Czechoslovak Mathematical Journal

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Czechoslovak Mathematical Journal, Vol. 44 (1994), No. 3, 443–459

Persistent URL: http://dml.cz/dmlcz/128483

Czechoslovak Mathematical Journal, 44 (119) 1994, Praha

OSCILLATION AND NONOSCILLATION OF NONHOMOGENEOUS THIRD ORDER DIFFERENTIAL EQUATIONS

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(Received July 17, 1992)

1. This paper is concerned with the study of oscillatory/nonoscillatory behaviour of solutions of nonhomogeneous third order differential equations of the form

$$y''' + a(t)y'' + b(t)y' + c(t)y = f(t), \tag{1.1}$$

where a, b, c and f are real-valued continuous functions on $[\sigma, \infty)$, $\sigma \in \mathbb{R}$, under the assumption that the associated homogeneous equation

$$y''' + a(t)y'' + b(t)y' + c(t)y = 0 \tag{1.2}$$

is oscillatory/nonoscillatory.

A great deal of work on oscillation theory of (1.2) has been done during the last several years (see Greguš [2], Hanan [3], Jones [4–6], Lazer [9], Parhi and Das [11,13] and the references therein). The first author and S. Parhi obtained sufficient conditions for oscillation and nonoscillation of (1.1) in [15–17]. However, the techniques employed here are different from the former ones.

A continuous real-valued function y on $[\sigma, \infty)$ is said to be oscillatory if it has arbitrarily large zeros in $[\sigma, \infty)$; otherwise, it is said to be nonoscillatory. Eq. (1.1) or (1.2) is said to be oscillatory if it has an oscillatory solution, and it is said to be nonoscillatory if all of its solutions are nonoscillatory.

Following Hanan [3], Eq. (1.2) is said to be of Class I or C_I if any solution $y(t)$ of the equation with $y(t_0) = y'(t_0) = 0$, $y''(t_0) > 0$, $t_0 > \sigma$, satisfies $y(t) > 0$ for $\sigma \leqslant t < t_0$. It is said to be of Class II or C_{II} if any solution $y(t)$ of it with $y(t_0) = y'(t_0) = 0$, $y''(t_0) > 0$, $t_0 \geqslant \sigma$ satisfies $y(t) > 0$ for $t > t_0$. We say that

* This work was done under a scheme supported by the U.G.C., India, under Grant No. F. 8-9/87(SR-III).

Eq. (1.1) has no solution with (2,2)-distribution of zeros if it has no solution with two consecutive double zeros.

The motivation for the present work had come from the work of Sitter and Teftellor [19] and from certain observations of the properties of solutions of the third order differential equations with constant coefficients of the form

$$y''' + ay'' + by' + cy = f \tag{1.3}$$

and the associated homogeneous equations

$$y''' + ay'' + by' + cy = 0, \tag{1.4}$$

where a, b, c and $f \in \mathbb{R}$ such that $f \neq 0$. Clearly, all solutions of (1.4) are nonoscillatory if and only if its characteristic equation

$$m^3 + am^3 + bm + c = 0 \tag{1.5}$$

has only real roots, say γ_i, $i = 1, 2, 3$. Consequently, the general solution of (1.3) for $c \neq 0$, is given by

$$y(t) = \frac{f}{c} + \sum_{i=1}^{3} \lambda_i \mathrm{e}^{\gamma_i t}, \quad \lambda_i \in \mathbb{R}$$

which is nonoscillatory. Hence nonoscillation of (1.4) implies nonoscillation of (1.3). On the other hand, oscillation of (1.4) need not imply the oscillation of (1.3). Indeed, a solution basis of

$$y''' - 2y' - 4y = 0$$

is given by

$$\left\{\mathrm{e}^{-t}\cos t, \mathrm{e}^{-t}\sin t, \mathrm{e}^{2t}\right\}.$$

So the general solution of the corresponding nonhomogeneous equation

$$y''' - 2y' - 4y = f,$$

where $f \in \mathbb{R}$ and $f \neq 0$, is of the form

$$y(t) = -\frac{f}{4} + \lambda_1 \mathrm{e}^{-t}\cos t + \lambda_2 \mathrm{e}^{-t}\sin t + \lambda_3 \mathrm{e}^{2t},$$

which is nonoscillatory for all real $\lambda_i, i = 1, 2, 3$. However, for $a \geqslant 0$ (or < 0), $b < 0$ and $c > 0$, oscillation of (1.4) implies the oscillation of (1.3). Indeed, oscillation of (1.4) implies that (1.5) admits two complex roots $\alpha + \mathrm{i}\beta$ and $\alpha - \mathrm{i}\beta$ and a negative real root γ. Clearly, $b = \gamma(\alpha+i\beta)+\gamma(\alpha+i\beta)+(\alpha+i\beta)(\alpha+i\beta)$ implies that $2\alpha\gamma < b$ and hence $\alpha > 0$. Consequently, $y(t) = \frac{f}{c} + \lambda e^{\alpha t}\cos\beta t$ is an oscillatory solution of (1.3). Thus, for $a \geqslant 0$ (< 0), $b < 0$ and $c > 0$, (1.3) is oscillatory if and only if (1.4) is oscillatory. Further, under these conditions on coefficients, Eq. (1.5) admits complex roots, that is, (1.4) is oscillatory if and only if

$$\frac{2a^3}{27} - \frac{ab}{3} + c - \frac{2}{3\sqrt{3}}\left(\frac{a^2}{3} - b\right)^{3/2} > 0.$$

2. Equations (1.1) and (1.2) may be written, respectively, as

$$(r(t)y'')' + q(t)y' + p(t)y = F(t) \tag{2.1}$$

and

$$(r(t)y'')' + q(t)y' + p(t)y = 0, \tag{2.2}$$

where $r(t) = \exp(\int_\sigma^t a(s)\,ds)$, $q(t) = b(t)r(t)$, $p(t) = c(t)r(t)$ and $F(t) = f(t)r(t)$. Let $\{u_1, u_2, u_3\}$ be a solution basis for (2.2) such that

$$W(u_1,u_2,u_3)(t) \equiv \begin{vmatrix} u_1(t) & u_2(t) & u_3(t) \\ u_1'(t) & u_2'(t) & u_3'(t) \\ r(t)u_1''(t) & r(t)u_2''(t) & r(t)u_3''(t) \end{vmatrix} = 1$$

