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OSCILLATION AND ASYMPTOTIC BEHAVIOR OF SYSTEMS OF ORDINARY LINEAR DIFFERENTIAL EQUATIONS

BY

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ABSTRACT. Conditions are established for oscillatory and asymptotic behavior for first-order matrix systems of ordinary differential equations, including Hamiltonian systems in the selfadjoint case. Asymptotic results of Hille, Shreve, and Hartman are generalized. Disconjugacy criteria of Ahlbrandt, Tomastik, and Reid are extended.

1. Introduction. The purpose of this paper is to establish conditions for oscillatory and asymptotic behavior for systems of linear homogeneous ordinary differential equations. A detailed definition of the systems involved is given in §2, and for background and motivation the reader may refer, for example, to [7], [11], or [15]. In §3 extensions are obtained of certain theorems of Hille [8], Shreve [14], and Hartman [7] on the asymptotic behavior of solutions, and partial converses of these are obtained in §4 for the selfadjoint case. Certain nonoscillation results of Reid ([9], [10]) are extended to an arbitrary non-selfadjoint system in §2. In §4 several nonoscillation results for selfadjoint systems are obtained, including extensions of results of Tomastik [16] and Ahlbrandt [1], [2], by means of an associated Riccati equation.

Matrix notation is used throughout. Matrices of one column are called vectors; any square identity matrix is denoted by I; and the zero matrix of any dimension is denoted by 0. The hermitian conjugate (complex conjugate of the transpose) of a matrix H is denoted by H^*, and H is called hermitian whenever $H^* = H$. If H and K are $n \times n$ hermitian matrices, then $H > K$ [$H \geq K$], indicates that $H - K$ is positive definite [positive semidefinite]. The symbol J is used throughout to denote a fixed subinterval (a_0, ∞), $a_0 \geq -\infty$, of the real line. A hermitian matrix $H = H(t)$ will be called nondecreasing [increasing] on a nondegenerate subinterval $J_0 \subseteq J$ if for $t_1, t_2 \in J_0$, $t_1 < t_2$ implies that $H(t_1) \leq H(t_2)$ [$H(t_1) < H(t_2)$]. A matrix has the properties of boundedness, continuity, differentiability, or integrability on a subinterval J_0

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if and only if each of its entries has the property on J_0. The symbols $\mathfrak{L}(J_0)$ and $\mathfrak{A}(J_0)$ will denote the classes of matrices which on arbitrary compact subintervals of J_0 are Lebesgue integrable and absolutely continuous, respectively. If a is an accumulation point of J_0, then we say that a matrix $H(t)$ on J_0 has a limit K at a provided each entry of $H(t)$ has the corresponding entry of K as a limit at a. The integral $\int_b^\infty H(s)ds$ is said to exist whenever each entry of $\int_b^t H(s)ds$ has a finite limit at ∞.

The eigenvalues of a hermitian $n \times n$ matrix $H(t) \in \mathfrak{L}(J_0)$ are real, and will be denoted by $\lambda_\nu[H(t)]$, $\nu = 1, 2, 3, \ldots, n$. It will always be understood that

$$\lambda_1[H(t)] \leqslant \lambda_2[H(t)] \leqslant \cdots \leqslant \lambda_n[H(t)]$$

for each t in J_0. Each $\lambda_\nu[H(t)]$ is in $\mathfrak{L}(J_0)$ whenever $H(t)$ is. If $H(t)$ is continuous on J_0, then so are the $\lambda_\nu[H(t)]$, $1 \leqslant \nu \leqslant n$. The matrix norm used throughout is $\|M\| = (\lambda_n[M^*M])^{1/2}$, where M is an arbitrary $m \times n$ complex matrix. The trace of a square matrix H, denoted by trace$[H]$, or simply tr$[H]$, is the sum of the main diagonal entries of H, and also equals the sum of the eigenvalues of H.

2. Formulation and reduction transformations. In this section we lay the groundwork and establish preliminaries for the later sections. An effort is made to present the problem in a quite general setting, and the main result of this section contained in Theorem 2.1 is established without variational techniques.

Consider the general matrix differential system

$$X' = A(t)X + B(t)Y, \qquad Y' = C(t)X + D(t)Y \tag{2.1}$$

on $J = (a_0, \infty)$, where $A(t)$, $B(t)$, $C(t)$, and $D(t)$ are complex matrices in $\mathfrak{L}(J)$ with sizes $r \times r$, $r \times n$, $n \times r$, and $n \times n$, respectively, and $0 < r \leqslant n$. If $X_0(t)$ and $Y_0(t)$ are $r \times k$ and $n \times k$ matrices, $k \geqslant 1$, then $\left[\begin{smallmatrix} X_0 \\ Y_0 \end{smallmatrix}\right]$, or alternatively (X_0, Y_0), will denote a solution of (2.1) provided $X_0(t)$ and $Y_0(t)$ are in $\mathfrak{A}(J)$ and satisfy (2.1) a.e. (almost everywhere) on J.

The formal adjoint of (2.1) is given by

$$U' = -A^*(t)U - C^*(t)V, \qquad V' = -B^*(t)U - D^*(t)V. \tag{2.2}$$

If (X_0, Y_0) and (U_0, V_0) are solutions of (2.1) and (2.2), respectively, one can readily see that $X_0^* U_0 + Y_0^* V_0$ is identically constant on J. Such a solution pair will be called a *conjugate pair* for (2.1) and its adjoint if this constant is zero. When $r = n$ and $D = -A^*$, $B = B^*$, and $C = C^*$, then (2.1) is a Hamiltonian system, and if (X_0, Y_0) is a solution of (2.1), then $(-Y_0, X_0)$ satisfies (2.2) and $X_0^* Y_0 - Y_0^* X_0$ is identically constant on J. In this case if the constant is zero, then the solution (X_0, Y_0) is called *self-conjugate*.

Let a and b be distinct points in J. Then b is called a *conjugate point of a* with respect to (2.1) if there exists an $(r + n) \times 1$ vector solution (x, y) of

(2.1) satisfying

$$\begin{bmatrix} I_n & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x(a) \\ y(a) \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ I_r & 0 \end{bmatrix} \begin{bmatrix} x(b) \\ y(b) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{2.3}$$

such that $x(t) \not\equiv 0$ on the interval between a and b. Here I_n and I_r denote the $n \times n$ and $r \times r$ identity matrices, respectively. Observe that when $r = n$, b is a conjugate point of a if and only if a is a conjugate point of b with respect to (2.1). In this case a and b are called *mutually conjugate* with respect to (2.1).

The point b is called an *adjoint conjugate point of a* with respect to (2.1) if there exists an $(r + n) \times 1$ vector solution (u, v) of (2.2) satisfying

$$\begin{bmatrix} 0 & I_r \\ 0 & 0 \end{bmatrix} \begin{bmatrix} u(a) \\ v(a) \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & I_n \end{bmatrix} \begin{bmatrix} u(b) \\ v(b) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{2.4}$$

such that $v(t) \not\equiv 0$ on the interval between a and b. As above, when $r = n$, b is an adjoint conjugate point of a if and only if a is an adjoint conjugate point of b, and the two are called *mutually adjoint conjugate* in this case. If, in addition, (2.1) is Hamiltonian, then a and b are mutually conjugate if and only if they are mutually adjoint conjugate. Standard results for two-point boundary value problems (see, for example, [7, Chapter 12]) imply that the boundary value problem consisting of (2.1) with (2.3) has the same number of linearly independent vector solutions as its adjoint boundary value problem given by (2.2) with (2.4).

Let X_0 and V_0 be fundamental matrices for

$$X' = AX \tag{2.5a}$$

and

$$V' = -D^* V, \tag{2.5b}$$

respectively, and transform (2.1) and (2.2) by

$$X = X_0 Z, \qquad Y = V_0^{*-1} W \tag{2.6a}$$

and

$$U = X_0^{*-1} S, \qquad V = V_0 R, \tag{2.6b}$$

respectively. The resulting reduced system is

$$Z' = X_0^{-1} B V_0^{*-1} W, \qquad W' = V_0^* C X_0 Z \tag{2.7}$$

and its adjoint

$$S' = -X_0^* C^* V_0 R, \qquad R' = -V_0^{-1} B^* X_0^{*-1} S. \tag{2.8}$$

If a and b are distinct points in J and if either $r = n$ or X_0 and V_0 are normalized to be I_r and I_n, respectively, at a, then b is a conjugate point (or an adjoint conjugate point) of a with respect to (2.1) if and only if it is such with respect to (2.7). Furthermore, if (2.1) is Hamiltonian, then (2.5a) and (2.5b) are identical. Taking $V_0 = X_0$, the resulting reduced system (2.7) is also

Hamiltonian. This special case of the following more general transformation shows that (2.1) and its adjoint can always be transformed to reduced systems such that boundary value problems are transformed to equivalent boundary value problems.

In (2.1) and (2.2) let $C = C_0 + C_1$. Suppose that (X_0, Y_0) is an $(r + n) \times r$ solution of

$$X' = AX + BY, \qquad Y' = C_0X + DY \tag{2.9}$$

and that (U_0, V_0) is an $(r + n) \times n$ solution of its adjoint

$$U' = -A^*U - C_0^*V, \qquad V' = -B^*U - D^*V \tag{2.10}$$

such that (X_0, Y_0) and (U_0, V_0) form a conjugate pair for (2.9) and (2.10), with X_0 and V_0 nonsingular on some subinterval $J_0 \subseteq J$. Restricting t to J_0 and transforming (2.1) and (2.2) by

$$X = X_0Z, \qquad Y = Y_0Z + V_0^{*-1}W \tag{2.11}$$

and

$$U = U_0R + X_0^{*-1}S, \qquad V = V_0R \tag{2.12}$$

one obtains, as before, the reduced system

$$Z' = GW, \qquad W' = QZ \tag{2.13}$$

and its adjoint

$$S' = -Q^*R, \qquad R' = -G^*S, \tag{2.14}$$

respectively, where G and Q are given by

$$G = X_0^{-1}BV_0^{*-1} \tag{2.15}$$

and

$$Q = V_0^*C_1X_0. \tag{2.16}$$

As before, by appropriate normalization of X_0 and V_0 at $a \in J_0$ if necessary, the transformation leaves the conjugate points of a in J_0 unchanged. If (2.1) is Hamiltonian, C_0 and C_1 are hermitian, and (X_0, Y_0) is self-conjugate, then taking $(U_0, V_0) = (-Y_0, X_0)$ yields a reduced Hamiltonian system, and for each t matrices G and Q have the same number of positive eigenvalues as B and C_1, respectively. To see that the transformations (2.5)–(2.8) is a special case of the transformation (2.9)–(2.16), note that if $C_0 \equiv 0$ and $C_1 \equiv C$ then $(X_0, 0)$ and $(0, V_0)$ form a conjugate pair for (2.9) and (2.10), where X_0 and V_0 are fundamental matrices for (2.5a) and (2.5b). Equations (2.11)–(2.14) are the same as (2.6a)–(2.8) in this case.

It is to be noted that if (X, Y) and (U, V) form a conjugate pair for (2.1) and its adjoint, then the images (Z, W) and (S, R) under transformations (2.11) and (2.12) form a conjugate pair for the reduced system (2.13) and its adjoint (2.14). Also, Z and R are nonsingular in J_0 precisely where X and V are, respectively.

The reduction transformations (2.11), (2.12) require a conjugate pair (X_0, Y_0), (U_0, V_0) for (2.9) and (2.10) with X_0 and V_0 nonsingular on J_0. However, for this it is sufficient that a solution (X_0, Y_0) of (2.9) exist with X_0 nonsingular on J_0, and sufficient conditions for this are established in Theorem 2.1 below. Let (X_0, Y_0) be such a solution of (2.9), and let (X_1, Y_1) be another solution of (2.9) such that

$$\begin{bmatrix} X_0 & X_1 \\ Y_0 & Y_1 \end{bmatrix}$$

is a fundamental matrix of solutions for (2.9). Then

$$\begin{bmatrix} U_1 & U_0 \\ V_1 & V_0 \end{bmatrix} = \begin{bmatrix} X_0 & X_1 \\ Y_0 & Y_1 \end{bmatrix}^{*-1}$$

is a fundamental matrix of solutions of (2.10), where

$$U_0 = -X_0^{*-1}Y_0^* V_0, \qquad V_0 = \left(Y_1^* - X_1^* X_0^{*-1} Y_0^*\right)^{-1} \tag{2.17}$$

and

$$U_1 = X_0^{*-1}(I - Y_0^* V_1), \qquad V_1 = -V_0 X_1^* X_0^{*-1}. \tag{2.18}$$

Therefore, (U_0, V_0) is a solution of (2.10) such that V_0 is nonsingular precisely where X_0 is, and (X_0, Y_0) and (U_0, V_0) form a conjugate pair for (2.9) and (2.10).

The following theorem extends to the arbitrary system (2.1) certain results of Reid ([9], [**10**]).

Theorem 2.1. *Let $a \in J = (a_0, \infty)$ and suppose that a has no conjugate points or adjoint conjugate points in (a, ∞) with respect to* (2.1). *Then there exists an $(n + r) \times r$ solution (X, Y) of* (2.1) *such that X is nonsingular on some terminal interval $(a_1, \infty) \subseteq (a, \infty)$. If $b > a$ is a point at which X is singular, then for some constant unit vector η, $X(t)\eta \equiv 0$ on $[a, b]$. If* (2.1) *is Hamiltonian, (X, Y) may be taken to be self-conjugate.*

Proof. Transform (2.1) and its adjoint by (2.6) with X_0 and V_0 normalized to be the respective identities at a. Then a has no conjugate points or adjoint conjugate points in (a, ∞) with respect to the reduced system

$$\begin{aligned} Z' &= GW, \qquad G = X_0^{-1} B V_0^{*-1}, \\ W' &= QZ, \qquad Q = V_0^* C X_0. \end{aligned} \tag{2.19}$$

Suppose that the only vector solution of (2.19) satisfying

$$\begin{bmatrix} I_n & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} z(a) \\ w(a) \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ I_r & 0 \end{bmatrix} \begin{bmatrix} z(b) \\ w(b) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{2.20}$$

for all $b > a$ (having the form $(0, w)$ on $[a, \infty)$) is the zero solution $(0, 0)$. Let (Z, W) be the $(n + r) \times r$ solution of (2.19) on $[a, \infty)$ with

$$Z(a) = 0 \quad \text{and} \quad W(a) = \begin{bmatrix} 0 \\ I_r \end{bmatrix}$$

and suppose that $\{t_\nu\}$ is an increasing sequence of points in (a, ∞) such that $t_\nu \to \infty$ and $Z(t_\nu)$ is singular. Then there is a sequence $\{\eta_\nu\}$ of constant unit vectors such that $Z(t_\nu)\eta_\nu = 0$. For each ν,

$$(z_\nu, w_\nu) = (Z(t)\eta_\nu, W(t)\eta_\nu)$$

is a vector solution of (2.19) satisfying (2.20) for $b = t_\nu$, in which case $z_\nu(t) = Z(t)\eta_\nu \equiv 0$ on $[a, t_\nu]$ since a has no conjugate points in (a, ∞) with respect to (2.19). Without loss of generality we may assume that $\{\eta_\nu\}$ converges. Then $\eta_\nu \to \eta$, where η is a constant unit vector. For any fixed t in (a, ∞)

$$Z(t)\eta = Z(t)\eta_\nu + Z(t)(\eta - \eta_\nu), \tag{2.21}$$

and as $\nu \to \infty$ the right-hand side of (2.21) tends to zero. Since $t \in (a, \infty)$ was arbitrary, $Z(t)\eta \equiv 0$ on (a, ∞). The vector solution $(Z(t)\eta, W(t)\eta) = (0, w)$ of (2.19) satisfies (2.20) for all $b > a$, whence $w \equiv 0$ on (a, ∞). In particular,

$$w(a) = \begin{bmatrix} 0 \\ \eta \end{bmatrix} = 0,$$

which contradicts the fact that η is a nonzero vector. Therefore, $Z(t)$ is nonsingular on some terminal interval $(a_1, \infty) \subseteq (a, \infty)$. Furthermore, if Z is singular at $b > a$ the above argument shows that $Z(t)\eta \equiv 0$ on $[a, b]$ for some constant unit vector η. The first member X of the corresponding solution (X, Y) of (2.1) also has these properties. Furthermore, if (2.1) is Hamiltonian, then (X, Y) is self-conjugate since $X(a) = 0$.

