Oscillation Criteria for Second-Order Nonlinear Dynamic Equations on Time Scales

Lynn Erbe[1], Allan Peterson[1], and S. H. Saker[2]

[1]Department of Mathematics and Statistics
University of Nebraska-Lincoln
Lincoln, NE 68588-0323

[2] Faculty of Mathematics and Computer Science,
Adam Mickiewicz University,
Matejki 48/49, 60–769 Poznan,
Poland

ABSTRACT: By means of generalized Riccati transformation techniques and generalized exponential functions, we give some oscillation criteria for the nonlinear dynamic equation

$$(p(t)x^{\Delta}(t))^{\Delta} + q(t)(f \circ x^{\sigma}) = 0,$$

on time scales. We also apply our results to linear and nonlinear dynamic equations with damping and obtain some sufficient conditions for oscillation of all solutions.

Key words: *Oscillation, dynamic equations, time scale, Riccati technique.*
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1. Introduction

In this paper, we shall consider the nonlinear dynamic equation

$$(p(t)x^{\Delta}(t))^{\Delta} + q(t)(f \circ x^{\sigma}) = 0, \tag{1.1}$$

on time scales, where p, q are positive, real–valued right–dense continuous functions, and $f : \mathbb{R} \to \mathbb{R}$ is continuous and satisfies

$$xf(x) > 0 \text{ and } |f(x)| \geq K|x| \text{ for } x \neq 0 \text{ for some } K > 0. \tag{1.2}$$

We shall also consider the two cases:

$$\int_{t_0}^{\infty} \frac{\Delta t}{p(t)} = \infty, \tag{1.3}$$

[1]Research was supported by NSF Grant 0072505, e-mail: lerbe@math.unl.edu, apeterso@math.unl.edu

[2]Permanent address is Department of Mathematics, Mansoura University, Mansoura 35516, Egypt, e-mail: shsaker@mans.edu.eg, shsaker@amu.edu.pl

and

$$\int_{t_0}^{\infty} \frac{\Delta t}{p(t)} < \infty. \tag{1.4}$$

By a solution of (1.1) we mean a nontrivial real–valued function x satisfying equation (1.1) for $t \geq t_0 \geq a$, for some $t_0 \geq a > 0$. A solution x of (1.1) is said to be oscillatory if it is neither eventually positive nor eventually negative, otherwise it is nonoscillatory. Equation (1.1) is said to be oscillatory if all its solutions are oscillatory. Our attention is restricted to those solutions of (1.1) which exist on some half line $[t_x, \infty)$ and satisfy $\sup\{|x(t)| : t > t_0\} > 0$ for any $t_0 \geq t_x$.

Much recent attention has been given to differential equations on time scales (or measure chains), and we refer the reader to the landmark paper of Hilger [13] for a comprehensive treatment of the subject. Since then several authors have expounded on various aspects of this new theory, see the survey paper by Agarwal, Bohner, O'Regan, and Peterson [1] and the references cited therein. A book on the subject of time scales, by Bohner and Peterson [4], summarizes and organizes much of time scale calculus.

In recent years there has been much research activity concerning the oscillation and nonoscillation of solutions of dynamic equations on time scales. We refer the reader to the papers [2], [3], [5]-[12].

In Došlý and Hilger [6], the authors consider the second order dynamic equation

$$(p(t)x^{\Delta}(t))^{\Delta} + q(t)x^{\sigma} = 0, \tag{1.5}$$

and give necessary and sufficient conditions for oscillation of all solutions on unbounded time scales. Often, however, the oscillation criteria require additional assumptions on the unknown solutions, which may not be easy to check.

In Erbe and Peterson [9], the authors consider the same equation and suppose that there exists $t_0 \in \mathbb{T}$, such that $p(t)$ is bounded above on $[t_0, \infty)$, $h_0 = \inf\{\mu(t) : t \in [t_0, \infty)\} > 0$, and showed via Riccati techniques that

$$\int_{t_0}^{\infty} q(t)\Delta t = \infty.$$

implies that every solution is oscillatory on $[t_0, \infty)$. It is clear that the results given in Erbe and Peterson [9], can not be applied when p is unbounded, $\mu(t) = 0$ and $q(t) = t^{-\alpha}$ when $\alpha > 1$. We refer also to the papers by Erbe and Peterson [9] and Erbe [7] for additional linear oscillation criteria, which also treat more general situations.

In Guseinov and Kaymakçalan [12], the authors consider the linear dynamic equation

$$x^{\Delta\Delta}(t) + \alpha(t)x^{\Delta}(t) + \beta(t)x(t) = 0, \tag{1.6}$$

and give some sufficient conditions for nonoscillation.

Recently Bohner and Saker [5] considered (1.1) and used Riccati techniques to give some sufficient conditions for oscillation when (1.3) or (1.4) hold. They obtain some sufficient conditions which guarantee that every solution oscillates or converges to zero.

In this paper we intend to use a generalized Riccati transformation technique to obtain several oscillation criteria for (1.1) when (1.3) or (1.4) holds. Our results improve the results given in Došlý and Hilger [6] and Erbe and Peterson [9] and complement the results in Bohner and Saker [5]. Applications to equations to which previously known criteria for oscillation are not applicable are given. The paper is organized as follows: In the next section we present some basic definitions concerning the calculus on time scales. In Section 3 we develop a generalized Riccati transformation technique to give some sufficient conditions for oscillation of all solutions of (1.1), subject to the condition (1.3). Also we present some conditions that ensure that all solutions are either oscillatory or convergent to zero when (1.4) holds. In Section 4, we will apply our results to the linear dynamic equation (1.5), (1.6) and also to nonlinear dynamic equations of the form

$$x^{\Delta\Delta}(t) + \alpha(t)x^{\Delta^{\sigma}}(t) + \beta(t)(f \circ x^{\sigma}) = 0 \tag{1.7}$$

to give some sufficient conditions for oscillation of all their solutions.