Then the general solution of (2.1) is given by

$$y(t) = \sum_{i=1}^{3} c_i u_i(t) + y_p(t), \tag{2.3}$$

where c_1, c_2, c_3 are constants and $y_p(t)$ is a particular solution of (2.1) and is given by

$$y_p(t) = \int_\sigma^t \begin{vmatrix} u_1(t) & u_2(t) & u_3(t) \\ u_1(s) & u_2(s) & u_3(s) \\ u_1'(s) & u_2'(s) & u_3'(s) \end{vmatrix} F(s)\,ds\,.$$

Clearly, $y_p(\sigma) = 0$, $y_p'(\sigma) = 0$ and $y_p''(\sigma) = 0$. Following Sitter and Teftcller [19], $W_i(t)$ denotes the determinant obtained from $W(u_1,u_2,u_3)(t)$ by replacing the ith-column with the vector $(0,0,1)^T$, $i = 1,2,3$. So

$$y_p(t) = \sum_{i=1}^{3} u_i(t) \int_\sigma^t F(s)W_i(s)\,ds$$

and

$$y(t)=\sum_{i=1}^{3}u_i(t)\Big[c_i+\int_\sigma^t F(s)W_i(s)\,\mathrm{d}s\Big].$$

Lemma 2.1. *If $y(t)$ is a solution of (2.1) given by (2.3), then*

$$\begin{aligned}(2.4)\qquad & r(t)W_i(t)y''(t)-r(t)W_i'(t)y'(t)\\ &+[q(t)W_i(t)+(r(t)W_i'(t))']y(t)=c_i+\int_\sigma^t F(s)W_i(s)\,\mathrm{d}s,\end{aligned}$$

$i=1,2,3$.

Proof. We may see that $S_i(t)=c_i+\int_\sigma^t F(s)W_i(s)\,\mathrm{d}s$, $i=1,2,3$, where $S_i(t)$ denotes the determinant obtained by replacing the ith-column of $W(u_1,u_2,u_3)(t)$ with the vector $(y(t),y'(t),r(t)y''(t))^T$. Indeed, for $i=1$ we write

$$S_1(t)=\begin{vmatrix} y(t) & u_2(t) & u_3(t)\\ y'(t) & u_2'(t) & u_3'(t)\\ r(t)y''(t) & r(t)u_2''(t) & r(t)u_3''(t)\end{vmatrix}$$

Then

$$S_1'(t)=\begin{vmatrix} y(t) & u_2(t) & u_3(t)\\ y'(t) & u_2'(t) & u_3'(t)\\ F(t) & 0 & 0\end{vmatrix}=F(t)W_1(t)$$

implies that

$$S_1(t)=S_1(\sigma)+\int_\sigma^t F(s)W_1(s)\,\mathrm{d}s.$$

But $S_1(\sigma)=c_1W(\sigma)=c_1$. Consequently, $S_1(t)=c_1+\int_\sigma^t F(s)W_1(s)\,\mathrm{d}s$. Expanding $S_1(t)$, we obtain (2.4) for $i=1$. Similarly, one may obtain (2.4) for $i=2$ and 3. □

Lemma 2.2. *Suppose that $W_1(t)\neq 0$ for $t\geqslant t_0>\sigma$. If $y(t)$ is a solution of (2.1) given by (2.3), then it is a solution of the second order nonhomogeneous equation*

$$(2.5)\qquad (R(t)y')'+Q(t)y=G(t),\quad t\geqslant t_0,$$

where

$$R(t)=\frac{1}{W_1(t)},\; Q(t)=\frac{q(t)W_1(t)+(r(t)W_1'(t))'}{r(t)W_1^2(t)}$$

and

$$G(t)=\Big[c_1+\int_\sigma^t F(s)W_1(s)\,\mathrm{d}s\Big]\Big/r(t)W_1^2(t).$$

Proof. Dividing (2.4) with $i=1$ by $r(t)W_1^2(t)$, we obtain (2.5). □

Lemma 2.3. *If $W_1(t) \neq 0$ for $t \geqslant t_0 > \sigma$, then $u_2(t)$ and $u_3(t)$ satisfy*

$$(R(t)y')' + Q(t)y = 0, \quad t \geqslant t_0 \tag{2.6}$$

where $R(t)$ and $Q(t)$ are the same as in Lemma 2.2.

Proof. Clearly, $u_2(t)$ and $u_3(t)$ are solutions of the second order differential equation

$$\begin{vmatrix} u_2(t) & u_3(t) & x \\ u_2'(t) & u_3'(t) & x' \\ r(t)u_2''(t) & r(t)u_3''(t) & r(t)x'' \end{vmatrix} = 0$$

Expanding this determinant we obtain (2.6). □

Theorem 2.4. *If $W_1(t) \neq 0$ for $t \geqslant t_0 > \sigma$ and*

$$(R(t)y')' + Q(t)y = G_c(t), \quad t \geqslant t_0, \tag{2.7}$$

is nonoscillatory for every constant c, where $R(t)$ and $Q(t)$ are the same as in Lemma 2.2 and

$$G_c(t) = \Big[c + \int_\sigma^t F(s)W_1(s)\,\mathrm{d}s\Big]/r(t)W_1^2(t),$$

then (2.1) is nonoscillatory.

The proof of the theorem follows from Lemma 2.2.

Remark 1. (i) The adjoint of (2.2) is given by

$$[(r(t)y')' + q(t)y]' - p(t)y = 0.$$

If $q(t)$ is differentiable, then it takes the form

$$(r(t)y')'' + q(t)y' + (q'(t) - p(t))y = 0. \tag{2.8}$$

It is easy to verify that $W_1(t)$ satisfies (2.8). (ii) $W_1(t)$ satisfies the equation

$$(r(t)z')' + q(t)z = g(t),$$

where $g(t) = r(t)(u_2'(t)u_3''(t) - u_3'(t)u_2''(t))$.

