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Oscillation criteria for second order non-linear difference equations

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Abstract. The paper is concerned with the study of oscillatory behaviour of solutions of the second order non-linear difference equation $\Delta(r_n \Delta u_n) + a_n f(u_n) = 0$. The sufficient conditions are given that all solutions (or all bounded solutions) of this equation are oscillatory. The present theorems are the discrete analogues of some results for differential equations due to I. V. Kamenev, A. G. Kartsatos, T. Kusano and H. Onose, Z. Opial, and others.

1. In this paper we are concerned with the oscillatory behaviour of solution of the second order non-linear difference equation

$$\Delta(r_n \Delta u_n) + a_n f(u_n) = 0, \quad n = 0, 1, 2, \ldots, \tag{1}$$

where Δ is the forward difference operator, i.e. $\Delta v_n = v_{n+1} - v_n$, $\{r_n\}$, $\{a_n\}$ are the real sequences and the following conditions are assumed to hold:

(I) $f: R \to R$ is continuous, $sf(s) > 0$ for $s \neq 0$,

(II) $r_n > 0$ for $n \geqslant n_0 \geqslant 0$, $R_n = \sum_{k=n_0}^{n-1} \frac{1}{r_k} \to \infty$, as $n \to \infty$.

As a special case we have

$$\Delta^2 u_n + a_n f(u_n) = 0, \quad \Delta^2 u_n = \Delta(\Delta u_n). \tag{2}$$

By a *solution* of (1) (similarly for equation (2)) we mean a real sequence $\{u_n\}$ ($u_n \not\equiv 0$) satisfying equation (1) for $n = 0, 1, 2, \ldots$ Let U_i ($i = 1, 2$) denote the family of solutions $\{u_n\}$ ($n \geqslant 0$) of equations (1) and (2), respectively.

Following paper [2], a real sequence $\{u_n\}$ ($n = 0, 1, \ldots$) is said to be *non-oscillatory* if there exists an $n_0 \geqslant 0$ such that for every $n \geqslant n_0$ either $u_n > 0$ or $u_n < 0$; otherwise it is said to be *oscillatory*.

The literature about the oscillatory behaviour of solutions of a non-linear difference equations is very scanty. Recently the problem of determining sufficient conditions for the oscillation of solutions of a non-linear second order difference equations has been studied in papers [2]–[4].

The purpose of the present paper is to derive several criteria for oscillation of all solutions (or all bounded solutions) of equation (1) (in particular of equation (2)). The result we obtain (Theorem 4) extends Opial's oscillation criterion for the linear differential equation [11] to equation (1). The remaining theorems of the paper are the discrete analogues of some results for a non-linear differential equations of second order due to Kartsatos [7], Kusano and Onose [8], Kamenev [6], and others [1], [5], [9]–[10]. For related results we refer the reader to a survey article by Wong [12].

2. Now we give two theorems in which the coefficient $\{a_n\}$ is allowed to take both positive and negative values for arbitrarily large values of n. These theorems are the discrete analogues of Kartsatos' results [7] for differential equations.

THEOREM 1. *Suppose that*

(i) $\{r_n\}$ *is a non-decreasing sequence for* $n \geqslant n_0$,

(ii) $\lim\limits_{n\to\infty} \dfrac{1}{r_n} \sum\limits_{k=n_0}^{n-1} k(\mu a_k^+ + a_k^-) = \infty,\ \mu > 0,$

where

$$a_k^+ = \max(a_k, 0), \quad a_k^- = \min(a_k, 0).$$

Then every bounded solution $\{u_n\} \in U_1$ *is either oscillatory or such that* $\liminf\limits_{n\to\infty} |u_n| = 0$.

