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OSCILLATING MULTIPLIERS ON LIE GROUPS AND RIEMANNIAN MANIFOLDS

GEORGIOS ALEXOPOULOS

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Abstract. We prove L^p estimates for oscillating spectral multipliers on Lie groups of polynomial volume growth and Riemannian manifolds of nonnegative curvature. We apply these results to obtain L^p estimates for the Riesz means of the Schrödinger operator.

Introduction and statement of the results. Oscillating spectral multipliers are multipliers of the type

$$m_{\alpha,\beta}(\lambda)=\psi(|\lambda|)|\lambda|^{-\beta/2}e^{i|\lambda|^{\alpha/2}}, \quad \alpha, \beta>0$$

with $\psi(\lambda)$ a C^∞ function, 0 for $|\lambda|\leq 1$ and 1 for $|\lambda|\geq 2$. They are interesting because of their intimate connection with the Cauchy problem for the Schrödinger and the wave equations. They are also interesting because they provide examples of operators that are given by "strongly singular kernels" (cf. [9]).

Oscillating multipliers have already been studied extensively in the context of $\boldsymbol{R}^n$ (cf. [9], [10], [21], [22], [23], [26]). Some of these results have been generalised to stratified nilpotent Lie groups (cf. [19]) and to rank one noncompact symmetric spaces (cf. [11]).

In this article we study the oscillating multipliers in the context of connected Lie groups of polynomial volume growth and Riemannian mamifolds of nonnegative Ricci curvature. More precisely:

(a) Lie groups of polynomial volume growth. We consider a connected Lie group G and we fix a right invariant Haar measure dg on G.

If A is a Borel measurable subset of G, then we set $|A|=dg$-measure(A).

We fix a choice of left invariant vector fields $X_1, \dots, X_k$ that generate, together with their successive Lie brackets $[X_{i_1}, [X_{i_2}, \dots, [X_{i_{s-1}}, X_{i_s}]\dots]]$, the Lie algebra of G. To those vector fields, we associate, in a canonical way, a left invariant distance $d_X(\cdot, \cdot)$ (see [28] for this and the other results from the geometry and the analysis on Lie groups used in this article) and we denote by $B_r(x)$ the associated ball of radius $r>0$ and centered at $x\in G$.

We know that there is $d\in \boldsymbol{N}$ such that

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(1) $$|B_r(x)| \sim r^d, \quad (r \to 0), \quad x \in G,$$

where by $f(t) \sim h(t)$, as $t \to t_0$ we mean that there is a constant $c > 0$ such that $c^{-1}h(t) \le f(t) \le ch(t)$ as $t \to t_0$.

We also know that either there is an integer $D \ge 0$ such that

(2) $$|B_r(x)| \sim r^D, \qquad (r \to \infty)$$

or there is $c > 0$ such that

$$|B_r(x)| \ge ce^{cr}, \qquad (r \to \infty).$$

In the first case we say that G has polynomial volume growth and in the second exponential volume growth.

In this article we shall assume that G has polynomial volume growth. Connected nilpotent Lie groups are examples of such groups. Lie groups of polynomial volume growth are unimodular.

We call d and D the local dimension and the dimension at infinity, respectively. The local dimension d depends on the choice of vector fields. The dimension at infinity D is independent of the choice of vector fields; it is a group invariant.

Notice that both of the situations $d \le D$ and $d > D$ are equally probable. For example when G is a simply connected nilpotent Lie group, then $d \le D$, and when G is compact, $D = 0$. Furthermore, if we start with a group G for which we have $d \le D$ we can always consider the group $G' = \boldsymbol{T}^{D-d+1} \times G$ (where $\boldsymbol{T} = \boldsymbol{R}/\boldsymbol{Z}$) which will have local dimension $d' = d + (D - d + 1) = D + 1$, dimension at infinity $D' = D$ and then of course $d' > D'$.

We denote by L the sub-Laplacian

$$L = -(X_1^2 + \cdots + X_k^2).$$

(b) Riemannian manifolds of nonnegative curvature. We consider a complete Riemannian manifold G of dimension n and we denote by $d(\cdot, \cdot)$ the Riemannian distance on G and by $B_r(x) = \{y \in M : d(x, y) < r\}$ the geodesic ball of radius $r > 0$ centered at $x \in M$. We also denote by $|B_r(x)|$ the volume of $B_r(x)$.