On the other hand, suppose there exist nonzero vector solutions of (2.19) of the form $(0, w)$ satisfying (2.20) for all $b > a$, and let $(0, [{0 \atop K_1}])$ be a basis for these. The matrix K_1 is an $r \times d$ constant matrix of rank $d > 0$ for some $d < r$. Let $(L_1, 0)$ be a basis for the vector solutions of the adjoint system

$$S' = -Q^*R, \qquad R' = -G^*S \tag{2.22}$$

satisfying

$$\begin{bmatrix} 0 & I_r \\ 0 & 0 \end{bmatrix} \begin{bmatrix} s(a) \\ r(a) \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & I_n \end{bmatrix} \begin{bmatrix} s(b) \\ r(b) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{2.23}$$

for all $b > a(r(b) \equiv 0$ on $(a, \infty))$. Since a has no conjugate points or adjoint conjugate points in (a, ∞) with respect to (2.19), we see that for fixed $b > a$ all the solutions of the boundary value problem (2.19) and (2.20) are of the form $(0, w)$, and all of the solutions of the boundary value problem (2.22) and (2.23) are of the form $(s, 0)$. Consequently there are constant basis matrices for the solution sets of the respective problems. Furthermore, the problems have the same number of linearly independent solutions on $[a, b]$, so these matrices have the same rank, say $d(b)$, depending on the right end-point b. The function $d(b)$ is nonincreasing with integer values ($d \leq d(b) \leq r$), and as b increases $d(b)$ becomes constant on some terminal subinterval of (a, ∞). Its value there is $d = \operatorname{rank}(K_1) = \operatorname{rank}(L_1)$. Therefore L_1 is also an $r \times d$ matrix of rank d. We may assume that $K_1^* K_1 = I_d = L_1^* L_1$. Let $L = [L_2 \vdots L_1]$, $K = [K_2 \vdots K_1]$, and

$$M = \begin{bmatrix} K_3 & 0 \\ 0 & K \end{bmatrix}$$

be unitary matrices of sizes $r \times r$, $r \times r$, and $n \times n$, respectively, and let (Z, W) be the $(n + r) \times r$ solution of (2.19) such that $Z(a) = [0 \vdots L_1]$ and

$$W(a) = \begin{bmatrix} 0 & 0 \\ K_2 & 0 \end{bmatrix}.$$

Let $Z_1 = L^* Z$ and $W_1 = M^* W$. Then

$$Z_1(a) = \begin{bmatrix} 0 & 0 \\ 0 & I_d \end{bmatrix},$$

and

$$Z_1' = \begin{bmatrix} G_{11} & 0 \\ 0 & 0_d \end{bmatrix} W_1, \quad W_1' = Q_1 Z_1, \tag{2.24}$$

where 0_d is the $d \times d$ zero matrix,

$$\begin{bmatrix} G_{11} & 0 \\ 0 & 0_d \end{bmatrix} = L^* G M,$$

and $Q_1 = M^* Q L$. (Note that G_{11} is a submatrix of $L^* G M$, not of G.) Then

$$Z_1 = \begin{bmatrix} Z_{11} & Z_{12} \\ 0 & I_d \end{bmatrix}$$

on $[a, \infty)$. Suppose that for some $b > a$, $Z(b)$ is singular. Then $Z_1(b)$ is singular, and for some constant unit vector $\left[\begin{smallmatrix} \eta_1 \\ \eta_2 \end{smallmatrix}\right]$ we have

$$0 = Z_1(b) \begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix} = \begin{bmatrix} Z_{11}(b)\eta_1 + Z_{12}(b)\eta_2 \\ \eta_2 \end{bmatrix},$$

whence $\eta_2 = 0$, $Z_{11}(b)\eta_1 = 0$, and η_1 is a unit vector. The vector solution (z, w) of (2.19) defined by

$$z(t) = Z(t) \begin{bmatrix} \eta_1 \\ 0 \end{bmatrix} = L \begin{bmatrix} Z_{11}(t)\eta_1 \\ 0 \end{bmatrix}$$

and

$$w(t) = W(t) \begin{bmatrix} \eta_1 \\ 0 \end{bmatrix}$$

satisfies (2.20), whence $z(t) \equiv 0$ on $[a, b]$, and $Z_{11}(\tau)\eta_1 \equiv 0$ on $[a, b]$. If $Z(t)$ is singular at an increasing sequence of points $\{t_\nu\}$, $t_\nu \to \infty$, then there is a sequence of unit vectors $\{\eta_\nu\}$ such that $Z_{11}(t_\nu)\eta_\nu = 0$. An argument similar to that in the first part of the proof yields that $Z_{11}(t)\eta \equiv 0$ on $[a, \infty)$ for some constant unit vector η. The vector solution (z, w) of (2.19) defined by

$$(z, w) = \left(Z(t)\begin{bmatrix} \eta \\ 0 \end{bmatrix}, W(t)\begin{bmatrix} \eta \\ 0 \end{bmatrix}\right)$$

satisfies (2.20) for all $b > a$. Consequently, $z(t) \equiv 0$ on $[a, \infty)$ and

$$w(t) \equiv w(a) = \begin{bmatrix} 0 & 0 \\ K_2 & 0 \end{bmatrix}\begin{bmatrix} \eta \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ K_2 \end{bmatrix}\eta.$$

However, $(0, [{0 \atop K_1}])$ is a basis for such solutions, whence $w(t) \equiv [{0 \atop K_1}]\alpha$ for some constant vector α. Since $K_1^* K_2 = 0$, this implies that $\eta = 0$, a contradiction. Therefore, Z is nonsingular on some terminal interval $(a_1, \infty) \subseteq (a, \infty)$. The solution (X, Y) of (2.1) corresponding to (Z, W) has the desired properties. Furthermore, if (2.1) is Hamiltonian, the reduced system (2.19) is also, and we may take $L_1 = K_1$, whence

$$Z^*(a)W(a) = \begin{bmatrix} 0 \\ K_1^* \end{bmatrix}\begin{bmatrix} K_2 \vdots 0 \end{bmatrix} = 0,$$

and since $X(a) = Z(a)$ and $Y(a) = W(a)$, this yields $X^*(a)Y(a) - Y^*(a)X(a) = 0$, so (X, Y) is self-conjugate. □

When $r = n$ in system (2.1) the coefficient matrices are square, and considering the boundary condition (2.3), we have already noted that point a is a conjugate point of b if and only if b is a conjugate point of a with respect to (2.1). If no two distinct points of a nontrivial subinterval J_0 are conjugate with respect to (2.1), the system is called *disconjugate* on J_0. The *order of abnormality* of (2.1) on a nontrivial subinterval $[a, c]$, $[a, c)$, or $[a, \infty)$ is the dimension of the space of vector solutions of the form $(0, y)$ on that subinterval (i.e., the solutions which satisfy (2.3) for all b in the subinterval). The system is *normal* on the subinterval whenever it has order of abnormality zero there. The system is called *oscillatory at* ∞, or simply *oscillatory*, if for any point $a \in J$ there is a point $b > a$ that is conjugate to a. Conversely, the system is nonoscillatory provided there exists some terminal interval $(a, \infty) \subseteq J$ on which it is disconjugate.

The following well-known theorem **[10]** provides a partial converse to Theorem 2.1.

THEOREM 2.2. *Suppose that* (2.1) *is Hamiltonian and that* $B(t) > 0$ *a.e. on* J. *Then* (2.1) *is nonoscillatory if and only if there exists a self-conjugate solution* (X_0, Y_0) *such that* X_0 *is nonsingular on some terminal subinterval* $[a, \infty) \subseteq J$.

PROOF. One direction is already proved in Theorem 2.1. Suppose (X_0, Y_0) is a self-conjugate solution of (2.1) such that X_0 is nonsingular on $[a, \infty)$, and transform (2.1) by (2.11) with $C_1(t) \equiv 0$, $C_0 = C$, and $(U_0, V_0) = (-Y_0, X_0)$ on J to obtain the reduced, equivalent system (2.13) on $[a, \infty)$, where $G(t) \geqslant 0$ a.e. and $Q(t) \equiv 0$ on $[a, \infty)$. It suffices to show that (2.13) is nonoscillatory. Suppose, on the contrary, that $b < c$ and that b and c are mutually conjugate points in $[a, \infty)$. Any vector solution (z, w) satisfying $z(b) = 0$ is of the form

$$z(t) = \int_b^t G(s)\, ds\, \eta, \qquad w(t) = \eta$$

for some constant vector η. Since c is conjugate to b, then for some $\eta \neq 0$ we have

$$z(c) = \int_b^c G(s)\, ds\, \eta = 0,$$

and $z(t) \not\equiv 0$ on $[b, c]$. This implies that

$$0 = \int_b^c \eta^* G(s) \eta\, ds,$$

and since $\eta^* G(t)\eta \geqslant 0$ a.e. on $[b, c]$ we must have $G(t)\eta \equiv 0$ a.e. on $[b, c]$, whence $z(t) \equiv 0$ on $[b, c]$. This is a contradiction. Thus, (2.13) is disconjugate on $[a, \infty)$ and so (2.1) is nonoscillatory. □

3. Asymptotic behavior. The results of this section concern a reduced system

$$X' = GY, \qquad Y' = QX \tag{3.1}$$

where $G(t)$ and $Q(t)$ are complex $r \times n$ and $n \times r$ matrices, respectively, in $\mathfrak{L}(J)$. Results by Hartman on systems of "type Z" **[7]**, where G and Q are complex valued continuous scalar functions, are extended to the system (3.1). The results of Hartman are based on a theorem by Wintner for the selfadjoint scalar equation of order two **[17]**. The method of proof of the following theorems involves an iteration scheme which has been applied to selfadjoint equations of order two, by Hille for the scalar equation **[8]** and by Shreve for the system corresponding to (3.1) with $G(t) \equiv I$ and $Q(t)$ hermitian on J **[14]**.

We have the following results.

THEOREM 3.1. *Suppose that, for some $a \in J = (a_0, \infty)$,*

$$\int_a^\infty Q(s)\, ds = \lim_{t \to \infty} \int_a^t Q(s)\, ds \tag{3.2}$$

exists (perhaps conditionally). Define $\Gamma(t)$ on J by

$$\Gamma(t) = \sup_{t < s < \infty} \left\| \int_s^\infty Q(\xi)\, d\xi \right\|, \tag{3.3}$$

and suppose that

$$\int_a^\infty \|G(s)\|\Gamma(s)\,ds < \infty. \tag{3.4}$$

Then there exist linearly independent solution matrices (X_0, Y_0) *and* (X_1, Y_1) *of* (3.1) *of sizes* $(r+n)\times r$ *and* $(r+n)\times n$, *respectively, such that as* $t\to\infty$ *we have the following*:

(i):

$$X_0(t)\to I \tag{3.5a}$$

and

$$Y_0(t)\to 0. \tag{3.5b}$$

In fact

$$\|X_0(t)-I\| < \tfrac{1}{2}\left\{\exp\left[2\int_t^\infty \|G(s)\|\Gamma(s)\,ds\right]-1\right\}$$

$$= O\left(\int_t^\infty \|G(s)\|\,\Gamma(s)\,ds\right), \tag{3.6a}$$

$$\|Y_0(t)\| < \Gamma(t)\exp\left\{2\int_t^\infty \|G(s)\|\Gamma(s)\,ds\right\} = O(\Gamma(t)) \tag{3.6b}$$

and

$$\int_a^t \|G(s)\|\,ds\,\|Y_0(t)\| = O\left(\Gamma(t)\int_a^t \|G(s)\|\,ds\right), \tag{3.6c}$$

where

$$\Gamma(t)\int_a^t \|G(s)\|\,ds \to 0. \tag{3.6d}$$

(ii):

$$\Gamma(t)X_1(t)\to 0 \tag{3.7a}$$

and

$$Y_1(t)\to I. \tag{3.7b}$$

In fact

$$\Gamma(t)\|X_1(t)\| = O\left(\Gamma(t)\int_a^t \|G(s)\|\,ds\right) \tag{3.8a}$$

and

$$\|Y_1(t)-I\| = O\left(\Gamma(t)\int_a^t \|G(s)\|\,ds\right) + O\left(\int_t^\infty \Gamma(s)\|G(s)\|\,ds\right). \tag{3.8b}$$

THEOREM 3.2. *Let a be a point in* $J = (a_0, \infty)$ *and define* $\Gamma(t)$ *on* $[a, \infty)$ *by*

$$\Gamma(t) = \sup_{a<s<t} \left\| \int_a^s G(\xi)\, d\xi \right\|. \tag{3.9}$$

Suppose that

$$\int_a^\infty \|Q(s)\|\, ds < \infty \tag{3.10}$$

and that

$$\int_a^\infty \Gamma(s)\|Q(s)\|\, ds < \infty. \tag{3.11}$$

(*Note that unless* $G(t) \equiv 0$ *a.e. on* $[a, \infty)$, (3.11) *implies* (3.10).) *Then there exist linearly independent solutions* (X_0, Y_0) *and* (X_1, Y_1) *of* (3.1) *of sizes* $(r + n) \times r$ *and* $(r + n) \times n$, *respectively, such that as* $t \to \infty$ *we have the following*:

(i):

$$X_0(t) \to I, \tag{3.12a}$$

$$Y_0(t) \to 0, \tag{3.12b}$$

and

$$\Gamma(t)\|Y_0(t)\| \to 0. \tag{3.12c}$$

In fact,

$$\|X_0(t) - I\| = O\left(\int_t^\infty \Gamma(s)\|Q(s)\|\, ds\right), \tag{3.13a}$$

$$\|Y_0(t)\| = O\left(\int_t^\infty \|Q(s)\|\, ds\right), \tag{3.13b}$$

and

$$\Gamma(t)\|Y_0(t)\| = O\left(\Gamma(t)\int_t^\infty \|Q(s)\|\, ds\right), \tag{3.13c}$$

where

$$\Gamma(t)\int_t^\infty \|Q(s)\|\, ds < \int_t^\infty \Gamma(s)\|Q(s)\|\, ds \to 0. \tag{3.13.d}$$

(ii):

$$\int_t^\infty \|Q(s)\|\, ds\|X_1(t)\| \to 0, \tag{3.14a}$$

and

$$Y_1(t) \to I \quad \text{as } t \to \infty. \tag{3.14b}$$

In fact,

$$\|X_1(t)\| = O(\Gamma(t)), \tag{3.15a}$$

$$\int_t^\infty \|Q(s)\|\, ds\, \|X_1(t)\| = O\left(\Gamma(t) \int_t^\infty \|Q(s)\|\, ds\right), \tag{3.15b}$$

and

$$\|Y_1(t) - I\| = O\left(\int_t^\infty \Gamma(s)\|Q(s)\|\, ds\right). \tag{3.15c}$$

Here the notation $f(t) = O(g(t))$ as $t \to \infty$, where f and g are nonnegative functions, means that there exist a point $a \in J$ and a constant $M_a > 0$ such that $f(t) < M_a g(t)$ for all $t > a$.

Proof of Theorem 3.1. A pair of integral equations equivalent to (3.1) is given by

$$X(t) = X(b) + \int_b^t G(s)Y(s)\, ds \tag{3.16a}$$

and

$$\begin{aligned} Y(t) &= Y(b) + \int_b^t Q(s)X(s)\, ds \\ &= Y(b) + \int_b^t Q(s)\, ds\, X(b) + \int_b^t Q(\xi) \int_b^\xi G(s)Y(s)\, ds\, d\xi \end{aligned} \tag{3.16b}$$

for $b, t \in J$. An integration by parts yields

$$\int_b^t Q(\xi) \int_b^\xi G(s)Y(s)\, ds\, d\xi = \int_b^t \int_s^t Q(\xi)\, d\xi G(s)Y(s)\, ds,$$

which suggests for (3.16b) the following integral equation:

$$Y(t) = -\int_t^\infty Q(\xi)\, d\xi + \int_t^\infty \int_t^s Q(\xi)\, d\xi\, G(s)Y(s)\, ds. \tag{3.17}$$

We shall prove that (3.17) has a solution by constructing a sequence $\{Y_\nu(t)\}$ which converges to such a solution.

Let

$$Y_0(t) = -\int_t^\infty Q(s)\, ds. \tag{3.18a}$$

By hypothesis this is well defined and absolutely continuous, and it satisfies $\|Y_0(t)\| < \Gamma(t)$ on $[a, \infty)$. Let

$$Y_{\nu+1}(t) = -\int_t^\infty Q(s)\, ds + \int_t^\infty \int_t^s Q(\xi)\, d\xi\, G(s)Y_\nu(s)\, ds, \tag{3.18b}$$

$\nu > 0$, and let

$$k(t) = 2\int_t^\infty \|G(s)\|\, \Gamma(s)\, ds. \tag{3.19}$$

Suppose that $Y_\nu(t)$, $0 < \nu < n - 1$, have been shown to be well defined, continuous, and bounded by

$$\|Y_\nu(t)\| < \Gamma(t)\left[1 + k(t) + \cdots + \frac{k^\nu(t)}{\nu!}\right]$$
$$< \Gamma(t)\exp\{k(t)\} < \Gamma(a)\exp\{k(a)\} \tag{3.20}$$

for t in $[a, \infty)$. When $\nu = n - 1$, the integrand on the right in (3.18b) satisfies

$$\left\|\int_t^s Q(\xi)d\xi\, G(s)Y_{n-1}(s)\right\| < 2\Gamma(t)\|G(s)\|\Gamma(s)\left[1 + k(s) + \cdots + \frac{k^{n-1}(s)}{(n-1)!}\right]$$
$$= -\Gamma(t)k'(s)\left[1 + k(s) + \cdots + \frac{k^{n-1}(s)}{(n-1)!}\right] \quad \text{a.e.,} \tag{3.21}$$

since, for $a < t < s$,

$$\left\|\int_t^s Q(\xi)d\xi\right\| = \left\|\int_t^\infty Q(\xi)d\xi - \int_s^\infty Q(\xi)d\xi\right\|$$
$$< \left\|\int_t^\infty Q(\xi)d\xi\right\| + \left\|\int_s^\infty Q(\xi)d\xi\right\| < 2\Gamma(t).$$

It is, therefore, integrable on $[a, \infty)$ uniformly in $t \in [a, \infty)$, and $Y_n(t)$ is well defined, continuous, and satisfies (3.20) for $\nu = n$. By induction this is true for all positive integers n.

