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OSCILLATION ESTIMATES RELATIVE TO p-HOMOGENEUOUS FORMS AND KATO MEASURES DATA

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We state pointwise estimate for the positive subsolutions associated to a p-homogeneous form and nonnegative Radon measures data. As a by-product we establish an oscillation's estimate for the solutions relative to Kato measures data.

1. Introduction.

The necessary part of the Wiener criterion for the regularity of boundary points in the case of nonlinear elliptic problems has been proved in a recent paper by Malỳ, [15] [16], using an estimate on positive subsolutions of the problem. The estimate has been generalized in the case of the subelliptic p-Laplacian in [8] and used to prove a Wiener criterion for relaxed Dirichlet problems for the subelliptic p-Laplacian, $p > 1$, [8] [9]. Moreover in [8] the estimate is used to prove by suitable comparison method the local Hölder continuity of the solution of a problem relative to the subelliptic p-Laplacian and to data that are

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measures in the corresponding Kato class. In [10], [4] a notion of p-homogeneous strongly local Dirichlet form is introduced and in [11] a Harnack inequality is proved for the corresponding harmonic functions in the Riemannian case.

The goal of the present paper is an extension of the above cited estimate to the case of problems relative to a p-homogeneous Riemannian strongly local Dirichlet form with measure data; we also use the estimate to prove a local Hölder continuity result (with respect to the intrinsic metric) in the case of data in suitable Kato classes. The result applies in particular to the case of weighted subelliptic p-Laplacian. We now state our framework and the result obtained.

1.1. Assumptions and preliminary results.

Firstly we describe the notion of strongly local p-homogeneous Dirichlet form, $p > 1$, as given in [4].

We consider a locally compact separable Hausdorff space X with a metrizable topology and a positive Radon measure m on X such that $\text{supp}[m] = X$. Let $\Phi : L^p(X, m) \to [0, +\infty]$, $p > 1$, be a l.s.c. strictly convex functional with domain D, i.e. $D = \{v : \Phi(v) < +\infty\}$, such that $\Phi(0) = 0$. We assume that D is dense in $L^p(X, m)$ and that the following conditions hold:

(H_1) D is a dense linear subspace of $L^p(X, m)$, which can be endowed with a norm $||.||_D$; moreover D has a structure of Banach space with respect to the norm $||.||_D$ and the following estimate holds

$$c_1||v||_D^p \leq \Phi_1(v) = \Phi(v) + \int_X |v|^p dm \leq c_2||v||_D^p$$

for every $v \in D$, where c_1, c_2 are positive constants.

(H_2) We denote by D_0 the closure of $D \cap C_0(X)$ in D (with respect to the norm $||.||_D$) and we assume that $D \cap C_0(X)$ is dense in $C_0(X)$ for the uniform convergence on X.

(H_3) For every $u, v \in D \cap C_0(X)$ we have $u \vee v \in D \cap C_0(X)$, $u \wedge v \in D \cap C_0(X)$ and

$$\Phi(u \vee v) + \Phi(u \wedge v) \leq \Phi(u) + \Phi(v)$$

We recall that we can define a Choquet capacity $cap(E)$. Moreover we can also define in a natural way the quasi-continuity of a function and prove that every function in D_0 is quasi-continuous and is defined quasi-everywhere (i.e. up to sets of zero capacity), [10].

The assumptions $(H_1)(H_2)(H_3)$ have a global character; now we will recall the definition of *strongly local Dirichlet functional* with a homogeneity degree $p > 1$. Let Φ satisfy $(H_1)(H_2)(H_3)$; we say that Φ is a *strongly local Dirichlet functional* with a homogeneity degree $p > 1$ if the following conditions hold:

(H_4) Φ has the following representation on D_0: $\Phi(u) = \int_X \alpha(u)(dx)$ where α is a non-negative bounded Radon measure depending on $u \in D_0$, which does not charge sets of zero capacity. We say that $\alpha(u)$ is the energy (measure) of our functional. The energy $\alpha(u)$ (of our functional) is convex with respect to u in D_0 in the space of measures, i.e. if $u, v \in D_0$ and $t \in [0, 1]$ then $\alpha(tu + (1-t)v) \leq t\alpha(u) + (1-t)\alpha(v)$, and it is homogeneous of degree $p > 1$, i.e. $\alpha(tu) = |t|^p \alpha(u)$, $\forall u \in D_0$, $\forall t \in \mathbf{R}$.

Moreover the following closure property holds: if $u_n \to u$ in D_0 and $\alpha(u_n)$ converges to χ in the space of measures then $\chi \geq \alpha(u)$.

(H_5) α is of strongly local type, i.e. if $u, v \in D_0$ and $u-v = constant$ on an open set A we have $\alpha(u) = \alpha(v)$ on A.

(H_6) $\alpha(u)$ is of Markov type, i.e. if $\beta \in C^1(\mathbf{R})$ is such that $\beta'(t) \leq 1$ and $\beta(0) = 0$ and $u \in D \cap C_0(X)$, then $\beta(u) \in D \cap C_0(X)$ and $\alpha(\beta(u)) \leq \alpha(u)$ in the space of measures.

Let $\Phi(u) = \int_X \alpha(u)(dx)$ be a strongly local Dirichlet functional with domain D_0. Assume that for every $u, v \in D_0$ we have

$$\lim_{t \to 0} \frac{\alpha(u + tv) - \alpha(u)}{t} = \mu(u, v)$$

in the weak* topology of $\mathcal{M}$ (where $\mathcal{M}$ is the space of Radon measures on X) uniformly for u, v in a compact set of D_0, where $\mu(u, v)$ is defined on $D_0 \times D_0$ and is linear in v. We say that $\Psi(u, v) = \int_X \mu(u, v)(dx)$ is a *strongly local p-homogeneous Dirichlet form*. We observe that (H_3) is a consequence of (H_1),(H_2), (H_4)-(H_6).

The strong locality property allow us to define the domain of the form with respect to an open set O, denoted by $D_0[O]$ and the local domain of the form with respect to an open set O, denoted by $D_{loc}[O]$. We recall that, given an open set O in X we can define a Choquet capacity $cap(E;O)$ with respect to the open set O for a set $E \subset \overline{E} \subset O$. Moreover the sets of zero capacity are the same with respect to O and to X.

We recall now some properties of strongly local (p-homogeneous) Dirichlet forms, which will be used in the following, [4] [10]:

(a) $\mu(u,v)$ is *homogeneous of degree* $p-1$ in u and *linear* in v; we have also $\mu(u,u) = p\alpha(u)$.

(b) *Chain rule* : if $u, v \in D_0$ and $g \in C^1(\mathbf{R})$ with $g(0) = 0$ and g' bounded on $\mathbf{R}$, then $g(u), g(v)$ belong to D_0 and

$$\mu(g(u), v) = |g'(u)|^{p-2} g'(u)\mu(u,v) \tag{1.1}$$

$$\mu(u, g(v)) = g'(v)\mu(u,v) \tag{1.2}$$

We observe that we have also a chain rule for α

$$\alpha(g(u)) = |g'(u)|^p \alpha(u) \tag{1.3}$$

(c) *Truncation property*: for every $u, v \in D_0$

$$\mu(u^+, v) = 1_{\{u>0\}}\mu(u,v) \tag{1.4}$$

$$\mu(u, v^+) = 1_{\{v>0\}}\mu(u,v) \tag{1.5}$$

where the above relations make sense, since u and v are defined quasi-everywhere.

(d) $\forall a \in \mathbf{R}^+$

$$|\mu(u,v)| \le \alpha(u+v) \le 2^{p-1}a^{-p}\alpha(u) + 2^{p-1}a^{p(p-1)}\alpha(v) \tag{1.6}$$

(e) *Leibniz rule* with respect to the second argument:

$$\mu(u, vw) = v\mu(u,w) + w\mu(u,v) \tag{1.7}$$

where $u \in D_0,\ v, w \in D_0 \cap L^\infty(X,m)$.

