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Oscillation, Nutation and Wobble of an Elliptical Rotating Earth with Liquid Outer Core

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Summary

We have constructed a general first-order theory describing those small oscillations of a rotating elliptical earth that are affected by the presence of a liquid outer core. The theory is applicable to free core oscillations and earth tides. Care has been taken to include the effects of the wobble or nutation due to the rotation of the outer core relative to the solid earth. On the basis of the theory the free spheroidal modes of degree 2 and order 1 have been investigated. We have searched for and listed undertones with periods less than 28 hr. No upper limit to the eigenperiods has been detected. It is shown that stable, unstable and neutral polytropic cores are capable of free oscillation. At a period close to the sidereal day the spheroidal mode is accompanied by rigid rotation of the liquid outer core with respect to the solid earth. This is the well-known diurnal wobble of the Earth. It appears probable that the diurnal wobble is one of a class of similar wobbles that involve large toroidal motions in the outer core. Finally, the amplitudes of the 18·6-yr principal nutations has been computed. Excellent agreement is found with observed values.

1. Introduction

The theoretical study of oscillations of a non-rotating spherical earth presents no major problems. The spheroidal and toroidal displacement fields are separable and the dynamical equations of equilibrium can be solved exactly. With the introduction of ellipticity and rotation however, the problem is no longer simple. A complete analytical solution is impossible because of the coupling between the various constituents of the displacement field in the liquid outer core and to a lesser extent in the solid earth.

Historically, the dynamic effects of the liquid core have been recognized through the Chandler wobble and astronomical nutations of the Earth. The discrepancy between the observed and calculated nutation amplitudes, in particular, can only be explained by consideration of the fluidity of the outer core. Nutations are associated with diurnal earth tides, which have periods falling close to one of the free core modes. In the present report, we show that this core mode is essentially a $T_1{}^1$ toroidal oscillation of the outer core relative to the solid earth. The possibility of this free mode for an incompressible fluid enclosed in a rigid elliptical shell was first considered by Poincaré (1910). Jeffreys (1948) pointed out its geophysical significance. When Poincaré's theory is applied to the Earth, it is found that the period of the Chandler wobble is shortened and the amplitude of the 18·66-yr principal nutation decreased, as compared to the corresponding values for a rigid earth model. Subsequent extensions of Poincaré's theory, allowing for the elasticity of the mantle and core, were given by Jeffreys (1949, 1950), Jeffreys & Vicente (1957a, b) and Molodensky (1961). Jeffreys & Vicente used a variational method that leads to some ambiguity in the degree of approximation and some puzzling results. For example, for P_1 tide with $n/\omega = 1/183$, the function ζ/ζ_0 was given by 0·9707 (Table 1, Jeffreys & Vicente 1957b). This gives rise to a correction for the nutation amplitude of the wrong sign. A result which is difficult to understand, however, is the ratio of Love numbers, k/h. Jeffreys & Vicente obtained a value of 0·412 for diurnal tides and 0·493 for semi-diurnal tides. Molodensky (1961), Shen (1975), and the present work, on the other hand, show that k/h is equal to about 0·495 for both diurnal and semi-diurnal tides. The present theory on diurnal earth tides and nutations is nearly identical with Molodensky's and agrees well with the observations on nutations. For a topical review of problems of the rotation of the Earth the reader is referred to the paper by Rochester (1973).

Free oscillation modes with periods longer than the fundamental elastic mode of the Earth have been ascribed to the liquid core (Alterman, Jarosch & Pekeris 1959). These modes are now called undertones by Smylie (1974). Smylie considered the effects of rotation on a spherical earth. But he limited himself to the effects of self coupling due to Coriolis force. Crossley (1975) extends Smylie's work to cover cross-couplings. But the effects of centrifugal force and ellipticity are neglected. Smith (1974) gives a more complete treatment but without numerical results.

One approach to the problem of earth dynamics is to consider rotation and ellipticity as perturbations to a spherical, non-rotating earth. The eigenfrequency and eigenfunction are expanded in power series of the ratio of the angular speed of rotation to the unperturbed eigenfrequency, or, the ellipticity. For the elastic oscillations of the Earth, the first-order perturbation methods yield adequate results (Dahlen 1968; Luh 1974; Review by Alterman, Eyal & Merzer 1974). This is because the power series converge rapidly. For long-period core oscillations the situation is different. The eigenfrequency may differ significantly from the unperturbed value so that the convergence of the power series become doubtful (Dahlen 1968). In this case very high order perturbation schemes must be used.

