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OSCILLATING AND ASYMPTOTIC PROPERTIES OF A CLASS OF FUNCTIONAL DIFFERENTIAL EQUATIONS WITH MAXIMA

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The present paper deals with some asymptotic and oscillating properties of functional differential equations of the form

$$x''(t) + \lambda\, x''(t - \tau) + f(t, \max_{s \in M(t)} x(s), \max_{s \in M(t)} x'(s)) = 0\,, \tag{1}$$

where $\tau > 0$ is a constant delay, $M(t) \subseteq [t_0, +\infty)$ when $t \in [t_0, +\infty)$, $t_0 \in \mathbb{R}^1$, and $\lambda > 0$ is an arbitrary constant.

Definition 1. As a solution of equation (1) we shall consider every function $x(t) \in \tilde{C}^1(I, \mathbb{R}^1)$, $I = [t_0 - \tau, +\infty)$, $t_0 \in \mathbb{R}^1$ which satisfies (1) almost everywhere when $t \geqq t_0$. (We shall denote by $\tilde{C}^k(I, \mathbb{R}^1)$ the space of the functions $\varphi(t) : I \mapsto \mathbb{R}^1$ possessing absolutely continuous derivatives up to order k inclusively.)

Definition 2. We shall call a continuous function $\varphi(t) : I \mapsto \mathbb{R}^1$ *oscillating* if it contains a sequence of zeros approaching $+\infty$. Otherwise, the function will be called non-oscillating.

Definition 3. We shall call a continuous function $\varphi(t) : I \mapsto \mathbb{R}^1$ *strongly oscillating* if there exists a sequence of points $\{t_i\}_{i=1}$ such that $\lim_{i \to +\infty} t_i = +\infty$ and $\varphi(t_i)\, \varphi(t_{i+1}) < 0$ for every i. Otherwise, the function $\varphi(t)$ will be called *strongly non-oscillating*.

Definition 4. [1], [2]. We shall call a continuous function $\varphi(t): I \mapsto \mathbb{R}^1$ $k(\varphi)$-*strongly oscillating* ($k(\varphi)$ *– oscillating*) if there exists a real number $k(\varphi)$ such that the function $\varphi(t) - k(\varphi)$ is strongly oscillating (oscillating).

Definition 5. We shall call a solution $x(t)$ of (1) *correct* if for every $t \in I$

$$\sup_{s \in [t, +\infty)} |x(s)| > 0\,.$$

Let us consider the following example:

$$x''(t) + x''(t - \pi) + \max_{s \in [t - \pi, t + \pi]} x(s) = 0\,, \tag{2}$$

$\tau = \pi$, $M(t) = [t - \pi, t + \pi]$. It is immediately verifiable that equation (2) has a solution $x(t) = \sin t - 1$, which is oscillating but is not strongly oscillating. On the other hand, by virtue of Theorem 1 of [2], the equation

$$x''(t) + x''(t - \pi) + x(t - \pi) = 0$$

has only strongly oscillating solutions, which shows that the maximum influences the asymptotic behaviour of functional differential equations of the neutral type.

Lemma 1. *Let the following conditions hold:*

1. *The function* $\varphi(t): I \mapsto \mathbb{R}^1$ *is continuous.*
2. *The function* $\varphi(t) + \lambda\,\varphi(t - \tau) \geqq c$ $(\varphi(t) + \lambda\,\varphi(t - \tau) \leqq -c)$ *when* $t \in I$, *where* c, τ *and* λ *are arbitrary positive constants.*

Let $s \in I$. *Then for the set*

$$A = \{t \mid s \leqq t \leqq s + 2\tau,\ \varphi(t - \tau) \geqq \beta > 0\}$$

$$(A = \{t \mid s \leqq t \leqq s + 2\tau,\ \varphi(t - \tau) \leqq -\beta < 0\})$$

the inequality mess $A \geqq \tau$ *holds, where* β *is a constant depending solely on* c, τ *and* λ.

The proof of Lemma 1 is given in [2].