(f) For any $f \in L^{p'}(X, \alpha(u))$ and $g \in L^p(X, \alpha(v))$ with $1/p + 1/p' = 1$, fg is integrable with respect to the absolute variation of $\mu(u,v)$ and $\forall a \in \mathbf{R}^+$

$$\begin{aligned} |fg|\mu(u,v)|(dx) &\le 2^{p-1}a^{-p}|f|^{p'}\alpha(u)(dx) \\ &\quad + 2^{p-1}a^{p(p-1)}|g|^p\alpha(v)(dx) \end{aligned} \tag{1.8}$$

(g) Properties (e) and (f) give a Leibniz inequality for α, that is: there exists a constant $C > 0$ such that

$$\alpha(uv) \leq C[|u|^p \alpha(v) + |v|^p \alpha(u)]$$

for every $u, v \in D_0 \cap L^\infty(X, m)$.

We assume that a distance d is defined on X, such that $\alpha(d) \leq m$ in the sense of the measures and

(i) The metric topology induced by d is equivalent to the original topology of X.

(ii) Denoting by $B(x, r)$ the ball of center x and radius r (for the distance d), for every fixed compact set K there exist positive constants c_0 and r_0 such that

$$m(B(x,r)) \leq c_0 m(B(x,s)) \left(\frac{r}{s}\right)^\nu \quad \forall x \in K \text{ and } 0 < s < r < r_0, \tag{1.9}$$

We assume without loss of generality $p < \nu$.

Remark 1.1. (a) Assume that

$$d(x, y) = \sup\{\varphi(x) - \varphi(y) : \varphi \in D \cap C_0(X), \alpha(\varphi) \leq m \text{ on } X\}$$

define a distance on X, which satisfies (i); then d is in $D_{loc}[X]$ and $\alpha(d) \leq m$; so we can use the above defined d as distance on X.

(b) We observe that from (i) and (ii) X has a structure of locally homogeneous space, [12]. Moreover the condition: for every fixed compact set K there exist positive constant c_1 and r_0 such that

$$0 < m(B(x, 2r)) \leq c_1 m(B(x, r)) \quad \forall x \in K \quad \text{and} \quad 0 < r < 2r_0$$

$c_1 < 1$, implies (ii) for a suitable ν.

(c) From the properties of d it follows that for any $x \in X$ there exists a function $\phi(.) = \phi(d(x, .))$ such that $\phi \in D_0[B(x, 2r)]$, $0 \leq \phi \leq 1$, $\phi = 1$ on $B(x, r)$ and

$$\alpha(\phi) \leq \frac{2}{r^p} m.$$

(d) From the assumptions on X and from (ii) the following property follows: for every fixed compact set K, such that the neighborhood of K of radius r_0 (for the distance d) is strictly contained in X, there exist a positive constant c_0', depending on c_0, such that $m(B(x,2r)-B(x,r)) \geq c_0' m(B(x,2r))$ for every $x \in K$ and $0 < r < \frac{r_0}{2}$.

We assume also that the following scaled *Poincaré inequality* holds: for every fixed compact set K there exist positive constants c_2, r_1 and $k \geq 1$ such that for every $x \in K$ and every $0 < r < r_1$

$$\int_{B(x,r)} |u - \overline{u}_{x,r}|^p m(dx) \leq c_2 r^p \int_{B(x,kr)} \mu(u,u)(dx) \tag{1.10}$$

for every $u \in D_{\mathrm{loc}}[B(x,kr)]$, where $\overline{u}_{x,r} = \frac{1}{m(B(x,r))} \int_{B(x,r)} u m(dx)$. A strongly local p-homogeneous Dirichlet form, such that the above assumptions hold, is called a *Riemannian Dirichlet form*. As proved in [17] the Poincaré inequality imply the following *Sobolev inequality* : for every fixed compact set K there exist positive constants c_3, r_2 and $k \geq 1$ such that for every $x \in K$ and every $0 < r < r_2$

$$\left(\frac{1}{m(B(x,r))} \int_{B(x,r)} |u|^{p^*} m(dx) \right)^{\frac{1}{p^*}} \leq \tag{1.11}$$

$$\leq c_3 \left(\frac{r^p}{m(B(x,r))} \int_{B(x,kr)} \mu(u,u)(dx) + \frac{r^p}{m(B(x,r))} \int_{B(x,r)} |u|^p m(dx) \right)^{\frac{1}{p}}$$

with $p^* = \frac{p\nu}{\nu - p}$ and c_3, r_2 depending only on c_0, c_2, r_0, r_1. We observe that we can assume without loss of generality $r_0 = r_1 = r_2$.

Remark 1.2. (a) From (1.10) we can easily deduce by standard methods that

$$\frac{1}{m(B(x,r))} \int_{B(x,r)} |u|^p m(dx) \leq c_2' \frac{r^p}{m(B(x,r) \cap \{u=0\})} \int_{B(x,kr)} \mu(u,u)(dx)$$

where c_2' is a positive constant depending only on c_2.

(b) From (a) it follows that for every fixed compact set K, such that the neighborhood of K of radius r_0 is strictly contained in X,

$$\int_{B(x,r)} |u|^p m(dx) \leq c_2^\star r^p \int_{B(x,r)} \mu(u,u)(dx)$$

for every $x \in K$, $u \in D_0[B(x,r)]$ and $0 < r < \frac{r_0}{2}$, where $c_2^\star$ depends only on c_2' and c_0.

(c) As a consequence of (d) Remark 1.1 and of the Poincaré inequality we have the following estimate on the capacity of a ball: for every fixed compact set K, such that the neighborhood of K of radius r_0 is strictly contained in X, there exists positive constants c_4 and c_5 such that

$$c_4 \frac{m(B(x,r))}{r^p} \leq cap(B(x,r), B(x,2r)) \leq c_5 \frac{m(B(x,r))}{r^p}$$

where $x \in K$ and $0 < 2r < r_0$.

Finally we give the definition of Kato space of measures, generalizing the definition given in [1] in the subelliptic framework:

Definition 1.1. Let σ be a Radon measure. We say that σ is in the Kato space $K(X)$ if

$$lim_{r \to 0} \eta_\sigma(r) = 0$$

where

$$\eta_\sigma(r) = \sup_{x \in X} \int_0^r \Big(\frac{|\sigma|(B(x,\rho))}{m(B(x,\rho))} \rho^p \Big)^{1/(p-1)} \frac{d\rho}{\rho}$$

Let $\Omega \subset X$ be an open set; $K(\Omega)$ is defined as the space of Radon measures σ on Ω such that the extension of σ by 0 out of Ω is in $K(X)$.

In section 3 we investigate the properties of the space $K(\Omega)$. In particular we prove that if Ω is a relatively compact open set of diameter $\frac{\bar{R}}{2}$, then

$$||\sigma||_{K(\Omega)} := \eta_\sigma(\bar{R})^{p-1}$$

is a norm on $K(\Omega)$ and, as in [3] for the bilinear case, we can prove that $K(\Omega)$ endowed with this norm is a Banach space. Moreover we prove that $K(\Omega)$ is contained in $D^{'}[\Omega]$, where $D^{'}[\Omega]$ denotes the dual of $D_0[\Omega]$.

1.2 Results.

We give now the result that we will prove in the following sections. Firstly we generalize to our case the pointwise estimate obtained in [2] in the subelliptic framework.