Let

$$\Delta Y_\nu(t) = Y_\nu(t) - Y_{\nu-1}(t), \qquad \nu > 1. \tag{3.22}$$

Then

$$\|\Delta Y_1(t)\| < \int_t^\infty \left\|\int_t^s Q(\xi)\, d\xi\right\| \|G(s)\|\, \Gamma(s)\, ds < \Gamma(t)k(t).$$

One can show inductively that

$$\|\Delta Y_n(t)\| < \Gamma(t)\frac{k^n(t)}{n!}, \qquad n > 1, \tag{3.23}$$

in which case the sequence converges uniformly on $[a, \infty)$ to a continuous limit $Y(t)$, with

$$\|Y(t)\| < \Gamma(t)\exp\left\{2\int_t^\infty \|G(s)\|\Gamma(s)\, ds\right\} < \Gamma(a)\exp\{k(a)\} \tag{3.24}$$

for $t \in [a, \infty)$. Since by (3.21) the norm of the integrand of the second term on the right in (3.18b) is bounded by $-\Gamma(t)k'(s)\exp\{k(s)\}$ a.e. for $s \in [a, \infty)$, uniformly for $t \in [a, \infty)$, by the Lebesgue dominated convergence theorem we may let $n \to \infty$ throughout (3.18b) to deduce that $Y(t)$ is a solution of (3.17). Then

$$Y'(t) = Q(t) - Q(t)\int_t^\infty G(s)Y(s)\, ds \quad \text{a.e.,}$$

and defining $X(t)$ by

$$X(t) = I - \int_t^\infty G(s)Y(s)\,ds \tag{3.25}$$

we have a solution $(X(t), Y(t))$ of (3.1) such that (3.5) is satisfied. Furthermore, using (3.19) and (3.24) we see that

$$\begin{aligned}\|X(t) - I\| &\le \int_t^\infty \|G(s)\|\,\|Y(s)\|\,ds\\ &\le \tfrac{1}{2}\int_t^\infty (-k'(s))\exp\{k(s)\}\,ds = \tfrac{1}{2}[\exp\{k(t)\} - 1]\\ &\le \tfrac{1}{2}\exp\{k(a)\}k(t).\end{aligned}$$

This along with (3.24) establishes (3.6a)–(3.6c).

Since $\Gamma(t)$ is a nonincreasing function (whose limit as $t \to \infty$ is zero) it is of bounded variation, and

$$\Gamma(t) - \Gamma(s) = \int_s^t d\Gamma. \tag{3.26}$$

Furthermore,

$$\begin{aligned}\infty > \int_b^\infty \Gamma(s)\|G(s)\|\,ds &\ge \int_b^t \Gamma(s)\|G(s)\|\,ds\\ = \int_b^t\left[\Gamma(t) - \int_s^t d\Gamma\right]\|G(s)\|\,ds &\ge \int_b^t\left(-\int_s^t d\Gamma\right)\|G(s)\|\,ds\end{aligned}$$

for all $t \ge b$. Since the last quantity is nondecreasing with t and is bounded above with limit

$$\int_b^\infty\left(-\int_s^\infty d\Gamma\right)\|G(s)\|\,ds = \int_b^\infty \Gamma(s)\|G(s)\|\,ds,$$

we see that

$$\lim_{t\to\infty} \Gamma(t)\int_b^t \|G(s)\|\,ds = 0. \tag{3.27}$$

This establishes (3.6d). Therefore (X, Y) is the solution (X_0, Y_0) of the conclusion.

To establish (ii) of the conclusion, we consider the integral equation

$$Y(t) = I + \int_b^t\int_s^t Q(\xi)\,d\xi\,G(s)Y(s)\,ds, \qquad b \ge a. \tag{3.28}$$

Define the function $\lambda(t)$ by

$$\lambda(t) = \int_b^t 2\Gamma(s)\|G(s)\|\,ds. \tag{3.29}$$

Then one can show by an induction argument that the sequence $\{Y_\nu(t)\}$ given by

$$Y_0(t) = I, \tag{3.30a}$$

$$Y_{\nu+1}(t) = I + \int_b^t \int_s^t Q(\xi)\, d\xi\, G(s) Y_\nu(s)\, ds, \qquad \nu > 0, \tag{3.30b}$$

is well defined, with each member being continuous on $[b, \infty)$ and bounded there by

$$\| Y_\nu(t) \| < 1 + \lambda(t) + \cdots + \frac{\lambda^\nu(t)}{\nu!} < \exp\{\lambda(t)\} < \exp\{\lambda(\infty)\}. \tag{3.30c}$$

Furthermore, the sequence converges uniformly on $[b, \infty)$ to a solution $Y(t)$ of (3.28) such that

$$\| Y(t) \| < \exp\left\{ 2 \int_b^t \Gamma(s) \| G(s) \|\, ds \right\}, \tag{3.31}$$

and

$$Y'(t) = Q(t) \int_b^t G(s) Y(s)\, ds.$$

Define $X(t)$ by

$$X(t) = \int_b^t G(s) Y(s)\, ds. \tag{3.32}$$

Then (X, Y) is a solution of (3.1). Furthermore,

$$\Gamma(t) \| X(t) \| < \Gamma(t) \int_b^t \| G(s) \| ds\, \exp\left\{ 2 \int_b^\infty \Gamma(s) \| G(s) \|\, ds \right\}, \tag{3.33}$$

and this together with (3.27) implies (3.7a) and (3.8a) hold for X.

For $t_2 > t_1 > b$, we have

$$Y(t_2) - Y(t_1) = \int_{t_1}^{t_2} Q(\xi)\, d\xi\, X(t_1) + \int_{t_1}^{t_2} \int_s^{t_2} Q(\xi)\, d\xi\, G(s) Y(s)\, ds$$

and

$$\begin{aligned} &\| Y(t_2) - Y(t_1) \| \\ &< 2\Gamma(t_1) \| X(t_1) \| + \int_{t_1}^{t_2} 2\Gamma(s) \| G(s) \| \exp\left\{ 2 \int_b^s \Gamma(\xi) \| G(\xi) \|\, d\xi \right\} ds \\ &= 2\Gamma(t_1) \| X(t_1) \| + \exp\left\{ 2 \int_b^{t_2} \Gamma(\xi) \| G(\xi) \|\, d\xi \right\} - \exp\left\{ 2 \int_b^{t_1} \Gamma(\xi) \| G(\xi) \|\, d\xi \right\} \\ &< 2\Gamma(t_1) \| X(t_1) \| + 2 \int_{t_1}^\infty \Gamma(s) \| G(s) \|\, ds\, \exp\left\{ 2 \int_b^\infty \Gamma(s) \| G(s) \|\, ds \right\}. \end{aligned} \tag{3.34}$$

This tends to zero as $t_1 \to \infty$, which implies that $Y(t)$ tends to a constant limit as $t \to \infty$. Since, from (3.28) and (3.31),

$$\|Y(t) - I\| < \int_b^t \left\| \int_s^t Q(\xi)\, d\xi \right\| \|G(s)\| \, \|Y(s)\| \, ds$$
$$< 2\int_b^\infty \Gamma(s)\|G(s)\| \, ds \exp\left\{2\int_b^\infty \Gamma(s)\|G(s)\| \, ds\right\},$$

$t > b$, we see that by choosing b large enough we may assume that this limit is nonsingular. Post multiplication of (X, Y) by the inverse of this constant limit yields a solution (X_1, Y_1) of (3.1) such that X_1 satisfies (3.7a) and (3.8a). In equation (3.34) replace (X, Y) by (X_1, Y_1), t_1 by t, and allow t_2 to tend to ∞. The resulting inequality yields (3.8b) which implies (3.7b). This completes the proof. □

The proof of Theorem 3.2 proceeds in a similar manner and will be omitted. Partial converses to Theorems 3.1 and 3.2 are provided by Theorem 4.7 in the next section on Hamiltonian systems.

4. Oscillation and asymptotic behavior of selfadjoint systems. This section concerns a reduced Hamiltonian system

$$X' = GY, \qquad Y' = -QX \tag{4.1}$$

where $G(t)$ and $Q(t)$ are complex, hermitian $n \times n$ matrices in $\mathfrak{L}(J)$.

Under the hypothesis that (4.1) is nonoscillatory and G and Q are only positive semidefinite, Theorem 4.1 establishes the existence of a self-conjugate solution (X_0, Y_0) of (4.1) with both members nonsingular on some terminal interval, thereby extending certain results of Ahlbrandt [2]. This also provides a direct proof of the known result (see, for example, [12, Theorem 5.4]) that (4.1) is nonoscillatory if and only if the system "reciprocal" ([4], [3]) to (4.1), defined by

$$U' = QV, \qquad V' = -GU, \tag{4.2}$$

is nonoscillatory. This result is also used in proving Theorems 4.2 and 4.3, which establish necessary conditions for nonoscillation of (4.1), extending results of Tomastik [16] and Ahlbrandt [1]. Theorems 4.4 and 4.5 establish sufficient conditions for nonoscillation of (4.1), and Theorem 4.6 provides partial converses to Theorems 3.1 and 3.2 on asymptotic behavior, extending results of Shreve [14].

Let $G(t)$ be an $n \times n$ matrix in $\mathfrak{L}(J)$. If $k(G; s)$ is the dimension of the space of constant vectors η such that $G(t)\eta \equiv 0$ a.e. on $[s, \infty)$, then $k(G; s)$ is nondecreasing in s, and $k_\infty(G) = \lim_{s\to\infty} k(G; s)$ will be called the *degree of degeneracy* of G. If $k_\infty(G) = 0$, then G will be called *nondegenerate*. Note that there exists some terminal subinterval $[s_0, \infty)$ such that $k(G; s) = k_\infty(G)$ for $s > s_0$. The space of constant n-vectors annihilated a.e. by G on $[s_0, \infty)$ will be called the *space of degeneracy* of G and has dimension $k_\infty(G)$.

LEMMA 4.1. *Let $G(t)$ be an $n \times n$ hermitian matrix in $\mathcal{L}(J)$ with $G(t) \geq 0$ a.e. on J. Then the following are equivalent*:

(i) *G is nondegenerate*;

(ii) *for each $a \in J$, $\int_a^t G(s)\, ds$ is eventually positive definite*;

(iii) *the system* (4.1) *is normal on every terminal subinterval* $[a, \infty) \subseteq J$.

PROOF. (i) $\Rightarrow$ (ii): A proof by contradiction follows. Suppose that $a \in J$ is a point for which $\mathcal{G}(t)$, given by

$$\mathcal{G}(t) = \int_a^t G(s)\, ds$$

is identically singular on $[a, \infty)$. Then there is an increasing sequence $\{t_\nu\}_{\nu=0}^{\infty}$ in $[a, \infty)$, $t_\nu \to \infty$ as $\nu \to \infty$, and a sequence of constant unit vectors $\{\eta_\nu\}_{\nu=0}^{\infty}$ such that $\mathcal{G}(t_\nu)\eta_\nu = 0$ for all $\nu \geq 0$. We may assume that $\eta_\nu \to \eta$, a constant unit vector, as $\nu \to \infty$. Then

$$0 = \eta_\eta^* \mathcal{G}(t_\nu)\eta_\nu = \int_a^{t_\nu} \eta_\eta^* G(s)\eta_\nu\, ds,$$

and since $G(t) \geq 0$ a.e. we see that $G(t)\eta_\nu \equiv 0$ a.e. on $[a, t_\nu]$ and that $\mathcal{G}(t)\eta_\nu \equiv 0$ on $[a, t_\nu]$.

Let $t \in [a, \infty)$ be arbitrary. Then

$$\mathcal{G}(t)\eta = \mathcal{G}(t)\eta_\nu + \mathcal{G}(t)(\eta - \eta_\nu),$$

and the right-hand side tends to zero as $\nu \to \infty$. Thus $\mathcal{G}(t)\eta \equiv 0$ on $[a, \infty)$, in which case $G(t)\eta \equiv 0$ a.e. on $[a, \infty)$ and G is degenerate.

(ii) $\Rightarrow$ (i): (by contradiction). Suppose G is degenerate. Then there is a point $a \in J$ and a constant unit vector η such that $G(t)\eta \equiv 0$ a.e. on $[a, \infty)$. Then

$$\eta^* \int_a^t G(s)\, ds\, \eta = \int_a^t \eta^* G(s)\eta\, ds \equiv 0$$

on $[a, \infty)$, whence $\int_a^t G(s)\, ds$ is identically singular for t in $[a, \infty)$.

(i) $\Rightarrow$ (iii): (by contradiction). Suppose (4.1) is abnormal on some terminal subinterval $[a, \infty)$ of J. Then there is a vector solution $(0, \eta)$ of (4.1) on $[a, \infty)$ with $\eta \neq 0$. Therefore, η is constant, and $G(t)\eta \equiv 0$ a.e. on $[a, \infty)$. This implies that G is degenerate.

(iii) $\Rightarrow$ (i): (by contradiction). Suppose G is degenerate. Then there is a point $a \in J$ and a constant unit vector η such that $G(t)\eta \equiv 0$ a.e. on $[a, \infty)$. Therefore, $(0, \eta)$ is a solution of (4.1) on $[a, \infty)$, and (4.1) is abnormal on $[a, \infty)$. □

THEOREM 4.1. *Let $G(t) \geq 0$ and $Q(t) \geq 0$ a.e. on $[a, \infty) \subseteq J$. The following are equivalent*:

(i) (4.1) *is nonoscillatory*;

(ii) (4.2) *is nonoscillatory*;

(iii) *there exists a* $(2n) \times n$ *self-conjugate solution* (X_0, Y_0) *of* (4.1) *such that* *both* $X_0(t)$ *and* $Y_0(t)$ *are nonsingular on some terminal interval* $[d, \infty) \subseteq [a, \infty)$;

(iv) *there exists a* $(2n) \times n$ *self-conjugate solution* (U_0, V_0) *of* (4.2) *such that both* $U_0(t)$ *and* $V_0(t)$ *are nonsingular on some terminal interval* $[d, \infty) \subseteq [a, \infty)$;

(v) *there exists a hermitian, nonsingular, absolutely continuous,* $n \times n$ *solution matrix* S *of*

$$S' = -Q - SGS \tag{4.3}$$

on some terminal interval $[d, \infty) \subseteq [a, \infty)$;

(vi) *there exists a hermitian, nonsingular, absolutely continuous,* $n \times n$ *solution matrix* W *of*

$$W' = G + WQW \tag{4.4}$$

on some terminal interval $[d, \infty) \subseteq [a, \infty)$.

Before presenting the proof, we should note that systems (4.1) and (4.2) are reduced cases of the more general system

$$X' = AX + GY, \qquad Y' = -QX - A^*Y \tag{4.5}$$

and its reciprocal

$$U' = -A^*U + QV, \qquad V' = -GU + AV \tag{4.6}$$

where G and Q are positive semidefinite a.e. on $[a, \infty)$. But since the reduction transformations

$$X = X_0Z, \qquad Y = X_0^{*-1}W \tag{4.7}$$

and

$$U = X_0^{*-1}S, \qquad V = X_0R \tag{4.8}$$

described in §2, where X_0 is a fundamental matrix for

$$X' = AX, \tag{4.9}$$

reduce the general systems (4.5) and (4.6) to (4.1) and (4.2), preserving the definiteness properties and oscillatory behavior of both systems, we see that the theorem actually applies to the more general systems.

Proof of Theorem 4.1. (iii) $\Leftrightarrow$ (iv): If (X_0, Y_0) is a solution as in part (iii), then (U_0, V_0) defined by $U_0(t) = Y_0(t)$, $V_0(t) = -X_0(t)$ is a solution of (4.2) satisfying (iv). The argument is reversible.

(iii) $\Leftrightarrow$ (v): Let (X_0, Y_0) be a solution of (4.1) as described in (iii). Then for $t > d$ the matrix S defined by $S(t) = Y_0(t)X_0^{-1}(t)$ is nonsingular, absolutely continuous, and by differentiating and using the equations (4.1) we see that S satisfies (4.3). S is hermitian since (X_0, Y_0) is self-conjugate.

Conversely, suppose S is as in (v), and let X_0 be a fundamental matrix for

$$X' = GSX \tag{4.10}$$

in $[d, \infty)$. Then defining $Y_0(t)$ by $Y_0(t) = S(t)X_0(t)$, we see by direct verification that (X_0, Y_0) satisfies (4.1), both members are nonsingular in $[d, \infty)$, and since S is hermitian, (X_0, Y_0) is self-conjugate.

By a parallel argument we see that (iv) $\Leftrightarrow$ (vi). Furthermore, by Theorem 2.2, (iii) $\Rightarrow$ (i) and (iv) $\Rightarrow$ (ii). To complete the proof it remains, therefore to show that (i) $\Rightarrow$ (iii).