2. Some Preliminaries on time scales

A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of the real numbers $\mathbb{R}$. On any time scale $\mathbb{T}$ we define the forward and backward jump operators by:

$$\sigma(t) := \inf\{s \in \mathbb{T} :\ s > t\}, \quad \rho(t) := \sup\{s \in \mathbb{T}, s < t\}, \tag{2.1}$$

where $\inf \Phi := \sup \mathbb{T}$ and $\sup \Phi = \inf \mathbb{T}$. A point $t \in \mathbb{T}$, $t > \inf \mathbb{T}$, is said to be left–dense if $\rho(t) = t$, right–dense if $t < \sup \mathbb{T}$ and $\sigma(t) = t$, left–scattered if $\rho(t) < t$ and right–scattered if $\sigma(t) > t$. The graininess function μ for a time scale $\mathbb{T}$ is defined by $\mu(t) := \sigma(t) - t$.

For a function $f : \mathbb{T} \to \mathbb{R}$ (the range $\mathbb{R}$ of f may be actually replaced by any Banach space) the (delta) derivative is defined by

$$f^{\Delta}(t) = \frac{f(\sigma(t)) - f(t)}{\sigma(t) - t}. \tag{2.2}$$

if f is continuous at t and t is right–scattered. If t is not right–scattered then the derivative is defined by

$$f^{\Delta}(t)=\lim_{s\to t}\frac{f(\sigma(t))-f(s)}{t-s}=\lim_{t\to\infty}\frac{f(t)-f(s)}{t-s}, \tag{2.3}$$

provided this limit exists. A function $f:[a,b]\to\mathbb{R}$ is said to be right–dense continuous if it is right continuous at each right–dense point and there exists a finite left limit at all left–dense points, and f is said to be differentiable if its derivative exists. A useful formula is

$$f(\sigma(t))=f(t)+\mu(t)f^{\Delta}(t). \tag{2.4}$$

We will make use of the following product and quotient rules for the derivative of the product fg and the quotient f/g (where $gg^{\sigma}\neq 0$) of two differentiable function f and g

$$(fg)^{\Delta} = f^{\Delta}g+f^{\sigma}g^{\Delta}=fg^{\Delta}+f^{\Delta}g^{\sigma}, \tag{2.5}$$

$$\left(\frac{f}{g}\right)^{\Delta} = \frac{f^{\Delta}g-fg^{\Delta}}{gg^{\sigma}}. \tag{2.6}$$

By using the product rule the derivative of $f(t)=(t-\alpha)^m$ for $m\in\mathbb{N}$, and $\alpha\in\mathbb{T}$ can be calculated as

$$f^{\Delta}(t)=\sum_{\nu=0}^{m-1}(\sigma(t)-\alpha)^{\nu}(t-\alpha)^{m-\nu-1}, \tag{2.7}$$

(see Theorem 1.24 in Bohner and Peterson [4]). For $a,b\in\mathbb{T}$, and a differentiable function f, the Cauchy integral of f^{Δ} is defined by

$$\int_a^b f^{\Delta}(t)\Delta t=f(b)-f(a).$$

An integration by parts formula reads

$$\int_a^b f(t)g^{\Delta}(t)\Delta t=[f(t)g(t)]_a^b-\int_a^b f^{\Delta}(t)g(\sigma(t))\Delta t, \tag{2.8}$$

and infinite integrals are defined as

$$\int_a^{\infty} f(t)\Delta t=\lim_{b\to\infty}\int_a^b f(t)\Delta t.$$

Note that in the case $\mathbb{T}=\mathbb{R}$ we have

$$\sigma(t)=\rho(t)=t,\quad f^{\Delta}(t)=f^{'}(t),\quad \int_a^b f(t)\Delta t=\int_a^b f(t)dt$$

and in the case $\mathbb{T}=\mathbb{Z}$ we have

$$\sigma(t)=t+1,\quad \rho(t)=t-1,\quad f^{\Delta}(t)=\Delta f(t)=f(t+1)-f(t),$$

and

$$\int_a^b f(t)\Delta t = \sum_{i=a}^{b-1} f(i),$$

if $a \le b$. We say that a function $p : \mathbb{T} \to \mathbb{R}$ is regressive provided

$$1 + \mu(t)p(t) \neq 0 \quad t \in \mathbb{T}.$$

Although we shall not make use of the fact, it turns out that the set of all regressive functions on a time scale $\mathbb{T}$ forms an Abelian group under the addition $\oplus$ defined by

$$p \oplus q := p + q + \mu pq.$$

We denote the set of all $f : \mathbb{T} \to \mathbb{R}$ which are rd–continuous and regressive by $\mathcal{R}$. If $p \in \mathcal{R}$, then we can define the exponential function by

$$e_p(t,s) = \exp\left(\int_s^t \xi_{\mu(\tau)}(p(\tau))\Delta\tau\right)$$

for $t \in \mathbb{T}$, $s \in \mathbb{T}^k$, where $\xi_h(z)$ is the cylinder transformation, which is given by

$$\xi_h(z) = \begin{cases} \frac{\log(1+hz)}{h}, & h \neq 0, \\ z, & h = 0. \end{cases}$$

Alternately, for $p \in \mathcal{R}$ one can define the exponential function $e_p(\cdot, t_0)$, to be the unique solution of the IVP

$$x^\Delta = p(t)x, \quad x(t_0) = 1.$$

We define

$$\mathcal{R}^+ := \{f \in \mathcal{R} : 1 + \mu(t)f(t) > 0, t \in \mathbb{T}\}.$$

For properties of this exponential function see Bohner and Peterson [4]. One such property that we will use is the formula

$$e_p(\sigma(t), t_0) = [1 + \mu(t)p(t)]e_p(t, t_0).$$

Also if $p \in \mathcal{R}$, then $e_p(t,s)$ is real-valued and nonzero on $\mathbb{T}$. If $p \in \mathcal{R}^+$, then $e_p(t, t_0)$ always positive.