Proposition 2.5. (i) *If* $p(t) > 0$ *and* $q(t) \leqslant 0$, *then* (2.2) *is of* C_I.

(ii) *If* $(p(t) < 0$ *and* $q(t) \leqslant 0$, *then* (2.2) *is of* C_{II}.

(iii) *If* $r'(t) \geqslant 0 (\leqslant 0)$ *and* $q \in C^1([\sigma, \infty), \mathbb{R})$ *isa such that* $2p(t) - q'(t) \geqslant 0 (\leqslant 0)$, *then* (2.2) *is of* $C_I(C_{II})$.

The proof in each case is straigthforward and hence is omitted.

Theorem 2.6. *Suppose that* (2.2) *is of* C_I *or* C_{II} *and* $F(t)$ *does not change sign for large* t. *If* (2.2) *is nonoscillatory, then* (2.1) *is nonoscillatory.*

Proof. As (2.2) is nonoscillatory, it follows from Theorem 4.7 due to Hanan [3] that (2.8) is nonoscillatory. So $W_1(t)$ is nonoscillatory. Suppose $W_1(t) \neq 0$ for $t \geqslant t_0 \geqslant \sigma$. Further, from Lemma 2.3 and the fact that (2.2) is nonoscillatory it is clear that (2.6) is nonoscillatory. Suppose that $F(t) > 0$ or < 0 for $t \geqslant t_1 \geqslant t_0$. For any constant c,

$$h(t) = c + \int_\sigma^t W_1(s)F(s)\,\mathrm{d}s$$

is nonoscillatory, because $h'(t) > 0$ or < 0 for $t \geqslant t_1$. Hence it is clear from Theorem 3 due to Keener [8] that (2.7) is nonoscillatory for every constant c. Thus the conclusion of the theorem follows from Theorem 2.4. □

Example. Consider the equation

$$y''' + \mathrm{e}^{-t}y' + 2\mathrm{e}^{-t}y = 3 + \mathrm{e}^t, \quad t \geqslant 0. \tag{2.9}$$

Theorem 2.2 due to Hanan [3] and Theorem 3.5 due to Lazer [9] imply that the homogeneous equation associated with (2.9) is of C_I and nonoscillatory. Consequently, by Theorem 2.6, all solutions of (2.9) are nonoscillatory. In particular, $y(t) = \mathrm{e}^t$ is a nonoscillatory solution of (2.9).

Remark 2. (i) Hanan in [3] and Lazer in [9] have obtained various sufficient conditions for nonoscillation of (2.2).

(ii) Although Keener [8] has proved his Theorem 3 for $r(t) = 1, p(t) \geqslant 0$ and $f(t) \geqslant 0$, his result holds good for

$$(r(t)y')' + p(t)y = f(t),$$

where $r(t) > 0$ and $f(t)$ does not change sign for large t. There is no sign restriction on $p(t)$.

If $F(t)$ is allowed to change sign for large t, then (2.7) is nonoscillatory provided (2.6) is nonoscillatory and

$$\int_{\sigma}^{t} G_c(s)\Phi(s)\,\mathrm{d}s \tag{2.10}$$

is nonoscillatory, where $\Phi(t)$ is a solution of (2.6) (see [14], Theorem 4.1). Hence we have the following result:

Theorem 2.7. *Suppose that* (2.2) *is of* C_I *or* C_{II} *and* (2.10) *is nonoscillatory, where* $\Phi(t)$ *is a solution of* (2.6). *Then* (2.2) *is nonoscillatory implies that* (2.1) *is nonoscillatory.*

The following result due to Leighton and Nehari [10, Lemma 1.2] is used in the sequel.

Lemma 2.8. *Let* u *and* $v \in C^1((a,b),\mathbb{R})$, *and let* $v(t)$ *be of constant sign in* (a,b). *If* α *and* $\beta(a<\alpha<\beta<b)$ *are consecutive zeros of* $u(t)$, *then there exists a nonzero constant* λ *such that the function* $u(t)-\lambda v(t)$ *has a double zero in* (α,β).

Theorem 2.9. *If* (2.2) *is of* C_I *and* C_{II}, *then it is nonoscillatory.*

Proof. Since (2.2) is of C_{II}, the solution $y(t)$ of (2.2) with initial conditions $y(\sigma)=y'(\sigma)=0, y''(\sigma)>0$ has the property that $y(t)>0$ for $t>\sigma$. If possible, let $z(t)$ be an oscillatory solution of (2.2). Let $\alpha_1,\beta_1,\alpha_2,\beta_2(\sigma<\alpha_1<\beta_1<\alpha_2<\beta_2)$ be succesive zeros of $z(t)$ such that $z(t)>0$ for $t\in(\alpha_1,\beta_1)\cup(\alpha_2,\beta_2)$. By Lemma 2.8, there exists non-zero constants λ_1 and λ_2 such that $z_1(t)=z(t)-\lambda_1 y(t)$ has a double zero at $t_1\in(\alpha_1,\beta_1)$ and $z_2(t)=z(t)-\lambda_2 y(t)$ has a double zero at $t_2\in(\alpha_2,\beta_2)$. Since $z(t)>0$ in (α_1,β_1) and (α_2,β_2) and $y(t)>0$ for $t>\sigma$, we have $\lambda_1>0$ and $\lambda_2>0$. If $\lambda_1>\lambda_2$, then $z_2(t_1)=z(t_1)-\lambda_2 y(t_1)>z(t_1)-\lambda_1 y(t_1)=z_1(t_1)=0$ and $z_2(\beta_1)=z(\beta_1)-\lambda_2 y(\beta_1)=-\lambda_2 y(\beta_1)<0$. Thus $z_2(t)$ is a solution of (2.2) with a zero in (t_1,β_1) and a double zero at t_2, which constradicts the assumption that (2.2) is of C_I. If $\lambda_1\leqslant\lambda_2$, then $z_1(t_2)=z(t_2)-\lambda_1 y(t_2)\geqslant z(t_2)-\lambda_2 y(t_2)=z_2(t_2)=0$ and $z_1(\beta_1)=z(\beta_1)-\lambda_1 y(\beta_1)=-\lambda_1 y(\beta_1)<0$. Hence $z_1(t)$ is a solution of (2.2) with a zero in $(\beta_1,t_2]$ and a double zero at t_1, a contradiction to the assumption that (2.2) is of C_{II}. Hence (2.2) cannot have an oscillatory solution. Thus (2.2) is nonoscillatory. This completes the proof of the theorem. □