Proof. Suppose that there exists a bounded non-oscillatory solution $\{u_n\} \in U_1$ and let $u_n > 0$ for $n \geqslant n_1 > n_0$ (a similar argument will also hold in the case of $u_n < 0$). If $\liminf\limits_{n\to\infty} u_n > 0$, then there are $n_2 \geqslant n_1$ and constants c_1, c_2 such that $0 < c_1 \leqslant u_n \leqslant c_2$ for $n \geqslant n_2$ and by (i) we have

$$0 < M_1 \leqslant f(u_n) \leqslant M_2 \quad \text{for } n \geqslant n_2. \tag{3}$$

From equation (1) we obtain

$$\sum_{k=n_2}^{n} k\Delta(r_k \Delta u_k) = -\sum_{k=n_2}^{n} k a_k f(u_k). \tag{4}$$

According to the summation by parts formula we may write

$$\sum_{k=n_2}^{n} k\Delta(r_k \Delta u_k) = n r_{n+1} \Delta u_{n+1} - n_2 r_{n_2} \Delta u_{n_2} - \sum_{k=n_2+1}^{n} r_k \Delta u_k$$

and

$$\sum_{k=n_2+1}^{n} r_k \Delta u_k = r_{n+1} u_{n+1} - r_{n_2+1} u_{n_2+1} - \sum_{k=n_2+1}^{n} u_{n+1} \Delta r_k.$$

Hence in view of (i) we have

$$\sum_{k=n_2}^{n} k\Delta(r_k \Delta u_k) \geqslant nr_{n+1}\Delta u_{n+1} - n_2 r_{n_2}\Delta u_{n_2} - c_2 r_{n+1}. \tag{5}$$

On the other hand, it follows from (3) that

$$\sum_{k=n_2}^{n} ka_k f(u_k) \geqslant \sum_{k=n_2}^{n} k(M_1 a_k^+ + M_2 a_k^-) = M_2 \sum_{k=n_2}^{n} k(\mu a_k^+ + a_k^-), \tag{6}$$

$$\mu = M_1/M_2.$$

By virtue of (4), (5) and (6) one can write

$$nr_{n+1}\Delta u_{n+1} - n_2 r_{n_2}\Delta u_{n_2} - c_2 r_{n+1} \leqslant -M_2 \sum_{k=n_2}^{n} k(\mu a_k^+ + a_k^-).$$

This in turn implies

$$\begin{aligned} n\Delta u_{n+1} - \gamma n_2 \Delta u_{n_2} - c_2 &\leqslant n\Delta n_{n+1} - c_2 - \frac{r_{n_2} n_2 \Delta u_{n_2}}{r_{n+1}} \\ &\leqslant -\frac{M_2}{r_{n+1}} \sum_{k=n_2}^{n} k(\mu a_k^+ + a_k^-), \end{aligned} \tag{7}$$

where

$$\gamma = \begin{cases} 0 & \text{if } \Delta u_{n_2} \leqslant 0, \\ 1 & \text{if } \Delta u_{n_2} > 0. \end{cases}$$

From (7) using (ii), we conclude that $n\Delta u_{n+1} \to -\infty$ as $n \to \infty$. Then there exsists an $n_3 \geqslant n_2$ such that $\Delta u_{n+1} \leqslant -1/n$ for $n \geqslant n_3$ and so we get

$$u_{n+1} \leqslant u_{n_3+1} - \sum_{k=n_3}^{n-1} 1/k, \quad n \geqslant n_3 + 1,$$

which gives $\lim_{n\to\infty} u_n = -\infty$. But this contradicts the fact that $\{u_n\}$ may be positive. Thus our assertion is true.

COROLLARY. *If* $\sum^{\infty} n(\mu a_n^+ + a_n^-) = \infty$ $(\mu > 0)$, *then every bounded solution* $\{u_n\} \in U_2$ *is oscillatory or* $\liminf_{n\to\infty} |u_n| = 0$.

THEOREM 2. *Assume that*

(i) $$\sum^{\infty} \frac{1}{r_n R_n} = \infty,$$

(ii) $$\sum^{\infty} R_{n+1}(\mu a_n^+ + a_n^-) = \infty, \quad \mu > 0.$$

Then the assertion of Theorem 1 *holds.*

Proof. Proceeding as in the proof of Theorem 1 we get inequalities (3). From equation (1) by (3) we have