3. Oscillation Criteria

In this section we give some new oscillation criteria for (1.1). Since we are interested in oscillatory behavior, we suppose that the time scale under consideration is not bounded above, i.e., it is a time scale interval of the form $[a, \infty)$. We start with the following auxiliary result.

Lemma 3.1. *Assume that* (1.3) *holds, and x solves* (1.1) *with $x(t) > 0$ for all $t > t_0$. Define $y = px^\Delta$. Then we have*

$$y^\Delta(t) < 0 \quad \text{and} \quad 0 \le y(t) \le \frac{x(t)}{\int_{t_0}^t \frac{\Delta s}{p(s)}}, \quad t > t_0 \tag{3.1}$$

and

$$0 \le \frac{x^\Delta(t)}{x(t)} \le \frac{1}{p(t)\int_{t_0}^t \frac{\Delta s}{p(s)}}, \quad t > t_0. \tag{3.2}$$

Proof. Let $t > t_0$. Then $x(\sigma(t)) > 0$, for $t > t_0$ and hence (1.1) implies that

$$y^\Delta(t) = (p(t)x^\Delta(t))^\Delta = -q(t)f(x^\sigma(t)) < 0, \quad t > t_0,$$

so that y is decreasing for $t > t_0$. Assume that there exists $t_1 > t_0$ such that $y(t_1) = c < 0$. Then

$$p(s)x^\Delta(s) = y(s) \le y(t_1) = c \quad s \ge t_1,$$

and therefore

$$x^\Delta(s) \le \frac{c}{p(s)} \quad s \ge t_1.$$

An integration from t_1 to $t > t_1$ now gives

$$x(t) = x(t_1) + \int_{t_1}^t x^\Delta(s)\Delta s \le x(t_1) + c\int_{t_1}^t \frac{\Delta s}{p(s)} \to -\infty \text{ as } t \to \infty.$$

a contradiction. Hence $y(t) = p(t)x^\Delta(t) \ge 0$ for all $t > t_0$. To show the last inequality in (3.1), note that

$$x(t) \ge x(t) - x(t_0) = \int_{t_0}^t \frac{y(s)\Delta s}{p(s)} \ge y(t)\left\{\int_{t_0}^t \frac{\Delta s}{p(s)}\right\},$$

for $t > t_0$. Since p is positive, the proof of (3.1) is complete, and (3.2) clearly follows from (3.1). □

Let $r \in \mathcal{R}$, assume that $p \cdot r$ is a differentiable function, and define the auxiliary functions

$$\begin{aligned}
C(t) &= C(t,t_0) := 1 + \frac{\mu(t)}{p(t)\int_{t_0}^t \frac{\Delta s}{p(s)}}, \quad Q_1(t) = Q_1(t,t_0) := \frac{1+\mu(t)r(t)}{p(t)e_r(t,t_0)}, \\
\psi(t) &= \psi(t,t_0)) := e_r(\sigma(t),t_0)\left[Kq(t) + \frac{1}{2}(p(t)r(t))^\Delta + \frac{r^2(t)p(t)}{4C(t)}\right], \\
Q(t) &= Q(t,t_0) := -\frac{r(t)(1+\mu(t)r(t))}{C(t)} + r(t),
\end{aligned}$$

for $t > t_0$. We also introduce the following condition

(A) There exists $M > 0$ such that $r(t)e_r(t, t_0)p(t) \leq M$ for all large t.

Theorem 3.1. *Assume that* (1.2), (1.3), *and (A) hold. Furthermore, assume that there exists $r \in \mathcal{R}^+$ such that $p \cdot r$ is differentiable and such that for any $t_0 \geq a$ there exists a $t_1 > t_0$ so that*

$$\limsup_{t\to\infty} \int_{t_1}^{t} H(s)\Delta s = \infty, \tag{3.3}$$

where

$$H(t) = H(t, t_0) = \psi(t) - \frac{Q^2(t)C(t)}{4Q_1(t)},$$

for $t > t_0$. Then equation (1.1) *is oscillatory on $[a, \infty)$.*

Proof. Suppose to the contrary that x is a nonoscillatory solution of (1.1). We will only consider the case where x is an eventually positive solution of (1.1), i.e., there exists $t_0 \geq a$ such that $x(t) > 0$ for $t > t_0$, since the other case is similar. Corresponding to this t_0, let $C(t, t_0)$, $Q_1(t, t_0)$, $Q(t, t_0)$, and $\psi(t, t_0)$ be defined as above. Note that all of the assumptions of Lemma 3.1 hold. Define the function w by

$$w(t) = e_r(t, t_0)\left[\frac{p(t)x^\Delta(t)}{x(t)} - \frac{1}{2}p(t)r(t)\right], \ t > t_0. \tag{3.4}$$