Initially we sought to extend Molodensky's theory to general harmonic oscillations of the Earth. However, this theory was formulated specifically for diurnal earth tides and required an Adams & Williamson (1923) core and was unsuitable for a general treatment. In the present report we construct a general theory which takes into account the dynamic effects of the liquid core. The set of ordinary differential equations that govern the motion within the elliptical rotating earth and the motions in space is derived. The logical organization of the theory is illustrated in Fig. 1. It is emphasized here that Euler's equation for angular momentum is necessary for those oscillation modes that involve motion of the axis of rotation within the Earth.

Uniform rotation occurs when the axis of maximum or minimum moment of inertia concides with the axis of angular momentum and the rotation axis. Internal redistribution of angular momentum or mass gives rise to excursions of the principal axis and rotation axis from the angular momentum vector. Motion of rotation axis in space accompanies motion within the Earth. For brevity we depart somewhat from the conventional definitions of wobble and nutation. The free motion is called wobble and a forced motion is called nutation. Thus wobble implies the free motion of the rotation axis within the Earth as well as the accompanying motion in space. Similarly nutation implies the forced motion of the rotation axis in space and the simultaneous motion of the rotation axis within the Earth. In this report we are primarily concerned with wobbles related to free oscillation, and, nutations caused by the tidal potential.

The real difficulty in dealing with the oscillations of an elliptical rotating earth lies

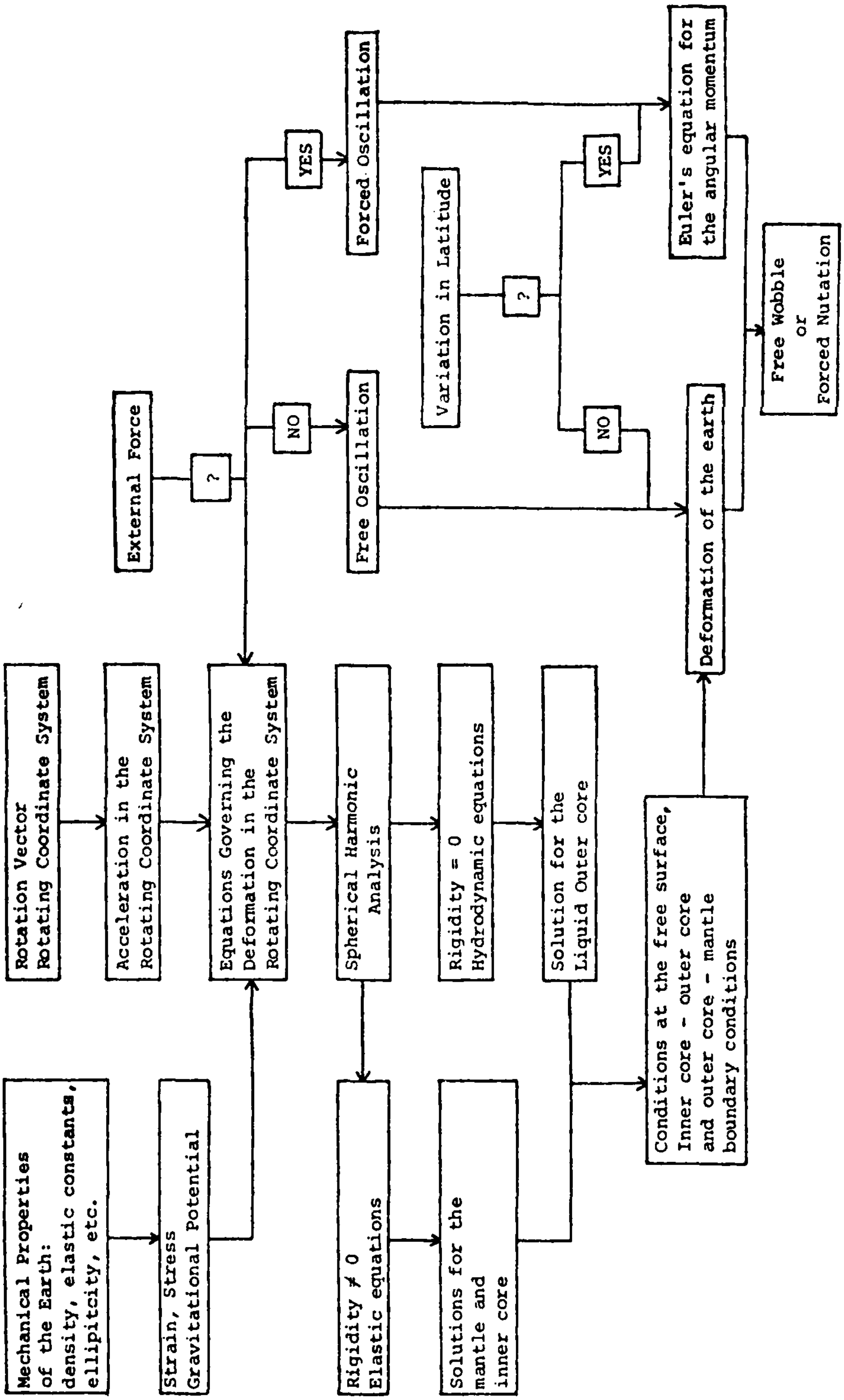

FIG. 1. Logical organization of the theory.

