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OSCILLATION CENTER THEORY AND PONDEROMOTIVE STABILIZATION OF LOW-FREQUENCY PLASMA MODES

P.L. Similon, A.N. Kaufman, and D.D. Holm

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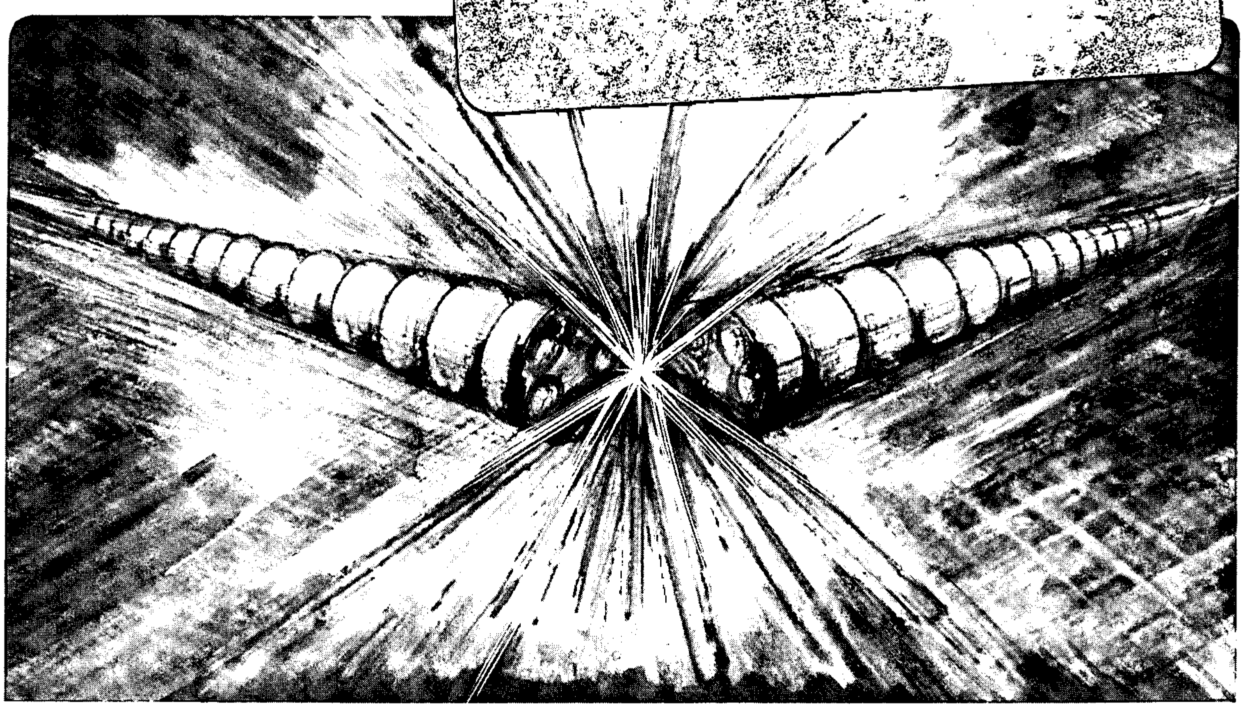

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Philippe L. Similon and Allan N. Kaufman

Lawrence Berkeley Laboratory
and
Physics Department
University of California
Berkeley, California 94720,

and

Darryl D. Holm

Center for Nonlinear Studies and Theoretical Division

Los Alamos National Laboratory
Los Alamos, New Mexico 87545

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Abstract

Nonlinear, nondissipative ponderomotive theory is developed in relation to recent experimental results showing that externally imposed RF fields can stabilize an axisymmetric mirror plasma. First, we reexamine the ponderomotive force problem, emphasizing self-consistency of the interaction between the plasma and high-frequency field, by using an averaged action principle for the antenna-plasma system, in a mixed Eulerian and Lagrangian representation. The averaged action principle yields self-consistent plasma and electromagnetic field dynamics on the oscillation-center time scale, and determines the condition for static plasma equilibrium to occur. This condition is expressed as a balance among plasma and magnetic pressure forces, including interchange, ponderomotive, and magnetization forces. Next, we study the spectral stability of such static equilibria in the low-frequency (MHD) approximation, by developing a ΔW principle which is modified to incorporate the various ponderomotive contributions, including particle effects, magnetization effects due to the RF field, and self-consistent adjustments of the RF field due to

displacements of the plasma away from equilibrium. The ponderomotive potential energy functional is related to the antenna inductance and antenna current amplitude. Finally, we develop the noncanonical Hamiltonian formulation for the system's dynamics. This formulation, entirely in terms of Eulerian fields, enables us to construct nonlinearly conserved functionals at any level of the several approximations we treat. Using these functionals, criteria are given for linearized Lyapunov stability in the two cases of multifluid and MHD equilibria. For MHD equilibria, the stability conditions determined this way are related to those found from the modified ΔW method.

INTRODUCTION

Recent experiments performed in the U.S. and Japan [1,2] have focused on the confinement properties of plasma in axisymmetric mirrors. These experiments show that an electromagnetic field emitted in the radio frequency (RF) range, can be used effectively in some situations to stabilize the flute instability, which would otherwise disrupt the plasma equilibrium in the axisymmetric mirror.

Axisymmetric mirrors are attractive fusion reactors because of the geometric simplicity of the machines, and because of their potentially good transport properties. Indeed, in axisymmetric static as well as time-varying RF fields, the total (kinetic plus magnetic) angular momentum of each particle is conserved, in the absence of collisions. This conservation law implies that the particles stay within a gyroradius of an axisymmetric, time-varying magnetic flux surface. The transport will thus be due exclusively to collisional processes, and be small at high temperature.

A series of experiments have been performed recently on the Phaedrus tandem mirror at the University of Wisconsin [1] in an axisymmetric configuration. To address the physics issues better, the mirror was operated with two R.F. antennas. The first emitted waves, at a frequency slightly below the ion gyrofrequency, which were absorbed in the plasma to provide the necessary heating. The waves emitted by the second antenna had a frequency somewhat above the ion gyrofrequency. It was observed that the fields emitted by this antenna have a stabilizing effect on the m=1, flute-like fluctuations of the plasma, and also that there was no appreciable resistive loading of the antenna. These experiments therefore suggest that RF fields can stabilize the flute instability, and that this stabilization is not related to dissipative mechanisms.

The low-frequency response due to nonlinear, nondissipative interaction between particles and high frequency fields is describable via ponderomotive theory, which is thus a likely candidate for the theoretical interpretation of the experiments [4]: the radial ponderomotive forces acting on the particles cause them to drift in the azimuthal direction; if these drifts balance the drifts due to the destabilizing interchange forces (e.g. unfavorable average curvature drifts, or unfavorable gravitational drifts), then the charge separation associated with the flute instability is reversed, and the plasma is stabilized by ponderomotive effects.

The aim of the present paper is to reexamine the ponderomotive force problem, with emphasis on the self-consistent treatment of the plasma and of the high-frequency field. One motivation is that, in the cold fluid limit, the expression for the ponderomotive force shows an apparent singularity at the ion gyrofrequency, an unphysical result, since the ponderomotive force is associated with radiation pressure. In fact, we shall show that such singular behavior does not occur: the plasma reacts back on the electromagnetic field, and shields its resonant component. Note that this particular effect is distinct from the one due to the finite transit time of the particles in the RF field [5], or due to the thermal distribution of the particles [6]. The shielding of the left-polarized (ion gyration sense) component of the RF field is just one aspect of the <u>interdependency</u> between the plasma dynamics and the wave dynamics. This interdependency appears strikingly in the <u>form</u> of the equations: the dielectric tensor that governs the propagation and the polarization of the waves determines at the same time the various ponderomotive effects (forces and magnetization). The need for a rigorous self-consistent analysis arises also in the study of the plasma <u>stability</u> in the presence of the high-frequency field. When the plasma position is perturbed, for instance during the instability, the dielectric properties of the medium in which the waves propagate are modified, and, consequently, the fields and ponderomotive forces are perturbed as well. Those changes in turn create additional forces (possibly destabilizing) affecting the dynamics of the plasma, and therefore its stability. Indeed, for the m=1 flute instability in axisymmetric mirrors, the contribution of the perturbed fields can be just as important as the more conventional ponderomotive terms [7,8]. One can show that the self-consistent response of the RF field is related to, and generalizes, the quasi-mode coupling of fluid theories [9,10].

In part I, we adopt a Lagrangian formulation. The starting point of the analysis is the classical action principle for the system composed of the plasma, the electromagnetic fields, and the imposed current circulating in the antenna. This formulation has the advantages of being global and compact, and easily subject to approximations. We use a separation of time scales for the evolution of the dynamical variables. There is, first, a slow component which varies on the time scale of the flute instability and which is, therefore, of direct interest. Second, there is a fast component which varies on the time scale of the fields emitted by the antenna, typically of the order of the ion gyroperiod. The particle motion can therefore be separated into the "slow" motion of its oscillation-center, and into a rapid oscillation [7,11]. Likewise, the electromagnetic field can be separated into the slowly varying background field (which includes here the flute perturbation), and into the high frequency field (the "wave field") with slowly varying amplitude. When the action is averaged over the fast time scale, there appears explicitly a term which is quadratic in the high-frequency field amplitude, and which generates by variation all of the various ponderomotive effects. The variation of the averaged action provides the coupled equations for each of the "slow" variables, as follows: a) for the RF field amplitude, there is a wave equation, driven by the imposed antenna current; b) for each oscillation center (ions and electrons), there is a modified Newton-Lorentz equation of evolution, which includes the respective ponderomotive forces; c) and for the background field, there is the set of Maxwell's equations whose sources include the oscillation center currents and a magnetization current term, due to the fast oscillations of the particles in the wave field.

Besides the algebraic simplicity of the derivation, the action method insures self-consistency and greatly clarifies the relations among ponderomotive forces, magnetization and RF wave polarization [11.5]. Moreover, the conservation laws for energy and momentum are automatically satisfied, being traceable, via Noether's theorem, to the symmetries of the averaged action.

Our method has been developed at this stage to describe nondissipative systems only. Thus we shall restrict the present work to a regime where resonances, collisions, etc., are negligible (for instance, when the RF frequency is in the range of, but still greater than, the ion gyrofrequency). This restriction limits the results available via the present approach (although some information about dissipation can already be gained by using the Kramers-Kronig relations). It also stimulates research to extend the method to the dissipative regime. Fortunately the experiments mentioned at the beginning of this introduction indicate that some effects are essentially nondissipative, and thus vindicates the approach.

For simplicity, we shall limit the analysis to ponderomotive effects in cold plasma. In fact, we shall argue that the plasma we consider is "cold" in its high-frequency response, and at the same time "warm" (i.e., with finite pressure) in its low-frequency behavior. We shall derive expressions for the ponderomotive terms and show the absence of singularity at the ion gyrofrequency. Furthermore, it will turn out that electron ponderomotive terms and magnetization terms will be just as important as the usually considered ion ponderomotive forces.

To study the stability of the system, it is convenient to work in the magnetohydrodynamic (MHD) approximation at low frequency, and to develop a modified ΔW variational principle [7]. This is possible, since the linearized equations for the (flute) perturbation are self-adjoint, as a consequence of the absence of dissipation in the system. Note that the self-adjointness holds even though the system is energetically open, since it has imposed external antenna current. In addition to the usual MHD terms in the ΔW variational principle, three distinct ponderomotive contributions will be included: a) The first contribution is produced by the equilibrium ponderomotive forces. It is the mechanism invoked in the heuristic argument mentioned at the beginning of this paper (i.e. the ponderomotive drifts oppose the interchange drifts); b) The second contribution involves the magnetization of the plasma due to the RF field, and is proportional to the gradient of the equilibrium magnetic field; c) The third contribution is due to the self-consistent variation of the ponderomotive forces in response to the plasma displacements, for instance during instability. This contribution is due primarily to the perturbation of the high frequency field, which depends also on boundary conditions.

The ponderomotive energy is a functional of the plasma parameters (densities, etc.), and of the magnetic field. Its value has a clear physical interpretation: it is the free energy that can be extracted from the high-frequency antenna current, and as such is related to the antenna impedance. This interpretation is valuable theoretically, because it helps to derive ponderomotive forces and determine their influence on equilibrium and stability for practical plasma configurations [12]. Experimentally it is especially valuable, because antenna impedance and currents are easily measurable quantities, and because it shows that the correlation of their variations with the plasma displacements provides direct information on ponderomotive forces.

In part II of this article, we present the Hamiltonian formulation of the problem, which illuminates the physics of the system from a somewhat different angle. In particular, it introduces in a natural way a family of constraints (the Casimir functionals) that are satisfied by the fields during the time evolution. Equilibrium states and their stability conditions can be obtained by demanding that the total energy be a conditional extremum, subject to the constraints. In other words, via the Hamiltonian formulation it is possible to construct a Lyapunov functional, whose stationary points are equilibrium states. Such an equilibrium is stable if the associated Lyapunov functional is extremum at that point [13,14].