(i) $\Rightarrow$ (iii): Consider first the case where the space of degeneracy of G has nontrivial intersection with the space of degeneracy of Q. Let D_1 be a constant matrix whose columns form an orthonormal basis for this intersection, and let $[e, \infty)$ be a subinterval of $[a, \infty)$ such that $G(t)D_1 \equiv 0$ and $Q(t)D_1 \equiv 0$ a.e. on $[e, \infty)$. Now by hypothesis, D_1 has rank $d > 0$. If $d = n$ then (I, I) is the desired solution of (4.1). If $0 < d < n$, let $D = [D_2 \vdots D_1]$ be a unitary constant matrix, and transform (4.1) by $Z = D^*X$, $W = D^*Y$. This results in the system

$$Z' = G_1W, \qquad W' = -Q_1Z \tag{4.11}$$

where

$$G_1 = D^*GD = \begin{bmatrix} G_{11} & 0 \\ 0 & 0_d \end{bmatrix},$$

and

$$Q_1 = D^*QD = \begin{bmatrix} Q_{11} & 0 \\ 0 & 0_d \end{bmatrix}$$

on $[e, \infty)$, where 0_d is the $d \times d$ zero matrix. Clearly, (4.1) and (4.11) are equivalent in oscillatory properties. Suppose b and c are mutually conjugate points for (4.11) in $[e, \infty)$. Then there exists a vector solution (z_0, w_0) of (4.11) such that $z_0(b) = 0 = z_0(c)$, but $z_0(t) \not\equiv 0$ on $[b, c]$. Now

$$z = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \quad \text{and} \quad w = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$$

where z_2 and w_2 are d-vectors, and for $t > e$ we see that

$$z_1' = G_{11}w_1, \qquad w_1' = -Q_{11}z_1 \tag{4.12}$$

and z_2 and w_2 are constant. In particular, $z_{02}(t) \equiv 0$ on $[b, c]$, in which case (z_{01}, w_{01}) is a solution of (4.12) such that $z_{01}(b) = z_{01}(c) = 0$, and $z_{01}(t) \not\equiv 0$ on $[b, c]$. Thus, b and c are mutually conjugate points with respect to (4.12). The converse is also true. Namely, if a and b are mutually conjugate points in $[e, \infty)$ with respect to (4.12), then reversing the above argument we see that they are mutually conjugate with respect to (4.11). Therefore (4.11) is nonoscillatory if and only if (4.12) is nonoscillatory. Furthermore, if (Z_{11}, W_{11}) is a self-conjugate solution matrix for (4.12) with both members nonsingular on some terminal subinterval of $[e, \infty)$, then the solution (Z, W) of (4.11) given by

$$Z(t)=\begin{bmatrix} Z_{11}(t) & 0 \\ 0 & I_d \end{bmatrix}, \qquad W(t)=\begin{bmatrix} W_{11}(t) & 0 \\ 0 & I_d \end{bmatrix}$$

is self-conjugate, with both members nonsingular on the same terminal subinterval. Note that the spaces of degeneracy of G_{11} and Q_{11} have only the trivial intersection. This argument shows that it suffices to prove the statement (i) $\Rightarrow$ (iii) for the case in which the spaces of degeneracy of G and Q have only the trivial intersection.

Let $\mathbb{S}_1$ and $\mathbb{S}_2$ denote the spaces of degeneracy of G and Q, respectively. In view of the above argument, we assume henceforth that $\mathbb{S}_1 \cap \mathbb{S}_2 = \{0\}$.

Suppose next that $\mathbb{S}_1$ and $\mathbb{S}_2$ are orthogonal, and let A_1 and A_2 be $n \times d_1$ and $n \times d_2$ matrices whose columns form orthonormal bases for $\mathbb{S}_1$ and $\mathbb{S}_2$, respectively. Then $d_\nu = \operatorname{rank} A_\nu = \dim \mathbb{S}_\nu$, $\nu = 1, 2$. Since $A_1^* A_2 = 0$, for some matrix A_3, $A = [A_2 \vdots A_3 \vdots A_1]$ is unitary. Transforming (4.1) by $Z = A^*X$ and $W = A^*Y$, we obtain the system

$$Z' = G_1 W, \qquad W' = -Q_1 Z, \tag{4.13}$$

where for t in some terminal interval $[b, \infty) \subseteq [a, \infty)$ on which (4.1) is disconjugate, we have

$$G_1 = \begin{bmatrix} G_{11} & 0 \\ 0 & 0_{d_1} \end{bmatrix}, \tag{4.14a}$$

$$Q_1 = \begin{bmatrix} 0_{d_2} & 0 \\ 0 & Q_{22} \end{bmatrix}, \tag{4.14b}$$

and G_{11} and Q_{22} are nondegenerate. Let (Z_0, W_0) be the solution matrix of (4.13) such that

$$Z_0(b) = \begin{bmatrix} 0 & 0 \\ 0 & I_{d_1} \end{bmatrix} \tag{4.15a}$$

and

$$W_0(b) = \begin{bmatrix} I_{n-d_1} & 0 \\ 0 & 0 \end{bmatrix}. \tag{4.15b}$$

Then as in the proof of Theorem 2.1 we see that (Z_0, W_0) is a self-conjugate solution of (4.13) with $Z_0(t)$ nonsingular on some terminal interval $[c, \infty) \subseteq [b, \infty)$. For $t > c$ the matrix S defined by $S(t) = W_0(t) Z_0^{-1}(t)$ is well defined, hermitian, and absolutely continuous;

$$S'(t) = -Q_1(t) - S(t) G_1(t) S(t) < 0,$$

and for $c < e < t$ we have

$$S(e) - S(t) = \int_e^t Q_1(\xi)\, d\xi + \int_e^t S(\xi) G_1(\xi) S(\xi) d\xi > 0. \tag{4.16}$$

From this we see that the eigenvalues of $S(t)$ are nonincreasing. If $-S'(t)$ is nondegenerate, then for any $e \geq c$ there is a $t > e$ such that

$$S(e) - S(t) = \int_e^t [-S'(\xi)]\, d\xi > 0,$$

by Lemma 4.1 ((i)$\Leftrightarrow$(ii) with $-S'(t)$ playing the role of $G(t)$), and this implies that $S(t)$ is nonsingular on some terminal interval. Then $W_0(t)$ is also nonsingular on that interval, and (X_0, Y_0) given by $(X_0, Y_0) = (AZ_0, AW_0)$ is the desired solution of (4.1), completing the proof. We shall now prove by contradiction that $-S'(t)$ is nondegenerate.

Suppose, on the contrary, that $-S'(t)$ is degenerate. Then for some $e \geq c$ and some constant unit vector η,

$$0 = \eta^*[-S'(t)]\eta = \eta^* Q_1(t)\eta + \eta^* S(t)G_1(t)S(t)\eta \tag{4.17}$$

for almost all $t \geq e$. Then $\eta^* Q_1(t)\eta \equiv 0$ a.e. on $[e, \infty)$, whence Q_1 is degenerate, and

$$\eta = \begin{bmatrix} \eta_1 \\ 0 \end{bmatrix} \tag{4.18}$$

for some unit d_2-vector η_1. Furthermore, $\eta^* S G_1 S \eta \equiv 0$ a.e. on $[e, \infty)$. Also, we have

$$SG_1S = W_0 Z_0^{-1} G_1 Z_0^{*\,-1} W_0^*,$$

and from (4.13), (4.14b), and (4.15b) we see that

$$W_0(t) = \begin{bmatrix} I_{d_2} & 0 \\ W_{21} & W_{22} \end{bmatrix} \tag{4.19}$$

for t in $[b, \infty)$. Therefore, $\eta^* W_0(t) = \eta^* = [\eta_1^*, 0]$, and we have

$$\eta^* Z_0^{-1} G_1 Z_0^{*\,-1} \eta \equiv 0 \quad \text{a.e. on } [e, \infty). \tag{4.20}$$

Define $\kappa(t)$ by

$$\kappa(t) = Z_0^{*\,-1}(t)\eta. \tag{4.21}$$

Then $G_1(t)\kappa(t) \equiv 0$ a.e. on $[e, \infty)$, κ is absolutely continuous there, and $\kappa(t) = -Z_0^{*\,-1} W_0^* G_1 \kappa \equiv 0$ on $[e, \infty)$. Therefore, κ is constant on $[e, \infty)$. Then $\kappa = [\begin{smallmatrix} 0 \\ \alpha \end{smallmatrix}]$ on $[e, \infty)$ for some d_1-vector α. Since $\eta \neq 0$ and $Z_0(t)$ is nonsingular on $[e, \infty)$, we see that $\alpha \neq 0$. Combining these results we find that the function $\phi(t)$, defined on $[a, \infty)$ by

$$\phi(t) = \eta - Z_0^*(t)\begin{bmatrix} 0 \\ \alpha \end{bmatrix}, \tag{4.22}$$

is absolutely continuous on $[b, \infty)$ with

$$\phi'(t) = -W_0^* G_1 \begin{bmatrix} 0 \\ \alpha \end{bmatrix} \equiv 0 \quad \text{a.e.}$$

there (recall (4.14)), and $\phi(t) \equiv 0$ on $[e, \infty)$. This implies that $\phi(t) \equiv 0$ on

$[b, \infty)$. In particular, at $t = b$ we have

$$\eta = \begin{bmatrix} \eta_1 \\ 0 \end{bmatrix} = Z_0^*(b)\begin{bmatrix} 0 \\ \alpha \end{bmatrix} = \begin{bmatrix} 0 \\ \alpha \end{bmatrix},$$

where η_1 and α are nonzero vectors of dimension d_2 and d_1, respectively, and $d_2 + d_1 < n$. This is a contradiction. Therefore, $W_0(t)$ is nonsingular on some terminal subinterval, and the corresponding solution $(X_0, Y_0) = (AZ_0, AW_0)$ of (4.1) is self-conjugate with both $X_0(t)$ and $Y_0(t)$ nonsingular on some terminal interval.

Now consider the remaining case where $\mathbb{S}_1 \cap \mathbb{S}_2 = \{0\}$ and $\mathbb{S}_1$ and $\mathbb{S}_2$ are not orthogonal. Let $[A_1 \vdots A_2]$ be a matrix whose columns form an orthonormal basis for $\mathbb{S}_2$, with A_1 spanning $\mathbb{S}_2 \cap \mathbb{S}_1^\perp$. Let A_4 be a matrix whose columns form an orthonormal basis for $\mathbb{S}_1 \cap \mathbb{S}_2^\perp$. $A_4^*[A_1 \vdots A_2] = 0$, and for some matrix A_3,

$$A = [A_1 \vdots A_2 \vdots A_3 \vdots A_4]$$

is unitary. Let d_ν denote the rank of A_ν, $\nu = 1, 2, 3, 4$. Since $\mathbb{S}_1$ and $\mathbb{S}_2$ are not orthogonal, $d_2 > 0$ and $d_3 > 0$. Transforming (4.1) by $Z = A^*X$, $W = A^*Y$, we have the system

$$Z' = G_1 W, \qquad W' = -Q_1 Z, \tag{4.23}$$

where for t in some terminal interval $[b, \infty) \subseteq [a, \infty)$, on which (4.1) is disconjugate and on which $k(G_1; t) = \dim \mathbb{S}_1$ and $k(Q_1; t) = \dim \mathbb{S}_2$ (see the definition of $k(G; t)$ before Lemma 4.1), we have

$$G_1 = \begin{bmatrix} G_{11} & G_{12} & G_{13} & 0 \\ G_{21} & G_{22} & G_{23} & 0 \\ G_{31} & G_{32} & G_{33} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \tag{4.24a}$$

and

$$Q_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & Q_{33} & Q_{34} \\ 0 & 0 & Q_{43} & Q_{44} \end{bmatrix} \tag{4.24b}$$

with

$$\begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix} \text{ and } \begin{bmatrix} Q_{33} & Q_{34} \\ Q_{43} & Q_{44} \end{bmatrix}$$

nondegenerate. G_1 and Q_1 are partitioned according to the partition of A. Let $\mathbb{S}_1'$ and $\mathbb{S}_2'$ denote the spaces of degeneracy of

$$\begin{bmatrix} G_{22} & G_{23} \\ G_{32} & G_{33} \end{bmatrix} \text{ and } \begin{bmatrix} 0 & 0 \\ 0 & Q_{33} \end{bmatrix},$$

respectively. Then $\mathcal{S}_1' \cap \mathcal{S}_2' = \{0\}$, $\mathcal{S}_1' \cap \mathcal{S}_2'^{\perp} = \{0\}$, and $\mathcal{S}_2' \cap \mathcal{S}_1'^{\perp} = \{0\}$, and $\mathcal{S}_1'$ and $\mathcal{S}_2'$ have dimensions greater than zero since $\mathcal{S}_1$ and $\mathcal{S}_2$ are not orthogonal. In fact, $\dim \mathcal{S}_2' = d_2 > 0$ and

$$\begin{bmatrix} I_{d_2} \\ 0 \end{bmatrix}$$

spans $\mathcal{S}_2'$. Let the columns of $K_1 = [{K_{11} \atop K_{12}}]$ form an orthonormal basis for $\mathcal{S}_1'$, where rank $K_1 = \dim \mathcal{S}_1' = d_5 > 0$, and where K_{11} and K_{12} are $d_2 \times d_5$ and $d_3 \times d_5$ matrices, respectively.

We shall now show that $d_5 = d_2 \leqslant d_3$ and that K_{11} and K_{12} both have rank d_2, so that K_{11} is invertible. Let α be a d_5-vector such that $K_{12}\alpha = 0$. Then $K_1\alpha = [{K_{11}\alpha \atop 0}]$ is in $\mathcal{S}_1' \cap \mathcal{S}_2' = \{0\}$. Then $K_1\alpha = 0$ and since K_1 has rank d_5, this implies that $\alpha = 0$. Therefore K_{12} has rank d_5, so $d_5 \leqslant d_3$. Now let α be a d_5-vector such that $K_{11}\alpha = 0$. Then $K_1\alpha = [{0 \atop K_{12}\alpha}]$ is in $\mathcal{S}_1' \cap \mathcal{S}_2'^{\perp} = \{0\}$, whence $\alpha = 0$ and so K_{11} has rank d_5, whence $d_5 \leqslant d_2$. Suppose $d_5 < d_2$. Then for some d_2-vector $l \neq 0$, $l^* K_{11} = 0$, whence $[{l \atop 0}]$ is in $\mathcal{S}_2' \cap \mathcal{S}_1'^{\perp} = \{0\}$. Then $l = 0$, a contradiction. Therefore $d_5 = d_2$, and K_{11} is a square invertible matrix.

Since K_{12} is a $d_3 \times d_2$ matrix of rank d_2, either $d_2 = d_3$ or there is a $d_3 \times (d_3 - d_2)$ matrix K_{22} whose columns are orthonormal such that $K_{22}^* K_{12} = 0$. Then

$$\begin{bmatrix} 0 & K_{11} \\ K_{22} & K_{12} \end{bmatrix}$$

is a $(d_2 + d_3) \times d_3$ matrix whose columns are orthonormal. There is a $(d_2 + d_3) \times d_2$ matrix $[{K_{31} \atop K_{32}}]$ whose columns are orthonormal such that

$$\begin{bmatrix} K_{31} & 0 & K_{11} \\ K_{32} & K_{22} & K_{12} \end{bmatrix}$$

is unitary. Here K_{31} is $d_2 \times d_2$ and K_{32} is $d_3 \times d_2$. In fact, both K_{31} and K_{32} have rank $d_2 > 0$. To see this, suppose that α is a d_2-vector such that $K_{32}\alpha = 0$. Then

$$\begin{bmatrix} K_{31} \\ K_{32} \end{bmatrix} \alpha = \begin{bmatrix} K_{31}\alpha \\ 0 \end{bmatrix}$$

is in $\mathcal{S}_1'^{\perp} \cap \mathcal{S}_2' = \{0\}$, whence $\alpha = 0$ since $[{K_{31} \atop K_{32}}]$ has rank d_2. Thus K_{32} has rank d_2. Since $K_{22}^* K_{32} = 0$, as a consequence of the unitarity of the large partitioned matrix, we see that $K_{32} = K_{12}B$ for some nonsingular $d_2 \times d_2$ matrix B. As a further consequence of unitarity of the large matrix, we have

$$0 = K_{31}^* K_{11} + K_{32}^* K_{12} = K_{31}^* K_{11} + B^* K_{12}^* K_{12},$$

whence

$$K_{31} = -K_{11}^{*-1} K_{12}^* K_{12} B,$$

which is a nonsingular matrix.

Now let (Z_0, W_0) be the $(2n) \times n$ solution matrix of (4.23) such that

$$Z_0(b) = \begin{bmatrix} 0_{d_1} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & K_{11} & 0 \\ 0 & 0 & 0 & K_{12} & 0 \\ 0 & 0 & 0 & 0 & I_{d_4} \end{bmatrix} \tag{4.25a}$$

and

$$W_0(b) = \begin{bmatrix} I_{d_1} & 0 & 0 & 0 & 0 \\ 0 & K_{31} & 0 & 0 & 0 \\ 0 & K_{32} & K_{22} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0_{d_4} \end{bmatrix}. \tag{4.25b}$$

Again, as in the proof of Theorem 2.1, we find that since (4.23) is disconjugate on $[b, \infty)$, (Z_0, W_0) is a self-conjugate solution of (4.23) with $Z_0(t)$ nonsingular on some terminal subinterval $[c, \infty) \subseteq (b, \infty)$.