Remark 3. (i) If $p(t)>0, q(t)\leqslant 0$ and $p(t)-q'(t)\leqslant 0$, then (2.2) is of C_I and C_{II}. Indeed, Proposition 2.5 (i) yields that (2.2) is of C_I. Next we show that the adjoint of (2.2), given by (2.8), is of C_I. If not, then $y(t)$ is a solution of (2.8) with $y(\alpha) = y'(\alpha) = 0, y''(\alpha) > 0$ and $y(t_0) = 0$ for some $\sigma < t_0 < \alpha$. Consequently, there exists a $\beta \in (t_0, \alpha)$ such that $y'(\beta) = 0$ and $y(t) > 0, y'(t) < 0$ for $t \in (\beta, \alpha)$. Multiplying (2.8) by $r(t)y'(t)$ and integrating the resulting identity from β to α we obtain

$$0 = -\int_\beta^\alpha ((r(t)y'(t))')^2 \, dt + \int_\beta^\alpha r(t)q(t)(y'(t))^2 \, dt$$
$$+ \int_\beta^\alpha r(t)(q'(t) - p(t))y(t)y'(t) \, dt < 0,$$

a contradiction. Hence (2.8) is of C_I. Consequently, by Lemma 2.9 due to Hanan [3], (2.2) is of C_{II}.

(ii) If $p(t) \leqslant 0, q(t) \leqslant 0$ and $p(t) - q'(t) \geqslant 0$, then (2.2) is of C_I and C_{II}. Indeed, it follows from Proposition 2.5 (ii) that (2.2) is of C_{II}. Now we show that the adjoint of (2.2), that is, (2.8) is of C_{II}. If not, suppose that $y(t)$ is a solution of (2.8) with the property $y(\alpha) = y'(\alpha) = 0, y''(\alpha) > 0$ and $y(t_0) = 0$ for some $t_0 > \alpha > \sigma$. Consequently, there exists a $\beta > \alpha$ such that $y'(\beta) = 0$ and $y(t) > 0, y'(t) > 0$ for $t \in (\alpha, \beta)$. Multiplying (2.8) by $r(t)y'(t)$ and integrating the resulting identity from α to β we obtain a contradiction. Hence, by Lemma 2.9 due to Hanan [3], (2.2) is of C_I.

Corollary 2.10. *Suppose that conditions of Theorem 2.9 are satisfied and $F(t)$ does not change sign for large t. Then (2.1) is nonoscillatory.*

The proof follows from Theorem 2.6 and 2.9.

Remark 4. In [16, Theorem 2.1], Parhi and Parhi have proved that $F(t) \geqslant 0$, $p(t) > 0$, $q(t) \leqslant 0$ and $p(t) - q'(t) \leqslant 0$ imply that all solutions of (2.1) are nonoscillatory. Remark 2(i) implies that Corollary 2.10 is a generalization of their result.

Lemma 2.11. *If $W_1(t)$ is nonoscillatory, then every solution of (2.7) is a solution of (2.1).*

Proof. From the discussions at the begining of this section it folows that $\lambda u_1(t) + y_p(t)$, $\lambda \in \mathbb{R}$, is a solution of (2.1) and (2.7). Since $\{u_2, u_3\}$ forms a fundamental set of solutions of (2.6) (see Lemma 2.3), the general solution of (2.7) is given by

$$\begin{aligned} y(t) &= \lambda u_1(t) + y_p(t) + \lambda_2 u_2(t) + \lambda_3 u_3(t) \\ &= \lambda u_1(t) + \lambda_2 u_2(t) + \lambda_3 u_3(t) + y_p(t), \lambda_2, \lambda_3 \in \mathbb{R}, \end{aligned}$$

which is a solution of (2.1). □

Theorem 2.12. *If $W_1(t)$ is nonoscillatory and, for some constant c, Eq. (2.7) has an oscillatory solution, then (2.1) is oscillatory.*

The proof follows from Lemma 2.11.

Theorem 2.13. *Suppose that (2.2) either is of C_I or is of C_{II} and oscillatory. Further, assume that the second order nonhomogeneous equation*

$$z'' + g_1(t)z = h_1(t), \tag{2.11}$$

where

$$g_1(t) = \frac{q(t)}{r(t)} + \frac{(r(t)W_1'(t))'}{r(t)W_1(t)} + \frac{1}{2}\frac{W_1''(t)}{W_1(t)} - \frac{3}{4}\frac{(W_1'(t))^2}{W_1^2(t)}$$

and

$$h_1(t) = \frac{1}{r(t)(W_1(t))^{3/2}} \int_\sigma^t F(s)W_1(s)\,\mathrm{d}s\,,$$

has an oscillatory solution. Then (2.1) is oscillatory.

Proof. Suppose that (2.2) is of C_I. Let the solution basis $\{u_1, u_2, u_3\}$ of (2.2) satisfy the initial conditions

$$\begin{aligned} &u_1(\sigma) = 1, \quad u_1'(\sigma) = 0, \quad u_1''(\sigma) = 0,\\ &u_2(\sigma) = 0, \quad u_2'(\sigma) = 1, \quad u_2''(\sigma) = 0,\\ &u_3(\sigma) = 0, \quad u_3'(\sigma) = 0, \quad u_3''(\sigma) = 1/r(\sigma). \end{aligned}$$

□

It may be easily verified that the Wronskian $W(u_1, u_2, u_3)(t) = 1$ and $W_1(t) = u_2(t)u_3'(t) - u_2'(t)u_3(t)$ is a solution of (2.8) with the properties $W_1(\sigma) = W_1'(\sigma) = 0$, $W_1''(\sigma) = 1$. From Lemma 2.9 due to Hanan [3], it follows that (2.8) is of C_{II}. Hence $W_1(t) > 0$ for $t > \sigma$. The transformation $y(t) = z(t)(W_1(t))^{1/2}$ transforms (2.7) with $c = 0$ to (2.11). Hence the given hypotheses imply that (2.7) with $c = 0$ has an oscillatory solution. Consequently, the oscillation of (2.1) follows from Theorem 2.12. Thus the proof of the theorem is completed when (2.2) is of C_I.