We assume that G has nonnegative Ricci curvature. This assumption implies, by the Bishop comparison theorem (cf. [5]), that there is a constant $c > 0$ such that

(3) $$|B_r(x)| \le cr^n, \quad r > 0, \quad \frac{|B_r(x)|}{|B_t(x)|} \le \left(\frac{r}{t}\right)^n, \quad r \ge t > 0.$$

We put $d = D = n$ and denote by L the Laplace-Beltrami operator on M.

In both of the above cases the operator L admits a selfadjoint extension on $L^2(G)$ which we also denote by L and hence a spectral resolution denoted by

$$L = \int_0^\infty \lambda dE_\lambda.$$

If $m(\lambda)$ is a bounded measurable function, then by using the spectral theorem we can define the operator

$$m(L)=\int_0^\infty m(\lambda)dE_\lambda ,$$

which, of course, will be bounded on $L^2(G)$.

We shall denote by $\check{m}(x, y)$ the Schwartz kernel of the operator $m(L)$. Observe that when G is a Lie group, the assumption that L is left invariant implies that $\check{m}(x, y)$ is also left invariant, i.e. $\check{m}(x, y)=\check{m}(zx, zy)$, $z\in G$.

The main result of this article is the following:

THEOREM 1. *Let G be either a connected Lie group of polynomial volume growth or a Riemannian manifold of nonnegative curvature. Let also d, D be as above.*

(a) *If $0<\alpha\leq 1$, then $m_{\alpha,\beta}(L)$ is bounded on L^p for $\beta>\alpha d\,|1/p-1/2|$, $1\leq p\leq\infty$.*

(b) *If $\alpha>1$ then $m_{\alpha,\beta}(L)$ is bounded on L^p for $\beta>\alpha\,|1/p-1/2|\max(d, D)$, $1\leq p\leq\infty$.*

Note that when $0<\alpha<1$ then it is only the local dimension that is taken into account. The reason for this is that, as it is the case in $\boldsymbol{R}^n$ and as we shall see in the course of the proof of the above theorem, when $0<\alpha<1$, the kernel $\check{m}_{\alpha,\beta}(x, y)$ is singular only near the diagonal.

When $\alpha=1$ (this case corresponds to the wave operator) then, according to what happens in $\boldsymbol{R}^n$, the critical index in the part (a) of Theorem 1 should have been $(d-1)\,|1/2-1/p|$ and not $d\,|1/2-1/p|$.

Applications to the Schrödinger equation. Let $f\in C_0^\infty(G)$ and denote by $u(t, x)$ the solution to the Schrödinger equation

$$\frac{\partial u}{\partial t}=iLu\,, \quad u(0, x)=f(x)\,.$$

Then we have

$$u(t, x)=e^{itL}f(x)\,.$$

We denote by $W^{p,s}$ the Sobolev space

$$W^{p,s}=\{f\colon \|(1+L)^{s/2}f\|_p<\infty\}\,.$$

We have the following theorem which generalises similar results of Brenner [4] and Ishii [14].

THEOREM 2. *Let $u(t, x)$ and $W^{p,s}$ be as above. Then for all $\varepsilon>0$ there is a constant $c_\varepsilon>0$ such that*

$$\|u(t,\cdot)\|_p\leq c_\varepsilon(1+|t|)^{\max(d,D)\,|1/p-1/2|+\varepsilon}\,\|f\|_{W^{p,\beta}}\,,\quad \beta>2\max(d, D)\left|\frac{1}{p}-\frac{1}{2}\right|,\quad p\geq 1\,.$$

The operator e^{itL} is bounded on L^p only for $p=2$. A possible substitute for this operator on L^p is its Riesz means

$$I_k(L)=kt^{-k}\int_0^t (t-s)^{k-1}e^{isL}ds\,, \quad k>0\,.$$

Of course we can also consider the more general operators

$$I_{k,\alpha}(L)=kt^{-k}\int_0^t (t-s)^{k-1}e^{isL^{\alpha/2}}ds\,, \quad k,\alpha>0\,.$$

These operators have been studied in the case of $\boldsymbol{R}^n$ by Miyachi [22] and Sjöstrand [25]. Their results have been recently generalised to Lie groups and Riemannian manifolds by Lohoué [17]. The following theorem improves some of the results of Lohoué [17].