Since $r \in \mathcal{R}^+$, $e_r(t, t_0) > 0$ which we will use in the proof below. With w defined as in (3.4), by the product rule (2.5) we have

$$\begin{aligned} w^\Delta(t) &= e_r^\Delta(t, t_0)\left[\frac{p(t)x^\Delta(t)}{x(t)} - \frac{1}{2}p(t)r(t)\right] \\ &+ e_r(\sigma(t), t_0)\left(\frac{p(t)x^\Delta(t)}{x(t)} - \frac{1}{2}p(t)r(t)\right)^\Delta. \end{aligned}$$

Hence

$$\begin{aligned} w^\Delta(t) &= r(t)w(t) \\ &+ e_r(\sigma(t), t_0)\left(\frac{x(t)\left(p(t)x^\Delta(t)\right)^\Delta - p(t)\left(x^\Delta(t)\right)^2}{x(t)x(\sigma(t))} - \frac{1}{2}(p(t)r(t))^\Delta\right) \\ &= r(t)w(t) + e_r(\sigma(t), t_0)\frac{\left(p(t)x^\Delta(t)\right)^\Delta}{x(\sigma(t))} \\ &- \frac{e_r(\sigma(t), t_0)}{p(t)} \cdot \frac{x(t)}{x(\sigma(t))}\left(\frac{p(t)\left(x^\Delta(t)\right)}{x(t)}\right)^2 \\ &-\frac{1}{2}e_r(\sigma(t), t_0)(p(t)r(t))^\Delta. \end{aligned}$$

From Lemma 3.1 we have that

$$\begin{aligned}\frac{x(\sigma(t))}{x(t)} &= \frac{x(t)+\mu(t)x^{\Delta}(t)}{x(t)} = 1+\mu(t)\frac{x^{\Delta}(t)}{x(t)}\\ &\leq 1+\frac{\mu(t)}{p(t)\int_{t_0}^{t}\frac{\Delta s}{p(s)}} = C(t),\end{aligned}$$

so we get that

$$\begin{aligned}w^{\Delta}(t) &\leq r(t)w(t) - e_r(\sigma(t),t_0)q(t)\frac{(f\circ x^{\sigma})(t)}{x(\sigma(t))}\\ &\quad -\frac{e_r(\sigma(t),t_0)}{p(t)C(t)}\left(\frac{p(t)\left(x^{\Delta}(t)\right)}{x(t)}\right)^2 - \frac{1}{2}e_r(\sigma(t),t_0)(p(t)r(t))^{\Delta}.\end{aligned}$$

From (3.4) we have that

$$\left(\frac{px^{\Delta}}{x}\right)^2 = \frac{w^2}{e_r^2} + \frac{pr}{e_r}w + \frac{1}{4}r^2p^2.$$

Using this last equation and (1.2) we get

$$\begin{aligned}w^{\Delta}(t) &\leq -e_r(\sigma(t),t_0)\left[Kq(t)+\frac{1}{2}(p(t)r(t))^{\Delta}+\frac{r^2(t)p(t)}{4C(t)}\right]+r(t)w(t)\\ &\quad -\frac{e_r(\sigma(t),t_0)w^2(t)}{C(t)p(t)e_r^2(t,t_0)} - \frac{r(t)e_r(\sigma(t),t_0)}{C(t)e_r(t,t_0)}w(t)\\ &= -e_r(\sigma(t),t_0)\left[Kq(t)+\frac{1}{2}(p(t)r(t))^{\Delta}+\frac{r^2(t)p(t)}{4C(t)}\right]\\ &\quad -\frac{e_r(\sigma(t),t_0)w^2(t)}{C(t)p(t)e_r^2(t,t_0)} - \left[\frac{r(t)e_r(\sigma(t),t_0)}{C(t)e_r(t,t_0)} - r(t)\right]w(t).\end{aligned} \tag{3.5}$$

Using the properties of the exponential function $e_r(t,t_0)$ and (3.5), we have

$$\begin{aligned}w^{\Delta}(t) &\leq -\psi(t) - \frac{(1+\mu(t)r(t))w^2(t)}{C(t)p(t)e_r(t,t_0)}\\ &\quad -\left[\frac{r(t)(1+\mu(t)r(t))}{C(t)} - r(t)\right]w(t).\end{aligned} \tag{3.6}$$

Hence,

$$w^{\Delta}(t) \leq -\psi(t) - \left[\sqrt{\frac{Q_1(t)}{C(t)}}w(t) - \frac{\sqrt{C(t)}Q(t)}{2\sqrt{Q_1(t)}}\right]^2 + \frac{Q^2(t)C(t)}{4Q_1(t)} \tag{3.7}$$

Then, (3.7) implies that

$$w^{\Delta}(t) \leq -\left[\psi(t) - \frac{Q^2(t)C(t)}{4Q_1(t)}\right]. \tag{3.8}$$

Let $t_1 > t_0$ be as in the statement of this theorem. Integrating (3.8) from t_1 to t, we obtain

$$w(t) - w(t_1) \leq -\int_{t_1}^{t} \left[\psi(s) - \frac{Q^2(s)C(s)}{4Q_1(s)}\right] \Delta s \tag{3.9}$$

which yields

$$\int_{t_1}^{t} \left[\psi(s) - \frac{Q^2(s)C(s)}{4Q_1(s)}\right] \Delta s \leq w(t_1) - w(t) \tag{3.10}$$

for all large t. Now by (3.4) and condition (A) we have

$$\begin{aligned} w(t) &= e_r(t, t_0)\left(\frac{p(t)x^{\Delta}(t)}{x(t)} - \frac{1}{2}p(t)r(t)\right) \\ &\geq -\frac{1}{2}p(t)r(t)e_r(t, t_0) \geq -\frac{1}{2}M, \end{aligned}$$

and therefore, it follows that the right hand side of (3.10) is bounded above. This contradicts (3.3) and proves the theorem.