The field Hamiltonian and the noncanonical Poisson bracket are derived, both for the multifluid and the MHD plasma models, and are similar to previous results but now include the ponderomotive energy. The Lyapunov functional is chosen as a combination of the Hamiltonian with the Casimirs of the Poisson bracket. We shall note the relation between the second variation of this Lyapunov functional and the ΔW energy principle. The Lyapunov stability method can in principle go beyond linear criteria, and give information about nonlinear stability. Research along these lines is under way.

We have published some of these results in letter form [7,8]. The present paper generalizes the earlier results and provides a more detailed discussion of the method.

PART I: LAGRANGIAN FORMULATION

A. Plasma Model and System Action

Consider a system composed of charged particles, of charge q and mass m (the species label will be always implied, and only rarely written explicitly), confined in a static magnetic field, and surrounded by a perfectly conducting vessel. This system may be considered as a model of a mirror machine, for instance.

In the Lagrangian picture of the system that we shall adopt, one keeps track of each particle and follows its evolving trajectory [11,15]. For this, the particles are labeled in some fixed but arbitrary reference state D by the point $z^0 \epsilon$ D, which includes species label. The evolution of the particles is determined by the field $\underline{r}(z^0,t)$, which maps the reference state to the actual configuration at time t.

In order to treat the electromagnetic field self-consistently, we consider it also as a component of the system. It is determined by the electromagnetic potentials $\underline{A}(\underline{x},t)$ and $\phi(\underline{x},t)$, viewed as independent field variables. The system is not isolated though, because it is coupled to the "outside world" by the imposed antenna current density $\underline{j}_a(\underline{x},t)$, by the static magnetic field, and by a species-dependent "gravitational" potential $\psi(\underline{x})$. The potential ψ is introduced here for convenience to provide a simple way to create interchange forces, and to mock up the unfavorable magnetic field curvature in two-dimensional geometry. Note also that taking the antenna current to be independent of the plasma state presupposes that the RF generator which drives the antenna has "infinite" impedance. This restriction can be removed, and does have consequences for stability [12], but will be maintained here for simplicity.

The total Lagrangian action S for the dynamics of this system is the sum of the actions of the particles S_p and of the electromagnetic fields S_{em}. The $\underline{j}\cdot\underline{A}$ coupling between these two components is determined by the plasma currents. Thus we get

$$S = S_p + S_{em} , \qquad (1)$$

with

$$S_p = \int dt \int dN[\frac{1}{2} m |\dot{\underline{r}}(z^o,t)|^2 + \frac{q}{c} \dot{\underline{r}}(z^o,t)\cdot\underline{A}(\underline{r}(z^o,t),t) - \psi(\underline{r}(z^o,t)) - q\, \phi(\underline{r}(z^o,t),t)] \qquad (2)$$

and

$$S_{em} = \int dx \int dt\, [\frac{1}{8\pi} |\nabla\phi(\underline{x},t) + \frac{1}{c}\frac{\partial}{\partial t} \underline{A}(\underline{x},t)|^2 - \frac{1}{8\pi} |\nabla\times\underline{A}(\underline{x},t)|^2 + \frac{1}{c} \underline{j}_a(\underline{x},t)\cdot\underline{A}(\underline{x},t)] . \qquad (3)$$

We use the following conventions: $\underline{x}$ denotes a set of coordinates for the two- or three-dimensional Euclidian space; dN is a measure on the reference state D which represents the number of particles contained in a small element of D in the neighborhood of the point z^0; thus the integral $\int dN$ represents a continuous sum over the particles. We will consider the plasma as a continuous medium, and adopt therefore a Vlasov picture, in which no discretization effects or "collisions" are present. Note that in the particle action, the electromagnetic fields are evaluated at the particle position, i.e., at $\underline{x} = \underline{r}(z^0,t)$.

The electromagnetic fields are defined via the potentials which can be chosen, without restriction, to satisfy the radiation gauge condition $\phi = 0$. Thus we have

$$\underline{B}(\underline{x},t) = \nabla \times \underline{A}(\underline{x},t)$$

$$\underline{E}(\underline{x},t) = -\frac{1}{c}\frac{\partial}{\partial t}\underline{A}(\underline{x},t). \quad (4)$$

The variation of the action with respect to the displacement field $\underline{r}(z^0,t)$, and with respect to the potential field $\underline{A}(\underline{x},t)$, leads, as is well known, to the Newton-Lorentz equations of motion for the particles, and to the Maxwell equations for the electromagnetic field, driven in this case by the plasma currents as well as by the antenna current.

B. Separation of time scales and averaging of the action

The separation of the time scales of the flute instability (the "slow" time scale, γ^{-1}) and of RF fields (the "fast" time scale, ω^{-1}, usually of the order of the ion gyroperiod) is in general an excellent approximation, (they are typically several orders of magnitude apart [1,2]) and may be exploited to get useful approximations of the exact formulation given by Eqns. (1) to (3). The high-frequency antenna current, $\mathrm{Re}\{\underline{j}_a(\underline{x})\ \exp(-i\ \omega\ t)\}$ is imposed and drives a high-frequency field in the plasma. One can therefore write the electromagnetic field as a superposition of: first, a background field $\underline{A}_s(\underline{x},t)$, evolving on the slow time scale; and second, a fast "wave" field $\underline{A}_f(\underline{x},t) = \mathrm{Re}\ \{\underline{A}_w(\underline{x},t)\ \exp(-i\omega t)\}$. The complex amplitude $\underline{A}_w(\underline{x},t)$ evolves on the slow time scale (and creates the "side bands" of "quasimode coupling" theories [9,10]).

Similarly, the motion of a particle is a superposition of a "slow" motion, the motion of its oscillation center $\underline{r}_{oc}(z^o,t)$, and of a "fast" oscillation driven by the RF wave, $\underline{r}_f(z^o,t) = \mathrm{Re}\ \{\underline{r}_w(z^o,t)\ \exp(-i\omega t)\}$. There is the possibility that some of the particles may be resonant with the RF wave, when $(\omega - k_\| v_\|)$ or $(\omega - \Omega_i)$ approaches γ. For such particles, the strict separation into "slow" and "fast" time scales is not possible, and the method described below does not apply as such. We shall avoid this problem by requiring that the regime of operation excludes any significant resonant effect [6].

The goal of the oscillation center theory can be summarized as follows: it is to perform a change of variables from the original fields $[\underline{r}(z^o,t), \underline{A}(\underline{x},t)]$ to new fields $[\underline{r}_{oc}(z^o,t), \underline{r}_w(z^o,t), \underline{A}_s(\underline{x},t), \underline{A}_w(\underline{x},t)]$, and to derive the dynamical equations for the new variables, which evolve on the slow time scale only. Thus, we decompose the various fields in the action (2,3) into sums of slow and fast components [16], and expand to second order in the (small) oscillating quantities. For instance,

$$\dot{\underline{r}}(z^o,t) = \dot{\underline{r}}_{oc}(z^o,t) + \mathrm{Re}\{(\dot{\underline{r}}_w - i\omega\ \underline{r}_w)\ \exp(-i\omega t)\},$$

and

$$\underline{A}(\underline{r}(z^o,t),t) = \underline{A}_s(\underline{r}_{oc}+\underline{r}_f) + \underline{A}_f(\underline{r}_{oc} + \underline{r}_f)$$

$$\simeq \underline{A}_s + \underline{r}_f \cdot \nabla\underline{A}_s + 1/2\ \underline{r}_f\underline{r}_f : \nabla\nabla\underline{A}_s + \underline{A}_f + \underline{r}_f \cdot \nabla\underline{A}_f,$$

where the field $\underline{A}_s$ and its derivatives are evaluated at $\underline{x} = \underline{r}_{oc}$, i.e. at the oscillation center position. Such expansions are valid when the amplitude of oscillation of a particle is small compared with typical scale lengths of the slow and the fast fields. This is usually well satisfied (for nonresonant particles), by a factor of order B_f/B_s.

The next step is an "average" of the action over the fast time scale [17]. When the fast and slow time scales are sufficiently separated, the time-integral of oscillating quantities vanishes, so that $\int dt\, a_s(t)\, b_f(t) \simeq 0$, and $\int dt\, a_f(t)\, b_f(t) \simeq 1/2\ \mathrm{Re}\{\int dt\, a_w^*(t)\, b_w(t)\ \}$, where a_s and a_f designate any slow and fast variables, a_w the amplitude, and the asterisk denotes complex conjugation.

After averaging, the approximate action takes the form:

$$S = S_p + S_{em} + S_{pd}, \tag{5}$$

where S_p and S_{em} have the same <u>form</u> (2)-(3) as before, but now the variables that figure in them are the oscillation-center, or slow variables. The new term in the action, S_{pd}, the "ponderomotive" action, is quadratic in the wave amplitude, and is

$$\begin{aligned} S_{pd} = \frac{1}{2}\, \mathrm{Re} \int dt \int dN\, \{ & \frac{1}{2} m\, |\dot{\underline{r}}_w - i\omega \underline{r}_w|^2 - \frac{1}{2} \underline{r}_w\, \underline{r}_w : \nabla\nabla\psi(\underline{r}_{oc}) \\ & + \frac{q}{c} [\frac{1}{2} \dot{\underline{r}}_{oc} \cdot \underline{r}_w^* \underline{r}_w : \nabla\nabla \underline{A}_s(\underline{r}_{oc}) + \dot{\underline{r}}_{oc} \cdot (\underline{r}_w^* \cdot \nabla \underline{A}_s) \\ & \quad + (\dot{\underline{r}}_w^* + i\omega \underline{r}_w^*) \cdot (\underline{A}_w(\underline{r}_{oc}) + \underline{r}_w \cdot \nabla\, \underline{A}_s(\underline{r}_{oc}))]\} \\ & + \frac{1}{2}\, \mathrm{Re} \int dt \int dx\, \{ \frac{1}{c}\, \underline{j}_a^* \cdot \underline{A}_w + |\frac{\partial \underline{A}_w}{\partial t} - i\omega \underline{A}_w|^2/8\pi c^2 \\ & \quad - |\underline{\nabla} \times \underline{A}_w|^2/8\pi \}. \end{aligned} \tag{6}$$

The action S (5,6) will generate all the needed equations, and S_{pd} will generate all the ponderomotive effects as well as the wave equation. Further inessential approximations can be made in order to simplify the expression for S_{pd}. The time evolution of the field amplitude, $\partial \underline{A}_w/\partial t$, is small compared to $\omega \underline{A}_w$ by a factor $\gamma/\omega << 1$ and can be neglected, and the "gravitational" field contribution ($\nabla\nabla\Psi$) will also be assumed negligible. As for the terms involving the oscillation center velocities, it is not possible to neglect them in general. There is nevertheless a useful particular case for which the expression of the action simplifies considerably: in the cold plasma approximation, the oscillation center velocities are small, as is the time evolution of the amplitudes $\underline{r}_w$:

$$d\underline{r}_{oc}/dt << \omega\, \underline{r}_w,$$

and

$$\dot{\underline{r}}_w \simeq k_{\parallel} v_{\parallel}\, \underline{r}_w << \omega\, \underline{r}_w \; . \qquad (7)$$

Under these conditions, the ponderomotive action (6) becomes

$$S_{pd} = \frac{1}{2} \mathrm{Re} \int dt \int dN \; \{\frac{1}{2} m\, \omega^2\, |\underline{r}_w|^2 + \frac{q}{c}\, i\omega(\underline{r}_w^* \cdot \underline{A}_w + \underline{r}_w \cdot \nabla \underline{A}_s \cdot \underline{r}_w^*)$$

$$+ \frac{1}{2} \mathrm{Re} \int dt \int dx \; \{\frac{1}{c}\, \underline{j}_a^* \cdot \underline{A}_w + \frac{1}{8\pi}\, [\frac{\omega^2}{c^2}\, |\underline{A}_w|^2 - |\nabla \times \underline{A}_w|^2]\} \; . \qquad (8)$$

For simplicity, we shall adopt here the cold plasma approximation (7). The action (5) must now be considered as a functional of the following independent fields (which are all slowly varying): (a) the oscillation center displacement field $\underline{r}_{oc}(z^o,t)$; (b) the oscillation amplitude $\underline{r}_w(z^o,t)$; (c) the background field $\underline{A}_s(\underline{x},t)$; and (d) the wave amplitude $\underline{A}_w(\underline{x},t)$.

Note that slow and fast variables can be considered independent, because of the separation of the time scales. (Fourier transformation of the fields in time shows that they parametrize disjoint parts of the fields.)