(8)
$$\sum_{k=n_2}^{n} R_{k+1}\Delta(r_k\Delta u_k) = -\sum_{k=n_2}^{n} R_{k+1}a_k f(u_k) \leqslant -M_2\sum_{k=n_2}^{n} R_{k+1}\left(\frac{M_1}{M_2}a_k^+ + a_k^-\right)$$

since

$$\sum_{k=n_2}^{n} R_{k+1}\Delta(r_k\Delta u_k) = R_{n+1}r_{n+1}\Delta u_{n+1} - R_{n_2}r_{n_2}\Delta u_{n_2} - u_{n+1} + u_{n_2}$$
$$\geqslant R_{n+1}r_{n+1}\Delta u_{n+1} + K,$$

where

$$K = -R_{n_2}r_{n_2}\Delta u_{n_2} - c_2 + u_{n_2}.$$

Therefore from (8) we obtain the inequality

$$R_{n+1}r_{n+1}\Delta u_{n+1} + K \leqslant -M_2\sum_{k=n_2}^{n} R_{k+1}(\mu a_k^+ + a_k^-), \quad \mu = M_1/M_2,$$

whence, in view of (ii), it follows that $\lim_{n\to\infty} r_nR_n\Delta u_n = -\infty$. So there exists an $n_3 \geqslant n_2$ such that $\Delta u_n \leqslant -1/r_nR_n$ for $n \geqslant n_3$, which, in view of (i), leads to the contradictory conclusion that $\lim_{n\to\infty} u_n = -\infty$. This completes the proof.

3. It will be assumed in the sequel that the coefficient $\{a_n\}$ in (1) and (2) is eventually non-negative, say for $n \geqslant n_0$.

THEOREM 3. *If $\sum^{\infty} R_n a_n = \infty$, then every bounded solution $\{u_n\} \in U_1$ is oscillatory.*

Proof. Let $\{u_n\} \in U_1$ be a bounded non-oscillatory solution of (1). Assume $u_n > 0$ for $n \geqslant n_1 > n_0$. It follows from equation (1) that $\Delta(r_n\Delta u_n) \leqslant 0$ and hence $\{r_n\Delta u_n\}$ is non-increasing for $n \geqslant n_1$. We first show that $r_n\Delta u_n \geqslant 0$ for $n \geqslant n_1$. In fact, if there existed $n_2 > n_1$ such that $r_{n_2}\Delta u_{n_2} = c < 0$, then $r_n\Delta u_n \leqslant c$ for $n \geqslant n_2$, i.e. $\Delta u_n \leqslant c/r_n$ for $n \geqslant n_2$, and hence

$$u_n \leqslant u_{n_2} + c\sum_{k=n_2}^{n-1}\frac{1}{r_k} \to -\infty, \quad \text{as } n \to \infty,$$

which contradicts the fact that $u_n > 0$ for $n \geqslant n_1$. Thus $r_n\Delta u_n \geqslant 0$ for $n \geqslant n_1$, i.e. $\{u_n\}$ is a non-decreasing sequence for $n \geqslant n_1$. From the above it follows that $u_n \to l$ as $n \to \infty$ $(0 < l < \infty)$ and by (I) $f(u_n) \to f(l) > 0$. Hence there exists an $n_3 \geqslant n_1$ such that $f(u_n) \geqslant f(l)/2$ for $n \geqslant n_3$, so that this fact and equation (1) imply

(9)
$$R_n\Delta(r_n\Delta u_n) + \tfrac{1}{2}f(l)R_na_n \leqslant 0, \quad n \geqslant n_3.$$

It is easy to see that

(10) $$R_n\Delta(r_n\Delta u_n) \geqslant \Delta(R_n r_n \Delta u_n) - r_n\Delta u_n \Delta R_n, \quad n \geqslant n_1.$$

From inequalities (9) and (10) we deduce

$$\sum_{k=n_3}^{n} \Delta(R_k r_k \Delta u_k) - \sum_{k=n_3}^{n} \Delta u_k + \tfrac{1}{2}f(l)\sum_{k=n_3}^{n} R_k a_k \leqslant 0, \quad n \geqslant n_3,$$

which implies

$$\tfrac{1}{2}f(l)\sum_{k=n_3}^{n} R_k a_k \leqslant u_{n+1} + R_{n_3} r_{n_3}\Delta u_{n_3} - u_{n_3}, \quad n \geqslant n_3.$$

Hence there exists a constant C such that

$$\sum_{k=n_3}^{n} R_k a_k \leqslant C \quad \text{for all } n \geqslant n_3,$$

contrary to the assumption of the theorem.