□

From Theorem 3.1, we can obtain different sufficient conditions for oscillation of all solutions of (1.1) by different choices of $r(t)$. For instance, let $r(t) = 0$, then $Q(t) = 0$, $e_r(t, t_0) = 1$, and $\psi(t) = Kq(t)$ and we get the following well–known result.

Corollary 3.1 (Leighton–Wintner Theorem)**.** *Assume that* (1.2) *and* (1.3) *hold. If*

$$\int_{a}^{\infty} q(s)\Delta s = \infty, \tag{3.11}$$

then equation (1.1) *is oscillatory on* $[a, \infty)$.

If $r(t) = \frac{1}{t}$, then $e_r(t, t_0) = \frac{t}{t_0}$ and it follows that condition (A) holds, provided p is bounded above, and so Theorem 3.1 yields the following result:

Corollary 3.2. *Assume p is bounded above, that* (1.2) *and* (1.3) *hold, and for any $t_0 \geq a$ there is a $t_1 > t_0$ such that*

(3.12)

$$\limsup_{t\to\infty} \int_{t_1}^{t} \left[\sigma(s)\left[Kq(s) + \left(\frac{p(s)}{2s}\right)^{\Delta} + \frac{p(s)}{4s^2C(s)}\right] - \frac{A^2(s)C(s)}{4B(s)}\right] \Delta s = \infty,$$

where

$$A(s) := \frac{-1}{sC(s)}\left(1 + \frac{1}{s}\mu(s) - C(s)\right), \quad B(s) := \frac{s + \mu(s)}{s^2p(s)}.$$

Then (1.1) *is oscillatory on* $[a, \infty)$.

If $p(t) = 1$ and $f(x) = x$, then equation (1.1) reduces to the linear dynamic equation

$$x^{\Delta\Delta}(t) + q(t)x^{\sigma} = 0, \tag{3.13}$$

for $t \in [a, \infty)$. From Theorem 3.1 we have the following oscillation criterion for equation (3.13) which improves some of the results in Bohner and Saker [5] and Erbe and Peterson [8].

Corollary 3.3. *Assume that* (1.2) *and* (1.3) *hold and for any $t_0 \geq a$ there is a $t_1 > t_0$ such that*

(3.14)
$$\limsup_{t\to\infty} \int_{t_1}^{t} \left[\sigma(s)\left[q(s) - \left(\frac{1}{2s\sigma(s)}\right) + \frac{1}{4s^2C_1(s)}\right] - \frac{A_1^2(s)C_1(s)}{4B_1(s)}\right]\Delta s = \infty,$$

where

$$A_1(s) = \frac{-1}{sC_1(s)}\left(1 + \frac{1}{s}\mu(s) - C_1(s)\right)$$
$$B_1(s) = \frac{s+\mu(s)}{s^2}, \quad C_1(s) = 1 + \frac{\mu(s)}{(s-t_0)}.$$

Then equation (3.13) *is oscillatory on* $[a, \infty)$.

Example 3.1. Consider the Euler–Cauchy dynamic equation

$$x^{\Delta\Delta} + \frac{\gamma}{t\sigma(t)}x^{\sigma} = 0, \tag{3.15}$$

for $t \in [a, \infty)$. Here $q(t) = \frac{\gamma}{t\sigma(t)}$. Then (3.14) in Corollary 3.3 reads

$$\limsup_{t\to\infty} \int_{t_1}^{t} \left[\left[\frac{\gamma}{s} - \frac{1}{2s} + \frac{\sigma(s)}{4s^2C_1(s)}\right] - \frac{A_1^2(s)C_1(s)}{4B_1(s)}\right]\Delta s = \infty. \tag{3.16}$$

If $\mathbb{T} = \mathbb{R}$, then the dynamic equation (3.15) is the second order Euler–Cauchy differential equation

$$x'' + \frac{\gamma}{t^2}x = 0, \ t \geq 1 \tag{3.17}$$

and in this case $\mu(s) = 0$, $\sigma(s) = s$, $C_1(s) = 1$ and $A_1(s) = 0$. Therefore (3.16) can be rewritten as

$$\limsup_{t\to\infty} \int_{t_1}^{t} \left[\frac{\gamma}{s} - \frac{1}{2s} + \frac{s}{4s^2}\right]\Delta s = \limsup_{t\to\infty} \int_{t_1}^{t} \left[\frac{\gamma - \frac{1}{4}}{s}\right]\Delta s = \infty.$$

provided that $\gamma > \frac{1}{4}$. Hence every solution of (3.17) oscillates if $\gamma > \frac{1}{4}$, which agrees with the well–known oscillatory behavior of (3.17), (see Li [15]).

If $\mathbb{T}=\mathbb{Z}$, then (3.15) is the second order discrete Euler–Cauchy difference equation

$$\Delta^2 x_t + \frac{\gamma}{t(t+1)} x_{t+1} = 0,\ t = 1, 2, ... \tag{3.18}$$

and we have $\mu(s) = 1$, $\sigma(s) = s + 1$, $C_1(s) = \frac{s-t_0+1}{s-t_0}$,

$$\frac{A_1^2(s)}{B_1(s)} = \frac{t_0^2}{s^2(s+1)(s-t_0+1)^2}.$$

Therefore (3.16) can be rewritten as

$$\limsup_{t\to\infty} \int_{t_1}^{t} \left[\left[\frac{\gamma}{s} - \frac{1}{2s} + \frac{s^2-1}{4s^3}\right] - \frac{t_0^2}{4s^2(s+1)(s-t_0)(s-t_0+1)}\right] \Delta s$$
$$= \limsup_{t\to\infty} \int_{t_1}^{t} \left[\frac{\gamma}{s} - \frac{1}{2s} + \frac{1}{4s}\right] \Delta s = \infty.$$

provided that $\gamma > \frac{1}{4}$. Hence every solution of (3.18) oscillates if $\gamma > \frac{1}{4}$, which agrees with the well–known oscillatory behavior of (3.18). It is known in Zhang and Cheng [16] that when $\mu \le 1/4$, (3.18) has a nonoscillatory solution. Hence, Theorem 3.1 and Corollary 3.3 are sharp. Note that the results in Došlý and Hilger [6] and Erbe and Peterson [9] cannot be applied to (3.15).