The evolution equations for these various fields are readily obtained by variation of the action (5). In particular, variation with respect to $\underline{r}_w(z^o,t)$ leads to the familiar equation for the small vibrations of a charged particle in a wave field:

$$- i\omega m\underline{r}_w = q\,[\underline{A}_w - \underline{r}_w x(\nabla x\underline{A}_s)]/c \quad . \tag{9}$$

This equation can be solved for $\underline{r}_w$, which can then be substituted into the ponderomotive action to yield $S_{pd} = - \int dt\ V$, where the <u>ponderomotive energy</u> V, quadratic in the wave amplitude, is

$$V = - \frac{1}{16\pi}\frac{\omega^2}{c^2}\int dx\ \underline{A}_w^* \cdot \underline{\underline{\varepsilon}} \cdot \underline{A}_w + \frac{1}{16\pi}\int dx\ |\nabla \times \underline{A}_w|^2$$

$$- \frac{1}{2c}\,\mathrm{Re}\int dx\ \underline{j}_a^* \cdot \underline{A}_w \ . \tag{10}$$

The quantity $\underline{\underline{\varepsilon}}$ that appears in the ponderomotive energy is the cold plasma dielectric tensor[18]. It is a hermitian operator, since we have consistently avoided resonances and dissipation.

Now, the variation of the action (5) with respect to the wave amplitude $\underline{A}_w$ yields the wave equation:

$$\nabla \times (\nabla \times \underline{A}_w) - \frac{\omega^2}{c^2}\underline{\underline{\varepsilon}} \cdot \underline{A}_w = \frac{4\pi}{c}\underline{j}_a \quad . \tag{11}$$

Note that the relation between of the wave equation (11) and the ponderomotive energy (10) makes it clear that the validity of equation (10) extends beyond the cold plasma fluid model: it is in some sense the <u>definition</u> of the dielectric tensor.

C. Ponderomotive forces and magnetization

The dynamical equations for oscillation centers and the equations for the slow fields are obtained by variation of the action with respect to $\underline{r}_{oc}(z^0,t)$ and with respect to $\underline{A}_s(\underline{x},t)$, respectively. Note that the ponderomotive action S_{pd} is a functional of those fields, through the dielectric tensor. For instance, in the cold plasma limit, $\underline{\varepsilon}$ is a function of particle densities $n_{oc}(\underline{x},t)$, and of magnetic field $\underline{B}_s(\underline{x},t)$ [18], which are themselves functionals of $\underline{r}_{oc}$ and $\underline{A}_s$:

$$n_{oc}(\underline{x},t) = \int dN\, \delta(\underline{x} - \underline{r}_{oc}(z^0,t)) \quad ,$$

$$\underline{B}_s(\underline{x},t) = \nabla \times \underline{A}_s(\underline{x},t) \ . \tag{12}$$

Therefore, using the chain rule for functional derivatives yields

$$\delta S_{pd} = - \int dt \int dx \left(\frac{\delta V}{\delta n_{oc}} \delta n_{oc} + \frac{\delta V}{\delta \underline{B}_s} \cdot \delta \underline{B}_s\right)$$

$$= \int dt \int dx \int dN \frac{\delta V}{\delta n_{oc}} \delta \underline{r}_{oc} \cdot \nabla\, \delta(\underline{x} - \underline{r}_{oc})$$

$$- \int dt \int dx \frac{\delta V}{\delta \underline{B}_s} \cdot \nabla \times \delta \underline{A}_s \ , \tag{13a}$$

which becomes, after integration by parts,

$$\delta S_{pd} = - \int dt \int dN\, \delta \underline{r}_{oc} \cdot \int dx\, \delta(\underline{x} - \underline{r}_{oc})\, \nabla\left(\frac{\delta V}{\delta n_{oc}}\right)$$

$$- \int dx \int dt\, \delta \underline{A}_s \cdot \left(\nabla \times \frac{\delta V}{\delta B_s}\right) \quad . \tag{13b}$$

Since the variation of the total action vanishes for all variations of the oscillation center trajectory $\delta \underline{r}_{oc}$, we find the Newton-Lorentz equation for oscillation centers:

$$m\, \ddot{\underline{r}}_{oc} = q(\underline{E}_s + \frac{1}{c}\, \dot{\underline{r}}_{oc} \times \underline{B}_s) - \nabla\psi + \underline{F}(\underline{r}_{oc}) \ , \tag{14}$$

with $\underline{F}(\underline{x}) = -\nabla \dfrac{\delta V}{\delta n(\underline{x})}$.

The oscillation center Eq. (14) includes the ponderomotive force $\underline{F}$, which derives from a potential $\delta V/\delta n$: the ponderomotive potential of a particle (of each species). More explicitly, the functional derivative of V is

$$\delta V/\delta n(\underline{x}) = - \frac{1}{16\pi} \frac{\omega^2}{c^2} \underline{A}_w^*(\underline{x}) \cdot \frac{\partial \underline{\underline{\varepsilon}}(n_{oc}, \underline{B}_s)}{\partial n_{oc}} \cdot \underline{A}_w(\underline{x}) \quad . \tag{15}$$

Similarly, the variation of the total action vanishes for all variations of the slow vector potential $\delta \underline{A}_s$. The resulting Maxwell equation for the slow fields,

$$\nabla \times \underline{B}_s - \frac{1}{c} \frac{\partial}{\partial t} \underline{E}_s = \frac{4\pi}{c} (\underline{J}_{oc} + \underline{J}_M) \quad , \tag{16}$$

shows that there are two distinct current contributions: the first is the oscillation-center current, $\underline{J}_{oc}$, due to the mean motion of the particles,

$$\underline{J}_{oc} = \int dN \, q \, \dot{\underline{r}}_{oc} \, \delta(\underline{x} - \underline{r}_{oc}), \tag{17}$$

and the second is a magnetization current $\underline{J}_M = c \, \nabla \times \underline{M}$.

The magnetization $\underline{M}$ of the plasma has its origin in the rapid oscillation of a particle, which causes it to describe an elliptic trajectory around its oscilation center; it is given by

$$\underline{M}(\underline{x}) = - \delta V/\delta \underline{B}_s(\underline{x}) = \frac{1}{16\pi} \frac{\omega^2}{c^2} \underline{A}_w^* (\underline{x}) \cdot \frac{\partial \underline{\underline{\varepsilon}}}{\partial \underline{B}_s (\underline{x})} \cdot \underline{A}_w(\underline{x}) \tag{18}$$

These equations and their derivation exhibit clearly the origin of the various ponderomotive terms (15) and (18), and their fundamental relations to the high-frequency wave equation (11).

In Appendix A, we generalize the present situation to allow a dependence of V (i.e., of the dielectric tensor) on the oscillation center velocity and background electric field. One thus gets additional terms: a Lorentz-like force (A.4) in the equation of motion (14), and a polarization current (A.5) in the Maxwell equation (16).

The oscillation center equations are potentially useful for interpreting results of particle simulations. Ultimately, they should be used to perform not particle, but oscillation center simulations, for which it would not be necessary to follow the evolution on the fast time scale. We expect that such a method would allow substantial economy in computation time.

D. Multifluid equations in RF field

In the cold plasma approximation, the plasma dynamics can be described by a complete set of fluid equations. The independent variables are the oscillation-center densities and currents (12, 17), and either the oscillation-center flux densities, $\underline{g}_{oc}$, or the fluid velocities, $\underline{u}_{oc}$:

$$\underline{g}_{oc}(\underline{x},t) = \int dN\ \delta(\underline{x}-\underline{r}_{oc}(z^o,t))\ \dot{\underline{r}}_{oc}(z^o,t)\ , \tag{19}$$

and

$$\underline{u}_{oc}(\underline{x},t) = \underline{g}_{oc}(\underline{x},t)/n_{oc}(\underline{x},t)\ . \tag{20}$$

From Eq. (14) and (16), one derives the complete set of fluid equations:

$$\frac{\partial n}{\partial t} + \nabla \cdot \underline{g} = 0, \tag{21a}$$

$$m\,\frac{\partial \underline{g}}{\partial t} + m\,\nabla\cdot(\underline{u}\,\underline{g}) = q\,(n\,\underline{E} + \frac{1}{c}\,g \times \underline{B}) - n\,\nabla\psi - n\,\nabla\,\frac{\delta V}{\delta n}\ , \tag{21b}$$

$$\frac{1}{c}\,\frac{\partial \underline{B}}{\partial t} + \nabla \times \underline{E} = 0\ , \tag{21c}$$

$$\nabla \times \underline{B} - \frac{1}{c}\,\frac{\partial \underline{E}}{\partial t} = \frac{4\pi}{c}\,(\underline{J}_{oc} + \underline{J}_M), \tag{21d}$$

where the oscillation-center current $\underline{J}_{oc}$ is expressed as $q\underline{g}$ (summed over species), and where we have omitted the "oc" and "s" subscripts.

Equations (21), together with the expression of the functional derivatives $\delta V/\delta n$ and $\delta V/\delta \underline{B}$ (15,18), and the wave equation (11), form a closed system, which can be used to study equilibrium and stability of the plasma in an antenna-induced RF field. Note that the dependence of the equations on the independent variables n and $\underline{B}$ is rather complicated, because the dielectric tensor, and therefore also the RF field, are functionals of n and $\underline{B}$. This interdependence of the RF field and the plasma state is a consequence of the self-consistent treatment of the equations, and, as we shall see shortly, is essential for the study of equilibria and their stability properties.

From Eq. (21b), it appears that the ponderomotive force per unit volume acting on particles of a species is given by $-n\nabla(\delta V/\delta n)$, in a form apparently distinct from the one obtained from fluid theories [10,19]. It has been shown [20], however, that the two forms are in fact equivalent, and that the discrepancy is due to a rather subtle difference in the definitions of force density. A generalization of Eq. (21b) may be found in Appendix A.

E. Comparison of ponderomotive contributions

The different ponderomotive terms can be evaluated from formulas (15) and (18). In order to keep the expressions as simple as possible, and to gain some information on the relative magnitude of these terms, we use a simplified form of the plasma dielectric tensor, valid for frequencies in the neighborhood of the ion gyrofrequency ($\omega \sim O(\Omega_i)$): the electron mass is neglected (a good approximation if $k_\parallel \sim k_\perp \sim \Omega_i/v_A \ll \omega_{pe}/c$ and if $j_{a\parallel} = 0$), as well as the displacement current (assuming $v_A \ll c$). Under these approximations, the parallel component of the electric field vanishes ("parallel" and "perpendicular" will always refer to the direction of the slow magnetic field), and the perpendicular dielectric tensor reduces to [18]

$$\underline{\underline{\varepsilon}}_\perp = S\, \underline{\underline{I}}_\perp + iD\, \underline{\underline{I}} \times \underline{B}/B \quad , \tag{22}$$

with

$$S = -\frac{\omega_{pi}^2}{\omega^2 - \Omega_i^2}$$

and

$$D = \frac{\Omega_i}{\omega} \frac{\omega_{pi}^2}{\omega^2 - \Omega_i^2} - \frac{\omega_{pe}^2}{\omega\,\Omega_e} . \tag{23}$$

In these expressions, the electron and ion densities appear implicitly in ω_{pe}^2 and ω_{pi}^2, respectively, and the background magnetic field appears in the signed gyrofrequencies Ω_e and Ω_i. The ion and electron ponderomotive potentials and the magnetization are expressed in terms of the derivatives of S and D with respect to n_i, n_e and B, respectively (15,18). As functions of the circularly right and left polarized components of the RF amplitude, A_+ and A_-, the ion and electron ponderomotive energy densities and the magnetization energy density are expressible as

$$n_i \frac{\delta V}{\delta n_i} = \frac{1}{16\pi} \frac{\omega^2}{c^2} \frac{\omega_{pi}^2}{\omega\,\Omega_i} [|A_+|^2 \frac{\Omega_i}{\omega - \Omega_i} + |A_-|^2 \frac{\Omega_i}{\omega + \Omega_i}] ,$$

$$n_e \frac{\delta V}{\delta n_e} = \frac{1}{16\pi} \frac{\omega^2}{c^2} \frac{\omega_{pi}^2}{\omega\,\Omega_i} [|A_+|^2 - |A_-|^2] , \tag{24}$$

$$B \frac{\delta V}{\delta B} = \frac{1}{16\pi} \frac{\omega^2}{c^2} \frac{\omega_{pi}^2}{\omega\Omega_i} [|A_+|^2 (\frac{\Omega_i^2}{(\omega - \Omega_i)^2} - 1) - |A_-|^2 (\frac{\Omega_i^2}{(\omega + \Omega_i)^2} - 1)], \tag{25}$$

where we have also used at this point the quasineutrality equation $\omega_{pe}^2/\Omega_e + \omega_{pi}^2/\Omega_i = 0$. In the form (24,25) the different effects can be easily compared: they all have the same order of magnitude, and must therefore all be retained. This conclusion extends even to the case when the wave frequency approaches the ion gyrofrequency, because the component A_+ then becomes very small. From the solution of the perpendicular wave equation,

$$[\nabla \times (\nabla \times \underline{A_W})]_\perp - \frac{\omega^2}{c^2} \underline{\varepsilon}_\perp \cdot \underline{A_W} = \frac{4\pi}{c} \underline{j}_{a\perp} , \tag{26}$$

we see that A_+ vanishes with $(\omega - \Omega_i)$, which shows that the plasma shields itself against right circularly polarized waves. This characteristic of the fields has the important consequence that (at least when the field gradients remain finite) the ponderomotive terms have no singularity at the ion gyrofrequency, despite the apparent poles of Eq. (24) [7]. Instead, rather unexpectedly, the ion ponderomotive force is dominated by the component of the field polarized in the opposite direction, A_-. More generally, as we shall see next, the absence of singularity is a consequence of the local conservation of total momentum, which includes both the particle and field momenta.