The proof of the case $u_n < 0$ is similar, and hence omitted.

COROLLARY 2. *If* $\sum^{\infty} na_n = \infty$, *then every bounded solution* $\{u_n\} \in U_2$ *is oscillatory.*

The next theorem extends Opial's oscillation criterion for linear differential equations [11] to difference equation (1).

THEOREM 4. *Suppose there exsist a differentiable function* $\varphi\colon R \to R$ *and a real sequence* $\{h_n\}$ *such that*

(i) $|f(s)| \geqslant |\varphi(s)|, \quad \varphi'(s) \geqslant \varepsilon, \quad \varepsilon > 0, \quad s\varphi(s) > 0$ *for* $s \neq 0$,

(ii) $h_n > 0$ *for* $n \geqslant n_0$

and

$$\limsup_{n\to\infty} \sum_{k=n_0}^{n} h_k\left[a_k - \frac{r_k}{4\varepsilon}\left(\frac{\Delta h_k}{h_k}\right)^2\right] = \infty.$$

Then every solution $\{u_n\} \in U_1$ *is oscillatory.*

Proof. Suppose there exsists a non-oscillatory solution $\{u_n\} \in U_1$ and let $u_n > 0$ for $n \geqslant n_1 > n_0$. As in the proof of Theorem 3, it follows that $\{u_n\}$ is non-decreasing, $r_n\Delta u_n \geqslant 0$ and $\{r_n\Delta u_n\}$ is a non-increasing sequence for $n \geqslant n_1$. In view of (i), from equation (1) we have

$$\Delta(r_n\Delta u_n) + a_n\varphi(u_n) \leqslant 0,$$

and so

(11) $$\frac{h_n\Delta(r_n\Delta u_n)}{\varphi(u_n)} \leqslant -a_n h_n, \quad n \geqslant n_1.$$

We define for $n \geqslant n_1$

(12) $$q_n = \frac{h_n v_n}{\varphi(u_n)}, \quad \text{where } v_n = r_n\Delta u_n.$$

Therefore,

$$\Delta q_n = \frac{h_n \Delta v_n}{\varphi(u_n)} + \frac{v_{n+1} \Delta h_n}{\varphi(u_{n+1})} - \frac{v_{n+1} h_n \Delta \varphi(u_n)}{\varphi(u_n)\varphi(u_{n+1})}. \tag{13}$$

By the mean value theorem and (i) we get

$$\Delta q_n \leqslant \frac{h_n \Delta v_n}{\varphi(u_n)} + \frac{v_{n+1} \Delta h_n}{\varphi(u_{n+1})} - \frac{\varepsilon h_n v_{n+1} \Delta u_n}{\varphi(u_n)\varphi(u_{n+1})}, \quad n \geqslant n_1. \tag{14}$$

Using the inequalities $v_{n+1} \leqslant v_n$, $\varphi(u_n) \leqslant \varphi(u_{n+1})$, $n \geqslant n_1$, and (11), we see from (14) that

$$\begin{aligned}\Delta q_n &\leqslant -a_n h_n + q_{n+1} \frac{\Delta h_n}{h_{n+1}} - q_{n+1}^2 \frac{\varepsilon h_n}{r_n h_{n+1}^2} \\ &= \frac{-\varepsilon h_n}{r_n h_{n+1}^2} \left[q_{n+1} - \frac{\Delta h_n r_n h_{n+1}}{2\varepsilon h_n} \right]^2 + \frac{r_n (\Delta h_n)^2}{4\varepsilon h_n} - a_n h_n.\end{aligned}$$

Hence

$$\Delta q_n \leqslant -h_n \left[a_n - \frac{r_n}{4\varepsilon} \left(\frac{\Delta h_n}{h_n} \right)^2 \right], \quad n \geqslant n_1. \tag{15}$$

Summing up both sides of (15) from n_1 to n, we obtain

$$q_{n+1} - q_{n_1} \leqslant -\sum_{k=n_1}^{n} h_k \left[a_k - \frac{r_k}{4\varepsilon} \left(\frac{\Delta h_k}{h_k} \right)^2 \right],$$

which yields

$$\sum_{k=n_1}^{n} h_k \left[a_k - \frac{r_k}{4\varepsilon} \left(\frac{\Delta h_k}{h_k} \right)^2 \right] \leqslant C_1 \quad (C_1 > 0) \text{ for } n \geqslant n_1,$$

and this contradicts assumption (ii).