We introduce the following notations:

$$(3) \qquad (Lx)(t) := x(t) + \lambda\, x(t - \tau),$$

$$f^*(t, u_0) = \inf_{\substack{v \in R^1 \\ |u| > u_0}} |f(t, u, v)| \quad \text{when} \quad u_0 > 0,$$

$$M^0(t) = \min_{s \in M(t)} s.$$

We shall say that the conditions (A) hold if the following conditions are satisfied:

A1. The function $f(t, u, v): I \times \mathbb{R}^2 \mapsto \mathbb{R}^1$ is continuous and $f(t, 0, 0) \equiv 0$ when $t \in I$.

A2. If $u \neq 0$, then $u f(t, u, v) > 0$ when $t \in I$.

A3. The set $M(t)$ is closed when $t \in I$ and $\lim_{t \to +\infty} M^0(t) = +\infty$.

Lemma 2. *Let the following be fulfilled:*

1. *Conditions* (A) *hold.*
2. *For every constant* $c > 0$, *the identity*

$$(4) \qquad \int_{t_0}^{+\infty} f^*(t, c)\, \mathrm{d}t = +\infty$$

holds.

Then every non-oscillating solution of equation (1) *satisfies* $\liminf_{t \to +\infty} |x(t)| = 0$.

Proof. Let us assume that there exists a non-oscillating solution of the equation

(1) with the property $\liminf_{t \to +\infty} |x(t)| \geqq c > 0$ and, to be more precise, let us assume that $x(t) < 0$ when $t \geqq t^*$, $t^* \in I$. By integrating (1) from t^* to $t > t^*$, we obtain

$$(5) \qquad [(Lx)(t)]' - [(Lx)(t^*)]' = -\int_{t^*}^{t} f(z, \max_{s \in M(z)} x(s), \max_{s \in M(z)} x'(s)) \, dz \geqq 0 .$$

If we assume that $[(Lx)(t)]' \geqq c_1 > 0$ when $t \geqq t^*$, then the following inequality holds when $t \geqq t^*$:

$$(Lx)(t) - (Lx)(t^*) \geqq c_1(t - t^*) ,$$

whence it follows that $(Lx)(t) > 0$ for sufficiently large values of t, which contradicts the assumption that $x(t) < 0$.

Therefore $[(Lx)(t)]' \leqq 0$ when $t \geqq t^*$ and, taking into account that the function $[(Lx)(t)]'$ is monotone increasing, we conclude that the integral present on the right-hand side of the equality (5) is convergent.

On the other hand, it follows from the fact that $\limsup_{t \to +\infty} x(t) \leqq -c < 0$ that there exists a point $\bar{t} \geqq t^*$ such that $x(t) \leqq -c/2$ when $t \geqq \bar{t}$. Hence, it follows from condition A3 that there exists a point $t_1 \geqq \bar{t}$ such that $\max_{s \in M(t)} x(s) \leqq -c/2$ when $t \geqq t_1$.

Consequently, from (3) we obtain

$$\left| \int_{t_1}^{+\infty} f(t, \max_{s \in M(t)} x(s), \max_{s \in M(t)} x'(s)) \, dt \right| \geqq \int_{t_1}^{+\infty} f^*(t, c/2) \, dt$$

and hence

$$\int_{t_1}^{+\infty} f^*(t, c/2) \, dt < +\infty ,$$

which contradicts equality 4. Thus Lemma 2 has been proved.

Theorem 1. *Let the following be fulfilled:*

1. *Conditions* (A) *hold.*
2. *Condition* 2 *of Lemma* 2 *holds.*
3. *When* $t \in I$ *the inequality*

$$\text{mess}\,(M(t) \cap [t, t + 2\tau]) \geqq \tau$$

holds.