F. Momentum conservation

Momentum conservation follows from the translational symmetries of the system [11]. For this problem, one statement of local momentum balance is obtained by summing the force densities acting at a given point $\underline{x}$ on the plasma, and relating it to the high-frequency field intensity. Perform the sum over species of the momentum equations (21b) and eliminate $\sum q\underline{g} = \underline{J}_{oc}$ by (21d); the force density obtained, which is explicitly dependent on the ponderomotive energy, is

$$\underline{F}(\underline{x}) \equiv - n \nabla \frac{\delta V}{\delta n} - \frac{1}{c} \underline{J}_M \times \underline{B} \tag{27}$$

$$= - \nabla[n \frac{\delta V}{\delta n} + \underline{B} \cdot \frac{\delta V}{\delta \underline{B}}] + \nabla \cdot (\underline{B} \frac{\delta V}{\delta \underline{B}})$$

$$+ \nabla n \frac{\delta V}{\delta n} + \nabla \underline{B} \cdot \frac{\delta V}{\delta \underline{B}}$$

$$= - \nabla(n \frac{\delta V}{\delta n} + \underline{B} \cdot \frac{\delta V}{\delta \underline{B}} + \frac{1}{16\pi} \frac{\omega^2}{c^2} \underline{A}_W^* \cdot \underline{\underline{\varepsilon}} \cdot \underline{A}_W)$$

$$+ \nabla \cdot (\underline{B} \frac{\delta V}{\delta \underline{B}}) + \frac{1}{16\pi} \frac{\omega^2}{c^2} 2 \operatorname{Re} (\nabla \underline{A}_W^* \cdot \underline{\underline{\varepsilon}} \cdot \underline{A}_W) \quad .$$

The right hand side of this equation can be modified by use of the wave equation (11), so that $\underline{F}$ can also be written as

$$\underline{F}(\underline{x}) = -\nabla(n \frac{\delta V}{\delta n} + \underline{B} \cdot \frac{\delta V}{\delta \underline{B}} + \frac{1}{16\pi} \frac{\omega^2}{c^2} \underline{A}_W^* \cdot \underline{\underline{\varepsilon}} \cdot \underline{A}_W + \frac{1}{16\pi} \underline{B}_W^* \cdot \underline{B}_W)$$

$$+ \nabla \cdot (\underline{B} \frac{\delta V}{\delta \underline{B}})$$

$$+ \frac{1}{8\pi} \nabla \cdot \mathrm{Re}[\underline{B}_W \underline{B}_W^* + \frac{\omega^2}{c^2} \underline{\underline{\varepsilon}} \cdot \underline{A}_W \underline{A}_W^*]$$

$$- \frac{1}{2c} \mathrm{Re} (\underline{j}_a^* \times \underline{B}_W - \nabla \cdot \underline{j}_a^* \underline{A}_W) \quad . \qquad (28)$$

In vacuum, this expression reduces to the divergence of the Maxwell stress tensor. The integration of Eq. (28) over the plasma volume yields the equality of the total ponderomotive force acting on the plasma, with the force acting on the boundary (the first and second terms of the RHS of Eq. (28)) and on the antenna (the last term of Eq. (28)).

G. Magnetohydrodynamic equations

We shall illustrate the use of the preceding formalism in the problem of ponderomotive stabilization of the flute instability in a mirror. The flute instability is well represented by the one-fluid magnetohydrodynamic equivalent of the equations (21), and is essentially a quasineutral perturbation of the plasma (in contrast to the RF field). The equations differ from the usual set of MHD equations by the additional ponderomotive force density in the momentum equation and by the magnetization $\underline{M}$ in Ampere's law. Thus we have

$$\frac{\partial n}{\partial t} + \nabla \cdot (n\underline{u}) = 0 \quad , \qquad 29a)$$

$$\frac{\partial \underline{B}}{\partial t} = \nabla \times (\underline{u} \times \underline{B}) \qquad (29b)$$

$$mn(\frac{\partial}{\partial t} \underline{u} + \underline{u} \cdot \nabla \underline{u}) = -\nabla \cdot \underline{\underline{P}} + \frac{1}{c} \underline{j} \times \underline{B} - n\nabla\psi - n \nabla \frac{\delta V}{\delta n} \quad , \qquad (29c)$$

$$\frac{4\pi}{c} \underline{j} = \nabla \times (\underline{B} - 4\pi \underline{M}) \quad , \qquad (29d)$$

where $\underline{P}$ is defined below. Note that the ponderomotive force in (29) must be the sum of the ion and electron ponderomotive forces: the fluctuation being quasineutral, a local change of ion density is accompanied by an equal change of electron density, and the change in ponderomotive energy involves their sum. More formally, if $\delta n_e = \delta n_i = \delta n$, one has $\delta V = \delta n_e \, \delta V/\delta n_e + \delta n_i \, \delta V/\delta n_i = \delta n \, (\delta V/\delta n_e + \delta V/\delta n_i)$.

For generality, we have introduced into Eq. (29) a (possibly anisotropic) pressure tensor $\underline{P}$. It is important for mirror devices, because it affects the equilibrium magnetic profile, and also because thermal energy is after all the source of free energy that drives the flute instability in mirrors. Formally, such thermal effects are incorporated into the theory, by adding to the action (1) an additional term S_{th} equal to

$$S_{th} = - \int dx \int dt \; U(n,B), \tag{30}$$

where U(n,B) is the internal energy density of the plasma (the specific entropy is assumed conserved by adiabaticity). The consequences of this thermal action S_{th} on the equations are easily drawn, and we shall omit the derivations here. One finds that the pressure tensor is

$$\underline{P} = \underline{I}\left(n \frac{\partial U}{\partial n} + \underline{B} \cdot \frac{\partial U}{\partial \underline{B}} - U\right) - \underline{B} \, \frac{\partial U}{\partial \underline{B}} \quad ,$$

and that the dielectric tensor is the warm-fluid plasma dielectric (as for instance in [21]).

Consider a tandem mirror with an elongated central cell. When the mode of interest is the flute instability, a two-dimensional model is justified for the equilibrium, as well as for the perturbed state of the plasma. The interchange forces due to the average curvature of the background magnetic field are modeled by a "gravitational" potential $\psi(\underline{x})$. The magnetic field is straight, and oriented in the $\hat{\underline{z}}$ (or "parallel") direction, along which the system is translationally symmetric. The antenna currents are in the perpendicular plane and emit RF waves with finite $k_\parallel$. The pressure is assumed isotropic, and is related to the internal energy of the plasma per unit volume U(n), by the thermodynamic relation $p = n\,dU/dn - U$, the derivative being taken at constant entropy.

With these assumptions, the static equilibrium equation which follows from (29) reduces to

$$\nabla\left(p + \frac{B^2}{8\pi}\right) + n\,\nabla\left(\psi + \frac{\delta V}{\delta n}\right) + B\,\nabla\left(\frac{\delta V}{\delta B}\right) = 0 \quad . \tag{31}$$

Equation (31) expresses the balance, at equilibrium, of the plasma and magnetic pressure forces with the interchange, ponderomotive, and magnetization forces.

For spectral stability analysis, the MHD equations are linearized around equilibrium, with a mode evolving in time as $\exp(\gamma t)$. It can be seen that, expressed in term of the plasma displacement $\underline{\xi}(\underline{x},t)$, the equations can be given a self-adjoint form. This is a consequence of the dissipationless nature of the physical mechanisms involved, and of the fact that the interactions with the external world can be reduced to the existence of a free energy source V. As a result, the equations can be expressed variationally, as in ordinary MHD. There are now some additional terms of ponderomotive origin. In Appendix C, we derive the ΔW variational principle for the growth rate γ:

$$\gamma^2 = \max_{\underline{\xi}(\underline{x})} \, [-\Delta W/N], \tag{32}$$

where

$$\Delta W \equiv \Delta W_{MHD} + \Delta W_p + \Delta W_M + \Delta W_A, \tag{33}$$

and the quantity

$$N = \int dx\; m\, n\, |\underline{\xi}|^2 \tag{34}$$

is a measure of plasma inertia. We define

$$\Delta W_{MHD} = \int dx\ [(\nabla \cdot \underline{\xi})^2 (\gamma_H p + \frac{B^2}{4\pi}) + 2\ \nabla \cdot \underline{\xi}\ \ \underline{\xi} \cdot \nabla(p + \frac{B^2}{8\pi})$$

$$- \underline{\xi} \cdot \nabla n\ \ \underline{\xi} \cdot \nabla\psi] \quad , \qquad (35)$$

which has the form of the usual ΔW for two-dimensional MHD (γ_H is the specific heat ratio, $\gamma_H\, p = n\ dp/dn = n^2\ d^2U/dn^2$). Also, the quantities ΔW_p, ΔW_M, and ΔW_A are three distinct RF contributions. The first RF contribution,

$$\Delta W_p = - \int dx \quad \underline{\xi} \cdot \nabla n \quad \underline{\xi} \cdot \nabla \frac{\delta V}{\delta n}, \qquad (36)$$

has its origin in the equilibrium ponderomotive drifts of the particles, and corresponds to the mechanism invoked in the introduction of this paper (the balance of ponderomotive drifts and of interchange drifts.) The second RF contribution,

$$\Delta W_M = - \int dx \quad \underline{\xi} \cdot \nabla B \quad \underline{\xi} \cdot \nabla \frac{\delta V}{\delta B}, \qquad (37)$$

involves the magnetization of the plasma due to the RF field: the energy of the magnetized medium, the plasma, changes as it is displaced in the quasi-static magnetic field gradient. The last RF contribution in the variational principle

$$\Delta W_A = \iint dx\ dx'\ \nabla \cdot (n\ \underline{\xi})\ \nabla' \cdot (n'\ \underline{\xi}')\ \delta^2 V/\delta n \delta n'$$

$$+ 2 \iint dx\ dx'\ \nabla \cdot (B\ \underline{\xi})\ \nabla' \cdot (n'\ \underline{\xi}')\ \delta^2 V/\delta B \delta n'$$

$$+ \iint dx\ dx'\ \nabla \cdot (B\ \underline{\xi})\ \nabla' \cdot (B'\ \underline{\xi}')\ \delta^2 V/\delta B \delta B' \qquad (38)$$

differs from the preceding two. Its expression involves the second functional derivatives of the ponderomotive energy V, considered as functional of the plasma density n and the slow field B (The notations n', ∇', etc. in (38) are short for $n(\underline{x}')$, $\partial/\partial \underline{x}'$, etc.) Because the first derivatives (15,18) involve the wave field $\underline{A}_w(\underline{x},t)$, which is a solution of the wave Eq. (11) and is therefore itself a functional of $n(\underline{x}')$ and $B(\underline{x}')$, the second derivatives of V are two-point terms depending on $\underline{x}$ and $\underline{x}'$. In terms of the hermitian Green's tensor $\underline{\underline{G}}(\underline{x},\underline{x}')$ of the wave equation (11), one has

$$\frac{\delta V^2}{\delta n\ \delta n'} = - \frac{1}{16\pi} \frac{\omega^2}{c^2} \underline{A}_w^* \cdot \frac{\partial^2 \underline{\underline{\varepsilon}}}{\partial n^2} \cdot \underline{A}_w\ \delta(\underline{x} - \underline{x}')$$

$$- \frac{1}{8\pi} \left(\frac{\omega^2}{c^2}\right)^2 \underline{A}_w^* \cdot \frac{\partial \underline{\underline{\varepsilon}}}{\partial n} \cdot \underline{\underline{G}} \cdot \frac{\partial \underline{\underline{\varepsilon}}}{\partial n'} \cdot \underline{A}_w' \quad , \tag{39}$$

and similar expressions for the other derivatives.

The term ΔW_A takes into account the self-consistent modification of the RF field due to infinitesimal displacements of the plasma away from its equilibrium position, which cause a perturbation of the ponderomotive forces. In practice, the calculation of ΔW_A does not necessarily require the knowledge of the Green's tensor $\underline{\underline{G}}(\underline{x},\underline{x}')$. Typically, as is usual with variational principles, one can get a good approximation of the maximum growth rate with a limited set of allowable displacement fields $\underline{\xi}(\underline{x})$, chosen to represent the most rapid instabilities, for instance the flute modes or the ballooning modes. It is then sufficient to evaluate the perturbation of the field $\underline{A}_w(\underline{x},t)$ (taking into account the boundary conditions) for each of these allowable degrees of freedom, a much easier task. A detailed evaluation of the various terms for the flute mode shows that the term ΔW_A is as large as the other ponderomotive contributions, and plays therefore a significant role in the stabilization. Those calculations will appear in a subsequent paper.