Next, suppose that Eq. (2.2) is of C_{II} and oscillatory. To complete the proof of the theorem it is sufficient to construct a solution basis $\{u_1, u_2, u_3\}$ of (2.2) such that $W_1(t) = u_2(t)u_3'(t) - u_2'(t)u_3(t) > 0$ for large t. Indeed, in that case we use the transformation $y(t) = z(t)(W_1(t))^{1/2}$ and proceed as above to arrive at the conclusion.

Let $y_1(t)$ be an oscillatory solution of (2.2). Let $\beta \geqslant \sigma$ be such that $y_1(\beta) \neq 0$. Suppose that $\langle t_n \rangle_{n=1}^{\infty}$ is a sequence of zeros of $y_1(t)$ in (β, ∞) such that $t_n \to \infty$ as

$n \to \infty$. Define a sequence $\langle x_n(t)\rangle_{n=1}^{\infty}$ of nontrivial solutions of (2.2) on $[\beta, \infty)$ with the boundary conditions

$$x_n(\beta) = x_n(t_n) = 0.$$

Then there exist real constants c_{1n}, c_{2n} and c_{3n} such that $x_n(t) = \sum_{i=1}^{3} c_{in} v_i(t)$ with $\sum_{i=1}^{3} c_{in}^2 = 1$, where $\{v_1, v_2, v_3\}$ is a solution basis of (2.2). We claim that the zeros of $y_1(t)$ and $x_n(t)$ separate each other in (β, t_n). If possible, let α_1 and $\alpha_2 (\alpha_1 < \alpha_2)$ be consecutive zeros of $y_1(t)$ in (β, t_n) and let $x_n(t) > 0$ or < 0 for $t \in [\alpha_1, \alpha_2]$. It follows from Lemma 2.8 that there exists a constant λ such that $y_1(t) - \lambda x_n(t)$ has a double zero in (α_1, α_2). This contradicts the fact that (2.2) is of C_{II} because the solution $y_1(t) - \lambda x_n(t)$ of (2.2) has a zero at $t = t_n$. Since $y_1(t)$ and $x_n(t)$ are linearly independent solutions of (2.2) and t_n is a common zero, Theorem 2.10 due to Hanan [3] yields that $x_n(t)$ cannot vanish at α_1 or α_2. Hence $x_n(t)$ has a zero in (α_1, α_2). Similarly, it may be shown that $y_1(t)$ has a zero between two consecutive zeros of $x_n(t)$ in (β, t_n). Thus our claim holds. Since the sequence $\langle c_{in}\rangle_{n=1}^{\infty}$, $i = 1, 2, 3$, is bounded, it admits a convergent subsequence $\langle c_{in_k}\rangle_{k=1}^{\infty}$, say, with limit c_i, $i = 1, 2, 3$. So $\{x_{n_k}\}$ converges uniformly to a solution $y_2(t) = \sum_{i=1}^{3} c_i v_i$ of (2.2). Thus $y_2(\beta) = 0$ and hence $y_1(t)$ and $y_2(t)$ are linearly independent. We claim that the zeros of $y_1(t)$ and $y_2(t)$ separate each other in (M, ∞) for some $M > \beta$. From Theorem 2.10 due to Hanan [3], it is clear that $y_1(t)$ and $y_2(t)$ can have at most one zero in common. Hence there exists $M > \beta$ such that $y_1(t)$ and $y_2(t)$ have no common zero in (M, ∞). Suppose that α_1 and $\alpha_2 \in (M, \infty)$ $(\alpha_1 < \alpha_2)$ are consecutive zeros of $y_1(t)$. Clearly, $y_2(t)$ does not vanish at α_1 and α_2. Suppose that $y_2(t) > 0$ or < 0 for $t \in [\alpha_1, \alpha_2]$. Without any loss of generality, we may take $y_2(t) > 0$ for $t \in [\alpha_1, \alpha_2]$. Then there exists an $\varepsilon > 0$ such that $y_2(t) \geqslant \varepsilon > 0$ for $t \in [\alpha_1, \alpha_2]$. Since the sequence $\langle x_{n_k}\rangle$ converges uniformly to y_2 on $[\alpha_1, \alpha_2]$, there exists an integer $N > 0$ such that $|y_2(t) - x_{n_k}(t)| < \varepsilon/2$ for $t \in [\alpha_1, \alpha_2]$ and $n_k > N$. Thus, for $n_k > N$ such that $t_{n_k} > \alpha_2$,

$$x_{n_k}(t) > y_2(t) - \varepsilon/2 \geqslant \varepsilon - \varepsilon/2 = \varepsilon/2 \quad \text{for} \quad t \in [\alpha_1, \alpha_2],$$