A similar argument can be used in the case of an eventually negative solution. Thus the proof is complete.

For the linear difference equation

$$\Delta^2 u_n + a_n u_n = 0, \tag{16}$$

we obtain from Theorem 4, by taking $r_n \equiv 1$, $\varphi(s) = s$, the following

COROLLARY 3. *Suppose that $a_n \geqslant 0$ for $n \geqslant n_0$ and that there exists a real sequence $h_n > 0$ for $n \geqslant n_0$ such that*

$$\limsup_{n \to \infty} \sum_{k=n_0}^{n} h_k [a_k - \tfrac{1}{4} (\Delta h_k / h_k)^2] = \infty; \tag{17}$$

then all solutions of (16) *are oscillatory.*

Remark 1. If $a_n \geqslant (1+\alpha)/4n^2$ fo. $n \geqslant n_0$ $(\alpha > 0)$, then, if we let $h_n = n$, the assertion of Corollary 3 holds.

The following theorem is the discrete analogue of the Kusano–Onose theorem ([8], Theorem 1) for differential equations with a retarded argument.

THEOREM 5. *Assume that the following conditions hold:*

(i) *there exist two non-decreasing continuous functions* $\varphi\colon R \to R$ *and* $\psi\colon (0, \infty) \to (0, \infty)$ *such that*

$$|f(s)| \geqslant |\varphi(s)|, \quad s\varphi(s) > 0 \quad \text{for } s \neq 0$$

and

$$\int_{\varepsilon}^{\infty} \frac{ds}{\varphi(s)\psi(s)} < \infty, \quad \int_{-\varepsilon}^{-\infty} \frac{ds}{\varphi(s)\psi(-s)} < \infty, \quad \varepsilon > 0, \tag{18}$$

(ii) *there exists a non-decreasing positive sequence* $\{\varrho_n\}$ *such that* $\{r_n \Delta\varrho_n\}$ *is non-increasing for* $n \geqslant n_0$ *and*

$$\sum^{\infty} \frac{\varrho_n a_n}{\psi(R_n)} = \infty. \tag{19}$$

Then every solution $\{u_n\} \in U_1$ *is oscillatory.*

Proof. Assume the contrary. Then as in Theorem 4 for a non-oscillatory solution $u_n > 0$, $n \geqslant n_1 > n_0$, we have $u_n \leqslant u_{n+1}$, $v_{n+1} \leqslant v_n$ for $n \geqslant n_1$, where $v_n = r_n \Delta u_n$.

Let us write

$$p_n = \frac{\varrho_n v_n}{\varphi(u_n)\psi(R_n)}, \quad n \geqslant n_1;$$

then

$$\Delta p_n = \frac{\varrho_n \Delta v_n}{\varphi(u_n)\psi(R_n)} + \frac{v_{n+1}\Delta\varrho_n}{\varphi(u_{n+1})\psi(R_{n+1})} - \frac{v_{n+1}\varrho_n \Delta[\varphi(u_n)\psi(R_n)]}{\varphi(u_n)\varphi(u_{n+1})\psi(R_n)\psi(R_{n+1})}.$$

Since $\Delta[\varphi(u_n)\psi(R_n)] \geqslant 0$, $v_n \geqslant 0$ for $n \geqslant n_1$ and by (i), (ii) we see from the above equality that

$$\Delta p_n \leqslant \frac{\varrho_n \Delta v_n}{\varphi(u_n)\psi(R_n)} + \frac{v_{n+1}\Delta\varrho_n}{\varphi(u_{n+1})\psi(R_{n+1})}, \quad n \geqslant n_1. \tag{20}$$

By the assumptions of the theorem we get from equation (1)

$$\frac{\varrho_n \Delta v_n}{\varphi(u_n)\psi(R_n)} \leqslant -\frac{\varrho_n a_n}{\psi(R_n)}, \quad n \geqslant n_1. \tag{21}$$