Then equation (1) *does not admit correct non-negative strongly non-oscillating solutions.*

Proof. Let us assume that there exists a correct strongly non-oscillating non-negative solution $x(t)$ of equation (1), $x(t) \geqq 0$ when $t \geqq t^*$, $t^* \in I$. It follows from equation (1) that $[(Lx)(t)]'$ is a monotone decreasing and non-negative function and hence from (5) we obtain

$$\int_{t^*}^{+\infty} f(t, \max_{s \in M(t)} x(s), \max_{s \in M(t)} x'(s)) \, dt < +\infty .$$

Since $[(Lx)(t)]' \geqq 0$ when $t \geqq t^*$, and $x(t)$ is a correct solution, there exists a point $\bar{t} \geqq t^*$ such that, when $t \geqq \bar{t}$, the inequality $(Lx)(t) \geqq c > 0$ holds.

Then, by virtue of Lemma 1, there exists a closed measurable set $E \subseteq [\bar{t}, +\infty)$ such that when $t \geqq \bar{t}$ the inequality mess $(E \cap [t, t + 2\tau]) \geqq \tau$ holds, and $x(t) \geqq \beta > 0$ when $t \in E$. It follows from condition A3 that there exists a point $t_1 \geqq \bar{t}$ such that $M(t) \subseteq [\bar{t}, +\infty)$ when $t \geqq t_1$.

Let $t \geqq t_1$ be an arbitrary point. Then, since the interval $[t, t + 2\tau]$ is a connected set and $E \cap [t, t + 2\tau]$ and $M(t) \cap [t, t + 2\tau]$ are closed sets, it follows from condition 3 of the theorem that

$$E \cap M(t) \cap [t, t + 2\tau] \neq \emptyset .$$

Therefore, since $x(t) \geqq \beta$ for $t \in E$, we can assert that for $t \geqq t_1$ the inequality $\max_{s \in M(t)} x(s) \geqq \beta$ holds. Using the inequality

$$\int_{t_1}^{+\infty} f(t, \max_{s \in M(t)} x(s), \max_{s \in M(t)} x'(s)) \, \mathrm{d}t \geqq \int_{t_1}^{+\infty} f^*(t, \beta) \, \mathrm{d}t$$

we obtain

$$\int_{t_1}^{+\infty} f^*(t, \beta) \, \mathrm{d}t < +\infty ,$$

which contradicts equality (4).

Thus Theorem 1 has been proved.

Theorem 1 proves that the maximum, in a sense, is a generator of oscillations for equations of the neutral type.

Theorem 2. *Let the conditions of Theorem* 1 *be fulfilled.*

Then each correct solution $x(t)$ of equation (1) *is $k(x)$ strongly oscillating, where $k(x) \leqq 0$ for each $x(t)$.*

Proof. It follows from Theorem 1 that (1) does not admit a correct non-negative strongly non-oscillating solution. Let us assume that there exists a correct non-positive strongly non-oscillating solution $x(t)$, $x(t) \leqq 0$ when $t \geqq t^*$, $t^* \in I$. Then, by virtue of Lemma 2, $\liminf_{t \to +\infty} |x(t)| = 0$. On the other hand, it follows from the proof of Lemma 2 that $[(Lx)(t)]' \leqq 0$ when $t \geqq t^*$ and since $x(t)$ is a correct solution, there exist a point $\bar{t} \geqq t^*$ and a constant $c > 0$ such that $(Lx)(t) \leqq -c < 0$ when $t \geqq \bar{t}$.

Hence we obtain from Lemma 1

$$\text{mess} \left(A = \{t \mid s \leqq t \leqq s + 2\tau, \ x(t - \tau) \leqq -\beta < 0\}\right) \geqq \tau$$

when $s \in [\bar{t}, +\infty)$ and, if we put $k(x) = -\beta$, we conclude that the function $x(t) + \beta$ is strongly oscillating.

This proves Theorem 2.

References

[1] *Norkin S. B.:* Осциляция решений дифференциальных уравнений с отклоняющимся аргументом. Дифференциальные уравнения с отклоняющимся аргументом, Наукова думка, Киев, 1977, 247—256.
[2] *A. I. Zahariev, D. D. Bainov:* Oscillating properties of the solutions of a class of neutral type functional differential equations. Bull. Austral. Math. Soc. vol. *22* (1980), 365—372.

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