Theorem 3.2. *Assume that* (1.2) *and* (1.3) *hold. Furthermore, assume that there exists a function $r \in \mathcal{R}^+$ such that $p \cdot r$ is differentiable and given any $t_0 \ge a$ there is a $t_1 > t_0$ such that*

$$\limsup_{t\to\infty} \frac{1}{t^m} \int_{t_1}^{t} (t-s)^m \left[\psi(s) - \frac{Q^2(s)C(s)}{4Q_1(s)}\right] \Delta s = \infty, \tag{3.19}$$

where m is a positive integer. Assume further that

$$\left(\frac{1}{t^m}\right) \int_{t_1}^{t} e_r^{\sigma}(s,t_0) p^{\sigma}(s) r^{\sigma}(s) \sum_{\nu=0}^{m-1} (\sigma(s)-t)^{\nu} (s-t)^{m-\nu-1} \Delta s \tag{3.20}$$

is bounded above. Then every solution of equation (1.1) *is oscillatory on* $[a,\infty)$.

Proof. We proceed as in the proof of Theorem 3.1. We may assume that (1.1) has a nonoscillatory solution x such that $x(t) > 0,\ x^{\Delta}(t) \ge 0,\ (p(t)x^{\Delta}(t))^{\Delta} \le 0$ for $t \ge t_1$. Define w by (3.4) as before, then as in the proof of Theorem 3.1 we obtain (3.8) so

$$\left[\psi(t) - \frac{Q^2(t)C(t)}{4Q_1(t)}\right] < -w^{\Delta}(t).$$

Pick $t_1 > t_0$ so that (3.19) holds. Note that

$$\int_{t_1}^{t}(t-s)^m\left[\psi(s)-\frac{Q^2(s)C(s)}{4Q_1(s)}\right]\Delta s \leq -\int_{t_1}^{t}(t-s)^m w^{\Delta}(s)\Delta s, \tag{3.21}$$

Using the integration by parts formula (2.8) gives

$$\begin{aligned}\int_{t_1}^{t}(t-s)^m w^{\Delta}(s)\Delta s &= (t-s)^m w(s)\big|_{t_1}^{t}\\ &+(-1)^{m+1}\int_{t_1}^{t}\sum_{\nu=0}^{m-1}(\sigma(s)-t)^{\nu}(s-t)^{m-\nu-1}w(\sigma(s))\Delta s,\end{aligned}$$

where we have used the power rule for differentiation (2.7). It follows that

$$\begin{aligned}\int_{t_1}^{t}(t-s)^m w^{\Delta}(s)\Delta s &= -(t-t_1)^m w(t_1)\\ &+\int_{t_1}^{t} w(\sigma(s))\sum_{\nu=0}^{m-1}(t-\sigma(s))^{\nu}(t-s)^{m-\nu-1}\Delta s.\end{aligned}$$

From (3.4) we get that

$$w(t) \geq -\frac{1}{2}p(t)r(t)e_r(t,t_0),$$

for $t \geq t_1$. It follows that

$$\begin{aligned}\int_{t_1}^{t}(t-s)^m w^{\Delta}(s)\Delta s &\geq -(t-t_1)^m w(t_1)\\ &-\frac{1}{2}\int_{t_1}^{t} p^{\sigma}(s)r^{\sigma}(s)e_r^{\sigma}(s,t_0)\sum_{\nu=0}^{m-1}(t-\sigma(s))^{\nu}(t-s)^{m-\nu-1}\Delta s.\end{aligned}$$

Then from (3.21) we have

$$\begin{aligned}&\int_{t_1}^{t}(t-s)^m\left[\psi(s)-\frac{Q^2(s)C(s)}{4Q_1(s)}\right]\Delta s\\ \leq\ &(t-t_1)^m w(t_1)+\frac{1}{2}\int_{t_1}^{t} e_r^{\sigma}(s,t_0)p^{\sigma}(s)r^{\sigma}(s)\sum_{\nu=0}^{m-1}(\sigma(s)-t)^{\nu}(s-t)^{m-\nu-1}\Delta s.\end{aligned}$$

Then

$$\begin{aligned}&\frac{1}{t^m}\int_{t_1}^{t}(t-s)^m\left[\psi(s)-\frac{Q^2(s)C(s)}{4Q_1(s)}\right]\Delta s\\ \leq\ &\left(\frac{t-t_1}{t}\right)^m w(t_1)+\frac{1}{2}\left(\frac{1}{t^m}\right)\int_{t_1}^{t} e_r^{\sigma}(s,t_0)p^{\sigma}(s)r^{\sigma}(s)\sum_{\nu=0}^{m-1}(\sigma(s)-t)^{\nu}(s-t)^{m-\nu-1}\Delta s,\end{aligned}$$

which gives a contradiction using (3.19) and (3.20). The proof is complete. $\square$

Note that if $r \in \mathcal{R}^+$ and $r(t) \leq 0$, then (3.20) holds. When $r(t) = 0$, then (3.19) reduces to

$$\limsup_{t\to\infty} \frac{1}{t^m} \int_{t_1}^{t} (t-s)^m q(s)\Delta s = \infty, \tag{3.22}$$

which can be considered as an extension of Kamenev type oscillation criteria for second order differential equations, (see Kamenev [14]).