THEOREM 3. (a) *If* $0<\alpha\leq 1$, *then* $I_{k,\alpha}(L)$ *is bounded on* L^p *for* $k>d|1/p-1/2|$, $1\leq p\leq\infty$.

(b) *If* $\alpha>1$ *then* $I_{k,\alpha}(L)$ *is bounded on* L^p *for* $k>|1/p-1/2|\max(d,D)$, $1\leq p\leq\infty$.

The basic ideas of the proofs. The proof of the above results is based on an idea, which is due to M. Taylor (see for example [6]) and which is the use of the finite propagation speed of the wave operator in order to obtain estimates for the kernel $\check{m}(x,y)$ of the operator $m(L)$ away from the diagonal.

More precisely, let $G_t(x,y)$ denote the kernel of the operator $\cos t\sqrt{L}$. Then $G_t(x,y)$ is also the fundamental solution of the wave equation

$$\left(\frac{\partial^2}{\partial t^2}+L\right)u(t,x)=0\,, \quad u(0,x)=f(x)\,, \quad \left(\frac{\partial}{\partial t}u\right)(0,x)=0$$

and therefore it has the property

(4) $$\operatorname{supp}(G_t)\subseteq\{(x,y)\colon d(x,y)\leq|t|\}\,.$$

In the case of subelliptic operators, this result was proved in [20].

The idea of M. Taylor is, roughly speaking, to write $m(L)=f(\sqrt{L})$ with f an even function. Then we have the formula

(5) $$f(\sqrt{L})=(2\pi)^{-1/2}\int_{-\infty}^{\infty}\hat{f}(t)\cos t\sqrt{L}\,dt\,,$$

which combined with (4) gives the formula

(6) $$\check{m}(x,y)=(2\pi)^{-1/2}\int_{|t|\geq d(x,y)}\hat{f}(t)G_t(x,y)dt\,,$$

which can be used to get estimates of the kernel $\check{m}(x,y)$ away from the diagonal.

Apart from (4) we shall also use the following estimate for the associated heat kernel $p_t(x, y)$ ($p_t(x, y)$ is the fundamental solution of the heat equation $(\partial/\partial t + L)u = 0$ as well as the kernel of the semigroup e^{-tL}, $t>0$):

$$p_t(x, y) \leq \frac{c}{|B_{\sqrt{t}}(x)|} \exp\left(-\frac{d(x, y)^2}{ct}\right), \qquad t>0 . \tag{7}$$

This estimate is proved in [27] when G is a Lie group of polynomial volume growth and in [8], [16] when G is a Riemannian manifold of nonnegative curvature.

Proof of Theorem 1. We start with some preliminary considerations. We state first the following:

Lemma 4 (cf. [13, pp. 237–238], [24, p. 88]). *Assume that the function $f(x) \in C(\boldsymbol{R})$ has compact support and that it possesses n continuous derivativs $f'(x), f''(x), \ldots, f^{(n)}(x)$. Let also $A = n + \varepsilon$ with $\varepsilon \in (0, 1]$ and set*

$$M_A(f) = \sup\left\{\frac{|f^{(n)}(x+t) - f^{(n)}(x)|}{t^{\varepsilon}}, t>0, x \in \boldsymbol{R}\right\}.$$

Then for every $\lambda > 0$ there is an even bounded integrable function $\psi_\lambda(x) \in C(\boldsymbol{R})$ such that for all $x \in \boldsymbol{R}$

$$\operatorname{supp}(\hat{\psi}) \subseteq [-\lambda, \lambda] \quad \text{and} \quad |f(x) - f * \psi(x)| \leq c M_A(f) \lambda^{-A} . \tag{8}$$

(c is a constant that depends only on n.)