Let $\Omega \subset X$ be a relatively compact open set. We denote by c_0, c_2, r_0 the constants appearing in (1.9) (1.10) relative to the compact set $\bar{\Omega}$. We assume that a neighborhood of Ω of radius $\frac{\bar{R}}{2}+r_0$ is strictly contained in X ($\bar{R} = 2diam\Omega$), that $\int_X \mu(u, v)(dx)$ is a Riemannian (p-homogeneous) Dirichlet form and that $u \in D_{loc}(\Omega)$ with $\int_\Omega \mu(u, u)(dx) < +\infty$ is a subsolution of the problem

$$\int_\Omega \mu(u, v) = \int_\Omega v\sigma(dx) \text{ for every } v \in D_0[\Omega], \quad supp(v) \subset \Omega \tag{1.12}$$

where $\sigma \in D^{'}[\Omega]$, i.e.

$$\int_\Omega \mu(u, v) \leq \int_\Omega v\sigma(dx) \text{ for every positive } v \in D_0[\Omega], supp(v) \subset \Omega \tag{1.13}$$

Theorem 1.1. *Let $u \in D_{loc}[\Omega]$ with $\int_\Omega \mu(u, u)(dx) < +\infty$ be a bounded subsolution of (1.12). For every $x_0 \in \bar{\Omega}$ and $r \leq \frac{r_0}{2}$*

$$\begin{aligned} & p - fine - limsup_{x\to x_0} u(x) \leq \\ & \leq C\Big(\frac{1}{m(B(x_0, r))}\int_{B(x_0,r)\cap\Omega\cap\{u>0\}} u^\gamma m(dx)\Big)^{\frac{1}{\gamma}} + \\ & +C\int_0^r \Big(\frac{\sigma(B(x_0, \rho))}{m(B(x_0, \rho))}\rho^p\Big)^{\frac{1}{(p-1)}} \frac{d\rho}{\rho} + C(1 + ||u||_{L^\infty})\times \\ & \times \int_0^{2r} \Big(cap(B(x_0, \rho)\backslash\Omega, B(x_0, 2\rho))\frac{\rho^p}{m(B(x_0, \rho))}\Big)^{\frac{1}{(p-1)}} \frac{d\rho}{\rho}. \end{aligned} \tag{1.14}$$

Here $p-1 < \gamma < \frac{\nu(p-1)}{\nu+1-p}$. Moreover if $B(x_0, 2r) \subset \Omega$, then the third term in the right hand side of (1.14) disappears and the result holds again for unbounded subsolutions of (1.12).

We observe that we have $p - fine - limsup_{x\to x_0} u(x) = u(x_0)$ q.e. in Ω. Let now σ be in $K(\Omega)$; we are able in this case to generalize the result obtained in [1] in the subelliptic case to our framework:

Theorem 1.2. *Let u be a solution of (1.13); then u is continuous in Ω. Moreover if $\sigma(B(x, r)) \leq C\frac{m(B_(x, r))}{r^{p-\epsilon}}$, $\epsilon > 0$, for all small $r \leq min(r_0, d(x, \partial\Omega))$, then u is locally Hölder continuous in Ω.*

2. Proof of Theorem 1.1.

This Section is devoted to prove Theorem 1.1. It is founded on some preliminaries Lemmas which we state and prove with accuracy. The proof follow the lines of the proof given in the Euclidean framework in [15], [16]. We observe that in the proof we denote always by C different structural constants.

2.1 Preliminaries Lemmas.

Lemma 2.1. Let $l \in [0, +\infty)$ and let φ be a non-negative bounded Borel measurable function on $\mathbb{R}$ which vanishes on $(-\infty, l)$. Let λ be the L^1-norm of φ. Let $\omega \in D_0(\Omega)$, $0 \leq \omega \leq 1$. Then

$$(2.1) \quad \int_\Omega \varphi(u)\omega^p \alpha(u)(dx) \leq \lambda p \int_{\Omega \cap \{u>l\}} \omega^{p-1} |\mu(u,\omega)|(dx) + \lambda\sigma(\{\omega > 0\})$$

Proof.. Let $\Psi(t) : \int_0^t \varphi(s)ds$, $L := \Omega \cap \{u > l\}$. Using the test function $\xi = \Psi(u)\omega^p$ in (1.13), we obtain

$$\int_L \varphi(u)\omega^p \alpha(u)(dx) \leq p \int_L \Psi(u)\omega^{p-1}|\mu(u,\omega)|(dx) + \int_L \Psi(u)\omega^p \; \sigma(dx)$$

and then, as $\Psi \leq \lambda$, (2.1) follows.

Let now $B = B(x_0, r)$, $0 < r < r_o$, be an open ball in X. Let φ, $\psi \in D_{loc}[B] \cap C(X)$, $\eta \in D_0[B]$ such that $0 \leq \eta \leq 1$, $0 \leq \varphi, \psi \leq 1$, $\eta\psi \in D_0[B \cap \Omega]$, $(1-\varphi)(1-\psi) = 0$ and $\alpha(\eta) \leq \frac{C}{r^p} m$. Let $l \geq 0$. Moreover we denote $\omega = \psi\eta$ and $\omega_0 = \omega\varphi$

Lemma 2.2. *(1) If $\delta > 0$, then*

$$\int_L \alpha(w_\delta)(dx) \leq C r^{-p} \int_E \Big(1 + \frac{u-l}{\delta}\Big)^\gamma m(dx) +$$

$$+ \delta^{1-p} \Big[M^{p-1} \int_L \alpha(\omega_0)(dx) + \sigma(B) \Big]$$

where

$$w_\delta = \left(\Big(1 + \frac{(u-l)^+}{\delta}\Big)^{\gamma/q} - 1 \right) \omega$$

where q is defined by the relation $\dfrac{1}{\gamma} = \dfrac{1}{q} + \dfrac{1}{p(p-1)}$

(2) There exists a constant $\kappa > 0$, depending only on the structural constants, such that either

$$\Big(\frac{1}{m(B)}\int_L (u-l)^\gamma \omega^q \, m(dx)\Big)^{(p-1)/\gamma} \le$$

$$\le C\frac{r^p}{m(B)}\Big[(1+||u||_\infty)^{p-1}\int_B \alpha(\omega_0)(dx) + \sigma(B)\Big]$$

if

$$\int_B \alpha(\omega_0)(dx) \neq 0$$

or

$$\left(\frac{1}{m(B)}\int_L (u-l)^\gamma \omega^q m(dx)\right)^{(p-1)/\gamma} \le C\frac{r^p}{m(B)}[\sigma(B)]$$

(where u may be unbounded) otherwise, provided that

$$m(E) \le \kappa c_0^\star m(B) \tag{2.2}$$

and

$$\int_E (u-l)^\gamma dx \le 2^\gamma c_0^\star \int_L (u-l)^\gamma \omega^q \; m(dx) \tag{2.3}$$

where $L := B \cap \Omega \cap \{u > l\}$, $E := L \cap \{\varphi < 1\}$ and $c_0^\star = 2^\nu c_0^{-1}$.

Proof. (**1**) We shall suppose $\int_B \alpha(\omega_0)(dx) \neq 0$ and $||u||_\infty < +\infty$ as otherwise the proof would be easier. We write $v = \dfrac{(u-l)^+}{\delta}$, $M = 1+||u||_\infty$ and $F := L \cap \{\varphi = 1\}$. Let us observe that $w_\delta = ((1+v)^{\frac{\gamma}{q}} - 1)\omega$ and

$$((1+v)^{\frac{\gamma}{q}} - 1)^p \le C\, min\{v^{p-\tau}, v^p\} \le C min\{(1+v)^\gamma, v^{p-1}\}$$

$$\alpha(w_\delta) = \left(\frac{\gamma}{q}\right)^p (1+v)^{-\tau}\alpha(v)$$

where $(p-1)\tau = \gamma$, $\omega = \eta$ on E, $\omega = \sigma$ on F. Then

$$\int_L \alpha(w_\delta)(dx) \leq$$
$$\leq C\Big[\int_L ((1+v)^{\frac{\gamma}{q}}-1)^p\alpha(\omega)(dx)+\int_L \omega^p\alpha(((1+v)^{\frac{\gamma}{q}}-1))(dx)\Big]$$
$$\leq C\Big[\int_L (1+v)^{\frac{\gamma}{q}}-1)^p\alpha(\omega)(dx)+\int_L \omega^p(1+v)^{-\tau}\alpha(v)(dx)\Big] \tag{2.4}$$
$$\leq C\Big[\int_E (1+v)^{\gamma}\alpha(\eta)(dx)+M^{p-1}\delta^{1-p}\int_F \alpha(\omega)(dx)\Big]$$
$$+\delta^{-p}\int_E \omega^p(1+v)^{-\tau}\alpha(u)(dx)+\delta^{-p}\int_F \omega^p(1+v)^{-\tau}\alpha(u)(dx)$$

where we take into account that $\alpha(\omega) \leq C(\alpha(\omega-\omega_0)+\alpha(\omega_0)) = C\alpha((\omega-\omega_0)^+)+C\alpha(\omega_0)) = C(\mathbf{1}_{(\omega-\omega_0)>0}\alpha(\omega-\omega_0)+\alpha(\omega_0))$, then $\alpha(\omega)\leq\alpha(\omega_0)$ on F (we recall that $\mathbf{1}_T$ denotes the characteristic function of the set T).