We again define $S(t)$ for $t > c$ by $S(t) = W_0(t)Z_0^{-1}(t)$, and observe that S is hermitian and satisfies (4.16) for $c < e < t$. As before, we find that either $W_0(t)$ is nonsingular on some terminal subinterval of $[c, \infty)$, or for some $e > c$ and some constant unit vector η, $\eta^*[-S'(t)]\eta \equiv 0$ a.e. on $[e, \infty)$. Then $\eta^* Q_1 \eta \equiv 0$ a.e. on $[e, \infty)$, whence

$$\eta = \begin{bmatrix} \eta_1 \\ \eta_2 \\ 0 \end{bmatrix}$$

for vectors η_1 and η_2 of dimension d_1 and d_2, respectively. Furthermore, $\eta^* SG_1 S\eta \equiv 0$ a.e. on $[e, \infty)$. From (4.23), (4.24b), and (4.25b), we see that

$$W_0(t) = \begin{bmatrix} I_{d_1} & 0 & 0 & 0 & 0 \\ 0 & K_{31} & 0 & 0 & 0 \\ W_{31} & W_{32} & W_{33} & W_{34} & W_{35} \\ W_{41} & W_{42} & W_{43} & W_{44} & W_{45} \end{bmatrix} \tag{4.26}$$

for all $t > b$, whence $\eta^* W_0(t) = (\eta_1^*, \eta_2^* K_{31}, 0) = \mu^*$, where

$$\mu = \begin{bmatrix} \mu_1 \\ \mu_2 \\ 0 \end{bmatrix}$$

and $\mu_1 = \eta_1$, $\mu_2 = K_{31}^* \eta_2$. Since η is a unit vector and K_{31} is nonsingular, μ is nonzero. Then

$$0 \equiv \eta^* SG_1 S\eta = \mu^* Z_0^{-1} G_1 Z_0^{*-1} \mu \quad \text{a.e. on } [e, \infty).$$

Let $\kappa(t)$ be defined by

$$\kappa(t) = Z_0^{*-1}(t)\mu \tag{4.27}$$

for $t > e$. Then $G_1(t)\kappa(t) \equiv 0$ a.e. on $[e, \infty)$ and κ is absolutely continuous there, with $\kappa'(t) = -Z_0^{*-1}W_0^*G_1\kappa = 0$ a.e. on $[e, \infty)$. Thus, κ is constant on $[e, \infty)$ and we must have

$$\kappa = \begin{bmatrix} 0 & \\ K_{11} & 0 \\ K_{12} & 0 \\ 0 & I_{d_4} \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix}$$

for some vectors α_1 and α_2 of dimension d_2 and d_4, respectively. Since μ is nonzero and $Z_0(t)$ is nonsingular on $[e, \infty)$, κ and $\alpha = \left[\begin{smallmatrix}\alpha_1\\ \alpha_2\end{smallmatrix}\right]$ are also nonzero. Combining these results we find that the function $\phi(t)$ defined on $[a, \infty)$ by

$$\phi(t) = \mu - Z_0^*(t) \begin{bmatrix} 0 & 0 \\ K_{11} & 0 \\ K_{12} & 0 \\ 0 & I_{d_4} \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} \tag{4.28}$$

is absolutely continuous on $[a, \infty)$ with

$$\phi'(t) = -W_0^*G_1 \begin{bmatrix} 0 & 0 \\ K_{11} & 0 \\ K_{12} & 0 \\ 0 & I_{d_4} \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} \equiv 0 \quad \text{a.e. on } [b, \infty),$$

and that $\phi(t) \equiv 0$ on $[e, \infty)$. (Recall that the constant vector represented by the product of the expressions in brackets is annihilated identically on $[b, \infty)$ by G_1 since b was chosen so that $k(G; t) = \dim \mathcal{S}_1$ for t in $[b, \infty)$.) This implies that $\phi(t) \equiv 0$ on $[b, \infty)$, and for $t = b$ we have

$$\mu = \begin{bmatrix} \mu_1 \\ \mu_2 \\ 0 \end{bmatrix} = Z_0^*(b) \begin{bmatrix} 0 & 0 \\ K_{11} & 0 \\ K_{12} & 0 \\ 0 & I_{d_4} \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} = \begin{bmatrix} 0 \\ \alpha_1 \\ \alpha_2 \end{bmatrix}.$$

However, $\left[\begin{smallmatrix}\mu_1\\ \mu_2\end{smallmatrix}\right]$ and $\left[\begin{smallmatrix}\alpha_1\\ \alpha_2\end{smallmatrix}\right]$ are $d_1 + d_2$ and $d_2 + d_4$ dimensional nonzero vectors, with $d_1 + d_4 + 2d_2 < n$. This is a contradiction. Consequently $W_0(t)$ is nonsingular on some terminal interval, and the corresponding solution $(X_0, Y_0) = (AZ_0, AW_0)$ of (4.1) is self-conjugate with both members nonsingular on some terminal subinterval. This completes the proof of the theorem. □

Let (X, Y) be a self-conjugate solution of (4.1) such that $X(t)$ is nonsingular on some terminal interval $[a, \infty) \subseteq J$, and suppose that $G(t) > 0$ a.e. on $[a, \infty)$. Then (X, Y) will be called a *principal* solution of (4.1) if there is a point $b \in [a, \infty)$ such that ([6], [7])

$$\lambda_1\left[\int_b^t X^{-1}(s)G(s)X^{*-1}(s)\,ds\right] \to \infty \quad \text{as } t \to \infty. \tag{4.29}$$

One can easily see that if (4.29) holds for one point $b_0 \in [a, \infty)$, then it holds for any point b in $[a, \infty)$. The solution (X, Y) will be called *nonprincipal* if there is a point $b \in [a, \infty)$ such that

$$\lim_{t\to\infty}\left\{\lambda_n\left[\int_b^t X^{-1}(s)G(s)X^{*-1}(s)\,ds\right]\right\} < \infty. \tag{4.30}$$

The following lemma follows as an application of the transformation discussed in §2.

LEMMA 4.2. *Let $G(t)$, $Q_0(t)$, and $Q(t)$ be hermitian $n \times n$ complex matrices in $\mathfrak{L}(J)$, and suppose that (X_0, Y_0) is a self-conjugate solution of*

$$X' = GY, \qquad Y' = -Q_0X \tag{4.31}$$

such that $X_0(t)$ is nonsingular on $[a, \infty) \subseteq J$. Let t be restricted to $[a, \infty)$ and let the system

$$X' = GY, \qquad Y' = -(Q_0 + Q)X \tag{4.32}$$

be transformed by

$$X = X_0Z, \qquad Y = Y_0Z + X_0^{*-1}W. \tag{4.33}$$

Then the resulting system in (Z, W) is given by

$$Z' = G_1W, \qquad W' = -Q_1Z, \tag{4.34}$$

where

$$G_1 = X_0^{-1}GX_0^{*-1}, \tag{4.35a}$$

and

$$Q_1 = X_0^*QX_0. \tag{4.35b}$$

Furthermore, the following hold:

(i) *$G_1(t)$ and $Q_1(t)$ are hermitian matrices in $\mathfrak{L}(J)$ and have the same number of positive eigenvalues at $t \in [a, \infty)$ as do $G(t)$ and $Q(t)$, respectively; and G_1 and G have the same degree of degeneracy.*

(ii) *System* (4.32) *is nonoscillatory iff* (4.34) *is nonoscillatory. If (X, Y) and (Z, W) are corresponding solutions of* (4.32) *and* (4.34), *respectively, on $[a, \infty)$, then $X(t)$ is singular iff $Z(t)$ is singular. (X, Y) is self-conjugate iff (Z, W) is self-conjugate. If $G(t) \gg 0$ a.e. on $[a, \infty)$, then (X, Y) is principal or nonprincipal iff (Z, W) is principal or nonprincipal, respectively.*

PROOF. We shall only verify that G and G_1 have the same degree of degeneracy. The rest of the proof follows directly from the properties of the transformation discussed in §2 and the definitions.

Suppose $G(t)\eta \equiv 0$ a.e. on $[c, \infty) \subseteq [a, \infty)$, where η is some constant vector. Then $(0, \eta)$ is a vector solution of (4.32) on $[c, \infty)$, and $(0, \mu)$ is a solution of (4.34) on $[c, \infty)$, where $\mu = X_0^*(t)\eta$. Differentiation of μ yields that $\mu' = Y_0^* G\eta \equiv 0$ a.e. on $[c, \infty)$. Therefore μ is constant, and $G_1(t)\mu \equiv 0$ a.e. on $[c, \infty)$. Conversely, if for some constant vector μ we have $G_1(t)\mu \equiv 0$ a.e. on $[c, \infty)$, then $\eta = X_0^{*-1}\mu$ is a constant on $[c, \infty)$ and $G(t)\eta \equiv 0$ a.e. there. Recall that $k(G; s)$ is the dimension of the space of constant vectors η such that $G(t)\eta \equiv 0$ a.e. on $[s, \infty)$. Since $X_0(t)$ is nonsingular on $[a, \infty)$, we see that $k(G; s) = k(G_1; s)$ for $s \geq a$. In particular, as $s \to \infty$ we see that the two functions have the same limiting value, which is the degree of degeneracy. □

LEMMA 4.3. *Let* $G(t) \geq 0$ *a.e. and suppose that* (4.1) *is nonoscillatory. Then* G *is nondegenerate if and only if there exists a principal solution of* (4.1).

PROOF. Since (4.1) is nonoscillatory it is disconjugate on some terminal subinterval of J, and by Theorem 2.1 there is a self-conjugate solution (X_0, Y_0) of (4.1) such that $X_0(t)$ is nonsingular on some terminal subinterval $[a, \infty) \subseteq J$. If we transform (4.1) by

$$X = X_0 Z, \qquad Y = Y_0 Z + X_0^{*-1} W \tag{4.36}$$

for $t \in [a, \infty)$, according to Lemma 4.2 we obtain a reduced system

$$Z' = G_1 W, \qquad W' = 0, \tag{4.37}$$

where $G_1 = X_0^{-1} G X_0^{*-1}$. According to Lemma 4.2 it suffices to consider a system of the form (4.37).

Suppose that G_1 is nondegenerate. Let

$$Z_N(t) = I + \int_a^t G_1(s)\, ds. \tag{4.38}$$

Then $Z_N(t)$ is hermitian and positive definite on $[a, \infty)$, and (Z_N, I) is a self-conjugate solution of (4.37). For each $b \in [a, \infty)$ there is a $t > b$ such that

$$\int_b^t G_1(s)\, ds > 0$$

so that

$$Z_N(t) - Z_N(b) = \int_b^t G_1(s)\, ds > 0.$$

Then $Z_N(t) > Z_N(b)$, and $Z_N^{-1}(b) > Z_N^{-1}(t)$. Let

$$\pi_0 = \lim_{t \to \infty} Z_N^{-1}(t). \tag{4.39}$$

Then $Z_N^{-1}(t) > \pi_0 \geq 0$ for all $t \geq a$, and

$$Z_N^{-1}(t) - \pi_0 > 0 \quad \text{for } t \in [a, \infty). \tag{4.40}$$

Now

$$(Z_N^{-1}(t))' = -(Z_N^{-1}G_1Z_N^{-1}) = -(Z_N^{-1}G_1Z_N^{*-1})$$

so that $Z_N^{-1}G_1Z_N^{*-1}$ is integrable on $[a, \infty)$ (this implies that (Z_N, I) is nonprincipal), and defining $\mathcal{F}(t)$ by

$$\mathcal{F}(t) = \int_t^\infty Z_N^{-1}(s)G_1(s)Z_N^{*-1}(s)\,ds \tag{4.41}$$

we see that

$$\mathcal{F}(t) = -\left[Z_N^{-1}(s)\right]_t^\infty = Z_N^{-1}(t) - \pi_0 > 0$$

for $t > a$, and $\mathcal{F}(t) \to 0$ as $t \to \infty$. Define $Z_p(t)$ by

$$Z_p(t) = Z_N(t)\mathcal{F}(t) = I - Z_N(t)\pi_0. \tag{4.42}$$

Then $(Z_p, -\pi_0)$ is a self-conjugate solution of (4.37) such that $Z_p(t)$ is nonsingular on $[a, \infty)$, and

$$Z_p^{-1}G_1Z_p^{*-1} = \mathcal{F}^{-1}Z_N^{-1}G_1Z_N^{*-1}\mathcal{F}^{-1} = -\mathcal{F}^{-1}\mathcal{F}'\mathcal{F}^{-1} = (\mathcal{F}^{-1})'.$$

Then

$$\int_b^t Z_p^{-1}(s)G_1(s)Z_p^{*-1}(s)\,ds = \mathcal{F}^{-1}(t) - \mathcal{F}^{-1}(b) > 0$$

for $t > b$ and all its eigenvalues are unbounded since $\mathcal{F}(t) \to 0$ as $t \to \infty$. Thus $(Z_p, -\pi_0)$ is a principal solution of (4.37).

Now suppose conversely that (Z_p, W_p) is a self-conjugate solution of (4.37) such that $Z_p(t)$ in nonsingular on some subinterval $[b, \infty) \subseteq [a, \infty)$, and that (Z_p, W_p) is principal. Let (4.37) be transformed by

$$Z = Z_pU, \qquad W = W_pU + Z_p^{*-1}V \tag{4.43}$$

for $t \in [b, \infty)$, and by Lemma 4.2 we obtain the system

$$U' = G_2V \qquad V' = 0 \tag{4.44}$$

where $G_2 = Z_p^{-1}G_1Z_p^{*-1}$. Since (Z_p, W_p) is a principal solution of (4.37),

$$\lambda_1\left[\int_b^t G_2(s)\,ds\right] \to \infty \quad \text{as } t \to \infty.$$

This implies that G_2 is nondegenerate, since if $G_2(t)\eta \equiv 0$ a.e. on some subinterval $[c, \infty) \subseteq [b, \infty)$ for some constant unit vector η, then

$$\eta^*\int_b^t G_2(s)\,ds\,\eta = \eta^*\int_b^c G_2(s)\,ds\,\eta,$$

and

$$\lambda_1\left[\int_b^t G_2(s)\,ds\right] < \eta^*\left(\int_b^c G_2(s)\,ds\right)\eta < \infty$$

for all $t > b$. By Lemma 4.2, since G_2 is nondegenerate, G_1 is nondegenerate. The lemma is proved. □

If H and K are $n \times n$ hermitian positive semidefinite matrices, then HK has the same eigenvalues as $L = H^{1/2}KH^{1/2}$, where $H^{1/2}$ is the hermitian positive semidefinite square root of H [5]. This follows from the fact that if A and B are arbitrary $n \times n$ complex matrices, AB has the same eigenvalues as BA. Taking $A = H^{1/2}$ and $B = H^{1/2}K$ the above statement follows. Since L is hermitian positive semidefinite, it has nonnegative real eigenvalues in which case so does HK. Then $\lambda_\nu[L] = \lambda_\nu[HK]$, $1 \leq \nu \leq n$. This is the sense in which (4.49a) and (4.50a) following are to be interpreted.

THEOREM 4.2. *Let $G(t) \geq 0$ and $Q(t) \geq 0$ a.e. on $[a, \infty) \subseteq J$. Define matrices $X_N(t)$, $Y_N(t)$, π_0 and π_1 by*

$$X_N(t) = I + \int_a^t G(s)\,ds. \tag{4.45a}$$

$$Y_N(t) = I + \int_a^t Q(s)\,ds, \qquad t \geq a. \tag{4.45b}$$

$$\pi_0 = \lim_{t\to\infty} X_N^{-1}(t). \tag{4.46a}$$

$$\pi_1 = \lim_{t\to\infty} Y_N^{-1}(t). \tag{4.46b}$$

(i) *If* (4.1) *is nonoscillatory, then*

$$\int_a^\infty \|(I - \pi_0 X_N(s))Q(s)(I - X_N(s)\pi_0)\|\,ds < \infty, \tag{4.47a}$$

$$\int_a^\infty \|(I - \pi_1 Y_N(s))G(s)(I - Y_N(s)\pi_1)\|\,ds < \infty, \tag{4.47b}$$

and

$$(\text{null } \pi_0) \perp (\text{null } \pi_1). \tag{4.47c}$$

(ii) *If* (4.1) *is nonoscillatory and, in addition, $G(t)$ is nondegenerate, then $I - X_N(t)\pi_0$ is nonsingular on $[a, \infty)$. Let matrices $\mathcal{G}_b(t)$ and $\mathcal{Q}(t)$ for $a \leq b \leq t$ be defined by*

$$\mathcal{G}_b(t) = \int_b^t (I - X_N(s)\pi_0)^{-1} G(s)(I - \pi_0 X_N(s))^{-1}\,ds, \tag{4.48a}$$

and

$$\mathcal{Q}(t) = \int_t^\infty (I - \pi_0 X_N(s)Q(s))(I - X_N(s)\pi_0)\,ds. \tag{4.48b}$$

Then for some $b \geq a$ we have

$$\lambda_\nu[\mathcal{G}_b(t)\mathcal{Q}(t)] < 1, \tag{4.49a}$$

and

$$\lambda_{n+1-\nu}[\mathcal{G}_b(t)]\lambda_\nu[\mathcal{Q}(t)] < 1 \tag{4.49b}$$

for $t \geq b$ *and* $1 \leq \nu \leq n$. *Furthermore,*

$$\limsup_{t\to\infty}\left(\lambda_\nu[\mathcal{G}_a(t)\mathcal{Q}(t)]\right) < 1, \tag{4.50a}$$

and

$$\limsup_{t\to\infty}\left(\lambda_{n+1-\nu}[\mathcal{G}_a(t)]\lambda_\nu[\mathcal{Q}(t)]\right) \leq 1 \tag{4.50b}$$

for $1 \leq \nu \leq n$.

(iii) *If* (4.1) *is nonoscillatory and, in addition, Q is nondegenerate, then the conclusion of* (ii) *holds with G, X_N, and π_0 interchanged with Q, Y_N, and π_1, respectively.*

PROOF. According to Theorem 4.1, system (4.1) is nonoscillatory iff its reciprocal (4.2) is nonoscillatory. Therefore, it will suffice to establish (4.47a) and (4.47c) of (i) and the conclusion of (ii). The remainder of the theorem will follow from these by applying them to the reciprocal system (4.2).