Following the standard procedure we consider a function $\phi \in C_0^\infty(\boldsymbol{R}^+)$ such that

$$\operatorname{supp}(\phi) \subseteq \left(\frac{1}{2}, 2\right), \quad \sum_{j \in \boldsymbol{Z}} \phi(2^j t) = 1 , \quad t>0 .$$

If $j \geq 0$, $j \in \boldsymbol{Z}$, then we set

$$m_j(\lambda) = m_{\alpha,\beta}(\lambda)\phi(2^{-j}\lambda) , \quad f_j(\lambda) = m_j(\lambda)\exp(2^{-j}\lambda) \quad \text{and} \quad h_j(\lambda) = f_j(\lambda^2) .$$

Observe that if $A > 0$ and $M_A(h_j)$ is as in Lemma 4 above, then there is $c > 0$ such that

$$M_A(h_j) \leq c 2^{-[\beta - (\alpha - 1)A]j/2} . \tag{9}$$

Also there is $c > 0$ such that

$$\| h_j \|_\infty \leq c 2^{-\beta j/2} . \tag{10}$$

Let $\check{m}_j(x, y)$ be the Schwartz kernel of the operator $m_j(L)$. Since

$$m_j(L) = f_j(L) e^{-2^{-j}L} , \quad f_j(L) = h_j(L^{1/2})$$

and since the operators $h_j(\sqrt{L})$ and $e^{-2^{-j}L}$ commute, we have

$$\check{m}_j(x, y) = h_j(\sqrt{L}) p_{2^{-j}}(x, y) = \bar{h}_j(\sqrt{L}) p_{2^{-j}}(x, y) \tag{11}$$

with the operators $h_j(\sqrt{L})$ and $\bar{h}_j(\sqrt{L})$ acting on the variables x and y, respectively.

We set $A_p(x)=\{y: 2^{p/2}\leq d(x,y)<2^{(p+1)/2}\}$, $p\in \boldsymbol{Z}$.

LEMMA 5. *For all $A>0$, there is a constant $c>0$ independent of y such that*

(i) $\|\check{m}_j(x,\cdot)\|_{L^1(B_{2^{-j/2}}(x))}\leq c\|h_j\|_\infty$.

(ii) $\|\check{m}_j(x,\cdot)\|_{L^1(A_p(x))}\leq c2^{-[\beta-(\alpha-1)A]j/2}2^{jd/4}2^{(d/4-A/2)p}$, $-j\leq p<0$.

(iii) $\|\check{m}_j(x,\cdot)\|_{L^1(A_p(x))}\leq c2^{-[\beta-(\alpha-1)A]j/2}2^{jd/4}2^{(D/4-A/2)p}$, $p\geq 0$.

Furthermore, the above estimates remain true, if we replace $\check{m}_j(x,\cdot)$ by $\check{m}_j(\cdot,x)$.

PROOF. The last assertion of the lemma will follow from (11) and the way the estimates (i), (ii) and (iii) will be proved.

Let us prove (i). It follows from (7) that

$$\|p_{2^{-j}}(x,\cdot)\|_2\leq c|B_{2^{-(j-1)/2}}(x)|^{-1/2}. \tag{12}$$

We also have

$$\|h_j(\sqrt{L})\|_{2\to 2}\leq\|h_j\|_\infty. \tag{13}$$

Hence, it follows from (11) that

$$\begin{aligned}\|\check{m}_j(x,\cdot)\|_{L^1(B_{2^{-j/2}}(x))}&\leq|B_{2^{-j/2}}(x)|^{1/2}\|\check{m}_j(x,\cdot)\|_2\\&\leq|B_{2^{-j/2}}(x)|^{1/2}\|h_j(\sqrt{L})\|_{2\to 2}\|p_{2^{-j}}(x,\cdot)\|_2\\&\leq c(|B_{2^{-j/2}}(x)|/|B_{2^{-(j-1)/2}}(x)|)^{1/2}\|h_j\|_\infty\end{aligned}$$

and from this, by using either (1) and (2) or (3) we get (i).