H. Axisymmetric geometry

To illustrate the above formalism, we specialize it to an axisymmetric system (including $\underline{j}_a$). Furthermore, we use the cold-plasma perpendicular dielectric tensor (22,23). This approximation is justified for a "moderately cold" plasma, i.e., in the following regime: the plasma is cold at high frequency: $|\gamma| + |k_{\parallel} v_{thi}| < |\omega - \Omega_i|$, $|k_{\parallel} v_{the}| < |\Omega_e|$, and is simultaneously warm at low frequency. Therefore, we do not exclude finite pressure in the low-frequency MHD equations, and preserve plasma diamagnetism (finite β) and interchange forces (driving flute modes). The radial component of the wave equation leads to

$$A_r(r) = i\, D(S - N_{\parallel}^2)^{-1} A_\theta(r) \tag{40}$$

which can then be eliminated from equations (10),(15), and (18) to give, in equilibrium,

$$V = \frac{1}{16\pi} \int dx \quad \{K^2 |A_\theta(r)|^2 + |(\frac{d}{dr} + \frac{1}{r}) A_\theta(r)|^2\}$$
$$- \frac{1}{2c} \int dx \ \mathrm{Re}\, [A_\theta^*(r) j_a(r)] \quad , \tag{41}$$

$$\frac{\delta V}{\delta n}(r) = \frac{1}{16\pi} |A_\theta|^2 \frac{\partial}{\partial n}(K^2) \quad ,$$
$$\frac{\delta V}{\delta B}(r) = \frac{1}{16\pi} |A_\theta|^2 \frac{\partial}{\partial B}(K^2) \quad , \tag{42}$$

and

$$\frac{d}{dr} \left(\frac{d}{dr} + \frac{1}{r}\right) A_\theta(r) = K^2 A_\theta - \frac{4\pi}{c} j_a(r), \tag{43}$$

where

$$K^2(n,B) \equiv -\frac{\omega^2}{c^2} \frac{(S - N_{\parallel}^2) - D^2}{S - N_{\parallel}^2} \tag{44}$$

(K is the attenuation wave number), and $N_{\parallel} = k_{\parallel} c/\omega$. From Eq. (23) one obtains

$$K^2(n,B) = \frac{\omega^2}{c^2} \quad \frac{[\omega_{pi}^2 + \Omega_i(\omega - \Omega_i)\, N_\parallel^2] \quad [\Omega_i(\omega + \Omega_i)\, N_\parallel^2 - \omega_{pi}^2]}{[\omega_{pi}^2 + N_\parallel^2 \; (\omega^2 - \Omega_i^2)] \quad \Omega_i^2} \quad . \tag{45}$$

It is therefore apparent that, for finite $k_\parallel$, there is no singularity in the ponderomotive potentials (42) at the ion gyrofrequency. It is only for $\omega < \Omega_i$ that a singularity can occur, when the denominator of Eq. (45) vanishes, i.e. when the wave undergoes an Alfvèn-ion-cyclotron (AIC) resonance:

$$\omega^2 = k_\parallel^2 \; v_A^2 \; (1 + k_\parallel^2 \; v_A^2/\Omega_i^2)^{-1} \tag{46}$$

Then the field A_θ becomes singular, and dissipative effects play an essential role.

The discussion above applies, strictly speaking, to the cold plasma limit only. For a warm plasma, at the ion gyrofrequency, the situation is somewhat complicated by ion thermal spread, the existence of Bernstein waves, and the possibility of dissipation. It remains true nevertheless that most of the conclusions extrapolate to this case; in particular there is no sudden reversal of the global ponderomotive effects across the boundary $\omega = \Omega_i$, and the A_- component of the field is dominant. This conclusion calls into question the interpretations of the experimental observation [1] that the stability of the plasma depends extremely sensitively on the frequency, in the neighborhood of Ω_i. We argue that ponderomotive effects are not directly related to this transition. We conjecture that dissipative effects and heating due to AIC resonance for $\omega < \Omega_i$ may be responsible for the observation.

I. Ponderomotive Energy Functional V

The ponderomotive energy V (10) is a functional of the plasma densities, $n(\underline{x})$, of the slow magnetic field, $B(\underline{x})$, and of the wave amplitude, $\underline{A}_w(\underline{x})$.

More precisely, for fixed n and B fields, it is quadratic in the field $\underline{A}_w$. Since the particular solution $\underline{A}_w(\underline{x})$ obeys the wave equation (11), and extremizes V for n and B fixed, it is a general property of such quadratic expressions that the value of V evaluated at $\underline{A}_w$ is equal to the opposite of its quadratic term, or in other words to half of its linear term. Therefore, after substitution of the solution $\underline{A}_w$ (function of $\underline{x}$ and t, functional of $n(\underline{x}')$ and $\underline{B}(\underline{x}')$), V is a functional of n and B only, and is (see also [22])

$$V = - \frac{1}{4c} \int dx \ \ \mathrm{Re}[\underline{j}_a^* \cdot \underline{A}_w] \tag{47}$$

Note that this integral must be evaluated at the antenna only, where $\underline{j}_a$ is different from zero. In fact, the ponderomotive energy V is simply related to the antenna impedance $Z(\omega)$ and to the antenna current amplitude I_a, by

$$V = - (1/4)\ I_a^* \cdot L \cdot I_a , \tag{48}$$

where the inductance L is defined as $L \equiv \mathrm{Im}\ Z(\omega)/\omega$. For a multiple antenna system, L must be interpreted as the symmetric inductance matrix, and I_a as the current array. The ponderomotive energy V is the RF contribution to the free energy of the system, the currents being maintained constant. The ponderomotive effects are the forces that act on the plasma, considered as a dielectric [23,24]. This interpretation allows an unequivocal experimental determination of the ponderomotive effects: the correlation of the variation of antenna impedance and current with the (generalized) displacement of the plasma leads to a direct evaluation of the corresponding (generalized) ponderomotive force.

PART II: HAMILTONIAN FORMULATION

We develop here the Hamiltonian formulation of the antenna-plasma system dynamics. This point of view complements the Lagrangian formulation, because it is formulated entirely in terms of Eulerian fields. It also allows the systematic construction of Lyapunov functionals, with which it is possible, in principle, to study the nonlinear stability of equilibrium states of the system.

A. Hamiltonian and Poisson Bracket

The derivation of the Hamiltonian and of the (noncanonical) Poisson bracket follows the usual procedure [25,26], starting from the action (5), considered as a functional of the low frequency fields only, $\underline{r}_{oc}(z^o,t)$ and $\underline{A}_s(\underline{x},t)$, the wave amplitude $\underline{A}_w$ being the solution of (11). The procedure is summarized as follows. First, one determines the fields $\underline{p}_{oc}$ and $\underline{\Pi}_s$ canonically conjugate to $\underline{r}_{oc}$ and $\underline{A}_s$; the Poisson bracket has its canonical form when it is expressed in terms of these variables. Second, the Hamiltonian is introduced via the Legendre transform of the Lagrangian, as a functional of $\underline{r}_{oc}(z^o,t)$, $\underline{p}_{oc}(z^o,t)$, $\underline{A}_s(\underline{x},t)$ and $\underline{\Pi}_s(\underline{x},t)$. Next, the Hamiltonian is expressed as a functional of the Eulerian fields $n(\underline{x},t)$, $\underline{B}(\underline{x},t)$, $\underline{g}(\underline{x},t)$, and $\underline{E}(\underline{x},t)$ (we shall omit their oc and s suffixes), each defined in terms of the canonical fields. Finally, the canonical Poisson bracket is mapped into a noncanonical Poisson bracket expressed in terms of the Eulerian fields.

By definition, the canonical momenta are functional derivatives of the action:

$$\underline{p}_{oc}(z^o,t) = \frac{\delta S}{\delta\left(\frac{\partial \underline{r}_{oc}}{\partial t}\right)} ,$$

and

$$\underline{\Pi}_s(\underline{x},t) = \frac{\delta S}{\delta(\frac{\partial \underline{A}s}{\partial t})} \quad . \tag{49}$$

The functional derivatives with respect to particle fields are defined here with respect to the measure dN: if $F[\underline{r}]$ is a functional of the particle field $\underline{r}(z^o,t)$, its derivative $\delta F/\delta \underline{r}$ is such that the differential δF is

$$\delta F = \int dN\, dt\, (\frac{\delta F}{\delta \underline{r}}) \cdot \delta \underline{r} \quad , \tag{50}$$

as a linear functional of the differential $\delta \underline{r}$. Consistently, the unit distribution $\delta(z^o,z^{o'})$ is such that for all functions $G(z^o)$ defined on the Lagrangian reference space, one has

$$G(z^o) = \int dN\, \delta(z^o, z^{o'})\, G(z^{o'}) \quad . \tag{51}$$

For a cold fluid, the dielectric tensor is a function of oscillation center densities and fluxes n and $\underline{g}$, and also of the background low frequency magnetic and electric fields $\underline{B}$ and $\underline{E}$. These fields are functionals of $\underline{r}_{oc}(z^o,t)$ and $\underline{A}_s(\underline{x},t)$, from (12,17,19). One finds therefore the conjugate momenta:

$$\begin{aligned} \underline{p}_{oc}(z^o,t) &= m\, \dot{\underline{r}}_{oc}(z^o,t) + \frac{q}{c}\, \underline{A}_s\, (\underline{r}_{oc},t) - \frac{\delta V}{\delta \underline{g}(\underline{x})} \Big|_{\underline{x} = \underline{r}_{oc}(z^o,t)} \;, \\ \underline{\Pi}_s(\underline{x},t) &= - \frac{1}{4\pi}\, \frac{1}{c}\, \underline{E}(\underline{x},t) + \frac{1}{c}\, \frac{\delta V}{\delta \underline{E}(\underline{x},t)} \end{aligned} \tag{52}$$

To find the Hamiltonian as a functional of the canonical variables, it is necessary to invert the equations (52), and obtain the velocity field $\dot{\underline{r}}_{oc}$, the flux $\underline{g}$, and the electric field $\underline{E} = (-1/c)\, \partial \underline{A}_s/\partial t$, as functionals of $\underline{r}_{oc}(z^o,t)$, $\underline{p}_{oc}(z^o,t)$, $\underline{A}_s(\underline{x},t)$ and $\underline{\Pi}_s(\underline{x},t)$. This operation can be done perturbatively to second order in the high-frequency field amplitude, consistently with the ordering we have made for the derivation of the ponderomotive action (5). As the simplest case, the dependence of the dielectric tensor on $\underline{g}$ and $\underline{E}$ will be neglected (see Appendix A). The terms omitted due to this simplification do not play a significant role unless the ambipolar electric field is large and the plasma is rotating at high velocity. The particle and field momenta therefore reduce to

$$\underline{p}_{oc}(z^o,t) = m\,\dot{\underline{r}}_{oc}(z^o,t) + \frac{q}{c}\underline{A}_s(\underline{r}_{oc},t) \quad ,$$
$$\underline{\Pi}_s(\underline{x},t) = \frac{1}{4\pi}\,\frac{1}{c^2}\,\frac{\partial \underline{A}_s}{\partial t} = -\frac{1}{4\pi}\,\frac{1}{c}\,\underline{E}_s(\underline{x},t) \quad . \tag{53}$$

The fields $\underline{r}_{oc}(z^o,t)$, $\underline{p}_{oc}(z^o,t)$, $\underline{A}_s(\underline{x},t)$, and $\underline{\Pi}_s(\underline{x},t)$, being canonically conjugate to each other, their Poisson brackets have the canonical form:

$$\{\underline{r}_{oc}(z^o),\, \underline{p}_{oc}(z^{o'})\} = \underline{\underline{I}}\,\delta(z^o,\, z^{o'}) \quad ,$$
$$\{\underline{A}_s(\underline{x}),\, \underline{\Pi}_s(\underline{x}')\} = \underline{\underline{I}}\,\delta(\underline{x} - \underline{x}') \quad . \tag{54}$$

The Hamiltonian is the Legendre transform of the Lagrangian L ($S \equiv \int dt\, L$):

$$H = \int dN\, \underline{p}_{oc}\cdot\dot{\underline{r}}_{oc} + \int dx\, \underline{\Pi}_s \cdot \frac{\partial \underline{A}_s}{\partial t} - L \quad . \tag{55}$$

The value of H is the total energy, including the free energy due to the high-frequency field V given in (10):

$$H = \int dN\, \{\tfrac{1}{2} m\, |\dot{\underline{r}}_{oc}|^2 + \psi(\underline{r}_{oc})\} + \int dx\, U(n,B)$$
$$+ \frac{1}{8\pi}\int dx\, \{\frac{1}{c^2}\left|\frac{\partial \underline{A}_s}{\partial t}\right|^2 + |\nabla \times \underline{A}_s|^2\} + V \quad . \tag{56}$$

As long as the fluid limit is valid, i.e., as long as all the particles at a given point $(\underline{x},t)$ have in common a unique velocity, one finds that the Hamiltonian (56) can also be expressed entirely in terms of the Eulerian fields $n(\underline{x},t)$, $\underline{B}(\underline{x},t)$, $\underline{g}(\underline{x},t)$ and $\underline{E}(\underline{x},t)$ [27]:

$$H = \int dx \; [\frac{1}{2} m\, n\, |\underline{u}(x)|^2 + n(\underline{x})\psi(\underline{x}) + U(n,B)]$$

$$+ \frac{1}{8\pi} \int dx \; [|\underline{E}(\underline{x})|^2 + |\underline{B}(\underline{x})|^2] + V \quad . \tag{57}$$

Now, it is possible to calculate the Poisson bracket of any two of these Eulerian fields: use their definitions (12,19,20,53) in terms of the canonical fields, the canonical relations (54), and elementary properties of the Poisson bracket (bilinearity, antisymmetry, and the derivative property: $\{F,G\} = \int dN\, \delta F/\delta z_i \, \{z_i, G\}$).