We estimate the third term in the right hand side of (2.4). We define

$$\varphi(t)=\begin{cases}\left(1+\dfrac{(t-l)^+}{\delta}\right)^{-\tau} & \text{if } t\geq l\\ 0 & \text{if } t<l\end{cases}$$

and we apply the Lemma 2.1. The L^1-norm of φ is bounded by $(\tau-1)^{-1}\delta$. We obtain

$$\int_L \omega^p(1+v)^{-\tau}\alpha(u)(dx)\leq C\delta\Big[\int_L \omega^{p-1}|\mu(u,\omega)|(dx)+\sigma(B)\Big]$$

We consider the integral in the right hand side and we split the integration on the domains E and F

$$\int_E \omega^{p-1}|\mu(u,\omega)|(dx)\leq$$
$$\leq C\delta\Big[\frac{\epsilon}{\delta}\int_L \omega^p(1+v)^{-\tau}\alpha(u)(dx)+(\frac{\epsilon}{\delta})^{1-p}\int_E (1+v)^{\gamma}\alpha(\omega)(dx)\Big]$$

As $\omega=\eta$ on E (E is an open set), then

$$\int_E \omega^{p-1}|\mu(u,\omega)|(dx)\leq \tag{2.5}$$
$$\leq C\Big[\epsilon\int_L \omega^p(1+v)^{-\tau}\alpha(u)(dx)+\delta^p\epsilon^{1-p}\int_E(1+v)^{\gamma}\alpha(\eta)(dx)\Big]$$

We use now (2.1) with φ as the characteristic function of the interval $[l, M]$ and ω_0 instead of ω. We obtain

$$\int_L \omega_0^p \alpha(u)(dx) = CM\Big[\int_L \omega_0^{p-1}|\mu(u,\omega_0)|(dx) + \sigma(B)\Big]$$
$$\leq C\epsilon \int_L \omega_0^p \alpha(u)(dx) + C\epsilon^{1-p}\delta\Big[M^{p-1}\int_L \alpha(\omega_0)(dx) + \sigma(B)\Big].$$

If ϵ is fixed small enough then it follows

$$\int_L \omega_0^p \alpha(u)(dx) \leq C\delta\Big[M^{p-1}\int_L \alpha(\omega_0)(dx) + \sigma(B)\Big]$$

Then

$$\int_F \omega^p (1+v)^{-\tau}\alpha(u)(dx) \leq C\delta\Big[M^{p-1}\int_L \alpha(\omega_0)(dx) + \sigma(B)\Big] \tag{2.6}$$

where we take into account that $\omega = \omega_0$ on F. By the same methods we obtain also

$$\int_F \omega^{p-1}|\mu(u,\omega)|(dx) \leq C\delta\Big[M^{p-1}\int_L \alpha(\omega_0)(dx) + \sigma(B)\Big] \tag{2.7}$$

From (2.4),...,(2.7) we get

$$\int_L \alpha(w_\delta)(dx) \leq$$
$$\leq C\int_E (1+v)^\gamma \alpha(\eta)(dx) + C\delta^{1-p}\Big[M^{p-1}\int_L \alpha(\omega_0)(dx) + \sigma(B)\Big]. \tag{2.8}$$

Since $\alpha(\eta) \leq Cr^{-p}m$, then

$$\int_L \alpha(w_\delta)(dx) \leq$$
$$\leq Cr^{-p}\int_E (1+v)^\gamma\, m(dx) + C\delta^{1-p}\Big[M^{p-1}\int_L \alpha(\omega_0)(dx) + \sigma(B)\Big]. \tag{2.9}$$

(**2**) We will use the same notations of the part (1).

Let

$$\delta := \Big(\frac{1}{km(B)}\int_L (u-l)^\gamma \omega^q m(dx)\Big)^{\frac{1}{\gamma}}$$

where $k > 0$ is a constant whose choice will be specified later. Let us observe that $k = m(B)^{-1}\int_L v^\gamma \omega^q m(dx)$. Then, by (2.2) we obtain

$$
\begin{aligned}
kc_0^\star m(B) = 2c_0^\star \int_L v^\gamma \omega^q dm &\le \int_L \omega^q\, m(dx) + +2c_0 \int_{L\cap\{2c_0^\star v^\gamma \ge 1\}} v^\gamma \omega^q m(dx) \\
&\le \frac{1}{2}\Big(|E| + \int_F \omega_0^q m(dx)\Big) + \int_L v^\gamma \omega^q m(dx) \\
&\le \frac{1}{2}k\, c_0^\star m(B) + \frac{1}{2}\int_F \omega_0^q m(dx) + \int_L v^\gamma \omega^q m(dx)
\end{aligned}
$$

Then

$$
\begin{aligned}
km(B) &\le C\Big[\int_L v^\gamma \omega^q m(dx) + \int_B \omega_0^q m(dx)\Big] \\
&\le C\Big[\int_L w_\delta^q m(dx) + \int_B \omega_0^q m(dx)\Big]
\end{aligned}
$$

Using the Sobolev inequality [17] we obtain

$$
\begin{aligned}
k^{p/q} &\le Cm(B)^{-p/q}\Big[\int_{B\cap\Omega} w_\delta^q m(dx) + \int_B \omega_0^q m(dx)\Big]^{p/q} \\
&\le C\frac{r^p}{m(B)}\Big[\int_{B\cap\Omega} \alpha(w_\delta)(dx) + \int_B \alpha(\omega_0)(dx)\Big]
\end{aligned}
\tag{2.10}
$$

By (2.9) and (2.10) we obtain

$$
\begin{aligned}
k^{\frac{p}{q}}\frac{m(B)}{r^p} &\le C\Big[\int_L \alpha(w_\delta)(dx) + \int_B \alpha(\omega_0)(dx)\Big] \\
&\le Cr^{-p}\int_E (1+v)^\gamma m(dx) \\
&\quad + C\delta^{1-p}\Big[M^{p-1}\int_B \alpha(\omega_0)(dx) + \sigma(B)\Big]
\end{aligned}
\tag{2.11}
$$

The assumptions (2.2) and (2.3) imply

$$
\begin{aligned}
\int_E (1+v)^\gamma m(dx) &\le C\Big[m(E) + \int_E v^\gamma m(dx)\Big] \\
&\le C\Big[m(E) + \int_L v^\gamma \omega^q m(dx)\Big] \le Ckm(B)
\end{aligned}
\tag{2.12}
$$

From (2.11) and (2.12) we obtain

$$
k^{\frac{p}{q}} \le C^* k + C\delta^{1-p} m(B)^{-1} r^p\Big[(M+\delta)^{p-1}\int_B \alpha(\omega_0)(dx) + \sigma(B)\Big]
$$

for some structural constant C^*. If $k < 1$ is so small that $k^{p/q} - C^*k > 0$, then we obtain

$$\Big[\frac{1}{km(B)}\int_L (u-l)^\gamma \psi^q \eta^q m(dx)\Big]^{(p-1)/\gamma} \leq$$

$$\leq \delta^{p-1} \leq C\frac{r^p}{m(B)}\Big[(M+\delta)^{p-1}\int_B \alpha(\omega_0)(dx) + \sigma(B)\Big]$$

So the proof of (2) is completed, since $\delta \leq CM$.