When $\mathbb{T} = \mathbb{R}$, then (3.22) becomes

$$\limsup_{t\to\infty} \frac{1}{t^m} \int_{t_1}^{t} (t-s)^m q(s) ds = \infty, \tag{3.23}$$

and when $\mathbb{T} = \mathbb{Z}$, then (3.22) becomes

$$\limsup_{t\to\infty} \frac{1}{t^m} \sum_{s=t_1}^{t-1} (t-s)^m q(s) = \infty, \tag{3.24}$$

We next give some sufficient conditions for the case when (1.4) holds, which guarantee that every solution of the dynamic equation (1.1) oscillates or converges to zero on $[a, \infty)$. The next result removes a monotonicity assumption on f in Bohner and Saker [5].

Theorem 3.3. *Assume that* (1.2) *and* (1.4) *hold and assume there exists* $r \in \mathcal{R}^+$ *such that* $p \cdot r$ *is differentiable and such that* (3.3) *holds. Furthermore, assume*

$$\int_{a}^{\infty} \frac{1}{p(t)} \int_{a}^{t} q(s)\Delta s \Delta t = \infty. \tag{3.25}$$

and let (A) hold. Then every solution of equation (1.1) *is either oscillatory or converges to zero on* $[a, \infty)$.

Proof. Let x be a nonoscillatory solution of (1.1) and, without loss of generality, suppose that $x(t) > 0$ for $t > t_0 \geq a$. There are two cases:

(1) $x^{\Delta}(t) > 0$ for all $t > t_0$, or
(2) there exists $t_1 > t_0$ with $x^{\Delta}(t_1) \leq 0$.

If (1) holds, then we proceed as in the proof of Theorem 3.1 (with w and the auxiliary functions C, ψ, Q_1, Q as defined before the statement of Theorem 3.1.) It follows that (3.10) holds and again, by condition (A) this leads to a contradiction. Therefore, (1) cannot hold.

Next consider case (2), that is, assume there exists $t_1 > t_0$ with $x^{\Delta}(t_1) \leq 0$, then since

$$\left(p(t)x^{\Delta}(t)\right)^{\Delta} = -q(t)f(x^{\sigma}(t)) \leq 0$$

for $t \geq t_1$, it follows that

$$p(t)x^{\Delta}(t) \leq p(t_1)x^{\Delta}(t_1) \leq 0$$

for $t \geq t_1$. Therefore $x^{\Delta}(t) \leq 0$ for all $t \geq t_1$, so $\lim_{t\to\infty} x(t) =: b \geq 0$ exists. we need to show that $b = 0$. If not, then $b > 0$ and hence $x(\sigma(t)) \geq b > 0$, for all $t \geq t_1$ and so from condition (1.2), we have

$$f(x(\sigma(t)) \geq Kx(\sigma(t)) \geq Kb > 0.$$

But this implies that

$$\big(p(t)x^{\Delta}(t)\big)^{\Delta} = -q(t)f(x^{\sigma}(t)) \leq -Kbq(t)$$

and so

$$p(t)x^{\Delta}(t) \leq p(t_1)x^{\Delta}(t_1) - Kb\int_{t_1}^{t} q(s)\Delta s \leq -Kb\int_{t_1}^{t} q(s)\Delta s,$$

for $t \geq t_1$. Hence, dividing by $p(t)$ and integrating gives

$$x(t) - x(t_1) \leq -Kb\int_{t_1}^{t} \frac{1}{p(s)}\int_{t_1}^{s} q(\tau)\Delta\tau\Delta s \to -\infty$$

as $t \to \infty$ which is a contradiction. Hence $b = 0$ and the proof is complete. □

In a similar manner, one may establish the following theorem.

Theorem 3.4. *Let all of the conditions of Theorem 3.3 hold with condition* (3.19) *replacing* (3.3). *Then every solution of equation* (1.1) *is oscillatory or converges to zero on* $[a, \infty)$.

4. Application to equations with damping

Our aim is to apply the results in Section 3, to give some sufficient conditions for oscillation of all solutions of the dynamic equations (1.6) and (1.7) with damping terms. We note that all of the results in Section 3, are true in the linear case, i.e., for the equations of the form (1.5), where the term $Kq(t)$ is replaced by $q(t)$.

Before stating our main results in this section we will need the following Lemmas, (see Bohner and Peterson [4]).

Lemma 4.1. *If* $\alpha, \beta \in C_{rd}$ *and*

$$1 - \mu(t)\alpha(t) + \mu^2(t)\beta(t) \neq 0, \quad t \in \mathbb{T}, \tag{4.1}$$

then the second order dynamic equation (1.6) *can be written in the self-adjoint form* (1.5), *where*

$$p(t) = e_{\gamma}(t, t_0), \quad q(t) = [1 + \mu(t)\gamma(t)]p(t)\beta(t) \tag{4.2}$$

$$\gamma(t) = \frac{\alpha(t) - \mu(t)\beta(t)}{1 - \mu(t)\alpha(t) + \mu^2(t)\beta(t)}. \tag{4.3}$$

Lemma 4.2. *If α is a regressive function, then the second order dynamic equation* (1.7) *can be written in the self-adjoint form*

$$(p(t)x^\Delta(t))^\Delta + q(t)f \circ x^\sigma = 0, \tag{4.4}$$

where

$$p(t) = e_\alpha(t, t_0) \quad \textit{and} \quad q(t) = \beta(t)p(t) \tag{4.5}$$

Now, by using the results in Section 3 and Lemma 4.1 we have the following results immediately.