which contradicts the fact that $x_{n_k}(t)$ has a zero between any two consecutive zeros of $y_1(t)$ in (β, t_{n_k}). Hence $y_2(t)$ has a zero in (α_1, α_2). Next we show that $y_1(t)$ has a zero between two consecutive zeros of $y_2(t)$. Let α_1 and $\alpha_2 \in (M, \infty)$, $\alpha_1 < \alpha_2$, be consecutive zeros of $y_2(t)$. We may assume, without any loss of generality, that $y_2(t) > 0$ for $t \in (\alpha_1, \alpha_2)$. Suppose that $y_1(t) \neq 0$ for $t \in [\alpha_1, \alpha_2]$. Since $y_1(t)$ is oscillatory it is possible to find β_1 and β_2 such that $\beta_1 < \alpha_1 < \alpha_2 < \beta_2, y_1(\beta_1) = 0 = y_1(\beta_2)$ and $y_1(t) \neq 0$ for $t \in (\beta_1, \beta_2)$. (If such a β_1 does not exist, then we choose M to be the zero of $y_2(t)$ which is just before the first zero of $y_1(t)$.) Since (2.2) is of C_{II}, the zeros α_1 and α_2 of $y_2(t)$ are simple. Hence it is possible to find a positive number $\varepsilon, t_1 \in (\beta_1, \alpha_1)$ and $t_2 \in (\alpha_1, \alpha_2)$ such that $y_2(t_1) < -\varepsilon$ and $y_2(t_2) > \varepsilon$. It follows from the definition of $y_2(t)$ that there exists an integer $N_1 > 0$ such that $|x_{nk}(t) - y_2(t)| < \varepsilon$ for $n_k \geqslant N_1$ and $t \geqslant M$. This in turn implies that $x_{n_k}(t_1) < y_2(t_1) + \varepsilon < 0$ and $x_{n_k}(t_2) > y_2(t_2) - \varepsilon > 0$ for $n_k \geqslant N_1$. Thus $x_{n_k}(t)$ has a zero in (t_1, t_2) for $n_k \geqslant N_1$. Similarly, there exist an integer $N_2 > 0$, $t_3 \in (\alpha_1, \alpha_2)$ and $t_4 \in (\alpha_2, \beta_2)$ such that $x_{n_k}(t)$ has a zero in (t_3, t_4) for $n_k \leqslant N_2$. Choosing n_k large enough such that $n_k > \max\{N_1, N_2\}$ and $t_{n_k} > \beta_2$, we see that $x_{n_k}(t)$ has two zeros in (β_1, β_2), a contradiction to the fact that the zeros of $x_{n_k}(t)$ and $y_1(t)$ are interlaced in (β, t_{n_k}). Hence $y_1(t)$ has a zero in (α_1, α_2). Thus we have shown that the zeros of $y_1(t)$ and $y_2(t)$ are interlaced in (M, ∞), $M > \beta$.

Next we claim that every linear combination of $y_1(t)$ and $y_2(t)$ is oscillatory. If possible, suppose that $\mu_1 y_1(t) + \mu_2 y_2(t)$ is nonoscillatory for some nonzero relas μ_1 and μ_2. Without any loss of generality, we may assume that $\mu_1 y_1(t) + \mu_2 y_2(t) > 0$ for $t \geqslant t_0 > M$. If t_1, t_2, t_3 $(t_1 < t_2 < t_3)$ are succesive zeros of $y_1(t)$ in $[t_0, \infty)$, then $\mu_2 y_2(t_i) > 0$, $i = 1, 2, 3$. This contradicts the fact that the zeros of $y_1(t)$ and $y_2(t)$ are interlaced in (M, ∞). Hence our claim holds. Now we show that $y_1(t)y_2'(t) - y_1'(t)y_2(t) \neq 0$ for $t > M$. Otherwise, there exists a $\gamma > M$ such that $y_1(\gamma)y_2'(\gamma) - y_1'(\gamma)y_2(\gamma) = 0$. Since the zeros of $y_1(t)$ and $y_2(t)$ are interlaced in (M, ∞), then $y_1(\gamma)$ and $y_2(\gamma)$ are not zeros simultaneously. Hence $v(t) = y_1(\gamma)y_2(t) - y_2(\gamma)y_1(t)$ is a nontrivial solution of (2.2) with $v(\gamma) = v'(\gamma) = 0$ and $v''(\gamma) \neq 0$. Consequently, (2.2) is of C_{II} impplies that $v(t) > 0$ or < 0 for $t > \gamma$, a contradiction to the fact that every linear combination of $y_1(t)$ and $y_2(t)$ is oscillatory. Hence $y_1(t)y_2'(t) - y_1'(t)y_2(t) \neq 0$ for $t > M$. We assume, without any loss of generality, that $y_1(t)y_2'(t) - y_1'(t)y_2(t) > 0$ for $t > M$.

Let $y_3(t)$ be a solution of (2.2) with $y_3(\beta) = y_3'(\beta) = 0, y_3''(\beta) = 1$. Clearly $y_1(t), y_2(t)$ and $y_3(t)$ are linearly independent. Hence $W(y_1, y_2, y_3)(t) = k \neq 0$. Now setting

$$u_1(t) = y_3(t), u_2(t) = y_1(t) \quad \text{and} \quad u_3(t) = y_2(t)$$

we see that $\{u_1, u_2, u_3\}$ is a solution basis of (2.2) with $W_1(t) = u_2(t)u_3'(t) - u_2'(t)u_3(t) > 0$ for large t.

Hence the theorem is proved.

Remark 5. Eq. (1.2) admits a nontrivial solution $y(t)$ with the property $y(\alpha) = y(\beta) = 0$ where $\sigma \leqslant \alpha < \beta$. Indeed, the solutions $y_1(t)$ and $y_2(t)$ of (2.2) with initial conditions

$$y_1(\alpha) = 0, y_1'(\alpha) = 0, y_1''(\alpha) = 1$$

$$y_2(\alpha) = 0, y_2'(\alpha) = 1, y_2''(\alpha) = 0$$

are linearly independent. If either $y_1(\beta) = 0$ or $y_2(\beta) = 0$, then there is nothing to prove. Otherwise,

$$y(t) = y_1(t) - \frac{y_1(\beta)}{y_2(\beta)}\, y_2(t)$$

is the required nontrivial solution of (2.2) with the property $y(\alpha) = y(\beta) = 0$.

Theorem 2.14. *Suppose that (2.2) is of C_I. If (2.1) does not admit a solution with $(2,2)$-distribution of zeros and $F(t)$ does not change sign for large t, then a necessary and sufficient condition for (2.1) to be oscillatory is that (2.11) has an oscillatory solution.*

Proof. If (2.1) is oscillatory, then by Theorem 2.6 (2.2) is oscillatory. From Theorem 2.12 due to Parhi and Das [12], it follows that $y_p(t)$ is oscillatory. Proceeding as in the first part of Theorem 2.13, we obtain $W_1(t) > 0$. Since $y_p(t)$ is a solution of (2.7) for $c = 0$, then $z(t) = y_p(t)(W_1(t))^{-1/2}$ is an oscillatory solution of (2.11). The sufficiency part follows from Theorem 2.13.