2.2 Proof of Theorem 1.2.

This proof follows the lines of the proof relative to the Euclidean case in [15], [16]. We can suppose $M = 1 + ||u||_\infty < +\infty$ without lost of generality otherwise the proof will be simpler. Let $B = B(x_0, r)$ and for any integer $j \geq 0$ let $r_j = 2^{-j}r$ and $B_j = B(x_0, r_j)$. Let $\eta_j \in D_0[B_j]$, $\eta_j = 1$ on B_{j+1} and $\alpha(\eta_j) \leq \dfrac{C}{r_j^p}m$. Let $g_j \in D_0[B_{j-1}] \cap C(X)$ such that $0 \leq g_j \leq 1$, $g_j = 1$ on $B_j \backslash \Omega$ and

$$\frac{1}{r_{j-1}^p}\int_X g_j^p m(dx) \leq c_2^\star \int_X \alpha(g_j)(dx) \leq Ccap(B_j\backslash\Omega, B_{j-1})$$

We denote

$$\psi_j = min(1, (2-3g_j)^+)$$

$$\varphi_j = min(1, 3g_j + 3g_{j-1}) \quad j \geq 1$$

and

$$L_j = B_j \cap \Omega \cap \{u \geq l_j\}, \quad E_j = L_j \cap \{\varphi_j < 1\}, \quad F_j = L_j \cap \{\varphi_j = 1\}.$$

Then

$$\int_{B_j} \alpha(\phi_j)(dx) \leq C(cap(B_{j-1}\backslash\Omega, B_{j-2}) + cap(B_j\backslash\Omega, B_{j-2})) \tag{2.13}$$

$$\int_{B_j} \alpha(\psi_j)(dx) \leq Ccap(B_j\backslash\Omega, B_{j-1})$$

We define recursively $l_0 = 0$

$$l_{j+1} = l_j + \Big(\frac{1}{km(B_j)}\int_{L_j}(u-l_j)^\gamma \psi_j^q \eta_j^q dm\Big)^{\frac{1}{\gamma}}$$

Let

$$\delta_j = l_{j+1} - l_j$$

For $j \geq 1$ we prove that either

$$(2.14)\quad \delta_j \leq \frac{1}{2}\delta_{j-1}+C\Big[\Big(\frac{\sigma(B_j)}{m(B_j)}r_j^p\Big)^{\frac{1}{p-1}}+M\Big(\frac{(cap(B_{j-1}\backslash\Omega, B_{j-2})}{m(B_j)}r_j^p\Big)^{1p-1}\Big]$$

if

$$\int_0^{2r}\Big(\frac{cap_p(B(x_0,\rho)\backslash\Omega, B(x_0,2\rho)}{m(B(x_0,\rho))}\rho^p\Big)^{\frac{1}{p-1}}\frac{d\rho}{\rho} \neq 0$$

and

$$\delta_j \leq \frac{1}{2}\delta_{j-1} + C\Big(\frac{\sigma(B_j)}{m(B_j)}r_j^p\Big)^{1/(p-1)}$$

otherwise. As the second case is easier we prove the first.

The proof is trivial when $\delta_j \leq \frac{1}{2}\delta_{j-1}$, so assume $\delta_{j-1} \leq 2\delta_j$. In this case, since $\psi_{j-1}\eta_{j-1} = 1$ on E_j, we have

$$m(E_j)\delta_{j-1}^{\gamma} \leq \int_{E_j}(l_j - l_{j-1})^{\gamma}\psi_{j-1}^q\eta_{j-1}^q m(dx) \leq$$

$$\leq \int_{E_j}(u - l_{j-1})^{\gamma}\psi_{j-1}^q\eta_{j-1}^q m(dx) = km(B_{j-1}) \leq kc_0^{\star}m(B_j)$$

and

$$\int_{E_j}(u - l_j)^{\gamma}m(dx) \leq \int_{L_{j-1}}(u - l_{j-1})^{\gamma}\psi_{j-1}^q\eta_{j-1}^q m(dx) =$$

$$= \delta_{j-1}^{\gamma}km(B_{j-1}) \leq 2^{\gamma}c_0^{\star}km(B_j)\delta_j^{\gamma} =$$

$$= 2^{\gamma}c_0^{\star}\int_{L_j}(u - l_j)^{\gamma}\psi_j^q\eta_j^q m(dx)$$

Thus (2.2) and (2.3) are verified and in virtue of Lemma 2.3 we obtain

$$\delta_j \leq C\Big[\Big(\frac{\sigma(B_j)}{m(B_j)}r_j^p\Big)^{\frac{1}{p-1}} + M\Big(\frac{cap(B_{j-1}\backslash\Omega, B_{j-2})}{m(B_j)}r_j^p\Big)^{\frac{1}{p-1}}\Big]$$

which proves (2.14). A summing routine machinery on (2.14), [15] [16]

[8] gives now

$$\lim_j l_j \leq C\Big[\Big(\frac{1}{m(B(x_0,r)}\int_{B(x_0,r)\cap\Omega\cap\{u>0\}} u^\gamma m(dx)\Big)^{\frac{1}{\gamma}}\Big]$$

$$+\,C\int_0^r \Big(\frac{\sigma(B(x_0,\rho)}{m(B(x_0,\rho))}\rho^p\Big)^{\frac{1}{p-1}}\frac{d\rho}{\rho} \tag{2.15}$$

$$+\,CM\int_0^{2r}\Big(\frac{cap(B(x_0,\rho)\backslash\Omega,\,B(x_0,2\rho))}{m(B(x_0,\rho)}\rho^p\Big)^{\frac{1}{p-1}}\frac{d\rho}{\rho}$$

so it remains to prove that p-fine-lim $\sup_{x\to x_0} u(x) \leq C\lim_j l_j$.

We denote $l = lim_j\, l_j$. For any $\epsilon > 0$ denote

$$w_j = (2^{\frac{\gamma}{q}}-1)^{-1}\Big[\Big(1+\frac{(u-l-\epsilon)^+}{\epsilon}\Big)^{\frac{\gamma}{q}}-1\Big]\psi_j\eta_j$$

on Ω and $w_j = 0$ elsewhere. Then $w_j \in D_0[B_j]$, $w_j \geq 1$ on $L^\epsilon_{j+1} := B_{j+1}\cap\Omega\cap\{u > l+2\epsilon\}$. Denote $E^\epsilon_j = L^\epsilon_j \cap \{\varphi_j < 1\}$. Using Lemma 2.2 we obtain

$$\text{cap}\,(L^\epsilon_{j+1}, B_j) \leq C\int_{B_j}\alpha(w_j)(dx) \leq Cr_j^{-p}\int_{E^\epsilon_j}\Big(1+\frac{u-l-\epsilon}{\epsilon}\Big)^\gamma m(dx)$$

$$+\,C\epsilon^{1-p}\Big[\sigma(L_j) + M^{p-1}\int_{B_j}(\alpha(\varphi_j)+\alpha(\psi_j)\Big)(dx)\Big]$$

It follows

$$\sum_{j=1}^\infty\Big(\frac{cap(L_{j+1},B_j)}{m(B_{j+1})}r_j^p\Big)^{1/(p-1)} \tag{2.16}$$

$$\leq C\sum_{j=1}^\infty\Big(\frac{1}{m(B_j)}\int_{E^\epsilon_j}\Big(1+\frac{u-l-\epsilon}{\epsilon}\Big)^\gamma m(dx)\Big)^{1/(p-1)} +$$

$$+C\epsilon^{-1}\Big[\sum_j\Big(\frac{\sigma(B_j)}{m(B_j)}r_j^p\Big)^{1/(p-1)}\Big]+$$

$$MC\epsilon^{-1}\Big[\sum_{j=1}^\infty\Big(\frac{cap_p(B_{j-1}\backslash\Omega,B_{j-2})}{m(B_j)}r_j^p + \frac{cap_p(B_j\backslash\Omega,B_{j-1})}{m(B_j)}r_j^p\Big)^{1/(p-1)}\Big]$$