The derivation follows the one in [25], and one obtains:

$$\{n(\underline{x}), \underline{g}(\underline{x}')\} = - \frac{1}{m} n(\underline{x}') \nabla\delta(\underline{x} - \underline{x}') \quad ,$$

$$\{\underline{g}(\underline{x}), \underline{g}(\underline{x}')\} = - \frac{1}{m} \underline{\nabla}\, \delta(\underline{x} - \underline{x}')\, \underline{g}(\underline{x}) - \frac{1}{m} \underline{g}(\underline{x}') \nabla\, \delta(\underline{x} - \underline{x}')$$

$$- \frac{1}{m} \delta(\underline{x} - \underline{x}')\, n(\underline{x}) \frac{q}{mc} \underline{B}(\underline{x}) \times \underline{\underline{I}} \quad ,$$

$$\{\underline{g}(\underline{x}), \underline{E}(\underline{x}')\} = 4\pi \frac{q}{m} n(\underline{x})\, \delta(\underline{x} - \underline{x}')\, \underline{\underline{I}} \quad ,$$

$$\{\underline{B}(\underline{x}), \underline{E}(\underline{x}')\} = - 4\pi c\, \nabla\, \delta(\underline{x} - \underline{x}') \times \underline{\underline{I}} \quad , \tag{59}$$

the other combinations being equal to zero. Remarkably, the Poisson bracket of any two of the Eulerian fields can be expressed entirely in terms of themselves. This closure property is ultimately a consequence of the symmetry of the system under permutation of particles of the same species: the evolution of the Eulerian fields, does not depend on the past history, i.e. the individual particle trajectories. Expressed in those variables, the Poisson bracket has lost its canonical form, and is even degenerate as will be emphasized later.

The Poisson bracket (59) is identical to the one for a multifluid plasma in absence of high-frequency fields [28,29], except that it is written now in terms of oscillation-center densities. The ponderomotive terms are included solely in the Hamiltonian (57), a consequence of the approximation (53).

Since the Hamiltonian (57) is a functional of the Eulerian fields only, the evolution of any functional F of $n(\underline{x},t)$, $\underline{B}(\underline{x},t)$, $\underline{g}(\underline{x},t)$ and $\underline{E}(\underline{x},t)$, is given by $dF/dt = \{F, H\}$, and can be determined from the expressions (59), and the derivative property of the Poisson bracket. Any reference to the Lagrangian fields now becomes unnecessary. In particular, for F in the set $\{n, \underline{B}, \underline{g}, \underline{E}\}$, one recovers the fluid equations (21).

B. Lyapunov Stability Analysis

The knowledge of the Hamiltonian and of the Poisson bracket makes it possible to apply Arnold's method of stability analysis [13,14], which is one of the most important practical outcomes of the Hamiltonian formulation. There is a simple and systematic way to construct Lyapunov functionals that are useful for deriving linear or nonlinear stability criteria, for nonstatic as well as static equilibria. The Lyapunov functional which is constructed here is a conserved quantity with respect to the nonlinear dynamics of the system (it is time independent under motion by (21).) In addition, it is stationary at the equilibrium state in functional space (its first variation evaluated at the equilibrium state, vanishes for any variation of the fields.) Therefore, if the equilibrium point turns out to be a local and strict minimum (or maximum) for the Lyapunov functional, one is guaranteed that the evolution of the perturbed equilibrium state will remain confined near that equilibrium in a certain norm, i.e. that the system is Lyapunov stable. (There are a few delicate points in this argument for infinite-dimensional systems such as this one [14].)

The construction of the Lyapunov functional involves a special class of functionals C (the Casimir functionals), which are characteristic of the degenerate Poisson bracket but are independent of the Hamiltonian. By definition, their Poisson bracket with any other functional F of the fields (here the Eulerian fields) vanishes: $\{C, F\} = 0$.

The Casimirs are quantities preserved by the evolution, whatever the Hamiltonian may be. They correspond to symmetries of the system, such as continuous transformations of the Lagrangian fields which do not affect the Eulerian fields (e.g., relabeling of the particles). One may also view the Casimirs as constraints on the dynamics, since they remain equal to their initial value.

Consider the functional H_C, constructed as the sum of the Hamiltonian H and of the various Casimirs, represented symbolically by C: $H_C = H + C$. It follows from the definition of a Casimir that this functional H_C can also be used as the Hamiltonian and that it generates exactly the same time evolution as H. In particular, H_c is a constant of motion. From the derivative property of the Poisson bracket, if H_C has a critical state in the functional space (where its differential vanishes), then this state is an equilibrium (i.e., does not evolve in time). Indeed, at the critical state $\delta H_C/\delta n = 0$, $\delta H_C/\delta B = 0$, etc., so that any functional F is stationary:

$$dF/dt = \{F, H\} = \{F, H_C\} = \int dx\, [\{F, n\}\, \delta H_C/\delta n + \{F, B\}\, \delta H_C/\delta B + \dots] = 0.$$

In summary, H_C satisfies the first requirements for the Lyapunov functional: its differential vanishes at the equilibrium state, and it is conserved in time during the evolution. The final requirement for Lyapunov stability is that the Lyapunov functional be definite in sign. The properties of H_C close to the equilibrium will determine the stability conditions: as discussed earlier, if H_C is locally convex, then the system will not be able to depart by more than a finite distance from its equilibrium point, as measured by a certain norm determined by H_C [13]. One can show [14] that the second variation of H_C evaluated at the equilibrium state is the Hamiltonian for the linearized dynamics. In particular, linear stability criteria follow from the conditions on the equilibrium for $\delta^2 H_C$ to be positive (or negative) definite. In this case $\delta^2 H_c$ can be used as a norm for linearized stability.

Casimir functionals for the Poisson bracket (59) are the same as for the multifluid plasma [30]. If we restrict the present discussion to the two-dimensional case, a first family is given by C_1, which is, for each species:

$$C_1 = \int dx\ n(\underline{x})\ \Phi(Z) \quad , \tag{60}$$

where $Z \equiv (w + \Omega)/n$, $w \equiv \hat{\underline{z}} \cdot \nabla x \underline{u}$ is the fluid vorticity, $\Omega = qB/mc$ is the gyrofrequency, and where Φ is an arbitrary function of its argument Z. These Casimir functionals generate displacements of the oscillation centers in Lagrangian phase space:

$$\delta \underline{r} = \{\underline{r}, C_1\} = -\frac{1}{m} \frac{1}{n(\underline{x})} \frac{d^2\Phi}{dZ^2}\ \hat{\underline{z}} \times \nabla Z \quad .$$

These particle displacements in the original phase space are along surfaces of constant Z, and exhibit one non-trivial symmetry of the system: the Eulerian fields (densities, etc.) are unaffected by such a "microscopic" transformation.

A second family of Casimirs is given by

$$C_2 = -\frac{1}{4\pi} \int dx\ \varphi(\underline{x})\ [\underline{\nabla} \cdot \underline{E} - 4\pi q n(\underline{x})] \quad , \tag{61}$$

where φ is an arbitrary function of $\underline{x}$, and where a sum over species is implied. These functionals only generate gauge transformations of the electromagnetic potential, by adding a time-independent gradient to the vector potential $\underline{A}$:

$$\delta \underline{A} = \{\underline{A}, C_2\} = - c \, \nabla\varphi \quad ,$$

$$\delta \underline{r} = \{\underline{r}, C_2\} = \underline{0} \quad ,$$

$$\delta \underline{\Pi} = \{\underline{\Pi}, C_2\} = \underline{0} \quad .$$

Using the functional H_C built from the Hamiltonian H (57) and the Casimirs C_1 and C_2, we find equilibrium equations by looking for critical states of H_C. The first variation of H_C is

$$\delta H_C = \int dx \, \{\delta n[\frac{1}{2} m \, |\underline{u}|^2 + \psi + \frac{\partial U}{\partial n} + \frac{\delta V}{\delta n} + \Phi(Z) + q\varphi]$$
$$+ n \, \delta Z \, \frac{d\Phi}{dZ} + \delta\underline{u} \cdot mn\underline{u}$$
$$+ \delta B \, [\frac{1}{4\pi} B + \frac{\partial U}{\partial B} + \frac{\delta V}{\delta B}] + \frac{1}{4\pi} \delta\underline{E} \cdot [\underline{E} + \nabla\varphi]\} \quad , \tag{62}$$

where

$$n \, \delta Z = - Z \, \delta n + (\Omega/B) \, \delta B + \hat{\underline{z}} \cdot \nabla \times \delta\underline{u}. \tag{63}$$

The condition that δH_C vanishes, for all variations δn, δB, $\delta\underline{u}$ and $\delta\underline{E}$, translates into a set of equilibrium equations:

$$\frac{1}{2} m \, |\underline{u}|^2 + \Psi + \frac{\partial U}{\partial n} + \frac{\delta V}{\delta n} + \Phi(Z) - Z \, \frac{d\Phi}{dZ} + q\varphi = 0 \quad ,$$

$$\frac{B}{4\pi} + \frac{\partial U}{\partial B} + \frac{\delta V}{\delta B} + (\Omega/B) \, \frac{d\Phi}{dZ} = 0 \quad ,$$

$$m \, n \, \underline{u} = \hat{\underline{z}} \times \nabla \, (\frac{d\Phi}{dZ}), \tag{64}$$

and

$$\underline{E} = - \nabla\varphi,$$

where the functional derivatives of V are given by (15,18), and where the wave amplitude $\underline{A}_w$ is the solution of equation (11).

Derivation of the linearized stability criteria requires knowledge of the second variation of H_C with respect to the Eulerian fields, which is readily obtained:

$$2\,\delta^2 H_C = \int dx\, [mn|\delta\underline{u} + \frac{\delta n}{n}\,\underline{u}|^2 + n\,\frac{d^2\Phi}{dZ^2}\,(\delta Z)^2 + |\delta\underline{E}|^2/4\pi + (\delta n)^2\,(\frac{\partial^2 U}{\partial n^2} - \frac{m}{n}\,|u|^2)$$

$$+ 2\,\delta n\,\delta B\,\frac{\partial^2 U}{\partial n\,\partial B} + (\delta B)^2\,(\frac{\partial^2 U}{\partial B^2} + \frac{1}{4\pi})] + 2\,\delta^2 V \quad . \qquad (65)$$

The second variation of V is

$$\delta^2 V = \frac{1}{16\pi}\,\frac{\omega^2}{c^2}\int dx\,\delta\underline{A}_w^* \cdot \underline{\underline{\varepsilon}} \cdot \delta\underline{A}_w$$

$$- \frac{1}{16\pi}\int dx\,|\nabla \times \delta\underline{A}_w|^2 - \frac{1}{16\pi}\,\frac{\omega^2}{c^2}\int dx\,\underline{A}_w^* \cdot \delta^2\underline{\underline{\varepsilon}} \cdot \underline{A}_w \quad , \qquad (66)$$

where $\delta\underline{A}_w$ is the solution of the linearized wave equation,

$$\nabla \times (\nabla \times \delta\underline{A}_w) - \frac{\omega^2}{c^2}\,\underline{\underline{\varepsilon}} \cdot \delta\underline{A}_w = \frac{\omega^2}{c^2}\,\delta\underline{\underline{\varepsilon}} \cdot \underline{A}_w, \qquad (67)$$

and

$$\delta\underline{\underline{\varepsilon}} = \frac{\partial\underline{\underline{\varepsilon}}}{\partial n}\,\delta n + \frac{\partial\underline{\underline{\varepsilon}}}{\partial B}\,\delta B.$$

The equilibrium state is linearly stable if $\delta^2 H_c$ is a positive definite quantity. All the terms, such as $d^2\Phi/dZ^2$, can be derived directly from the equilibrium Eq. (64). Only the term $\delta^2 V$ presents some difficulty, because, as can be seen from (66), it couples field perturbations at different points $\underline{x}$ and $\underline{x}'$ (it involves $\delta\underline{A}_w$). It is nevertheless possible to find <u>sufficient</u> stability criteria, by replacing $\delta^2 V$ by a lower bound. In Appendix C, we derive such a bound:

$$\delta^2 V \geq - \frac{1}{16\pi}\,\lambda_{max}\,(\frac{\omega^2}{c^2})^2 \int dx\,\underline{A}_w^* \cdot \delta\underline{\underline{\varepsilon}} \cdot \delta\underline{\underline{\varepsilon}} \cdot \underline{A}_w$$

$$- \frac{1}{16\pi}\,\frac{\omega^2}{c^2}\int dx\,\underline{A}_w^* \cdot \delta^2\underline{\underline{\varepsilon}} \cdot \underline{A}_w \, ,$$

where λ_{max} is given by Eq. (C.7) (it is possibly infinite, in which case this method does not work.) Using this inequality in Eq. (65), and collecting terms, one finds

$$\delta^2 H_c \geq \int dx \begin{bmatrix} \delta n(\underline{x}) \\ \delta \underline{u}(x) \\ \delta B(\underline{x}) \\ \delta \underline{E}(\underline{x}) \end{bmatrix}^t Q(\underline{x}) \begin{bmatrix} \delta n(x) \\ \delta \underline{u}(x) \\ \delta B(\underline{x}) \\ \delta \underline{E}(\underline{x}) \end{bmatrix},$$

where $Q(\underline{x})$ is a matrix of terms which depend only on the equilibrium (64). We therefore conclude that if the equilibrium is such that $Q(\underline{x})$ is a positive definite matrix at every point $\underline{x}$, then this equilibrium is linearly stable.