To prove (ii) and (iii) we observe that if $z\in A_p(x)$, $p\geq -j$, then it follows from (5) and (6) that

$$\begin{aligned}\check{m}_j(x,y)&=[h_j(\sqrt{L})p_{2^{-j}}(x,\cdot)](z)\\&=(2\pi)^{-1/2}\int_{-\infty}^{+\infty}\hat{h}_j(t)\cos t\sqrt{L}\,p_{2^{-j}}(x,\cdot)](z)dt\\&=(2\pi)^{-1/2}\int_{-\infty}^{+\infty}\hat{h}_j(t)\{\cos t\sqrt{L}\,[p_{2^{-j}}(x,\cdot)\mathbf{1}_{\{y:d(x,y)\leq 2^{p/2-1}\}}\\&\quad+p_{2^{-j}}(x,\cdot)\mathbf{1}_{\{y:d(x,y)>2^{p/2-1}\}}]\}(z)dt\\&=(2\pi)^{-1/2}\int_{|t|\geq 2^{p/2-1}}\hat{h}_j(t)\{\cos t\sqrt{L}\,[p_{2^{-j}}(x,\cdot)\mathbf{1}_{\{y:d(x,y)\leq 2^{p/2-1}\}}]\}(z)dt\\&\quad+(2\pi)^{-1/2}\int_{-\infty}^{+\infty}\hat{h}_j(t)\{\cos t\sqrt{L}\,[p_{2^{-j}}(x,y)\mathbf{1}_{\{y:d(x,y)>2^{p/2-1}\}}]\}(z)dt\\&=(2\pi)^{-1/2}\int_{-\infty}^{+\infty}[\hat{h}_j(t)-\hat{h}_j(t)\hat{\psi}_{j,p}(t)]\{\cos t\sqrt{L}\,[p_{2^{-j}}(x,\cdot)\mathbf{1}_{\{y:d(x,y)\leq 2^{p/2-1}\}}]\}(z)dt\end{aligned}$$

$$+(2\pi)^{-1/2}\int_{-\infty}^{+\infty}\hat{h}_j(t)\{\cos t\sqrt{L}\,[p_{2^{-j}}(x,\cdot)\mathbf{1}_{\{y:d(x,y)>2^{p/2-1}\}}]\}(z)dt\,,$$

where $\psi_{j,p}$ is an even function as in Lemma 7 satisfying

$$\operatorname{supp}(\hat{\psi}_{j,p})\subseteq[-2^{p/2-2},2^{p/2-2}]\quad\text{and}\quad|h_j(\lambda)-h_j*\psi_{j,p}(\lambda)|\leq cM_A(h_j)2^{-Ap/2}\,.$$

Hence

$$\check{m}_j(x,z)=\{(h_j-h_j*\psi_{j,p})(\sqrt{L})[p_{2^{-j}}(x,\cdot)\mathbf{1}_{\{y:d(x,y)\leq 2^{p/2-1}\}}]\}(z)$$
$$+\{h_j(\sqrt{L})[p_{2^{-j}}(x,\cdot)\mathbf{1}_{\{y:d(x,y)>2^{p/2-1}\}}]\}(z)$$

and from this we have (always for $p\geq -j$)

$$\begin{aligned}\|\check{m}_j(x,\cdot)\|_{L^1(A_p(x))}&\leq|A_p(x)|^{1/2}\|h_j-h_j*\psi_{j,p}\|_\infty\|p_{2^{-j}}(x,\cdot)\|_2\\&\quad+|A_p(x)|^{1/2}\|h_j\|_\infty\|p_{2^{-j}}(x,\cdot)\mathbf{1}_{\{y:d(x,y)>2^{p/2-1}\}}\|_2\\&\leq(|B_{2^{(p+1)/2}}(x)|/|B_{2^{(-j+1)/2}}(x)|)^{1/2}M_A(h_j)2^{-Ap/2}\\&\quad+|B_{2^{(p+1)/2}}(x)|^{1/2}\|h_j\|_\infty\|p_{2^{-j}}(x,\cdot)\mathbf{1}_{\{y:d(x,y)>2^{p/2-1}\}}\|_\infty^{1/2}\,.\end{aligned}$$