We take into account the following estimate

$$\sum_{j=1}^{\infty}\Big(\frac{1}{m(B_j)}\int_{E'_j}(1+\frac{(u-l-\epsilon)}{\epsilon})^{\gamma}m(dx)\Big)^{\frac{1}{p-1}} \leq$$

$$\leq C\sum_{j=1}^{\infty}\Big(\frac{1}{m(B_j)}\int_{E'_j}\epsilon^{-\gamma}(u-l_{j-1})^{\gamma}m(dx)\Big)^{\frac{1}{p-1}} \leq$$

$$C\sum_{j=1}^{\infty}\Big(\frac{1}{m(B_j)}\int_{L_{j-1}}\epsilon^{-\gamma}(u-l_{j-1})^{\gamma}\eta_{j-1}\psi_{j-1}\ m(dx)\Big)^{\frac{1}{p-1}} \leq$$

$$\leq C\sum_{j=1}^{\infty}(k\epsilon^{-\gamma}\delta_{j-1}^{\gamma})^{\frac{1}{p-1}} < +\infty$$

As the remaining part in the right hand side of (2.16) is finite, then the set $\Omega \cap \{u > l + 2\epsilon\}$ is p-thin at x_0 for any $\epsilon > 0$, so we have p-fine-lim $sup_{x\to x_0}u(x) \leq C \lim_j\ l_j$.

3. Kato Classes of Measures.

Given $\sigma \in K(\Omega)$ we denote again by σ the extension by 0 of σ to Ω. In this section we will prove for sake of completeness some properties of the measures in the Kato class $K(\Omega)$ (relative to the form under consideration):

Proposition 3.1. *Let $\sigma \in K(\Omega)$. Then $|\sigma|(\Omega) < +\infty$.*

Proof. Let $\bar{R} = 2\ diam\,\Omega$. We assume $\bar{R} \leq r_0$. By the definition of $K(\Omega)$, there exists $r^\star > 0$ such that

$$C_0\Big(\frac{|\sigma|(B(x,r^\star))}{m(B(x,r^\star))}(r^\star)^p\Big)^{\frac{1}{(p-1)}} \leq \eta_\sigma(r^\star) \leq 1$$

for every $x \in X$, where C_0 is a structural constant. Let x_1, x_2,..., x_n be such that $\bar{\Omega} \subseteq \bigcup_{i=1}^{n} B(x_i, r^\star)$. Due to the homogeneous structure of X we have that n can be chosen less than $C_1\left(\frac{\bar{R}}{r^\star}\right)^{\nu}$, where C_1 is a structural constant. Then

$$|\sigma|(\Omega) \leq \sum_{i=1}^{n} |\sigma|(B(x_i, r^\star)) \leq C_0^{-1} \frac{n}{(r^\star)^p} m(B_{\bar{R}})$$
$$\leq C_0^{-1} C_1 \left(\frac{\bar{R}}{r^\star}\right)^\nu \frac{1}{(r^\star)^p} m(B_{\bar{R}})$$

The result in the general case follows by a covering argument.

Proposition 3.2. Let $\sigma \in K(\Omega)$. Then $\eta_\sigma(\bar{R}) < +\infty$, where $\bar{R} = 2diam(\Omega)$.

Proof. By the definition of $K(\Omega)$, there exists $r^\star > 0$ such that $\eta_\sigma(r^\star) \leq 1$ for every x. Then we have

$$\eta_\sigma(\bar{R}) \leq \eta_\sigma(r^\star) + sup_{x\in X} \int_{r^\star}^{\bar{R}} \left(\frac{|\sigma|(B(x, \rho))}{m(B(x, \rho))} \rho^p\right)^{1/(p-1)} \frac{d\rho}{\rho} \leq$$
$$\leq 1 + C(r^\star)[|\sigma|(\Omega)]^{1/(p-1)}$$

In virtue of Proposition 3.2, the definition $||\sigma||_{K(\Omega)} := \eta_\sigma(\bar{R})^{(p-1)}$ is well posed and it is easy to verify that $|| \cdot ||_{K(\Omega)}$ is a norm in $K(\Omega)$.

Proposition 3.3. *The space* $K(\Omega)$ *is a Banach space for the norm* $||.||_{K(\Omega)}$.

Proof. Let σ_k be a Cauchy sequence in $K(\Omega)$. For every fixed $\epsilon > 0$ there exists k_ϵ such that for $h, k \geq k_\epsilon$

$$\eta_{\sigma_h - \sigma_k}(\bar{R}) \leq \epsilon$$

We have

$$(|\sigma_h - \sigma_k|(\Omega))^{\frac{1}{(p-1)}} \leq C \left(\frac{m(B(x_0, \bar{R}))}{\bar{R}^p}\right)^{\frac{1}{(p-1)}} \eta_{\sigma_h - \sigma_k}(\bar{R}) \leq$$
$$\leq C \left(\frac{m(B(x_0, \bar{R}))}{\bar{R}^p}\right)^{\frac{1}{(p-1)}} \epsilon$$

where x_0 is a fixed point in Ω and C a structural constant. Then σ_k is a Cauchy sequence in the Radon measures. Since the space of the bounded Radon measures is complete, the sequence σ_k converges to a bounded Radon measure σ, which is zero out of Ω.

We prove that $\sigma \in K(\Omega)$. For every x and $r > 0$ we have

$$\int_r^{\bar{R}} \left(\frac{|\sigma|(B(x,\rho))}{m(B(x,\rho))}\rho^p\right)^{1/(p-1)} \frac{d\rho}{\rho} \leq \lim_{k\to 0} \int_0^{\bar{R}} \left(\frac{|\sigma_k|(B(x,\rho))}{m(B(x,\rho))}\rho^p\right)^{1/(p-1)} \frac{d\rho}{\rho}$$
$$= \lim_{k\to 0} \eta_{\sigma_k}(\bar{R})$$

then

$$\int_0^{\bar{R}} \left(\frac{|\sigma|(B(x,\rho))}{m(B(x,\rho))}\rho^p\right)^{1/(p-1)} \frac{d\rho}{\rho} \leq \lim_{k\to 0} \eta_{\sigma_k}(\bar{R})$$

so $\eta_\sigma(\bar{R}) \leq \lim_{k\to 0} \eta_{\sigma_k}(\bar{R})$ is finite. By the same methods we can prove that

$$\eta_\sigma(r) \leq \lim_{k\to 0} \eta_{\sigma_k}(r)$$

for every $r > 0$. For every $\epsilon > 0$ there exists k_ϵ such that

$$\eta_\sigma(r) \leq \eta_{\sigma_{k_\epsilon}}(r) + \epsilon$$

Since $\sigma_{k_\epsilon} \in K(\Omega)$, there exists r_ϵ such that

$$\eta_{\sigma_{k_\epsilon}}(r_\epsilon) \leq \epsilon$$

Then for $0 < r < r_\epsilon$

$$\eta_\sigma(r) \leq 2\epsilon$$

so

$$\lim_{r\to 0} \eta_\sigma(r) = 0$$

and $\sigma \in K(\Omega)$.

We prove now that σ_k converges to σ in $K(\Omega)$. By the same methods used above we obtain that

$$\int_r^{\bar{R}} \left(\frac{|\sigma_k - \sigma|(B(x,\rho))}{m(B(x,\rho))}\rho^p\right)^{1/(p-1)} \frac{d\rho}{\rho} \leq$$
$$\leq \left[\frac{1}{p-1}\frac{\bar{R}^p}{m(B(x,r))}|\sigma_k - \sigma|(B(x,\bar{R}))\right]^{\frac{1}{p-1}}$$

We recall that σ_k converges to σ in the Radon measures and that σ_k is a Cauchy sequence in $K(\Omega)$. Then for every $\epsilon > 0$ there exists k_ϵ such that for $k \geq k_\epsilon$

$$|\sigma_k - \sigma|(X) \leq \epsilon$$

and for $h, k \geq k_\epsilon$

$$\eta_{\sigma_k - \sigma_h}(\bar{R}) \leq \epsilon$$

We have for $k \geq k_\epsilon$

$$\eta_{\sigma_k - \sigma}(\bar{R}) \leq \left[\frac{1}{p-1} \sup_{x \in B(x_0, 2\bar{R})} \frac{\bar{R}^p}{m(B(x,r))} \epsilon \right]^{\frac{1}{p-1}} + \eta_\sigma(r) + \eta_{\sigma_k}(r)$$

Since $\inf_{x \in B(x_0, 2\bar{R})} m(B(x,r)) \geq \delta(r) > 0$ the above relation implies that

$$\lim_{k \to \infty} \sigma_k = \sigma$$

in $K(\Omega)$. Then $K(\Omega)$ is a Banach space.