We shall prove the theorem in three parts. Initially, suppose that (4.1) is nonoscillatory and that

$$\lambda_1\left[\int_a^t G(s)\,ds\right] \to \infty \quad \text{as } t \to \infty.$$

(The latter condition implies that G is nondegenerate.) According to Theorem 4.1 there exists a self-conjugate solution (X_0, Y_0) of (4.1) such that both members are nonsingular on some terminal subinterval $[b, \infty) \subseteq [a, \infty)$. Furthermore, for

$$S(t) = Y_0(t)X_0^{-1}(t), \qquad t \geq b,$$

we see that $S(t)$ is hermitian and nonsingular on $[b, \infty)$, and that S and S^{-1} satisfy (4.3) and (4.4), respectively, on $[b, \infty)$. Then

$$S^{-1}(t) - S^{-1}(b) = \int_b^t G(\xi)\,d\xi + \int_b^t S^{-1}(\xi)Q(\xi)S^{-1}(\xi)\,d\xi \tag{4.51}$$

for $t \geq b$. Since the second integral term is positive semidefinite, we have

$$S^{-1}(t) - S^{-1}(b) \geq \int_b^t G(\xi)d\xi,$$

and since

$$\lambda_1\left[\int_b^t G(\xi)\,d\xi\right] \to \infty \quad \text{as } t \to \infty$$

we see that $S^{-1}(t)$ and $S(t)$ are positive definite on $[b, \infty)$, and $S(t) \to 0$ as $t \to \infty$. Furthermore, $S(t)$ satisfies

$$S(b) \geq S(b) - S(t) = \int_b^t Q(\xi)\,d\xi + \int_b^t S(\xi)G(\xi)S(\xi)\,d\xi, \tag{4.52}$$

and since $Q(t) \geq 0$ and $S(t)G(t)S(t) \geq 0$, this implies that $\|Q(t)\|$ and $\|S(t)G(t)S(t)\|$ are integrable on $[b, \infty)$, whence

$$S(t) = \int_t^\infty Q(\xi)\, d\xi + \int_t^\infty S(\xi)G(\xi)S(\xi)\, d\xi \tag{4.53}$$

for $t \geq b$. In the case being considered, since the eigenvalues of $\int_a^t G(\xi)\, d\xi$ are all unbounded, $\pi_0 = 0$. Since $\int_a^\infty Q(\xi)\, d\xi$ exists, π_1 is nonsingular. This establishes (4.47a) and (4.47c) for this case, and also establishes the corollary immediately following this theorem.

Equations (4.53) and (4.51) yield

$$S(t) \geq \int_t^\infty Q\, d\xi = \mathfrak{Q}(t)$$

and

$$S^{-1}(t) - S^{-1}(b) \geq \int_b^t G\, d\xi = \mathcal{G}_b(t).$$

Thus

$$S^{-1}(t) \geq S^{-1}(b) + \mathcal{G}_b(t) \geq S^{-1}(b) > 0,$$

so

$$\left(S^{-1}(b) + \mathcal{G}_b(t)\right)^{-1} \geq S(t) \geq \mathfrak{Q}(t) \quad \text{for } t \geq b, \tag{4.54}$$

Then $\lambda_\nu[\mathfrak{Q}(t)] \leq 1/\{\lambda_{n+1-\nu}[S^{-1}(b) + \mathcal{G}_b(t)]\}$, and since $\lambda_\nu[\mathcal{G}_b(t) + S^{-1}(b)] > \lambda_\nu[\mathcal{G}_b(t)]$ for all $\nu = 1, 2, \ldots, n$, this establishes (4.49b). Also, from (4.54) we have

$$I \geq \left(S^{-1}(b) + \mathcal{G}_b(t)\right)^{1/2}\mathfrak{Q}(t)\left(S^{-1}(b) + \mathcal{G}_b(t)\right)^{1/2}$$

which yields

$$\begin{aligned} 1 &\geq \lambda_\nu\left[\left(S^{-1}(b) + \mathcal{G}_b(t)\right)^{1/2}\mathfrak{Q}(t)\left(S^{-1}(b) + \mathcal{G}_b(t)\right)^{1/2}\right] \\ &= \lambda_\nu\left[\mathfrak{Q}^{1/2}(t)\left(S^{-1}(b) + \mathcal{G}_b(t)\right)\mathfrak{Q}^{1/2}(t)\right] \\ &= \lambda_\nu\left[\mathfrak{Q}^{1/2}(t)S^{-1}(b)\mathfrak{Q}^{1/2}(t) + \mathfrak{Q}^{1/2}(t)\mathcal{G}_b(t)\mathcal{G}^{1/2}(t)\right] \\ &\geq \lambda_\nu\left[\mathfrak{Q}^{1/2}(t)\mathcal{G}_b(t)\mathfrak{Q}^{1/2}(t)\right] = \lambda_\nu\left[\mathcal{G}_b(t)\mathfrak{Q}(t)\right] \end{aligned}$$

for $1 \leq \nu \leq n$. This establishes (4.49a).

To establish (4.50), we note that

$$\begin{aligned} S^{-1}(t) \geq S^{-1}(b) + \int_b^t G\, d\xi &= S^{-1}(b) - \int_a^b G\, d\xi + \int_a^t G\, d\xi \\ &= \mathcal{G}_a(t) + \left(S^{-1}(b) - \mathcal{G}_a(b)\right) > 0 \quad \text{for } t \geq b. \end{aligned}$$

Thus,

$$\left(\mathcal{G}_a(t) + S^{-1}(b) - \mathcal{G}_a(b)\right)^{-1} \geq S(t) \geq \mathfrak{Q}(t) \quad \text{for } t \geq b, \tag{4.55}$$

and this implies that

$$\lambda_\nu[\mathcal{Q}(t)] < 1/\{\lambda_{n+1-\nu}[\mathcal{G}_a(t) - \mathcal{G}_a(b) + S^{-1}(b)]\}.$$

Since

$$\lambda_\nu[\mathcal{G}_a(t) - \mathcal{G}_a(b) + S^{-1}(b)] > \lambda_\nu[\mathcal{G}_a(t)] - \|\mathcal{G}_a(b) - S^{-1}(b)\|,$$

we see that

$$1 > \lambda_\nu[\mathcal{Q}(t)](\lambda_{n+1-\nu}[\mathcal{G}_a(t)] - \|\mathcal{G}_a(b) - S^{-1}(b)\|),$$

and since $\lambda_\nu[\mathcal{Q}(t)] \to 0$ as $t \to \infty$, this establishes (4.50b). Equation (4.50a) is established by a similar argument.

In the next part of the proof we shall relax the hypothesis on $G(t)$, supposing only that it is nondegenerate. Let $X_N(t)$ be defined as in (4.45a). Then as in the proof of Lemma 4.3, we see that (X_N, I) is a self-conjugate, nonprincipal solution of

$$X' = GY, \qquad Y' = 0, \tag{4.56}$$

where $X_N(t)$ is nonsingular on $[a, \infty)$. Furthermore, if we define $X_p(t)$ by

$$X_p(t) = I - X_N(t)\pi_0, \tag{4.57}$$

we see, as in Lemma 4.3, that $(X_p, -\pi_0)$ is a principal, self-conjugate solution of (4.56) such that $X_p(t)$ is also nonsingular on $[a, \infty)$. Let (4.1) be transformed by

$$X = X_p Z, \qquad Y = -\pi_0 Z + X_p^{*-1} W. \tag{4.58}$$

Applying Lemma 4.2 we see that the resulting system

$$Z' = G_1 W, \qquad W = -Q_1 Z, \tag{4.59}$$

where $G_1 = X_p^{-1} G X_p^{*-1}$ and $Q_1 = X_p^* Q X_p$, is nonoscillatory; and $G_1(t) > 0$ and $Q_1(t) > 0$ a.e. on $[a, \infty)$, G_1 is nondegenerate, and

$$\lambda_1\left[\int_a^t G_1(\xi)\, d\xi\right] \to \infty \quad \text{as } t \to \infty$$

since $(X_p, -\pi_0)$ is principal. Therefore, the argument of the first part of the proof applies to system (4.59), establishing (4.47a) and the conclusion of (ii).

In this third and final part of the proof, it remains only to establish (4.47a) and (4.47c) under the hypothesis that (4.1) is nonoscillatory. We shall first establish (4.47a). Since the case in which G is nondegenerate has been treated, we assume that m, the degree of degeneracy of G, is positive. Then there exists an $n \times m$ matrix A_1 whose columns form an orthonormal basis for the space of degeneracy of G. Furthermore, $G(t)A_1 \equiv 0$ a.e. in some subinterval $[b, \infty) \subseteq [a, \infty)$. If $m = n$, then A_1 is nonsingular, $G(t) \equiv 0$ a.e. on $[b, \infty)$, and $\pi_0 = X_N^{-1}(b)$. Then the integrand of (4.47a) is zero on $[b, \infty)$. The integral is therefore bounded, establishing (4.47a). This leaves the case $0 < m < n$. In this case, there is a matrix A_2 such that $A = [A_2 \vdots A_1]$ is unitary. Let (4.1) be transformed by $U = A^*X$ and $V = A^*Y$. The resulting system

$$U' = G_1V, \qquad V' = -Q_1U, \tag{4.60}$$

where $G_1 = A^*GA$ and $Q_1 = A^*QA$, is also Hamiltonian and nonoscillatory. Let G_1 and Q_1 be partitioned according to the partition of A. Then for $t > b$ we have

$$G_1(t) = \begin{bmatrix} G_{11} & 0 \\ 0 & 0_m \end{bmatrix}, \tag{4.61}$$

where 0_m is the $m \times m$ zero matrix, and G_{11} is an $(n-m) \times (n-m)$ nondegenerate, hermitian matrix.

Since (4.1) is nonoscillatory, there is a self-conjugate solution (X_0, Y_0) of (4.1) such that $X_0(t)$ is nonsingular on some terminal interval $[d, \infty) \subseteq [b, \infty)$. Let (U_0, V_0) be the corresponding solution of (4.60). Let U and V in (4.60) be partitioned according to the partition on G_1. Then

$$U = \begin{bmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{bmatrix},$$

and for $t > b$ we see that U_{21} and U_{22} are constant for any solution (U, V) of (4.60). In particular, since $U_0(t)$ is nonsingular on $[d, \infty) \subseteq [b, \infty)$, there is a nonsingular matrix B such that

$$U_0B = Z_0 = \begin{bmatrix} Z_{0_{11}} & Z_{0_{12}} \\ 0 & I_m \end{bmatrix} \quad \text{on } [d, \infty). \tag{4.62}$$

Thus $(Z_0, W_0) = (U_0, V_0)B$ is a self-conjugate solution of (4.60) such that $\det Z_0(t) = \det Z_{0_{11}}(t)$ is nonzero on $[d, \infty)$. In fact $(Z_{0_{11}}, W_{0_{11}})$ is a solution of

$$Z_{11}' = G_{11}W_{11}, \qquad W_{11}' = -Q_{11}Z_{11}, \tag{4.63}$$

where $(Z_{0_{11}}, W_{0_{11}})$ is self-conjugate on $[d, \infty)$ since (Z_0, W_0) is. Since G_{11} and Q_{11} are positive semidefinite and G_{11} is nondegenerate, the argument of the second part applies to system (4.63).

Let

$$Z_N(t) = I + \int_a^t G_{11}\, d\xi, \quad P_0 = \lim_{t\to\infty} Z_N^{-1}(t)$$

and $Z_p(t) = I - Z_N(t)P_0$. Then we have

$$\int_a^\infty \|Z_p^*(\xi)Q_{11}(\xi)Z_p(\xi)\|\, d\xi < \infty. \tag{4.64}$$

Let $X_N(t)$ be defined as in (4.45a), and for $t > d$ we have

$$A^*X_N(t)A = \begin{bmatrix} Z_N(t) & C \\ C^* & D \end{bmatrix},$$

where $C = \int_a^d G_{12}\, d\xi$, and $D = I + \int_a^d G_{22}\, d\xi$ is nonsingular. Then

$$A^*\pi_0 A = \lim_{t\to\infty}\left[A^* X_N^{-1}(t) A\right]$$
$$= \lim_{t\to\infty}\begin{bmatrix} Z_N^{-1}(t)\left[I + C\left(D - C^* Z_N^{-1}(t)C\right)^{-1} C^* Z_N^{-1}(t)\right] & -Z_N^{-1}(t)C\left(D - C^* Z_N^{-1}(t)C\right)^{-1} \\ -\left(D - C^* Z_N^{-1}(t)C\right)^{-1} C^* Z_N^{-1}(t) & \left(D - C^* Z_N^{-1}(t)C\right)^{-1}\end{bmatrix}$$
$$= \begin{bmatrix} P_0[I + CEC^*P_0] & -P_0CE \\ -EC^*P_0 & E\end{bmatrix}$$

where $E = (D - C^*P_0C)^{-1}$. This yields

$$A^*(I - X_N(t)\pi_0)A = I - (A^*X_N(t)A)(A^*\pi_0 A) = \begin{bmatrix} Z_p(t)H & Z_p(t)K \\ 0 & 0\end{bmatrix}$$

where $H = I + CEC^*P_0$ and $K = -CE$. Therefore

$$A^*(I - \pi_0X_N)Q(I - X_N\pi_0)A = \begin{bmatrix} H^* & 0 \\ K^* & 0\end{bmatrix}\begin{bmatrix} Z_p^*Q_{11}Z_p & 0 \\ 0 & 0\end{bmatrix}\begin{bmatrix} H & K \\ 0 & 0\end{bmatrix}$$

and since A is unitary and $\|Z_p^*(t)Q_{11}(t)Z_p(t)\|$ is integrable on $[a, \infty)$ by (4.64), we see that $\|(I - \pi_0X_N(t))Q(t)(I - X_N(t)\pi_0)\|$ is integrable on $[a, \infty)$. This establishes (4.47a)

To establish (4.47c), let $X_N(t)$, $Y_N(t)$, π_0, and π_1 be defined as in (4.45) and (4.46). Then $Y_N(t)$ is hermitian, positive definite, and has nondecreasing eigenvalues. If either π_0 or π_1 is nonsingular, then (4.47c) holds. Suppose both are singular, and that ξ and η are in null π_0 and null π_1, respectively. Then $\pi_0\xi = 0$ and $\pi_1\eta = 0$. With (4.47a) this implies that

$$\xi^*Q(t)\xi = \xi^*(I - \pi_0X_N)Q(I - X_N\pi_0)\xi$$

is integrable on $[a, \infty)$, whence $\xi^*Y_N(t)\xi$ is uniformly bounded there, say by $K > 0$. Since

$$\xi^*\eta = \xi^*Y_N^{1/2}(t)Y_N^{-1/2}(t)\eta,$$

the Cauchy-Schwartz inequality yields

$$|\xi^*\eta|^2 < (\xi^*Y_N(t)\xi)\left(\eta^*Y_N^{-1}(t)\eta\right) < K\left(\eta^*Y_N^{-1}(t)\eta\right)$$

for all $t > a$, and since

$$\lim_{t\to\infty}\left(\eta^*Y_N^{-1}(t)\eta\right) = \eta^*\pi_1\eta = 0,$$

we see that $\xi^*\eta = 0$. This establishes (4.47c) and completes the proof of the theorem. □

An immediate corollary to the first part of the proof of this theorem is the following.

COROLLARY 4.1. *Let $G(t) \geqslant 0$ and $Q(t) \geqslant 0$ a.e. in $[a, \infty) \subseteq J$ and suppose that*

$$\lambda_1\left[\int_a^t G(s)\, ds\right] \to \infty \quad \text{as } t \to \infty.$$

Then (4.1) *is nonoscillatory if and only if $Q(t)$ is integrable on $[a, \infty)$ and there exists an absolutely continuous, hermitian [positive definite] matrix $S(t)$ such that on some terminal subinterval of $[a, \infty)$, $S(t)G(t)S(t)$ is integrable and $S(t)$ satisfies* (4.53) *there.*

This extends a result of Ahlbrandt (Theorem 3.2 of [2]) in that it does not require Q to be nondegenerate, and it therefore resolves a question posed by that author [2].

If we consider Theorem 4.2 from the standpoint of testing for oscillation, we see that (4.47a) implies (4.47c). Also, the case $\nu = n$ of (4.49a) implies the rest of (4.49) and (4.50). Therefore, increasingly stronger tests are provided by (4.47c), (4.47a), and (4.50a) with $\nu = n$. To see that these provide strictly stronger necessary conditions for nonoscillation, but that (4.50a) does not provide a sufficient condition for nonoscillation, consider the following oscillatory systems

$$x' = t^{\alpha_k - 1}y, \qquad y' = -t^{-\beta_k - 1}x, \qquad 1 < k < 4, \quad t > 1, \tag{4.65.k}$$

where (α_k, β_k) is $(0, 0)$, $(-\frac{1}{4}, -\frac{3}{4})$, $(-\frac{1}{4}, -\frac{1}{4})$, and $(-1, -1)$ for $k = 1, 2, 3$, and 4, respectively. System (4.65.1) fails (4.47c). System (4.65.2) satisfies (4.47c) but fails (4.47a). System (4.65.3) satisfies (4.47a) but fails (4.50a). System (4.65.4) satisfies even (4.50a), yet it is oscillatory.