Now (ii) and (iii) follow from (1), (2), (3), (7) and (11) and the observation that there are constants $c, C, C'>0$ such that

$$\begin{aligned}&|B_{2^{(p+1)/2}}(x)|^{1/2}\|h_j\|_\infty\|p_{2^{-j}}(x,\cdot)\mathbf{1}_{\{y:d(x,y)>2^{p/2-1}\}}\|_\infty^{1/2}\\&\quad\leq c'(|B_{2^{(p+1)/2}}(x)|/|B_{2^{(-j+1)/2}}(x)|)^{1/2}2^{-j\beta/2}e^{-C'2^{j+p}}\leq c2^{-j\beta/2}e^{-C2^{j+p}}\,.\end{aligned}$$

PROOF OF THEOREM 1. We observe that it follows from (10) and Lemma 5, (i) that

$$\|\check{m}_j(x,\cdot)\|_{L^1(B_{2^{-j/2}}(x))}\leq c2^{-\beta j/2}\,. \tag{14}$$

It follows from (9) and Lemma 5, (ii) that if $-j\leq p<0$ then

$$\begin{aligned}\|\check{m}_j(x,\cdot)\|_{L^1(A_p(x))}&\leq c2^{-[\beta+(1-\alpha)A]j/2}2^{jd/4}2^{(d/4-A/2)p}\\&=c2^{-[\beta+(1-\alpha)A-d/2]j/2}2^{(d/4-A/2)p}\end{aligned}$$

and therefore if we chose $A=d/2-\varepsilon$, $\varepsilon>0$ then

$$\begin{aligned}\sum_{-j\leq p\leq 0}\|\check{m}_j(x,\cdot)\|_{L^1(A_p(x))}&\leq c2^{-[\beta+(1-\alpha)A-d/2]j/2}\\&=c2^{-[\beta+(1-\alpha)(d/2-\varepsilon)-d/2]j/2}\\&=c2^{-[\beta-\alpha d/2]j/2}2^{\varepsilon(1-\alpha)j/2}\,.\end{aligned}\tag{15}$$

Finally it follows from (9) and Lemma 5, (iii) that if $p\geq 0$, then

$$\begin{aligned}\|\check{m}_j(x,\cdot)\|_{L^1(A_p(x))}&\leq c2^{-(\beta+(1-\alpha)A)j/2}2^{jd/4}2^{(D/4-A/2)p}\\&=c2^{-(\beta+(1-\alpha)A-d/2)j/2}2^{(D/4-A/2)p}\end{aligned}$$

and therefore if we chose $A=D/2+\varepsilon$, $\varepsilon>0$, then

$$\sum_{p\geq 0}\|\check{m}_j(x,\cdot)\|_{L^1(A_p(x))}\leq c2^{-(\beta+(1-\alpha)A-d/2)j/2}$$
$$=c2^{-[\beta+(1-\alpha)(D/2+\varepsilon)-d/2]j/2}$$
$$=c2^{-[\beta+(1-\alpha)D/2-d/2]j/2}2^{-\varepsilon(1-\alpha)j/2}$$

and from this, that for all $\varepsilon>0$

$$(16)\qquad \sum_{p\geq 0}\|\check{m}_j(x,\cdot)\|_{L^1(A_p(x))}\leq\begin{cases} c2^{-\varepsilon j/2}, & 0<\alpha<1\\ c2^{-[\beta-d/2]j/2}, & \alpha=1\\ 2^{-[\beta-\alpha\max(d,D)/2]j/2}2^{\varepsilon j/2}, & \alpha>1\,.\end{cases}$$

In the case $0<\alpha\leq 1$, $\beta>d\alpha/2$ and the case $\alpha>1$, $\beta>(\alpha/2)\max(d,D)$, Theorem 1 follows from the fact that (14), (15) and (16) imply that the kernel $\check{m}_{\alpha,\beta}(x,y)$ of the operator $m_{\alpha,\beta}(L)$ is integrable:

$$\sup_{x\in G}\|\check{m}_{\alpha,\beta}(x,\cdot)\|_1\leq\sup_{x\in G}\sum_{j\geq 0}\|\check{m}_j(x,\cdot)\|_1<\infty\,.$$