In the next Proposition we prove that $K(\Omega)$ is contained in $D^{'}[\Omega]$, where $D^{'}[\Omega]$ denotes the dual of $D_0[\Omega]$. Moreover we are also able to estimate $||\sigma||_{D'[\Omega]}$ in term of the norm $||\sigma||_{K(\Omega)}$ as described in the following Theorem.

Theorem 3.1. *Let* $\sigma \in K(\Omega)$. *Then* $\sigma \in D^{'}[\Omega]$ *and*

$$||\sigma||_{D'[\Omega]} \leq C[|\sigma|(\Omega)\eta_\sigma(\bar{R})]^{(p-1)/p} \tag{3.1}$$

Proof. First step. We prove (3.1) in a fixed ball $B = B(x_0, r)$, $r \leq \dfrac{\bar{r_0}}{2}$ supposing $\sigma \in D^{'}[B]$. Let $w \in D_0[B]$ be the solution of the problem

$$\int_B \mu(w, v)(dx) = \int_B v\sigma(dx) \tag{3.2}$$

for every $v \in D_0[B]$. By Theorem 1 applied to $w^\pm$ we have

$$\sup_B |w| \leq C\left[\left(\frac{1}{m(B)} \int_B |w|^p \, m(dx)\right)^{\frac{1}{p}} + \eta_\sigma(2r)\right] \tag{3.3}$$

By (3.3) and the Poincaré inequality we obtain

$$\sup_B |w| \leq C\left[\left(\frac{r^p}{m(B)} \int_B \alpha(w)(dx)\right)^{1/p} + \eta_\sigma(2r)\right] \tag{3.4}$$

By (3.2) and (3.4) we have

$$\int_B \alpha(w)(dx) \leq C|\sigma|(B)\left(\frac{r^p}{m(B)} \int_B \alpha(w)(dx)\right)^{\frac{1}{p}} + C|\sigma|(B)\eta_\sigma(2r) \tag{3.5}$$

Applying Young's inequality to the first term in the right hand-side of (3.4) gives

$$\int_B \alpha(w)(dx) \leq C|\sigma|(B)\eta_\sigma(2r) \tag{3.6}$$

From (3.2) we obtain

$$||\sigma||_{D'[B]} \leq \Big(\int_B \alpha(w)(dx)\Big)^{\frac{(p-1)}{p}}$$

so from (3.5) we conclude the proof of the first step.

Second step. We can assume without loss of generality σ positive. Fix a small number $s > 0$. We consider a finite covering of Ω by balls $B_i = B_{x_i,s}$ with $s/2 \leq d(x_i, x_j) \leq s \leq \frac{r}{8}$ and such that every point of Ω is covered by at most M balls, where M is independent on s. Define

$$\sigma_s = \sum_i \frac{\sigma(B_i)}{m(B_i)} \mathbf{1}_{B_i} m$$

where $\mathbf{1}_{B_i}$ denotes the characteristic function of B_i. For any arbitrary $x \in \Omega$ and $\rho > 0$ we have:

(a) If $\rho \geq s$, then

$$\sigma_s(B(x, \rho)) \leq M\sigma(B(x, 2\rho))$$

(b) If $\rho < s$, then

$$\sigma_s(B(x, \rho)) \leq C\frac{\sigma(B(x, 4s))}{m(B(x, 4s))} m(B(x, \rho))$$

for a structural constant C.

In fact if $\rho \geq s$, then $\sigma_s(B(x, \rho)) \leq \sum_{B_i \cap B(x,\rho) \neq \emptyset} \sigma(B_i) \leq M\sigma(B(x, 2\rho))$.

If $\rho < s$ then for any i such that $B_i \cap B(x, \rho) \neq \emptyset$ we have $B(x_i, s) \subset B(x, 4s) \subset B(x_i, 8s)$, so by the duplication property

$$m(B(x_i, s)) \geq Cm(B(x_i, 8s)) \geq Cm(B(x, 4s)).$$

Then $\sigma_s(B(x, \rho)) \leq \sum_{B_i \cap B(x,\rho) \neq \emptyset} \frac{\sigma(B(x, 4s))}{m(B(x, 4s))} m(B(x, \rho))$

It follows that for any arbitrary $x \in \Omega$ we have

$$\int_0^r \left(\frac{\sigma_s(B(x,\rho))}{m(B(x,\rho))}\rho^p\right)^{\frac{1}{(p-1)}} \frac{d\rho}{\rho} \leq C \int_0^{2r} \left(\frac{\sigma(B(x,\rho))}{m(B(x,\rho))}\rho^p\right)^{\frac{1}{(p-1)}} \frac{d\rho}{\rho}$$

After extraction of a subsequence we have that σ_s converges weakly* to a measure $\chi \geq \sigma$. Then we have

$$||\sigma||_{D'[B]} \leq C||\chi||_{D'[B]} \leq C[\sigma(\Omega)\eta_\sigma(2r)]^{\frac{p-1}{p}}$$

and the proof is concluded.

The proof in the general case follows by a covering argument.

4. Proof of Theorem 1.2.

Lemma 4.1. *Let $u \in L^\infty(B(x,r),m)$, where $B(x,2r) \subset \Omega$, $2r \leq r_0$. Assume that there exist positive constants C, K and L such that for each s and t with $\frac{1}{2} \leq s < t \leq 1$*

$$\sup_{B(x,sr)} |u| \leq \frac{C}{(t-s)^L}\left(\frac{1}{m(B(x,tr))}\int_{B(x,tr)} |u|^d m(dx)\right)^{\frac{1}{d}} + K$$

for a certain $d > 0$. Then for every fixed $q > 0$ we have

$$\sup_{B(x,\frac{1}{2}r)} |u| \leq C_q\left[\left(\frac{1}{m(B(x,r))}\int_{B(x,r)} |u|^q m(dx)\right)^{\frac{1}{q}} + K\right]$$

where C_q is a structural constant depending on q,C and L.

Proof. If $q \geq d$ the result follows directly from the assumptions. Let us assume $q < d$. If

$$\left(\frac{1}{m\left(B\left(x,\frac{2}{3}r\right)\right)}\int_{B\left(x,\frac{2}{3}r\right)} |u|^d m(dx)\right)^{\frac{1}{d}} \leq K$$

we have

$$\sup_{B(x,\frac{1}{2}r)} |u| \leq (6^L C + 1)K$$

and the result follows.

If

$$\Big(\frac{1}{m(B(x,\frac{2}{3}r))}\int_{B(x,\frac{2}{3}r)}|u|^d m(dx)\Big)^{\frac{1}{d}} \geq K$$

we have that for each s and t with $\dfrac{2}{3} \leq s < t \leq 1$

$$\sup_{B(x,sr)}|u| \leq \frac{C_1}{(t-s)^L}\Big(\frac{1}{m(B(x,tr))}\int_{B(x,tr)}|u|^d m(dx)\Big)^{\frac{1}{d}} \tag{4.1}$$

where C_1 is a constant depending on C and c_0 (the constant in the duplication inequality). From (4.1) the result follows by the same proof as in Lemma 5.2 [5].

We prove now a Harnack type inequality, which generalizes the one given in [11] in the case $\sigma = 0$; we think that this inequality can have an interest in itself.