C. Two-dimensional magnetohydrodynamic plasma model

A similar analysis can be achieved with the two-dimensional magnetohydrodynamic plasma model, which has the advantage of being simpler than the multifluid model, and leads to more manageable expressions. In addition, it is possible to make the connection between the ΔW (33) and the stability criteria provided by the Casimir method.

For the single fluid model, the Eulerian fields that parametrize the phase space are the plasma density $n(\underline{x})$, the plasma fluid velocity $\underline{u}(\underline{x})$, perpendicular to the unit vector $\hat{\underline{z}}$, and the axial magnetic field $\underline{B}(\underline{x}) = B(\underline{x})\,\hat{\underline{z}}$.

Within the same approximations as before, the Hamiltonian is the functional

$$H = \int dx\ [\frac{1}{2}\ m\ n\ |\underline{u}(\underline{x})|^2 + n(\underline{x})\ \psi(\underline{x}) + U(n,\ B)]$$

$$+ \frac{1}{8\pi} \int dx |B|^2 + V \quad , \tag{68}$$

whose value is interpreted as the total free energy of the system, including the ponderomotive energy V, and the plasma internal energy U.

The MHD Poisson bracket is [31]:

$$\{n(\underline{x}),\ \underline{g}(\underline{x}')\} = -\frac{1}{m}\ \frac{\partial}{\partial \underline{x}}\ [n(\underline{x})\ \delta(\underline{x} - \underline{x}')] \quad ,$$

$$\{\underline{g}(\underline{x}),\ \underline{g}(\underline{x}')\} = -\frac{1}{m}\ \frac{\partial}{\partial \underline{x}}\ \delta(\underline{x} - \underline{x}')\ \underline{g}(\underline{x}) - \frac{1}{m}\ \underline{g}(\underline{x}')\ \frac{\partial}{\partial \underline{x}}\ \delta(\underline{x} - \underline{x}') \quad , \tag{69}$$

$$\{B(\underline{x}),\ \underline{g}(\underline{x}')\} = -\frac{1}{m}\ \frac{\partial}{\partial \underline{x}}\ [B(\underline{x})\ \delta(\underline{x} - \underline{x}')] \quad .$$

The evolution of a functional F is given by dF/dt = {F, H}, and in particular one finds the MHD equations

$$\frac{\partial n}{\partial t} + \nabla \cdot (n\ \underline{u}) = 0 \quad ,$$

$$\frac{\partial B}{\partial t} + \nabla \cdot (B\ \underline{u}) = 0 \quad ,$$

$$m\ n\ (\frac{\partial \underline{u}}{\partial t} + \underline{u} \cdot \underline{\nabla}\ \underline{u}) = -\nabla(\frac{B^2}{8\pi}) - B\ \nabla\ (\frac{\delta V}{\delta B} + \frac{\partial U}{\partial B}) - n\ \nabla(\psi + \frac{\delta V}{\delta n} + \frac{\partial U}{\partial n}) \tag{70}$$

Pursuing the discussion as before, we determine the family of Casimir functionals that is associated with the Poisson bracket [32]:

$$C = \int dx\ n\ \Phi(Y) \quad , \tag{71}$$

where Φ is an arbitrary function of its argument Y = B/n. The existence of those Casimirs is related to the fact that for such a two-dimensional ideal flow, the ratio B/n is purely convected, as the magnetic flux is "frozen" to the fluid.

The Lyapunov functional is $H_C = H + C$ and admits critical points in functional space when

$$\delta H_C = \int dx \; \{\delta n[\frac{1}{2} m \; |\underline{u}|^2 + \psi + \frac{\partial U}{\partial n} + \frac{\delta V}{\delta n} + \Phi(Y)]$$

$$+ n \; \delta Y \frac{d\Phi}{dY} + \delta\underline{u} \cdot m \; n \; \underline{u} + \delta B[\frac{1}{4\pi} B + \frac{\partial U}{\partial B} + \frac{\delta V}{\delta B}] = 0$$

for all variations of the fields,taking into account

$$n \; \delta Y = \delta B - Y \; \delta n. \tag{72}$$

The resulting equilibrium equations are

$$\underline{u} = \underline{0} \quad ,$$

$$\psi + \frac{\partial U}{\partial n} + \frac{\delta V}{\delta n} + \Phi(Y) - Y \frac{d\Phi}{dY} = 0 \quad ,$$

and

$$\frac{1}{4\pi} B + \frac{\partial U}{\partial B} + \frac{\delta V}{\delta B} + \frac{d\Phi}{dY} = 0 \quad . \tag{73}$$

The linear stability criterion is provided as before by the requirement that the second variation of H_C be positive definite. One finds in the present case

$$2 \; \delta^2 H_C = \int dx \; m \; n \; |\delta\underline{u}|^2 + \int dx \; n \frac{d^2\Phi}{dY^2} (\delta Y)^2$$

$$+ \int dx \; \{(\delta n)^2 \frac{\partial^2 U}{\partial n^2} + 2 \; \delta n \; \delta B \frac{\partial^2 U}{\partial n \partial B} + (\delta B)^2 [\frac{\partial^2 U}{\partial B^2} + \frac{1}{4\pi}]\} + 2 \; \delta^2 V. \tag{74}$$

The first and third terms of (74) are positive. For stability, it is sufficient that the sum of the second and fourth terms are positive definite. They can be evaluated using

$$n \frac{d^2\Phi}{dY^2} = \frac{n^2}{|\nabla Y|^2} [- \nabla n \cdot \nabla(\psi + \frac{\partial U}{\partial n} + \frac{\delta V}{\delta n}) - \nabla B \cdot \nabla(\frac{B}{4\pi} + \frac{\partial U}{\partial B} + \frac{\delta V}{\delta B})] \quad ,$$

which follows from the equilibrium Eq. (73), and the expression (66) (or a lower bound, as shown in Appendix C) for $\delta^2 V$.

Relation to ΔW

One must note that with such stability criteria, the more constraints imposed on the field evolution, the stronger will be the stability conditions. This is because the second variation of the Lyapunov functional is more easily positive definite when the number of degrees of freedom of the field variations δn, δB, etc. is more limited. To illustrates this remark, take into account the fact that the variations of δn and δB are not independent, but are related each to the small plasma displacement $\underline{\xi}(\underline{x},t)$ from the equilibrium position. The plasma and the magnetic flux are both convected and, to first order in $\underline{\xi}$, the variations of density and magnetic field are $\delta n = -\nabla\cdot(n\underline{\xi})$ and $\delta B = -\nabla\cdot(B\underline{\xi})$, while the perturbation of the velocity field is $\delta\underline{u} = \partial\underline{\xi}/\partial t$. After substitution of these expressions, $\delta^2 H_C$ becomes a quadratic functional of the displacement field $\underline{\xi}$. The term $d^2\Phi/dY^2$ can be evaluated, by noting that the equilibrium equations (73) imply

$$\underline{\xi} \cdot \nabla Y \frac{d^2\Phi}{dY^2} = -\underline{\xi} \cdot \nabla \left(\frac{B}{4\pi} + \frac{\partial U}{\partial B} + \frac{\delta V}{\delta B}\right)$$

and

$$\underline{\xi} \cdot \nabla Y \frac{d^2\Phi}{dY^2} = -\underline{\xi} \cdot \nabla \left(\psi + \frac{\partial U}{\partial n} + \frac{\delta V}{\delta n}\right) \quad ,$$

from which the term of (74) becomes

$$\int dx\, n \frac{d^2\Phi}{dY^2} |\underline{\xi} \cdot \nabla Y|^2 = -\int dx\, \underline{\xi} \cdot \nabla B \;\; \underline{\xi} \cdot \nabla \left(\frac{B}{4\pi} + \frac{\partial U}{\partial B} + \frac{\delta V}{\delta B}\right)$$

$$- \int dx\, \underline{\xi} \cdot \nabla n \;\; \underline{\xi} \cdot \nabla \left(\psi + \frac{\partial U}{\partial n} + \frac{\delta V}{\delta n}\right) \quad .$$

The ponderomotive contributions in this term are precisely ΔW_M and ΔW_P introduced earlier(36),(37). Similarly, for these variations of n and B, one finds that the second variation of the ponderomotive energy is $2\ \delta^2 V = \Delta W_A$, where ΔW_A was given in (38).

Putting all together:

$$2\ \delta^2 H_C = \int dx\ mn\ \left|\frac{\partial \underline{\xi}}{\partial t}\right|^2 + \Delta W[\underline{\xi}] \quad . \tag{75}$$

The condition that $\delta^2 H_C$ be positive definite is now a necessary and sufficient stability condition, which is precisely the same stability condition as produced by the ΔW variational principle: $\Delta W > 0$ (33).

CONCLUSIONS

We have presented a theoretical framework for the the study of ponderomotive forces in magnetized plasma and their effect on stability. The oscillation center theory is derived from a mixed Lagrangian and Eulerian representation of the action of the original system, including the electromagnetic fields and the antenna, when averaged over the fast time scale determined by the oscillations of the RF field.

Such a global description ensures the self-consistency of the interactions between plasma, low-frequency fields, and high-frequency fields, and thus implies the conservation of energy and momentum in the absence of dissipation. We have determined that despite apparent singularities appearing in the expressions of the ponderomotive effects at the ion gyrofrequency, self-consistency ensures that their contribution remains finite, not only because of thermal dispersion [6], but also because of the back reaction of the plasma that polarizes the fields. The mixed representation of the average action leads by Hamilton principle to the set of equations obeyed by the oscillation centers, which includes ponderomotive forces, as well as the equations obeyed by the low-frequency fields, the latter of which include magnetization currents due to the RF fields. The stability of plasma equilibria can be studied by using a ΔW variational principle, which contains, in addition to the MHD terms, three ponderomotive contributions, due respectively to ponderomotive forces, magnetization, and self-consistant modifications of the RF field due to plasma perturbations. The total ponderomotive energy is the free energy of the system due to the high-frequency field, and is as such related to the antenna inductance.

The equations are also given a Hamiltonian form. For the multifluid plasma model as well as for the magnetohydrodynamic plasma models, the evolution of the Eulerian fields (densities, momentum densities, electo-magnetic fields, etc.), is given by a Hamiltonian functional (the total free energy of the system) and a noncanonical Poisson bracket. The Poisson brackets being degenerate, there are Casimir functionals, which are used with the Hamiltonian to construct Lyapunov functionals for the nonlinear system. The critical points of the Lyapunov functionals are equilibria of the plasma in the presence of RF field, and the conditions on these equilibria for convexity of the associated Lyapunov functional will ensure nonlinear stability.

This formalism is suitable for applications. It has been used in particular to study the RF stabilization of the flute modes in mirrors. For instance, it has been shown [12] that high $k_{\|}$ RF fields are stabilizing, a conclusion that has implications for antenna design for optimal stabilization. These results, along with the evaluation of intensity threshold, etc., will appear in a future publication. The formalism lends itself easily to generalizations. Work is in progress to determine the influence of the RF generator impedance on stability, because of the feedback effect that it produces. We investigate also the limitations on RF intensity due to parametric instabilities. The Lyapunov functional may lead to useful results on nonlinear stability of flute modes, and on the possible existence of bifurcation points for the equilibria. Finally, a complete study of ponderomotive effects requires the coupling of an antenna computational code with a stability code.