Theorem 4.1. *Let p, q be defined as in* (4.5) *and ssume that* (1.3) *holds. Furthermore, assume that there exists a $r \in \mathcal{R}$ with r differentiable such that* (3.3) *holds with*

$$\psi(t) = e_r(\sigma(t), t_0)\left[q(t) + \frac{1}{2}(p(t)r(t))^\Delta + \frac{r^2(t)p(t)}{4C(t)}\right]. \tag{4.6}$$

Then equation (1.6) *is oscillatory on* $[a, \infty)$.

The proof follows from Lemma 4.1 and Theorem 3.1 and hence is omitted.

Corollary 4.1. *Assume that* (1.3) *and* (3.11) *hold, where p and q are as defined in* (4.2). *Then equation* (1.6) *is oscillatory on* $[a, \infty)$.

Corollary 4.2. *Assume that* (1.3) *and* (3.12) *hold except that the term $Kq(t)$ is replaced by $q(t)$, where p and q are as defined in* (4.2). *Then equation* (1.6) *is oscillatory on* $[a, \infty)$.

Theorem 4.2. *Assume that* (1.3) *holds. Furthermore, assume that there exists $r \in \mathcal{R}$ with r differentiable such that* (3.17) *holds, where p, q and ψ are as defined by* (4.2) *and* (4.6) *respectively, and m is odd integer. Then* (1.6) *is oscillatory on* $[a, \infty)$.

Theorem 4.3. *Assume that all the assumption of Theorem 4.1 hold except that the condition* (1.3) *is replaced by* (1.4). *If* (3.25) *holds, then every solution of equation* (1.6) *is oscillatory or converges to zero on* $[a, \infty)$.

Theorem 4.4. *Assume that all the assumption of Theorem 4.2 hold except that the condition* (1.3) *is replaced by* (1.4). *If* (3.25) *holds, then every solution of equation* (1.6) *is oscillatory or converges to zero on* $[a, \infty)$.

Oscillation criteria for equation (1.7) are now elementary consequences of the oscillation results in Theorems 4.1-4.4. The details are left to the reader.

References

[1] R. P. Agarwal, M. Bohner, D. O'Regan, and A. Peterson, Dynamic equations on time scales: A survey, J. Comp. Appl. Math., Special Issue on "Dynamic Equations on Time Scales", edited by R. P. Agarwal, M. Bohner, and D. O'Regan, (Preprint in Ulmer Seminare 5) 141(1-2), (2002), 1–26.

[2] E. Akın, L. Erbe, B. Kaymakçalan, and A. Peterson, Oscillation results for a dynamic equation on a time scale, J. Diff. Eqns. Appl. 7, (2001), 793–810.

[3] M. Bohner, O. Došlý, and W. Kratz, An oscillation theorem for discrete eigenvalue problems, Rocky Mountain J. Math, 2002. To appear.

[4] M. Bohner and A. Peterson, Dynamic Equations on Time Scales: An Introduction with Applications, Birkhäuser, Boston, 2001.

[5] M. Bohner and S. H. Saker, Oscillation of second order nonlinear dynamic equations on time scales, preprint.

[6] O. Došlý and S. Hilger, A necessary and sufficient condition for oscillation of the Sturm–Liouville dynamic equation on time scales, Special Issue on "Dynamic Equations on Time Scales", edited by R. P. Agarwal, M. Bohner, and D. O'Regan, J. Comp. Appl. Math. 141(1-2), (2002) 147–158.

[7] L. Erbe, Oscillation criteria for second order linear equations on a time scale, Canadian Applied Mathematics Quarterly, 9 (2001), 1–31.

[8] L. Erbe and A. Peterson, Positive solutions for a nonlinear differential equation on a measure chain, Math. Comput. Modelling, Boundary Value Problems and Related Topics, 32(5-6), (2000), 571–585.

[9] L. Erbe and A. Peterson, Riccati equations on a measure chain, In G. S. Ladde, N. G. Medhin, and M. Sambandham, editors, *Proceedings of Dynamic Systems and Applications*, volume 3, pages 193–199, Atlanta, 2001. Dynamic publishers.

[10] L. Erbe and A. Peterson. Oscillation criteria for second order matrix dynamic equations on a time scale, Special Issue on "Dynamic Equations on Time Scales", edited by R. P. Agarwal, M. Bohner, and D. O'Regan, J. Comput. Appl. Math., 141(1-2), (2002), 169–185.

[11] L. Erbe, A. Peterson, and P. Rehak, Comparison Theorems for Linear Dynamic Equations on Time Scales, Journal of Mathematical Analysis and Applications, preprint.

[12] G. Sh. Guseinov and B. Kaymakçalan, On a disconjugacy criterion for second order dynamic equations on time scales, Special Issue on "Dynamic Equations on Time Scales", edited by R. P. Agarwal, M. Bohner, and D. O'Regan, J. Comput. Appl. Math., 141(1-2), (2002), 187–196.

[13] S. Hilger, Analysis on measure chains — a unified approach to continuous and discrete calculus, Results Math. 18, (1990), 18–56.

[14] I. V. Kamenev, An integral criterion for oscillation of linear differential equations of second order, Mat. Zametki, 23, (1978), 249–251.

[15] H. J. Li, Oscillation criteria for second order linear differential equations, J. Math. Anal. Appl., 194, (1995), 312–321.

[16] G. Zhang and S. S. Cheng, A necessary and sufficient oscillation condition for the discrete Euler equation, PanAmerican Math. J. 9(4), (1999), 29–34.