Hence the proof of the theorem is complete. □

Remark 6. (i) (See Theorem 2.8 [12].) Suppose that $p(t) \geqslant 0, p'(t) \geqslant 0, F(t) \geqslant 0$ and $F'(t) \leqslant 0$. If

$$(r(t)z')' + q(t)z = 0 \tag{2.12}$$

is nonoscillatory, then (2.2) is of C_I and (2.1) does not admit a solution with (2,2)-distribution of zeros.

(ii) (See Theorem 2.10 [12].) Suppose that $p(t) \leqslant 0$, $p'(t) \geqslant 0$, $F(t) \geqslant 0$, $F'(t) \geqslant 0$, $r'(t) \geqslant 0$ and $2p(t) - q'(t) > 0$. If (2.12) is nonoscillatory, then (2.2) is of C_I and (2.1) does not admit a solution with (2,2)-distribution of zeros.

Corollary 2.15. *Suppose that $p(t) \geqslant 0$, $p'(t) \geqslant 0$, $q(t) \leqslant 0$, $F(t) > 0$, $F'(t) \leqslant 0$. Then a necessary and sufficient condition for (2.1) to be oscillatory is that Eq. (2.11) has an oscillatory solution.*

Proof. Theorem 2.8 due to Parhi and Das [12] implies that (2.2) is of C_I and (2.1) does not admit a solution with $(2,2)$-distribution of zeros. Hence the proof follows from Theorem 2.14. □

Remark 7. If p, q, r and F are real constants such that $q \leqslant 0$ or > 0, $p > 0$ and $F > 0$ and (2.2) is oscillatory, then (2.11) has an oscillatory solution. Indeed, in this case (2.1), (2.2) and (2.8) are reduced, respectively, to

$$y''' + q_1 y' + p_1 y = F_1,$$

$$y''' + q_1 y' + p_1 y = 0, \tag{2.13}$$

$$y''' + q_1 y' - p_1 y = 0, \tag{2.14}$$

$t \geqslant \sigma$, where $p_1 = p/r$, $q_1 = q/r$ and $F_1 = F/r$. The characteristic equations of (2.13) and (2.14) are given respectively by

$$m^3 + q_1 m + p_1 = 0, \tag{2.15}$$
$$n^3 + q_1 n - p_1 = 0.$$

Since (2.13) is assumed to be oscillatory, Eq. (2.15) admits two complex roots, say $\alpha + \mathrm{i}\beta$ and $\alpha - \mathrm{i}\beta$, and a real root, say γ. Without any loss of generality, we may assume that $\beta > 0$. Clearly,

$$\{\mathrm{e}^{\gamma t}/k, \mathrm{e}^{\alpha t}\cos\beta t, \mathrm{e}^{\alpha t}\sin\beta t\} \tag{2.16}$$

forms a basis for the solution space of (2.13), where $k \neq 0$ is the value of the Wronskian of $\{\mathrm{e}^{\gamma t}/k, \mathrm{e}^{\alpha t}\cos\beta t, \mathrm{e}^{\alpha t}\sin\beta t\}$. Writing $u_1(t) = \mathrm{e}^{\gamma t}/k, u_2(t) = \mathrm{e}^{\alpha t}\cos\beta t$, and $u_3(t) = \mathrm{e}^{\alpha t}\sin\beta t$, we see that $W(u_1, u_2, u_3)(t) \equiv 1$ and $W_1(t) = u_2(t)u_3'(t) - u_2'(t)u_3(t) = \beta\mathrm{e}^{2\alpha t} > 0$. Since $p_1 > 0$, we have $\gamma < 0$ and hence $(\alpha + \mathrm{i}\beta) + (\alpha - \beta) + \gamma = 0$ implies that $\alpha > 0$. Consequently, $g_1(t) = q_1 + 3\alpha^2, h_1(t) = W_1^{-3/2}(t)F_1\int_\sigma^t W_1(s)\,\mathrm{d}s > 0$ and

$$h_1'(t) = \frac{F_1\mathrm{e}^{-\alpha t}}{2\beta^{1/2}}[3\mathrm{e}^{2\alpha(\sigma-t)} - 1] < 0$$

for sufficiently large t. Clearly, $u_2(t)$ and $u_3(t)$ are linearly independent oscillatory solutions of

$$\begin{vmatrix} u_2(t) & u_3(t) & x \\ u_2'(t) & u_3'(t) & x' \\ u_2''(t) & u_3''(t) & x'' \end{vmatrix} = 0$$

that is, of

$$\left(\frac{x'}{W_1(t)}\right)' + \left(\frac{W_1''(t) + q_1W_1(t)}{W_1^2(t)}\right)x = 0. \tag{2.17}$$

Since the transformation $x(t) = z(t)(W_1(t))^{1/2}$ transforms (2.17) to

$$z'' + g_1(t)z = 0 \tag{2.18}$$

this equation is oscillatory and hence $g_1(t) > 0$. From Theorem 2.4 due to Skidmore and Leighton [18], it follows that (2.11) has an oscillatory solution. Hence we have the following proposition.

Proposition 2.16. *Consider* (2.1) *and* (2.2) *with p, q, r and F are constants such that $p > 0$ and $F > 0$. Then* (2.2) *is oscillatory implies that* (2.1) *is oscillatory.*

P r o o f. If $q < 0$, then the proof follows from Corollary 2.15 and Remark 7 (we may note that this fact has been observed at the beginning of the paper).

If $q \geqslant 0$ then the proof follows from Proposition 2.5 (iv) and Theorem 2.13. □

3. In this section we obtain sufficient conditions in terms of coefficients and the forcing terms for Eq. (1.1) to be oscillatory. First we state a result due to present authors [12] to be used in the sequel.