In view of the monotonicity of $\{v_n\}$ and $\{r_n \Delta\varrho_n\}$, from (20) and (21) we may write

$$\Delta p_n \leqslant -\frac{\varrho_n a_n}{\psi(R_n)} + r_{n_1}\Delta\varrho_{n_1} \frac{\Delta u_n}{\varphi(u_{n+1})\psi(R_{n+1})}, \quad n \geqslant n_1. \tag{22}$$

Since $\Delta v_n \leqslant 0$, $n \geqslant n_1$, it follows that $u_n \leqslant u_{n_1} + v_{n_1} R_n$ and hence by condition (II) we conclude that there is a constant $\beta \geqslant 1$ such that

(23) $$u_n \leqslant \beta R_n, \quad n \geqslant n_1.$$

Returning to (22) and noting that φ and ψ are non-decreasing, we obtain by (23) the following estimation:

(24) $$\Delta p_n \leqslant -\frac{\varrho_n a_n}{\psi(R_n)} + r_{n_1} \Delta \varrho_{n_1} \frac{\Delta u_n}{\varphi(u_{n+1}/\beta)\psi(u_{n+1}/\beta)}, \quad n \geqslant n_1.$$

For $u_n/\beta \leqslant s \leqslant u_{n+1}/\beta$ we have

$$[\varphi(s)\psi(s)]^{-1} \geqslant [\varphi(u_{n+1}/\beta)\psi(u_{n+1}/\beta)]^{-1},$$

and so

$$\int_{u_n/\beta}^{u_{n+1}/\beta} \frac{ds}{\varphi(s)\psi(s)} \geqslant \frac{1}{\beta} \frac{\Delta u_n}{\varphi(u_{n+1}/\beta)\psi(u_{n+1}/\beta)}.$$

Now substituting in (24) and summing up both sides from n_1 to n we obtain

(25) $$\sum_{k=n_1}^{n} \frac{\varrho_k a_k}{\psi(R_k)} + \sum_{k=n_1}^{n} \Delta p_k \leqslant \beta r_{n_1} \Delta \varrho_{n_1} \int_{u_{n_1}/\beta}^{u_{n+1}/\beta} \frac{ds}{\varphi(s)\psi(s)}, \quad n \geqslant n_1.$$

Since $p_n \geqslant 0$ for $n \geqslant n_1$, it follows from (25), by condition (18), that

$$\sum^{\infty} \frac{\varrho_n a_n}{\psi(R_n)} \leqslant C_2, \quad C_2 = \text{const},$$

which contradicts (19). A similar argument is used in the case of a negative solution.

Remark 2. Putting $\psi(s) \equiv 1$, Theorem 5 gives the discrete analogue of Kamenev's result [6] for differential equations.

Taking in particular $\psi(s) \equiv 1$ and $\varrho_n = R_n$, we have the following

COROLLARY 4. *Suppose that*

(i) *there exists a non-decreasing continuous function* $\varphi\colon R \to R$ *such that* $|f(s)| \geqslant |\varphi(s)|$, $s\varphi(s) > 0$, $s \neq 0$ *and*

$$\int_{\varepsilon}^{\infty} \frac{ds}{\varphi(s)} < \infty, \quad \int_{-\varepsilon}^{-\infty} \frac{ds}{\varphi(s)} < \infty, \quad \varepsilon > 0,$$

(ii) $\sum^{\infty} R_n a_n = \infty$.

Then every solution $\{u_n\} \in U_1$ *is oscillatory.*

COROLLARY 5. *Assume that there is an* $\alpha > 0$ *such that*

(26) $$\sum^{\infty} R_n^{1-\alpha} a_n = \infty.$$

Then all solutions of the equation

(27) $$\Delta(r_n \Delta u_n) + a_n u_n = 0 \quad (\{a_n\} \text{ is eventually non-negative}),$$

are oscillatory.

Proof. Apply Theorem 5 to the particular case where $\varphi(s) = s$, $\psi(s) = s^{\alpha}$, $\varrho_n = R_n$.

COROLLARY 6. *If* $a_n \geqslant 0$, $n \geqslant n_0$ *and* $\sum^{\infty} R_n a_n = \infty$, *then all solutions of the equation*

(28) $$\Delta(r_n \Delta u_n) + a_n |u_n|^{\alpha} \operatorname{sgn} u_n = 0, \quad \alpha > 1,$$

are oscillatory.