The necessary condition for nonoscillation expressed by (4.47c) is equivalent to a result of Tomastik [16], which is proved for G and Q positive definite. Theorem 4.2 therefore extends this result and presents a more algebraic proof as opposed to the geometric proof in that paper. The theorem also extends results of Ahlbrandt [1], which are established under the additional hypothesis that

$$\lambda_1\left[\int_a^t G(\xi)\, d\xi\right] \to \infty \quad \text{as } t \to \infty.$$

For example, system (4.1) for

$$G(t) = \begin{bmatrix} t^{-1/4} & 0 \\ 0 & t^{-5/4} \end{bmatrix} \tag{4.66a}$$

and

$$Q(t) = \begin{bmatrix} t^{-5/4} & 0 \\ 0 & t^{-1/4} \end{bmatrix} \tag{4.66b}$$

is oscillatory since (4.47a) is not satisfied, yet the results of neither [16] nor [1] apply to indicate this, since

$$\pi_0 = \begin{bmatrix} 0 & 0 \\ 0 & \frac{1}{5} \end{bmatrix} \quad \text{and} \quad \pi_1 = \begin{bmatrix} \frac{1}{5} & 0 \\ 0 & 0 \end{bmatrix},$$

so that null π_0 and null π_1 are spanned by $\{\left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right]\}$ and $\{\left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right]\}$, respectively, and are therefore orthogonal, and

$$\lambda_1\left[\int_1^t G(\xi)\, d\xi\right] = 4(1 - t^{-1/4})$$

is uniformly bounded on $[1, \infty)$.

We shall precede the next theorem by some preliminary remarks. Suppose $Q(t)$ is hermitian on J and is in $\mathfrak{L}(J)$. Then there exists a hermitian matrix $R(t) > 0$ on J with $R(t)$ in $\mathfrak{L}(J)$ and $R^2(t) = Q^2(t)$ on J. That is, R is the positive semidefinite "square root" of $Q^2(t) > 0$ on J. Let $Q_+(t)$ and $Q_-(t)$ be defined on J by

$$Q_+(t) = [R(t) + Q(t)]/2 \tag{4.67a}$$

and

$$Q_-(t) = [R(t) - Q(t)]/2. \tag{4.67b}$$

Then Q, R, Q_+, and Q_- are similar matrices for all t in J. $R(t)$, $Q_+(t)$, and $Q_-(t)$ are positive semidefinite on J and are in $\mathfrak{L}(J)$. Fix $t \in J$. The eigenvalues of $Q(t)$ are real. The eigenvalues of $R(t)$ are $|\lambda_\nu[Q(t)]|$, where $\lambda_\nu[Q(t)]$, $1 < \nu < n$, are the eigenvalues of $Q(t)$. The positive eigenvalues of $Q_+(t)$ are the positive eigenvalues of $Q(t)$. The positive eigenvalues of $Q_-(t)$ are the positive eigenvalues of $-Q(t)$. In this sense, $Q_+(t)$ and $Q_-(t)$ are the "positive" and "negative" parts of $Q(t)$, and

$$Q(t) = Q_+(t) - Q_-(t) \tag{4.68}$$

for all $t \in J$. The decomposition (4.68) is minimal in the sense that if $A(t)$ and $B(t)$ are hermitian with $A(t) > 0$ and $B(t) > 0$ a.e. on J, and

$$Q(t) = A(t) - B(t), \tag{4.69}$$

then $A(t) > Q_+(t)$ and $B(t) > Q_-(t)$ a.e. on J.

We shall also use the fact that if $G(t) > 0$ and $Q(t) < 0$ a.e. on $[a, \infty) \subseteq J$, then (4.1) is disconjugate on $[a, \infty)$. The following theorem applies Lemma 4.2 to extend Theorem 4.2 to apply to (4.1) when $Q(t)$ is not necessarily positive semidefinite.

THEOREM 4.3. *Let $G(t) \geq 0$ a.e. on $[c, \infty) \subseteq J$. Let*

$$Q(t) = Q_1(t) - Q_2(t) \tag{4.70}$$

be a decomposition of Q such that Q_1 and Q_2 are hermitian matrices in $\mathfrak{L}(J)$, and $Q_1(t) \geq 0$ and $Q_2(t) \geq 0$ a.e. on $[c, \infty)$. Let (X_0, Y_0) be a self-conjugate solution of

$$X' = GY, \qquad Y' = Q_2 X \tag{4.71}$$

such that $X_0(t)$ is nonsingular on some interval $[a, \infty) \subseteq [c, \infty)$. Let $\hat{G}(t)$ and $\hat{Q}(t)$ be defined for $t \geq a$ by

$$\hat{G}(t) = X_0^{-1}(t) G(t) X_0^{*-1}(t) \tag{4.72a}$$

and

$$\hat{Q}(t) = X_0^*(t) Q_1(t) X_0(t). \tag{4.72b}$$

Define $X_N(t)$, $Y_N(t)$, π_0, and π_1 by

$$X_N(t) = I + \int_a^t \hat{G}(s)\, ds, \tag{4.73a}$$

$$Y_N(t) = I + \int_a^t \hat{Q}(s)\, ds, \tag{4.73b}$$

and

$$\pi_0 = \lim_{t \to \infty} X_N^{-1}(t), \tag{4.74a}$$

$$\pi_1 = \lim_{t \to \infty} Y_N^{-1}(t). \tag{4.74b}$$

(i) *If* (4.1) *is nonoscillatory, then*

$$\int_a^\infty \|(I - \pi_0 X_N(s)) \hat{Q}(s) (I - X_N(s)\pi_0)\|\, ds < \infty, \tag{4.75a}$$

$$\int_a^\infty \|(I - \pi_1 Y_N(s)) \hat{G}(s) (I - Y_N(s)\pi_1)\|\, ds < \infty, \tag{4.75b}$$

and

$$(\text{null } \pi_0) \perp (\text{null } \pi_1). \tag{4.75c}$$

(ii) *If* (4.1) *is nonoscillatory and, in addition, $G(t)$ is nondegenerate, then $I - X_N(t)\pi_0$ is nonsingular on $[a, \infty)$. Let matrices $\mathcal{G}_b(t)$ and $\mathcal{Q}(t)$ for $a < b < t$ be defined by*

$$\mathcal{G}_b(t) = \int_b^t (I - X_N(s)\pi_0)^{-1} \hat{G}(s) (I - \pi_0 X_N(s))^{-1}\, ds \tag{4.76a}$$

and

$$\mathcal{Q}(t) = \int_t^\infty (I - \pi_0 X_N(s)) \hat{Q}(s) (I - X_N(s)\pi_0)\, ds. \tag{4.76b}$$

Then for some $b > a$ we have

$$\lambda_\nu[\mathcal{G}_b(t)\mathcal{Q}(t)] < 1, \tag{4.77a}$$

and

$$\lambda_{n+1-\nu}[\mathcal{G}_b(t)]\lambda_\nu[\mathcal{Q}(t)] < 1 \quad \textit{for } t > b \textit{ and } 1 \leqslant \nu \leqslant n. \tag{4.77b}$$

Furthermore,

$$\limsup_{t\to\infty}(\lambda_\nu[\mathcal{G}_a(t)\mathcal{Q}(t)]) < 1 \tag{4.78a}$$

and

$$\limsup_{t\to\infty}(\lambda_{n+1-\nu}[\mathcal{G}_a(t)]\lambda_\nu[\mathcal{Q}(t)]) < 1 \quad \textit{for } 1 \leqslant \nu \leqslant n. \tag{4.78b}$$

(iii) *If* (4.1) *is nonoscillatory and, in addition, $\hat{Q}$ is nondegenerate, then the conclusion of* (ii) *holds with $\hat{G}$, X_N, and π_0 interchanged with $\hat{Q}$, Y_N, and π_1, respectively.*

Proof. Since (4.71) is disconjugate on $[c, \infty)$, there exists a self-conjugate solution (X_0, Y_0) such that $X_0(t)$ is nonsingular on some terminal subinterval $[a, \infty) \subseteq [c, \infty)$, as hypothesized. Transforming (4.1) by

$$X = X_0Z, \qquad Y = Y_0Z + X_0^{*-1}W, \qquad t > a, \tag{4.79}$$

and applying Lemma 4.2, we obtain the system

$$Z' = \hat{G}W, \qquad W' = -\hat{Q}Z \tag{4.80}$$

where $\hat{G}$ and $\hat{Q}$ are given by (4.72). System (4.80) is nonoscillatory, $\hat{G}(t) > 0$ and $\hat{Q}(t) > 0$ a.e. on $[a, \infty)$, and $\hat{G}$ is nondegenerate iff G is nondegenerate. The conclusion now follows by applying Theorem 4.2 to (4.80). □

Theorems 4.2 and 4.3 established necessary conditions for nonoscillation. The next two theorems establish sufficient conditions for nonoscillation.

Theorem 4.4. *Let $G(t) > 0$ for almost all t in J and let $G(t)$ and $Q(t)$ satisfy the hypothesis of Theorem* 3.1 *or Theorem* 3.2. *Then* (4.1) *is nonoscillatory.*

Proof. In either case there is a solution (X_0, Y_0) of (4.1) such that $X_0(t) \to I$ and $Y_0(t) \to 0$ as $t \to \infty$. Thus $X_0^*(t)Y_0(t) \to 0$, which implies that (X_0, Y_0) is self-conjugate. Clearly, $X_0(t)$ is nonsingular on some terminal subinterval $[a, \infty) \subseteq J$, and by Theorem 2.2 the system is nonoscillatory. □

Suppose $G(t)$ and $Q(t)$ are $n \times n$ hermitian matrices in $\mathcal{L}(J)$ and that $G(t) > 0$, and $Q(t) > 0$ a.e. on $[a, \infty) \subseteq J$. Suppose, also, that $\int_a^\infty \|Q(s)\|\, ds < \infty$. Let $\mathcal{G}(t)$ and $\mathcal{Q}(t)$ be defined by

$$\mathcal{G}(t) = \int_a^t G(s)\, ds \tag{4.81a}$$

and

$$\mathcal{Q}(t) = \int_t^\infty Q(s)\, ds. \tag{4.81b}$$

An integration by parts yields

$$\int_a^t G(s)\mathcal{Q}(s)\,ds = \mathcal{G}(t)\mathcal{Q}(t) + \int_a^t \mathcal{G}(s)Q(s)\,ds \tag{4.82a}$$

and

$$\int_a^t G(s)\int_s^t Q(\xi)\,d\xi\,ds = \int_a^t \mathcal{G}(s)Q(s)\,ds. \tag{4.82b}$$

Recall that if H and K are $n \times n$, hermitian, positive semidefinite matrices, then HK has real nonnegative eigenvalues. Therefore, trace$[HK] \geq 0$. Suppose that

$$\lim_{t\to\infty} \int_a^t \operatorname{tr}[\mathcal{G}(s)Q(s)]\,ds = \int_a^\infty \operatorname{tr}[\mathcal{G}(s)Q(s)]\,ds < \infty. \tag{4.83}$$

Then applying (4.82b) we see that

$$\begin{aligned}\lim_{t\to\infty} \int_a^t \operatorname{tr}\left[G(s)\int_s^t Q(\xi)\,d\xi\right] ds &= \int_a^\infty \operatorname{tr}\left[G(s)\int_s^\infty Q(\xi)\,d\xi\right] ds\\ &= \int_a^\infty \operatorname{tr}[G(s)\mathcal{Q}(s)]\,ds\\ &= \int_a^\infty \operatorname{tr}[\mathcal{G}(s)Q(s)]\,ds < \infty\end{aligned}$$

also. Thus (4.83) implies that

$$\lim_{t\to\infty} \int_a^t \operatorname{tr}[G(s)\mathcal{Q}(s)]\,ds = \int_a^\infty \operatorname{tr}[G(s)\mathcal{Q}(s)]\,ds < \infty, \tag{4.84}$$

and that

$$\int_a^\infty \operatorname{tr}[G(s)\mathcal{Q}(s)]\,ds = \int_a^\infty \operatorname{tr}[\mathcal{G}(s)Q(s)]\,ds. \tag{4.85}$$

Furthermore, applying this to (4.82a) yields

$$\lim_{t\to\infty} \operatorname{tr}[\mathcal{G}(t)\mathcal{Q}(t)] = 0. \tag{4.86}$$

On the other hand, suppose that (4.84) holds. Then since $\int_a^t \operatorname{tr}[\mathcal{G}(s)Q(s)]\,ds$ is an increasing function, and $\operatorname{tr}[\mathcal{G}(t)\mathcal{Q}(t)] \geq 0$, (4.82a) implies that (4.83) holds, since

$$\begin{aligned}\int_a^\infty \operatorname{tr}[G(s)\mathcal{Q}(s)]\,ds &\geq \operatorname{tr}[\mathcal{G}(t)\mathcal{Q}(t)] + \int_a^t \operatorname{tr}[\mathcal{G}(s)Q(s)]\,ds\\ &\geq \int_a^t \operatorname{tr}[\mathcal{G}(s)Q(s)]\,ds \quad \text{for all } t \geq a.\end{aligned}$$

This in turn implies (4.85) and (4.86) again. These remarks establish all but (ii)(b) of the following theorem.

THEOREM 4.5. *Let* $G(t) \geq 0$ *and* $Q(t) \geq 0$ *a.e. on* $[a, \infty) \subseteq J$, *and suppose that*

$$\int_a^\infty \|Q(s)\|\, ds < \infty. \tag{4.87}$$

Then

(i) (4.83) *holds if and only if* (4.84) *holds.*

(ii) *If either* (4.83) *or* (4.84) *holds, then*

(a) (4.85) *and* (4.86) *hold, and*

(b) (4.1) *and* (4.2) *are nonoscillatory.*

Before proceeding with the proof, let us compare these last two theorems. Theorem 4.5 assumes that $Q(t)$ is positive semidefinite as well as $G(t)$. However, the remaining part of the hypothesis is weaker than that of Theorem 4.4. In fact, provided both are positive semidefinite, then the hypothesis of Theorem 3.1 implies that (4.84) holds and the hypothesis of Theorem 3.2 implies that (4.83) holds. Therefore, as a test for nonoscillation of (4.1) with both G and Q positive semidefinite, Theorem 4.5 is stronger than Theorem 4.4.

To see that the test is strictly stronger in this case, consider system (4.1) where

$$G(t) = \begin{bmatrix} t^{1-\varepsilon/2}\, 2[\varepsilon \ln t + 2]^{-1} & 0 \\ 0 & t^{1-\eta/2}\, 2[\eta \ln t + 2]^{-1} \end{bmatrix} \tag{4.88a}$$

and

$$Q(t) = \begin{bmatrix} t^{-1-\varepsilon} & 0 \\ 0 & t^{-1-\eta} \end{bmatrix}, \tag{4.88b}$$

where $0 < \varepsilon < \eta/2 < 1$ and $t \geq 1$. Then

$$\mathcal{G}(t) = \int_1^t G(s)\, ds = \begin{bmatrix} t^{\varepsilon/2} \ln t & 0 \\ 0 & t^{\eta/2} \ln t \end{bmatrix}. \tag{4.89a}$$

Since $\|Q(t)\| = t^{-1-\varepsilon}$ is integrable on $[1, \infty)$, Q satisfies the first parts of the hypotheses of both theorems, and

$$\mathcal{Q}(t) = \int_t^\infty Q(s)\, ds = \begin{bmatrix} \varepsilon^{-1} \tau^{-\varepsilon} & 0 \\ 0 & \eta^{-1} t^{-\eta} \end{bmatrix}. \tag{4.89b}$$

Since neither

$$\|\mathcal{G}(t)\|\, \|Q(t)\| = (\ln t) t^{-1-\varepsilon+\eta/2} \tag{4.90a}$$

nor

$$\|G(t)\|\, \|\mathcal{Q}(t)\| = 2[\varepsilon^2 \ln t + 2\varepsilon]^{-1} t^{1-3\varepsilon/2} \tag{4.90b}$$

is integrable on $[1, \infty)$. Theorem 4.4 does not indicate nonoscillation. However, since

$$\operatorname{tr}[\mathcal{G}(t)Q(t)] = (\ln t)(t^{-1-\varepsilon/2} + t^{-1-\eta/2}) \tag{4.91}$$

is integrable on $[1, \infty)$, Theorem 4.5 indicates that the system is nonoscillatory.

PROOF OF THEOREM 4.5. In view of Theorem 4.1 and the remarks preceding the statement of the theorem, it will suffice to show that equation (4.4) has a hermitian, positive definite solution on some terminal subinterval of J.

The hypothesis implies that $\int_a^t \operatorname{tr}[(I + \mathcal{G}(s))Q(s)]\, ds$ is uniformly bounded on $[a, \infty)$. Then

$$\begin{aligned}\lambda_n[(I + \mathcal{G}(t))Q(t)] &= \lambda_n[(I + \mathcal{G}(t))^{1/2}Q(t)(I + \mathcal{G}(t))^{1/2}] \\ &= \|(I + \mathcal{G}(t))^{1/2}Q(t)(I + \mathcal{G}(t))^{1/2}\|\end{aligned}$$

is integrable on $[a, \infty)$. Let η be a real number greater than 2, and let $b > a$ be large enough so that

$$\int_b^\infty \|(I + \mathcal{G}(s))^{1/2}Q(s)(I + \mathcal{G}(s))^{1/2}\|\, ds < 1 - 2/\eta. \tag{4.92}$$

We shall complete the proof by showing that

$$S(t) = I + \int_b^t G(\xi)\, d\xi + \int_b^t S(\xi)Q(\xi)S(\xi)\, d\xi \tag{4.93}$$

has an absolutely continuous, hermitian, positive definite solution on $[b, \infty)$.