The rest of the cases of Theorem 1 follows by interpolation:
Let $0<t<1$, $1/p=t/1+(1-t)/2$, i.e. $t=2/p-1$. Then, by interpolation, we have

$$\|m_j(L)\|_{p\to p}\leq\|m_j(L)\|_{1\to 1}^{t}\,\|m_j(L)\|_{2\to 2}^{1-t}\leq\left(\sup_{x\in G}\|\check{m}_j(x,\cdot)\|_1\right)^{t}\|m_j\|_{\infty}^{1-t}\,.$$

Hence it follows from (14), (15) and (16) and the fact that

$$\|m_j\|_{\infty}\leq 2^{-\beta j/2}$$

that there is $c>0$ such that

$$\|m_j(L)\|_{p\to p}\leq\begin{cases} c2^{-[\beta-\alpha d(1/p-1/2)]j/2}2^{\varepsilon j/2}, & 0<\alpha<1\\ c2^{-[\beta-\alpha d(1/p-1/2)]j/2}, & \alpha=1\\ c2^{-[\beta-\alpha(1/p-1/2)\max(d,D)]j/2}2^{\varepsilon j/2}, & \alpha>1\,.\end{cases}$$

Theorem 1 follows from the above estimates and the fact that

$$\|m_{\alpha,\beta}(L)\|_{p\to p}\leq\sum_{j\geq 0}\|m_j(L)\|_{p\to p}\,.$$

PROOF OF THEOREM 2. Let us consider the multiplier

$$m(\lambda)=(1+|\lambda|)^{-\beta/2}e^{it|\lambda|}\,.$$

The proof of Theorem 2 is reduced to proving that $m(L)$ is bounded on L^p for $\beta>2\max(d,D)|1/p-1/2|$, $p\geq 1$ with operator norm

$$\| m(L) \|_{p\to p} \le c_\varepsilon(1+|t|)^{\max(d,D)|1/p-1/2|+\varepsilon} .$$

To do this we consider a C^∞ function $\psi(\lambda)$, which is 0 for $|\lambda|\le 2$ and 1 for $|\lambda|\ge 3$ and we put

$$m_0(\lambda)=(1-\psi(|\lambda|))(1+|\lambda|)^{-\beta/2}e^{it|\lambda|} .$$

We also consider a function ϕ as in the proof of Theorem 1 and for $j\in N$ we set

$$m_j(L)=\phi(2^{-j}L)\psi(L)(1+L)^{-\beta/2}e^{itL} .$$

As in the proof of Theorem 1 we set

$$f_j(\lambda)=m_j(\lambda)\exp(2^{-j}\lambda) , \qquad h_j(\lambda)=f_j(\lambda^2) .$$

If $M_A(h_j)$ is defined as in Lemma 4, then we have

(17) $$M_A(h_j)\le c(1+|t|)^A 2^{-(\beta-A)j/2} .$$

From here on the proof of Theorem 2 is exactly the same as the proof of Theorem 1. The only difference is that instead of using the estimate (9), we use the estimate (17) above.

Proof of Theorem 3. We have

$$kt^{-k}\int_0^t (t-s)^{k-1}e^{is|\lambda|^{\alpha/2}}ds=k\int_0^1 (1-s)^{k-1}e^{is(t^{2/\alpha}|\lambda|)^{\alpha/2}}ds , \qquad k,\alpha>0 .$$

So, by replacing, if necessary, the operator L by the operator $t^{2/\alpha}L$, we may assume that $t=1$. Let

$$m(\lambda)=k\int_0^1 (1-s)^{k-1}e^{is\lambda}ds .$$

Then

$$I_{k,\alpha}(L)=m(L^{\alpha/2}) .$$

As has been shown in [25], [29]

$$m(\lambda)=C_k\psi_1(\lambda)\lambda^{-k}e^{i\lambda}+\tilde m(\lambda) ,$$

where $\tilde m(\lambda)$ is a smooth function such that

$$\frac{d^n}{d\lambda^n}\tilde m(\lambda)=O(\lambda^{-n-1}) , \qquad (\lambda\to\infty) ,$$

$\psi_1(\lambda)$ is a C^∞ function, which is 0 for $|\lambda|\le 1$ and 1 for $|\lambda|\ge 2$ and

$$C_k=k\Gamma(k)e^{-\pi ik/2} .$$

If we put $f(\lambda)=\tilde m(|\lambda|^{\alpha/2})$, then the operator $f(L)$, hence also the operator $\tilde m(L^{\alpha/2})$, is bounded on L^p, $1 \leq p \leq \infty$. This is proved by considering functions $\psi(\lambda)$ and $\phi(\lambda)$ as in the proof of Theorem 2, setting

$$m_0(L) = (1-\psi)(L)f(L), \quad m_j(L) = \phi(2^{-j}L)\psi(L)f(L), \quad j \in \boldsymbol{N}$$

and then working as in the proof of Theorem 1. We omit the details.