Proof. Apply Theorem 5 to the particular case where $\varphi(s) = |s|^{\alpha} \operatorname{sgn} s$, $\alpha > 1$, $\psi(s) = 1$, $\varrho_n = R_n$.

A close look at the proof of Theorem 5 ensures the validity of the following

THEOREM 6. *Let the assumptions of Theorem 5 be satisfied with the exception of condition* (18). *Then every bounded solution* $\{u_n\} \in U_1$ *is oscillatory.*

THEOREM 7. *Assume that conditions* (i) *of Theorem* 1 *and* (i) *of Corollary* 4 *hold. If there exists a non-decreasing positive sequence* $\{\gamma_n\}$ *such that* $\{\Delta \gamma_n\}$ *is non-increasing for* $n \geqslant n_0$ *and*

$$\sum^{\infty} \frac{\gamma_n a_n}{r_n} = \infty,$$

then every solution $\{u_n\} \in U_1$ *is oscillatory.*

Proof. Assume that there exists a non-oscillatory solution $\{u_n\} \in U_1$. We may suppose that $u_n > 0$ for $n \geqslant n_1 > n_0$ since a similar argument holds when $\{u_n\}$ is negative. Then by the assumptions we observe that $\{\Delta u_n\}$ is non-increasing, $\Delta u_n \geqslant 0$ for $n \geqslant n_1$ and

$$\frac{\gamma_n \Delta^2 u_n}{\varphi(u_n)} \leqslant -\frac{\gamma_n a_n}{r_n}, \quad n \geqslant n_1.$$

Now if we put $p_n = \gamma_n \Delta u_n / \varphi(u_n)$, $n \geqslant n_1$, then the rest of the proof follows analogously to that of Theorem 5; therefore we omit the details.

4. We now adduce examples which illustrate our principal results.

The oscillatory nature of the equation

(29) $$\Delta^2 u_n + n^{-2} u_n = 0, \quad n = 1, 2, \ldots,$$

may be inferred from Theorem 4. Especially Corollary 3 is applicable since condition (17) holds for $h_n = n$, and hence all non-trivial solutions of (29) are oscillatory. Corollary 4 and Theorem 7 exclude the linear case because of assumption (i) of Corollary 4 and so cannot be applied to equation (29). Also, it is obvious that the oscillation of (29) cannot be derived from Corollary 5. However, if we consider equation (16), where

$$a_n \geqslant cn^{-1-\varepsilon}(\ln n)^{-1}, \quad 0 < \varepsilon < 1,\ 0 < c = \text{const},$$

then the oscillation of this equation follows already from Corollary 5, since in this case $\sum^{\infty} n^{\varepsilon} a_n = \infty$, i.e., condition (26) is satisfied.

We wish to remark that α in condition (26) of Corollary 5 cannot be omitted. The next example justifies this remark, namely the equation

$$(30) \qquad \Delta^2 u_n + \frac{2\sqrt{n+2}-\sqrt{n+1}-\sqrt{n+3}}{\sqrt{n+1}} u_n = 0$$

has the non-oscillatory solution $u_n = \sqrt{n+1}$. It is easy to verify that for (30) we have $\sum^{\infty} na_n = \infty$.

However, Corollary 5 is still true even for $\alpha = 0$ for the bounded solutions of equation (27), as we proved in Theorem 3.

The oscillation of the equation

$$(31) \qquad \Delta(n\Delta u_n) + n^{-1}(\ln n)^{-2} u_n^3 = 0, \quad n = 2, 3, \ldots,$$

may be derived from Theorem 5. In particular, letting $\varphi(s) = s^3$, we see that all the hypotheses of Corollary 4 are satisfied. It is clear that Theorem 4 does not apply to equation (31). Obviously, if for equation (28) we put

$$r_n \leqslant c_1 n, \quad a_n \geqslant c_2/n(\ln n)^2 \ln_2 n \ldots \ln_p n$$

($c_1 > 0, c_2 > 0, \ln_1 n = \ln n, \ln_p n = \ln \ln_{p-1} n, p = 2, 3, \ldots$), then all solutions of equation (28) are oscillatory. This conclusion follows immediately from Corollary 6.

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