Proposition 4.2. *Let u be a positive solution of (1.12) and $B(x,r) \subset B(x,(4k+12)r) \subset B(x,R) \subset \Omega$, $2r \leq r_0$. Then*

$$\sup_{B(x,r)} u \leq C_1 \inf{}_{B(x,r)} u + C_2\bigg(\Big(\frac{1}{m(B(x,R))}\int_{B(x,R)} u^p + \eta(\bar{R})\Big)^2 + 1\bigg)\eta_\sigma(2r)$$

Proof. In the proof we indicate by C possibly different structural constants. Let u be a positive solution of (1.12). We apply Theorem 1.1 with $\gamma = p$, taking into account that u is a subsolution of (1.12) with σ replaced by $|\sigma|$; we obtain

$$\sup_{B(x,r)} u \leq C\Big(\frac{1}{m(B(x,2r))}\int_{B(x,2r)} u^p m(dx)\Big)^{\frac{1}{p}} + C\eta_\sigma(2r) \tag{4.2}$$

where $B(x,4r) \subset \Omega$. We consider now a fixed ball $B(x,r)$ such that $B(x,4r) \subset \Omega$ and $\dfrac{1}{2} \leq s < t \leq 1$. We consider a finite covering of $B(x,sr)$ by balls

$$B(x_i, \frac{(t-s)}{2} r) = B_i$$

$x_i \in B\left(x, \dfrac{(t+s)}{2} r\right)$. We apply (4.1) to every ball B_i and we obtain

$$\sup_{B_i} u \leq C\Big(\frac{1}{m(2B_i)}\int_{2B_i} u^p m(dx)\Big)^{\frac{1}{p}} + C\eta_\sigma(2r)$$

where $2B_i = B(x_i, (t-s)r)$

There exists $\bar{x}$ in the ball $B_{\bar{i}}$ such that $sup_{B(x,sr)} u - \eta_\sigma(2r) \le u(\bar{x})$. Then

$$\sup_{B(x,sr)} u \le C\Big(\frac{1}{m(2B_{\bar{i}})}\int_{2B_{\bar{i}}} u^p\, m(dx)\Big)^{\frac{1}{p}} + C\eta_\sigma(2r) \le$$

$$\le C\left(\frac{t}{(t-s)}\right)^{\frac{\nu}{p}}\Big(\frac{1}{m(B(x,tr))}\int_{B(x,tr)} u^p m(dx)\Big)^{\frac{1}{p}} + C\eta_\sigma(2r)$$

Then by Lemma 4.1 for every $q > 0$ there exists a structural constant C_q depending on q such that

$$\sup_{B(x,\frac{1}{2}r)} u \le C_q\Big[\Big(\frac{1}{m(B(x,r))}\int_{B(x,r)} u^q m(dx)\Big)^{\frac{1}{q}} + \eta_\sigma(2r)\Big] \tag{4.3}$$

where C_q is a structural constant depending on q. Assume that $inf_{B(x,\frac{\bar{R}}{2})} u \ge 1$ then $\dfrac{1}{u}$ is a a subsolution of (1.12) with σ replaced by $|\sigma|$ in $B\left(x, \dfrac{\bar{R}}{2}\right)$. Then we have again

$$\sup_{B(x,\frac{1}{2}r)} \frac{1}{u} \le C_q\Big[\Big(\frac{1}{m(B(x,r))}\int_{B(x,r)} u^{-q} m(dx)\Big)^{\frac{1}{q}} + \eta_\sigma(2r)\Big] \tag{4.4}$$

Moreover for any $v \in D_0[B(x, \dfrac{R}{2})]$ bounded we have

$$\int\limits_{B(x,\frac{R}{2})} v^p \mu(logu, logu)(dx) = \int\limits_{B(x,\frac{R}{2})} v^p(\frac{1}{u})^p \mu(u,u)(dx) =$$

$$= \frac{1}{(1-p)}\int\limits_{B(x,\frac{R}{2})}\mu\left(u, \left(\frac{1}{u}\right)^{(p-1)} v^p\right)(dx) - \frac{p}{(1-p)}\int\limits_{B(x,\frac{R}{2})}\Big(\frac{v}{u}\Big)^{(p-1)}\mu(u,v)(dx) =$$

$$= \frac{1}{(1-p)}\int\limits_{B(x,\frac{R}{2})}\left(\frac{1}{u}\right)^{p-1} v^p\sigma(dx) - \frac{p}{(1-p)}\int\limits_{B(x,\frac{R}{2})}\left(\frac{v}{u}\right)^{(p-1)}\mu(u,v)(dx)$$

From the above relation,taking into account that $inf_{B(x,\frac{\bar{R}}{2})} u \ge 1$ and choosing v as a cut-off function between balls we obtain that u is a locally bounded mean variation function in $B(x, \frac{R}{2})$, i.e.

$$\frac{1}{m(B(y,s))}\int_{B(y,s)} |u-\bar{u}_{y,s}|m(dx) \le C(1+||\sigma||_{K(\Omega)})$$

$B(y,2s) \subset B\left(x, \dfrac{R}{2k}\right)$, $s \le r_0$, where C is a structural constant. As in [5] there exists a suitable fixed $q \in (0,1)$ such that

$$(\frac{1}{m(B(x,r))}\int_{B(x,r)} u^q m(dx))(\frac{1}{m(B(x,r))}\int_{B(x,r)} u^{-q} m(dx)) \le C \tag{4.5}$$

From (4.3) (4.4) (4.5) we obtain

$$\sup_{B(x,\frac{1}{2}r)} u \le C \ \inf{}_{B(x,\frac{1}{2}r)} u +$$

$$+C[(\inf{}_{B(x,\frac{1}{2}r)} u)(\frac{1}{m(B(x,r))}\int_{B(x,r)} u^q m(dx))^{\frac{1}{q}} + 1]\eta_\sigma(2r)$$

To remove the assumption $\inf{}_{B(x,\frac{\bar{R}}{2})} u \ge 1$ we apply the above inequality to $(u+1)$ and we obtain

$$\sup_{B(x,\frac{1}{2}r)} u \le C \inf{}_{B(x,\frac{1}{2}r)} u +$$

$$+C\{(\inf{}_{B(x,\frac{1}{2}r)} u + 1)[(\frac{1}{m(B(x,r))}\int_{B(x,r)} u^q m(dx))^{\frac{1}{q}} + 1] + 1\}\eta_\sigma(2r)$$

Using now Theorem 1.1 we obtain

$$\sup_{B(x,\frac{1}{2}r)} u \le C \inf{}_{B(x,\frac{1}{2}r)} u + C[(\sup_{B(x,\frac{\bar{R}}{2})} u + 1)^2 + 1]\eta_\sigma(2r) \le$$

$$\le C \inf{}_{B(x,\frac{1}{2}r)} u + C\left(\left(\frac{1}{m(B(x,R))}\int_{B(x,R)} u^p + \eta(\bar{R}) + 1\right)^2 + 1\right)\eta_\sigma(2r)$$

We are now in condition to conclude the proof of Theorem 1.2. We apply the result in Proposition 4.2 to $M_2 - u$ and to $u - m_2$ where $M_2 = sup_{B(x,2r)} u$ ($m_2 = inf_{B(x,2r)} u$) and we obtain

$$osc_{B(x,r)} u \le \theta\, osc_{B(x,2r)} u$$
$$+ 2C_2\left[\left(\frac{1}{m(B(x,R))}\int\limits_{B(x,R)} u^p + \eta(\bar{R}) + 1\right)^2 + 1\right]\eta_\sigma(2r)$$

where $0 < \theta < 1$ is a structural constant. By standard methods (see Lemma 8.23 [14]) we obtain

$$
\begin{aligned}
(4.6)\qquad osc_{B(x,r)}u &\le C\left(\frac{r}{s}\right)^{\left(-\frac{lg(\theta)}{lg2}\right)} osc_{B(x,s)}u \\
&\quad + 2C_2\left[\left(\frac{1}{m(B(x,R))}\int_{B(x,R)} u^p + \eta(\bar{R}) + 1\right)^2 + 1\right]\eta_\sigma(2s)
\end{aligned}
$$

for $r < 2r < s < R$. From (4.6) the result of Theorem 1.2 easily follows.

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