Theorem 3.1 (Theorem 2.6 [12]). *Suppose that* (1.2) *is of* C_I *and* (1.1) *does not admit a solution with* (2, 2)*-distribution of zeros. If* (1.2) *is oscillatory, then* (1.1) *is oscillatory.*

Theorem 3.2. *Suppose that* $b(t) \leqslant 0$, $c(t) > 0$, $c'(t) \geqslant 0$, $f(t) \geqslant 0$, $f'(t) \leqslant 0$, *and* $2b(t) - a'(t) \leqslant 0$. *Then* (1.2) *is of* C_I *and* (1.1) *does not admit a solution with* (2, 2)*-distribution of zeros.*

Proof. The proof that (1.2) is of C_I is straightforward and hence is omitted. Let $y(t)$ be a solution of (1.1) with consecutive double zeros at $t = \alpha$ and $t = \beta$. Suppose that $y(t) > 0$ for $t \in (\alpha, \beta)$. Multiplying (1.1) by $y'(t)$ and then integrating the resulting identity from α to β we have

$$0 > \int_\alpha^\beta \Big[-(y''(t))^2 + \frac{(2b(t) - a'(t))}{2}(y'(t))^2 - \frac{c'(t)}{2} y^2(t) \Big] \,dt$$
$$= -\int_\alpha^\beta f'(t)y(t)\,dt > 0,$$

a contradiction.

Now suppose that $y(t) < 0$ for $t \in (\alpha, \beta)$. Then there exists a point $\gamma \in (\alpha, \beta)$ such that $y'(\gamma) = 0$ and $y'(t) > 0$ for $t \in (\gamma, \beta)$. Eq. (1.1) may be written as (2.1). Multiplying (2.1) by $y'(t)$ and integrating the resulting identity from γ to β we obtain

$$0 = \int_\gamma^\beta \Big[r(t)(y''(t))^2 - q(t)(y'(t))^2 - p(t)y(t)y'(t)$$
$$+ r(t)f(t)y'(t) \Big] \,dt > 0,$$

a contradiction. Hence the proof is completed. □

Theorem 3.3. *Suppose that* $a(t) \geqslant 0, b(t) \leqslant 0, c(t) > 0$ *and* $b(t) - a'(t) \leqslant 0$. *If*

$$\int_\sigma^\infty \Big[\frac{2a^3(t)}{27} - \frac{a(t)(b(t) - a'(t))}{3} + c(t) - \frac{2}{3\sqrt{3}} \Big(\frac{a^2(t)}{3} - (b(t) - a'(t)) \Big)^{3/2} \Big] \,dt = \infty, \tag{3.1}$$

then equation (1.2) *is oscillatory.*

Proof. Let $u(t)$ be a nonoscillatory solution of (1.2). By Lemma 2.2 due to Erbe [1], $u(t)u'(t) \geqslant 0$ or $\leqslant 0$ and $u(t) > 0$ for large t, say, for $t \geqslant t_0 \geqslant \sigma$. In view of Lemma 2.3 due to Erbe [1], to complete the proof of the theorem it is enough to show that $u(t)u'(t) \geqslant 0$ is not possible.

Suppose that $u(t)u'(t) \geqslant 0$ for $t \geqslant t_0 \geqslant \sigma$. Clearly $z(t) = u'(t)/u(t)$ satisfies the Riccati equation

$$z'' + 3zz' + a(t)z' = -\big[z^3 + a(t)z^2 + b(t)z + c(t)\big]. \tag{3.2}$$

Integrating (3.2) from t_0 to t we have

$$\begin{aligned} z'(t) + \frac{3z^2(t)}{2} + a(t)z(t) &= z'(t_0) + \frac{3z^2(t_0)}{2} + a(t_0)z(t_0) \\ &\quad - \int_{t_0}^{t} [z^3(s) + a(s)z^2(s) + (b(s) - a'(s))z(s) + c(s)]\,\mathrm{d}s\,. \end{aligned} \tag{3.3}$$

The minimum of $[z^3(s) + a(s)z^2(s) + (b(s) - a'(s))z(s) + c(s)]$ for positive $z(s)$ is given by

$$\frac{2a^3(s)}{27} - \frac{a(s)(b(s) - a'(s))}{3} + c(s) - \frac{2}{3\sqrt{3}}\Big(\frac{a^2(s)}{3} - (b(s) - a'(s))\Big)^{3/2}$$

Subsituting this value into (3.3) we see that $\lim_{t\to\infty} z'(t) = -\infty$. Consequently, $z(t) < 0$ for large t, a contradiction.

This completes the proof of the theorem. □

Corollary 3.4. *Suppose that* $a(t) \geqslant 0$, $b(t) \leqslant 0$, $c(t) > 0$, $b(t) - a'(t) \leqslant 0$, $c'(t) \geqslant 0$, $f(t) > 0$ *and* $f'(t) \leqslant 0$. *If* (3.1) *holds, then* (1.1) *is oscillatory.*

The proof of the Corollary follows from Theorems 3.1, 3.2 and 3.3.

Theorem 3.5. *Suppose that* $a(t) \leqslant 0$, $b(t) \leqslant 0$, $c(t) > 0$ *and* $b(t) - a'(t) \leqslant 0$. *If*

$$\int_{\sigma}^{\infty} \left[\frac{2a^3(t)}{27} - \frac{a(t)b(t)}{3} + c(t) - \frac{2}{3\sqrt{3}}\Big(\frac{a^2(t)}{3} - (b(t) - a'(t))\Big)^{3/2}\right] \mathrm{d}t = \infty. \tag{3.4}$$

then (1.2) is oscillatory.

This is Theorem 2.1 in [13] due to the present authors.

Corollary 3.6. *Suppose that* $a(t) \leqslant 0$, $b(t) \leqslant 0$, $c(t) > 0$, $b(t) - a'(t) \leqslant 0$. $c'(t) \geqslant 0$, $f(t) \geqslant 0$ *and* $f'(t) \leqslant 0$. *If* (3.4) *holds, then* (1.1) *is oscillatory.*

The proof of the Corollary follows from Theorems 3.1, 3.2 and 3.5.

Acknowledgement. The authors are thankful to the referee for his remarks.

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