Define $\Gamma(t)$ for $t \geq b$ by

$$\Gamma(t) = I + \int_b^t G(\xi)\, d\xi. \tag{4.94}$$

Γ is hermitian, positive definite, and absolutely continuous with nondecreasing eigenvalues on $[b, \infty)$. Let (4.93) be transformed by

$$R(t) = \Gamma^{-1/2}(t)S(t)\Gamma^{-1/2}(t), \tag{4.95}$$

which yields the equivalent integral equation

$$\begin{aligned}R(t) = I + \int_b^t &\Gamma^{-1/2}(t)\Gamma^{1/2}(s) \\ &\times [R(s)\Gamma^{1/2}(s)Q(s)\Gamma^{1/2}(s)R(s)]\Gamma^{1/2}(s)\Gamma^{-1/2}(t)\, ds.\end{aligned} \tag{4.96}$$

It will suffice to show that (4.96) has an absolutely continuous, hermitian, positive definite solution on $[b, \infty)$.

Define $\omega(t)$ on $[b, \infty)$ by

$$\omega(t) = \int_b^t \|\Gamma^{1/2}(s)Q(s)\Gamma^{1/2}(s)\|\, ds. \tag{4.97}$$

Let A, B, and C be hermitian matrices with $A \geq 0$ and $B \geq C \geq 0$. Then for

$1 \geq \nu \geq n$ we have

$$\lambda_\nu[BA] = \lambda_\nu[A^{1/2}BA^{1/2}] = \lambda_\nu[A^{1/2}(C + (B - C))A^{1/2}]$$
$$= \lambda_\nu[A^{1/2}CA^{1/2} + A^{1/2}(B - C)A^{1/2}] \geq \lambda_\nu[A^{1/2}CA^{1/2}] = \lambda_\nu[CA].$$

Then for $a \leq b \leq t$,

$$I + \mathcal{G}(t) = I + \int_b^t G(\xi)\, d\xi + \int_a^b G(\xi)\, d\xi$$
$$= \Gamma(t) + \int_a^b G(\xi)\, d\xi \geq \Gamma(t),$$

and inequality (4.92) combined with the definition of $\omega(t)$ given by (4.97) yields

$$\omega(t) \leq \lim_{\xi \to \infty} \omega(\xi) \leq 1 - 2/\eta < 1 \quad \text{for all } t \text{ in } [b, \infty). \tag{4.98}$$

Also, for $t \geq s \geq b$, $\Gamma(t) \geq \Gamma(s)$, and

$$\|\Gamma^{-1/2}(t)\Gamma^{1/2}(s)\|^2 = \lambda_n[\Gamma^{1/2}(s)\Gamma^{-1}(t)\Gamma^{1/2}(s)] = \lambda_n[\Gamma(s)\Gamma^{-1}(t)]$$
$$\leq \lambda_n[\Gamma(t)\Gamma^{-1}(t)] = \lambda_n[I] = 1,$$

yielding

$$\|\Gamma^{-1/2}(t)\Gamma^{1/2}(s)\| = \|\Gamma^{1/2}(s)\Gamma^{-1/2}(t)\| \leq 1. \tag{4.99}$$

Consider the sequence $\{R_\nu(t)\}_{\nu=0}^{\infty}$ defined inductively by

$$R_0(t) = I \tag{4.100a}$$

and for $\nu \geq 0$,

$$R_{\nu+1}(t) = I + \int_b^t \Gamma^{-1/2}(t)\Gamma^{1/2}(s)$$
$$\times [R_\nu(s)\Gamma^{1/2}(s)Q(s)\Gamma^{1/2}(s)R_\nu(s)]\Gamma^{1/2}(t)\, ds. \tag{4.100b}$$

Then $\|R_0(t)\| = 1$, R_0 is absolutely continuous, hermitian, and positive definite on $[b, \infty)$. Suppose $R_\nu(t)$, $0 \leq \nu \leq n-1$, have been shown to be well defined, absolutely continuous, hermitian, and positive definite on $[b, \infty)$, with

$$\|R_\nu(t)\| \leq 1/(1 - \omega(t)) \quad \text{for } t \in [b, \infty). \tag{4.101}$$

Then for $\nu = n - 1$ in the integrand on the right-hand side of (4.100b) we see that the integrand is norm bounded by $\omega'/(1 - \omega)^2 = (1/(1 - \omega))'$ uniformly for t in $[b, \infty)$, whence the integral converges absolutely as $t \to \infty$. Therefore, R_n is well defined, absolutely continuous, hermitian, and positive definite on $[b, \infty)$, and

$$\|R_n(t)\| \leq 1 + \left[\frac{1}{1 - \omega(s)}\right]_{s=b}^{s=t} = \frac{1}{1 - \omega(t)},$$

whence by induction the sequence is well defined, each member is absolutely continuous and positive definite and bounded by (4.101) on $[b, \infty)$.

Define $\Delta_\nu(t)$ by

$$\Delta_\nu(t) = R_\nu(t) - R_{\nu-1}(t), \qquad \nu \geq 1. \tag{4.102}$$

Then $\|\Delta_1(t)\| \leq \omega(t) \leq \eta\omega(t)$ for t in $[b, \infty)$. Suppose that it has been shown that

$$\|\Delta_\nu(t)\| \leq \frac{(\eta\omega(t))^\nu}{\nu!}, \qquad 1 \leq \nu \leq n-1. \tag{4.103}$$

Then since

$$\begin{aligned}\Delta_{\nu+1}(t) &= R_{\nu+1}(t) - R_\nu(t)\\ &= \int_b^t \Gamma^{-1/2}(t)\Gamma^{1/2}(s)\big\{\big[(R_\nu - R_{\nu-1})\Gamma^{1/2}Q\Gamma^{1/2}R_\nu\big]\\ &\qquad + \big[R_{\nu-1}\Gamma^{1/2}Q\Gamma^{1/2}(R_\nu - R_{\nu-1})\big]\big\}\Gamma^{1/2}(s)\Gamma^{-1/2}(t)\,ds,\end{aligned}$$

we see that

$$\begin{aligned}\|\Delta_n(t)\| &\leq \int_b^t 2\|\Delta_{n-1}(s)\|\frac{\omega'}{1-\omega}\,ds \leq \int_b^t 2\frac{(\eta\omega(s))^{n-1}}{(n-1)!}\left(\frac{\omega'}{1-\omega}\right)ds\\ &\leq \frac{2\eta^{n-1}}{(n-1)!\,(1-\omega(\infty))}\int_b^t \omega^{n-1}\omega'\,ds\\ &= \frac{2}{\eta(1-\omega(\infty))}\,\frac{(\eta\omega(t))^n}{n!} < \frac{(\eta\omega)^n}{n!},\end{aligned}$$

where $\omega(\infty) = \lim_{\xi\to\infty}\omega(\xi) < 1 - 2/\eta$ by (4.98). By induction (4.103) holds for all $\nu \geq 1$, and this implies that the sequence $\{R_\nu\}$ converges uniformly on $[b, \infty)$ to a continuous limit R. Since the integrand on the right in (4.100b) is norm bounded for almost all s in $[b, \infty)$ by $\omega'(s)/(1-\omega(s))^2$, uniformly for t in $[b, \infty)$, the Lebesgue Dominated Convergence Theorem allows us to let ν tend to ∞ in (4.100b) to get that R satisfies (4.96). R has the desired properties, and the proof is complete. □

The following theorem provides partial converses to Theorems 3.1 and 3.2, and it extends results of [14] which deal only with $G(t) \equiv I$.

THEOREM 4.6. *Let $G(t)$ and $Q(t)$ be nondegenerate and positive semidefinite a.e. on $[a, \infty) \subseteq J$, with*

$$\lambda_1\left[\int_a^t G(s)\,ds\right] \to \infty \quad \text{as } t \to \infty. \tag{4.104a}$$

Suppose that (4.1) *has a self-conjugate solution (X_0, Y_0) such that*

$$X_0(t) \to I \quad \text{as } t \to \infty. \tag{4.104b}$$

(i) *If either*

$$\lambda_n\left[\int_a^t G(\xi)\,d\xi\right] = O\left(\lambda_1\left[\int_a^t G(\xi)\,d\xi\right]\right), \tag{4.105a}$$

or

$$\lambda_n[Q(t)] = O(\lambda_1[Q(t)]) \quad a.e.\ as\ t \to \infty, \tag{4.105b}$$

then

$$\int_a^\infty \left\| \int_a^s G(\xi)\,d\xi \right\| \|Q(s)\|\,ds < \infty. \tag{4.105c}$$

(ii) *If either*

$$\lambda_n\left[\int_t^\infty Q(\xi)\,d\xi\right] = O\left(\lambda_1\left[\int_t^\infty Q(\xi)\,d\xi\right]\right), \tag{4.106a}$$

or

$$\lambda_n[G(t)] = O(\lambda_1[G(t)]) \quad a.e.\ as\ t \to \infty, \tag{4.106b}$$

then

$$\int_a^\infty \|G(s)\| \left\| \int_s^\infty Q(\xi)d\xi \right\| ds < \infty. \tag{4.106c}$$

PROOF. Condition (4.104b) with Theorem 2.2 implies that (4.1) is nonoscillatory. Therefore, all three parts of the conclusion of Theorem 4.2 hold. In particular, $\int_a^\infty \|Q\|\,d\xi < \infty$. Furthermore, $X_0(\tau)$ is nonsingular on some terminal subinterval $[b, \infty) \subseteq [a, \infty)$, and $S(t)$ defined by

$$S(t) = Y_0(t)X_0^{-1}(t) \quad \text{for } t \in [b, \infty) \tag{4.107}$$

is hermitian and positive definite on $[b, \infty)$ and satisfies

$$S^{-1}(t) = S^{-1}(b) + \int_b^t G(\xi)\,d\xi + \int_b^t S^{-1}(\xi)Q(\xi)S^{-1}(\xi)\,d\xi \tag{4.108}$$

for $t > b$.

Let $\mathcal{G}(t)$ be defined by

$$\mathcal{G}(t) = \int_b^t G(\xi)\,d\xi. \tag{4.109}$$

An integration by parts of $X_0' = GY_0$ yields

$$X_0(t) - X_0(b) = \int_b^t GY_0\,d\xi = \mathcal{G}(t)Y_0(t) + \int_b^t \mathcal{G}(\xi)Q(\xi)X_0(\xi)\,d\xi.$$

Postmultiplication by X_0^{-1} and rearrangement yield

$$I - \mathcal{G}(t)S(t) = \left[X_0(b) + \int_b^t \mathcal{G}(\xi)Q(\xi)X_0(\xi)\,d\xi\right]X_0^{-1}(t). \tag{4.110}$$

Since all the terms on the right-hand side of (4.108) are positive semidefinite, we have $S^{-1}(t) > \mathcal{G}(t) \geq 0$ for $t \geq b$, which implies that $I \geq I - S^{1/2}\mathcal{G}S^{1/2} > 0$, and $0 \leq \operatorname{trace}[I - \mathcal{G}(t)S(t)] \leq n$ for all $t \in [b, \infty)$. Therefore, the trace of the quantity on the right-hand side of (4.110) is nonnegative and bounded uniformly by n on $[b, \infty)$. Since $X_0^{-1}(t) \to I$, $X_0(b)X_0^{-1}(t)$ is norm bounded on $[b, \infty)$, and this yields that

$$\left|\operatorname{trace}\left[\int_b^t \mathcal{G}(s)Q(s)X_0(s)X_0^{-1}(t)\,ds\right]\right|$$

is uniformly bounded on $[b, \infty)$, say by K. Then

$$K \geq \left|\operatorname{Real}\operatorname{tr}\left[\int_b^t \mathcal{G}(s)Q(s)X_0(s)X_0^{-1}(t)\,ds\right]\right| \tag{4.111}$$

uniformly on $[b, \infty)$.

Since $X_0(s)X_0^{-1}(t) \to I$ as s and $t \to \infty$, there is a matrix B such that $\|B(s, t)\| \to 0$ as s and $t \to \infty$, and $X_0(s)X_0^{-1}(t) = I + B(s, t)$. Furthermore,

$$|\operatorname{tr}[\mathcal{G}QB]| \leq n\|\mathcal{G}QB\| \leq n\|\mathcal{G}\|\,\|Q\|\,\|B\|,$$

whence

$$\begin{aligned}\operatorname{Real}\operatorname{tr}[\mathcal{G}(s)Q(s)X_0(s)X_0^{-1}(t)] &= \operatorname{Real}\operatorname{tr}[\mathcal{G}(s)Q(s) + \mathcal{G}(s)Q(s)B(s, t)]\\ &= \operatorname{tr}[\mathcal{G}(s)Q(s)] + \operatorname{Real}\operatorname{tr}[\mathcal{G}(s)Q(s)B(s, t)]\\ &\geq \operatorname{tr}[\mathcal{G}(s)Q(s)] - n\|\mathcal{G}(s)\|\,\|Q(s)\|\,\|B(s, t)\|,\end{aligned}$$

since the trace of $\mathcal{G}(s)Q(s)$ is real. If A and B are hermitian, positive semidefinite matrices, then

$$\lambda_n[AB] \geq \lambda_\nu[A]\lambda_{n+1-\nu}[B] \quad \text{for } 1 \leq \nu \leq n.$$

Furthermore, (4.105) implies that there is an interval $[d, \infty) \subseteq [c, \infty)$ and a constant $L > 0$ such that either

$$\lambda_n[\mathcal{G}(t)] \leq L\lambda_1[\mathcal{G}(t)],$$

or

$$\lambda_n[Q(t)] \leq L\lambda_1[Q(t)] \quad \text{a.e. for } t \in [d, \infty).$$

Therefore

$$\operatorname{tr}[\mathcal{G}(t)Q(t)] \geq \lambda_n[\mathcal{G}(t)Q(t)] \geq (1/L)\|\mathcal{G}(t)\|\,\|Q(t)\| \tag{4.112}$$

for almost all $t \geq d$. Let $e \geq d$ be so large that $\|B(s, t)\| \leq 1/(2nL)$ for $s, t \geq e$. Then for almost all $s, t \geq e$, we have

$$\begin{aligned}\operatorname{Real}\operatorname{tr}[\mathcal{G}(s)Q(s)X_0(s)X_0^{-1}(t)] &\geq \|\mathcal{G}(t)\|\,\|Q(s)\|\left(\frac{1}{L} - n\|B(s, t)\|\right)\\ &\geq \frac{1}{2L}\|\mathcal{G}(s)\|\,\|Q(s)\| \geq 0.\end{aligned}$$

This together with (4.111) yields (4.105c).

Let (X_1, Y_1) be the self-conjugate solution of (4.1) defined by

$$X_1(t) = X_0(t)\int_b^t X_0^{-1}(\xi)G(\xi)X^*{}_0^{-1}(\xi)\,d\xi \tag{4.113a}$$

$$Y_1(t) = Y_0(t)\int_b^t X_0^{-1}(\xi)G(\xi)X_0^{*-1}(\xi)\,d\xi + X_0^{*-1}(t). \tag{4.113b}$$

Since $X_0^{-1}(t) \to I$,

$$\left\|\int_b^t X_0^{-1}GX_0^{*-1}\,d\xi\right\| = O\left(\left\|\int_b^t G\,d\xi\right\|\right) \quad \text{as } t \to \infty.$$

Also, the hypothesis of Theorem 3.2 is satisfied, and so (3.12c) holds, whence

$$Y_1(t) \to I \quad \text{as } t \to \infty. \tag{4.114}$$

Then $Y_1(t)$ is nonsingular on some interval $[\beta, \infty) \subseteq [b, \infty)$.

We define $W(t)$ for $t > \beta$ by

$$W(t) = X_1(t)Y_1^{-1}(t). \tag{4.115}$$

Then $W(t)$ is hermitian, positive definite on some subinterval $[\gamma, \infty) \subseteq [\beta, \infty)$, and

$$W^{-1}(t) = \int_t^\infty Q\,d\xi + \int_t^\infty W^{-1}GW^{-1}\,d\xi \text{ for } t > \gamma. \tag{4.116}$$

Define $\mathcal{Q}(t)$ by

$$\mathcal{Q}(t) = \int_t^\infty Q\,d\xi. \tag{4.117}$$

Integration of $Y_1' = -QX_1$ by parts yields

$$Y_1(t) = Y_1(\gamma) + \mathcal{Q}(t)X_1(t) - \mathcal{Q}(\gamma)X_1(\gamma) - \int_\gamma^t \mathcal{Q}(\xi)G(\xi)Y_1(\xi)\,d\xi,$$

and

$$I - \mathcal{Q}(t)W(t) = \left[Y_1(\gamma) - \mathcal{Q}(\gamma)X_1(\gamma) - \int_\gamma^t \mathcal{Q}(\xi)G(\xi)Y_1(\xi)\,d\xi\right]Y_1^{-1}(t). \tag{4.118}$$

From (4.116) we see that $0 < \operatorname{trace}[I - \mathcal{Q}(t)W(t)] < n$ uniformly on $[\gamma, \infty)$, and since

$$(Y_1(\gamma) - \mathcal{Q}(\gamma)X_1(\gamma))Y_1^{-1}(t)$$

is norm bounded in t uniformly on $[\gamma, \infty)$, we obtain

$$\left|\operatorname{trace}\int_\gamma^t \mathcal{Q}(\xi)G(\xi)Y_1(\xi)Y_1^{-1}(t)\,d\xi\right| < M \tag{4.119}$$

for some positive constant M and all $t > \gamma$. The remaining part of the proof of (ii) parallels that of (i) above. □

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