Let us put now $\psi(\lambda) = \psi_1(|\lambda|^{\alpha/2})$ and

$$h(\lambda) = \psi(\lambda)\,|\lambda|^{-k\alpha/2} e^{i|\lambda|^{\alpha/2}}.$$

Then to prove Theorem 3 it is enough to prove that the operator $h(L)$ is bounded on L^p, for those $p \geq 1$ that satisfy $k > d|1/p - 1/2|$ when $0 < \alpha \leq 1$ and $k > |1/p - 1/2|\max(d, D)$, when $\alpha > 1$, which of course is a consequence of Theorem 1.

FINAL REMARKS. Let us put ourselves in the context of Theorem 1, (a). Then as we can see from the proof of that theorem, the kernel $\check{m}_{\alpha,\beta}(x, y)$ is integrable away from the diagonal and it is singular only near the diagonal.

So, it is natural to ask whether, at least when G is a Riemannian manifold (the case where the situation should be more manageable), the operator $m_{\alpha,\beta}(L)$ for $0 < \alpha < 1$ is bounded on H^1 when $\beta = \alpha D/2$ and on L^p when $\beta = \alpha D|(1/p) - (1/2)|$, $1 < p < \infty$.

In the context of $\boldsymbol{R}^n$, this end point result has been proved in [10]. The basic ingredients of that proof are very good L^∞ estimates for the kernel $\check{m}_{\alpha,\beta}(x, y)$ and its gradient $\nabla\check{m}_{\alpha,\beta}(x, y)$, the Hardy-Littlewood-Sobolev theorem and the appropriate estimates on the norm of the operator $L^{i\gamma}$, $\gamma \in \boldsymbol{R}$, on H^1.

Very good L^∞ estimates for $\check{m}_{\alpha,\beta}(x, y)$ and $\nabla\check{m}_{\alpha,\beta}(x, y)$ can be obtained as follows:

First we use the formulas (4) and (5) to express $\nabla\tilde{m}_{\alpha,\beta}(x, y)$ in terms of the kernel $G_t(x, y)$ of the operator $\cos t\sqrt{L}$ for $t < a$. Next we use the Hadamard parametrix construction (cf. [2]) to obtain an asymptotic expansion for the kernel $G_t(x, y)$. The desired estimates will follow by a calculation similar to the one that was carried out for example in [2, pp. 6–7] and [12, pp. 5–6]. The appropriate estimates for the Fourier transform $\hat{f}(t)$ of the function $f(t) = m_{\alpha,\beta}(t^2)$ have been proved in Theorem 9 of [29]. The asymptotic expansion for $G_t(x, y)$ and hence the estimates for $\check{m}_{\alpha,\beta}(x, y)$ will be "uniformly good" if we assume for example that G has bounded C^∞ geometry (this condition could be weakened).

In order to have the Hardy-Littlewood-Sobolev theorem available, as is shown in [15], we need to make the additional assumption that there is a constant c_D independent of $x \in G$ such that $|B_r(x)| \geq cr^D$, $r > 0$.

Finally, the desired estimate for the norm of the operator $L^{i\gamma}$, $\gamma \in \boldsymbol{R}$, on H^1 follows from the main result of [1] which is stated for Lie groups with polynomial volume growth but is also valid for Riemannian manifolds of nonnegative Ricci curvature, since the same proof also works in that context.

Once G satisfies these additional conditions, then the proof of the above mentioned end point result in the context of $\boldsymbol{R}^n$ in [10] can also be made to work on G. We believe that with these indications the interested readers will be able to supply a proof for themselves.

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Université de Paris-Sud
Mathématiques, Bât. 425
91405 Orsay Cedex
France