We note furthermore that the equations of motion derived here for the slowly varying fields and fluid variables should provide substantial computational economy on the simulation of plasma dynamics in interaction with a high-frequency R.F. field, in comparison to a straight simulation including the high-frequency components.

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APPENDIX A

Velocity-dependent ponderomotive force, and polarization current

We generalize here the functional dependence of the ponderomotive energy V, to include dependence on oscillation-center flux $\underline{g}_{oc}(\underline{x},t)$, Eq. (19), and on slow electric field $\underline{E}_s(\underline{x},t)$. The effects of these terms are an additional ponderomotive force on guiding centers, and a polarization current density.

The expression for δS_{pd}, Eq. (13 a), will have two additional terms:

$$- \int dt \int dx \left(\frac{\delta V}{\delta \underline{g}_{oc}} \cdot \delta \underline{g}_{oc} + \frac{\delta V}{\delta \underline{E}_s} \cdot \delta \underline{E}_s\right), \tag{A.1}$$

where

$$\delta \underline{g}_{oc} = \int dN\, \delta \dot{\underline{r}}_{oc}\, \delta(\underline{x} - \underline{r}_{oc}) - \nabla \cdot \int dN\, \delta \underline{r}_{oc}\, \dot{\underline{r}}_{oc}\, \delta(\underline{x} - \underline{r}_{oc}) \tag{A.2}$$

and

$$\delta \underline{E}_s = - \frac{1}{c} \frac{\partial}{\partial t} \delta \underline{A}_s \quad .$$

Integration by parts leads to the additional terms in δS_{pd} (13b):

$$- \int dt \int dN\, \delta \underline{r}_{oc} \cdot \int dx\, \delta(\underline{x} - \underline{r}_{oc}) \left\{- \frac{\partial}{\partial t} \frac{\delta V}{\delta \underline{g}_{oc}} + \underline{u}_{oc} \times \left(\nabla \times \frac{\delta V}{\delta \underline{g}_{oc}}\right)\right\}$$

$$- \int dt \int dx\, \delta \underline{A}_s \cdot \frac{1}{c} \frac{\partial}{\partial t} \frac{\delta V}{\delta \underline{E}_s} \quad , \tag{A.3}$$

with $\underline{u}_{oc}$ given by (20).

As a consequence, the Newton equation for oscillation centers (14) is modified by an additional force

$$\underline{F}_g(\underline{x},t) = - \frac{\partial}{\partial t} \frac{\delta V}{\delta \underline{g}_{oc}(\underline{x},t)} + \underline{u}_{oc} \times \left(\nabla \times \frac{\delta V}{\delta \underline{g}_{oc}}\right) \quad . \tag{A.4}$$

This force has the same structure as the Lorentz force, $\delta V/\delta \underline{g}$ (a "ponderomotive momentum") playing the role of the electromagnetic vector potential.

Maxwell equation (16) is modified by an additional polarization current density:

$$\underline{J}_p(\underline{x},t) = \frac{1}{c}\frac{\partial}{\partial t}\frac{\delta V}{\delta \underline{E}_s(\underline{x},t)} \, . \tag{A.5}$$

The fluid equations (21) also change, accordingly.

APPENDIX B

Construction of ΔW

We outline here the derivation of ΔW (Eq. 33), in the two-dimensional case.

Let us first introduce the total potential energy of the system V^*, the sum of the magnetic, internal, gravitational and ponderomotive energies:

$$V^* \equiv \int dx \left(\frac{B^2}{8\pi} + U + n\psi\right) + V \ . \tag{B.1}$$

The momentum equation (29) can therefore be rewritten as

$$mn \left(\frac{\partial}{\partial t} \underline{u} + \underline{u} \cdot \nabla \underline{u}\right) = - n\nabla \frac{\delta V^*}{\delta n} - B \nabla \frac{\delta V^*}{\delta B} \ , \tag{B.2}$$

and the static equilibrium equation (31) becomes

$$n \nabla \frac{\delta V^*}{\delta n} + B \nabla \frac{\delta V^*}{\delta B} = 0 \quad . \tag{B.3}$$

Assuming a perturbation of the plasma given by the displacement field $\underline{\xi}(\underline{x}) \exp(\gamma t)$, one has $\tilde{\underline{u}} = \gamma \underline{\xi}$, where the tilde designates perturbed quantities. From the continuity equation and the Ohm's law (29), one finds $\tilde{n} = - \nabla \cdot (\underline{\xi} n)$ and $\tilde{B} = - \nabla \cdot (\underline{\xi} B)$. The linearized Eq. (B.2) gives

$$mn \gamma^2 \underline{\xi} = - \tilde{n} \nabla \frac{\delta V^*}{\delta n} - \tilde{B} \nabla \frac{\delta V^*}{\delta B}$$

$$- n \nabla \left(\widetilde{\frac{\delta V^*}{\delta n}}\right) - B \nabla \left(\widetilde{\frac{\delta V^*}{\delta B}}\right). \tag{B.4}$$

Multiplication by $- \underline{\xi}$ and integration over space leads to

$$- \gamma^2 \int mn \, |\underline{\xi}|^2 \, dx = \Delta W \quad ,$$

where

$$\Delta W = - \int dx \left[\nabla \cdot (n\underline{\xi}) \ \underline{\xi} \cdot \nabla \frac{\delta V^*}{\delta n} + \nabla \cdot (B\underline{\xi}) \ \underline{\xi} \cdot \nabla \frac{\delta V^*}{\delta B}\right]$$

$$+ \int dx \left[n \ \underline{\xi} \cdot \nabla \left(\widetilde{\frac{\delta V^*}{\delta n}}\right) + B \ \underline{\xi} \cdot \nabla \left(\widetilde{\frac{\delta V^*}{\delta B}}\right)\right]. \tag{B.5}$$

Making use of the equilibrium Eq. (B.3) and of one integration by parts, one finds

$$\Delta W = - \int dx \ [\underline{\xi} \cdot \nabla n \ \underline{\xi} \cdot \nabla \frac{\delta V^*}{\delta n} + \underline{\xi} \cdot \underline{\nabla} B \ \underline{\xi} \cdot \nabla \frac{\delta V^*}{\delta B}]$$

$$- \int dx \ [\nabla \cdot (n\underline{\xi}) \ (\widetilde{\frac{\delta V^*}{\delta n}}) + \nabla \cdot (B\underline{\xi})(\widetilde{\frac{\delta V^*}{\delta B}})] \quad . \qquad (B.6)$$

Explicitely, from (B.1), the functional derivatives of V* are, assuming U depending on n only,

$$\frac{\delta V^*}{\delta n} = \frac{dU}{dn} + \psi + \frac{\delta V}{\delta n} \quad ,$$

$$\frac{\delta V^*}{\delta B} = \frac{1}{4\pi} B + \frac{\delta V}{\delta B} \ , \qquad (B.7)$$

and their perturbations are

$$(\widetilde{\frac{\delta V^*}{\delta n}}) = - \nabla \cdot (n \ \underline{\xi}) \ \frac{d^2U}{dn^2}$$

$$- \int dx' \ \nabla' \cdot (n' \ \underline{\xi}') \ \frac{\delta^2 V}{\delta n(\underline{x}) \ \delta n(\underline{x}')}$$

$$- \int dx' \ \nabla' \cdot (B' \ \underline{\xi}') \ \frac{\delta^2 V}{\delta n(\underline{x}) \ \delta B(\underline{x}')} \quad ,$$

$$(\widetilde{\frac{\delta V^*}{\delta B}}) = - \nabla \cdot (B\underline{\xi}) \ \frac{1}{4\pi}$$

$$- \int dx' \ \nabla' \cdot (n'\underline{\xi}') \ \frac{\delta^2 V}{\delta B(\underline{x}) \ \delta n(\underline{x}')}$$

$$- \int dx' \ \nabla' \cdot (B'\underline{\xi}') \ \frac{\delta^2 V}{\delta B(\underline{x}) \ \delta B(\underline{x}')} \quad . \qquad (B.8)$$

With these expressions, ΔW (B.6) can readily be cast on the form (33 to 38). It is easy to show from (B.6) that the variational principle (32) restitutes the linearized equations (B.4).

APPENDIX C

Lower bound for $\delta^2 V$

The expression of $\delta^2 V$ is given by Eq. (66). We derive here a lower bound for $\delta^2 V$, which has simpler form, and is useful for the formulation of sufficient stability criteria.

It is convenient at this point to introduce the eigenvalues $1/\lambda_n$ and the eigenvectors $\underline{a}_n(\underline{x})$ of the wave equation operator (the left-hand side of Eq. 67). They satisfy

$$\nabla \times (\nabla \times \underline{a}_n) - \frac{\omega^2}{c^2} \underline{\underline{\varepsilon}} \cdot \underline{a}_n = \frac{1}{\lambda_n} \underline{a}_n \quad . \tag{C.1}$$

Since the wave operator is Hermitian, the values λ_n are real, possibly infinite, and the set of eigenvectors is complete and may be chosen orthonormal. In terms of this base, Eq. (67) may be solved, to give

$$\delta \underline{A}_W(\underline{x}) = \sum_n \underline{a}_n(\underline{x}) \lambda_n \delta s_n, \tag{C.2}$$

where

$$\delta s_n \equiv \int dx\, \underline{a}_n^* \cdot \frac{\omega^2}{c^2} \delta \underline{\underline{\varepsilon}} \cdot \underline{A}_W \quad . \tag{C.3}$$

The second variation of V (66) becomes

$$\delta^2 V = - \frac{1}{16\pi} \sum_n \lambda_n |\delta s_n|^2$$

$$- \frac{1}{16\pi} \frac{\omega^2}{c^2} \int dx\, \underline{A}_W^* \cdot \delta^2 \underline{\underline{\varepsilon}} \cdot \underline{A}_W \quad . \tag{C.4}$$

Defining the maximum value of λ_n,

$$\lambda_{max} \equiv \max_n \lambda_n, \tag{C.5}$$

we find

$$\delta^2 V \geq - \frac{1}{16\pi} \lambda_{max} \sum_n |\delta s_n|^2 - \frac{1}{16\pi} \frac{\omega^2}{c^2} \int dx \, \underline{A}_W^* \cdot \delta^2 \underline{\underline{\varepsilon}} \cdot \underline{A}_W \, ,$$

that is

$$\delta^2 V \geq - \frac{1}{16\pi} \lambda_{max} \int dx \, \left(\frac{\omega^2}{c^2}\right)^2 \underline{A}_W^* \cdot \delta\underline{\underline{\varepsilon}} \cdot \delta\underline{\underline{\varepsilon}} \cdot \underline{A}_W$$

$$- \frac{1}{16\pi} \frac{\omega^2}{c^2} \int dx \, \underline{A}_W^* \cdot \delta^2 \underline{\underline{\varepsilon}} \cdot \underline{A}_W \quad . \tag{C.6}$$

The right hand side of Eq. (C.6) is a lower bound, which couples fluctuations evaluated at the <u>same</u> point. Note finally that because (C.1) is hermitian, λ_{max} obeys a variational principle:

$$\lambda_{max} = \max_{\{\underline{a}(\underline{x})\}} \frac{\int dx \, |\underline{a}(\underline{x})|^2}{\int dx \, [|\nabla \times \underline{a}|^2 - \frac{\omega^2}{c^2} \underline{a}^* \cdot \underline{\underline{\varepsilon}} \cdot \underline{a}]} \quad . \tag{C.7}$$

Of course, λ_{max} may possibly be infinite, when ω is an eigenfrequency of the equilibrium system. On the other hand, if the eigenvalues of the dielectric tensor $\underline{\underline{\varepsilon}}$ are everywhere smaller than a negative constant, then λ_{max} exists and is finite.

As a particular example appropriate for MHD stability, take for $\underline{\underline{\varepsilon}}$ the perpendicular cold plasma dielectric tensor (22,32), choose $\omega > \Omega_i$, fix $k_\parallel$, and consider an axisymmetric geometry (Eqs. 40 to 45). Equation (C.7) becomes:

$$\lambda_{max} = \max_{\{a(r)\}} \frac{\int r \, dr \, |a(r)|^2}{\int r \, dr \{ |(\frac{d}{dr} + \frac{1}{r}) \, a(r)|^2 + K^2(r) |a(r)|^2 \}} \tag{C.8}$$

and shows that λ_{max} exists, is finite and positive, if $K^2(r)$ (Eq. (45)) is positive everywhere, i.e., if $k_\parallel$ is large enough so that $N_\parallel^2 > \omega_{pi}^2/\Omega_i(\omega + \Omega_i)$ at any radius.

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LAWRENCE BERKELEY LABORATORY
TECHNICAL INFORMATION DEPARTMENT
UNIVERSITY OF CALIFORNIA
BERKELEY, CALIFORNIA 94720