)$. Furthermore, there exists a unique $\alpha \in \,]a,b[$ such that $\psi(\alpha) = \phi(\alpha)$. Hence, $\psi([a,\alpha]) = \phi([\alpha,b])$ and $\psi([\alpha,b]) = \phi([a,\alpha])$. It is then clear that $\gamma := \psi|_{[a,\alpha]}$ is a $[a,\alpha] \to \phi([\alpha,b])$ bijection. Note that γ has the same type of monotonicity as ψ and that $\gamma^{-1} = \psi^{-1}|_{\phi([\alpha,b])}$. Taking into account that $\psi = \phi \circ \psi^{-1} \circ \phi$, Eq. (4) is easily verified. ∎

Consider the family $(f_n)_{n\in\mathbb{N}_0}$ of $[0,1] \to [0,1]$ bijections, defined by $f_n(x) = \sqrt[n]{1-(1-x)^n}$. It is easily verified that all these bijections form symmetrical pairs with $\mathcal{N} : [0,1] \to [0,1] : x \mapsto 1-x$. Unfortunately, the function f_∞, defined by

$$f_\infty(x) = \begin{cases} 0 & \text{, if } x = 0\,, \\ 1 & \text{, if } x \in \,]0,1]\,, \end{cases}$$

is not $\mathcal{N}$-symmetric. However, as verified on this example, the $\mathcal{N}$-orthosymmetry of the bijections f_n is passed on to f_∞.

Theorem 26. *The limit of a pointwisely converging sequence of monotone ϕ-orthosymmetrical $[a,b] \to [c,d]$ functions $(f_n)_{n\in\mathbb{N}}$ is always a monotone ϕ-orthosymmetrical $[a,b] \to [c,d]$ function.*

Proof Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of monotone ϕ-orthosymmetrical $[a,b] \to [c,d]$ functions pointwisely converging to a function f. Clearly f is a monotone $[a,b] \to [c,d]$ function. To demonstrate the ϕ-orthosymmetry of f we need to distinguish three cases (see Theorems 23 and 24). Firstly, if there exists a number $N \in \mathbb{N}$ such that all functions f_n, with $n \geq N$, equal $\boldsymbol{c}$ or such that all functions f_n, with $n \geq N$, equal $\boldsymbol{d}$, then $f = \boldsymbol{c}$ or $f = \boldsymbol{d}$, ensuring the ϕ-orthosymmetry of f. Secondly, if there exists a number $N \in \mathbb{N}$ such that all functions f_n, with $n \geq N$, equal ϕ, then f necessarily equals ϕ which is trivially ϕ-orthosymmetrical. Thirdly, there exits a number $N \in \mathbb{N}$ such that all functions f_n, with $n \geq N$, differ from $\boldsymbol{c}$ and $\boldsymbol{d}$ and such that the monotonicity of these functions f_n is the opposite of that of ϕ. From Theorem 23 we then know that

$$\sup\{t \in [c,d] \mid f_n(\phi^{-1}(t)) > \phi(x)\} \leq f_n(x) \leq \inf\{t \in [c,d] \mid f_n(\phi^{-1}(t)) < \phi(x)\}\,,$$

for every $x \in [a,b]$ and every $n \geq N$. The latter implies that for $n \geq N$ it holds that $f_n(\phi^{-1}(t)) \leq \phi(x)$ whenever $t \in \,]f_n(x), d]$ and that $f_n(\phi^{-1}(t)) \geq \phi(x)$ whenever $t \in [c, f_n(x)[$. Suppose now that there exists a number $t \in \,]f(x), d]$ such that $f(\phi^{-1}(t)) > \phi(x)$. As $\lim_{n\to\infty} f_n = f$, there exists a natural number $N_1 > N$ such that for every $n \geq N_1$ it holds that $t \in \,]f_n(x), d]$. Furthermore, there exists a natural number $N_2 > N$ such that $f_n(\phi^{-1}(t)) > \phi(x)$, for every $n \geq N_2$. Combining both results we obtain the contradiction that there exists for every $n \geq \max(N_1, N_2)$ a number $t \in \,]f_n(x), d]$ such that $f_n(\phi^{-1}(t)) > \phi(x)$. Consequently, it holds that $f(\phi^{-1}(t)) \leq \phi(x)$ whenever $t \in \,]f(x), d]$. In a similar way, it is shown that $f(\phi^{-1}(t)) \geq \phi(x)$ whenever $t \in [c, f(x)[$. Hence,

$$\sup\{t \in [c,d] \mid f(\phi^{-1}(t)) > \phi(x)\} \leq f(x) \leq \inf\{t \in [c,d] \mid f(\phi^{-1}(t)) < \phi(x)\}\,,$$

for every $x \in [a,b]$, or, in other words $\underline{f}^\phi \leq f \leq \underline{f}_\phi$. Applying Corollary 6 finishes the proof. ∎

From Theorem 23, it then follows that a sequence of monotone, ϕ-orthosymmetrical $[a,b] \to [c,d]$ functions $(f_n)_{n\in\mathbb{N}}$ can never converge to $\boldsymbol{\alpha}$ if $\alpha \in \,]c,d[$.

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