in the numerical solution of the infinite set of coupled ordinary differential equations. Truncation of the coupling sequence is inevitable for a numerical solution. The problem is essentially the same as the one encountered in perturbation methods. Convergence of the truncation scheme has not been established. In the present work, to make numerical solutions possible, the hydrodynamic equations are simplified to the extent that the effects of rotation and ellipticity in the solid earth can be neglected. However, it is emphasized that such an approximation must be considered incomplete.

There is incomplete knowledge concerning the density stratification of the liquid outer core. The radial variation of density may be gravitationally unstable, neutral or stable (Pekeris & Accad 1972). In a non-rotating earth oscillation is only possible with a stable stratification. We have computed the spectra of free spheroidal core oscillation of degree 2 and order 1 and the tidal Love numbers for diurnal earth tides. The results show that all three types of polytropic cores are capable of free oscillation. The periods however, depend on the nature of density stratification of the liquid core. Observation of the periods should prove to be of diagnostic value in choosing between various core models.

The free spheroidal oscillations of degree 2 and order 1 are associated with wobbles, and diurnal tides are associated with nutations. Due to the existence of a nearly diurnal free oscillation the diurnal tides and nutations exhibit resonance effects.

There has been a recent upsurge of interest in the diurnal wobble of the earth. (Rochester, Jensen & Smylie 1974). Predictions of the diurnal wobble are based on rather simple calculations on rotating fluids in oblate containers (Poincaré 1910; Toomre 1974). We show that the diurnal wobble is the consequence of toroidal motion of the entire outer core associated with a tesseral spheroidal oscillation of the Earth. We also confirm Toomre's speculation on the possibility of a set of other wobbles of the Earth due to toroidal motion in the core.

Finally, we have computed the amplitudes of nutations for different core modes and compared with observed values. The results for neutral cores agree with those of Molodensky.

2. Equations of motion

To describe the dynamic behaviour of the Earth, consider a cartesian reference frame (x_1, x_2, x_3) 'rotating with the Earth' in space at an angular velocity $\Omega = (\Omega_1, \Omega_2, \Omega_3)$. There are several methods of attaching the cartesian frame to the Earth (Munk & Macdonald 1960). Because of the simple form of Euler's equations we choose to attach the rotating frame to the principal axes of the Earth. The polar axis of figure of the Earth is chosen as the z axis.

If the Earth is subject only to diurnal rotation, the vector Ω can be written as

$$\Omega = (0, 0, \omega). \tag{1}$$

This is valid for all free or forced oscillations of the Earth with polar axial symmetry. However, the axis of rotation is disturbed by a class of tesseral spheroidal oscillations. In this report we are particularly concerned with spheroidal oscillations of the Earth of degree 2 and order 1. The rotation vector then becomes

$$\Omega = (\omega\varepsilon \cos \sigma t, \omega\varepsilon \sin \sigma t, \omega). \tag{2}$$

Here σ is the angular frequency of a spheroidal oscillation of degree 2, order 1, and ε is the amplitude of the related wobble or a constant proportional to the amplitude of the related nutation if the oscillation is forced.

The theoretical treatment of the problems of deformation of the Earth presents some difficulties due to the existence of the liquid core. However, for small oscillations, the Lagrangian and Eulerian formulations are equivalent. Then the deformation of the Earth can be described by the displacement vector $\mathbf{u} = (u_1, u_2, u_3)$ of each material particle.

The equation of motion in tensor notation is

$$\rho \frac{d^2 u_i}{dt^2} = \rho F_i + \rho \frac{\partial W}{\partial x_i} + \frac{\partial}{\partial x_j} T_{ji},\ i = 1, 2, 3, \tag{3}$$

where ρ is the mass density, $\mathbf{F} = (F_1, F_2, F_3)$ the external force density. W the potential of self gravitation, and T_{ij} the stress tensor.

The acceleration $d^2\mathbf{u}/dt^2$ is in an inertial reference frame. For motion in a rotating co-ordinate system, the acceleration in the inertial frame is given by

$$\frac{d^2\mathbf{u}}{dt^2} = \frac{\partial^2\mathbf{u}}{\partial t^2} + 2\boldsymbol{\Omega}\times\frac{\partial\mathbf{u}}{\partial t} + \frac{\partial\boldsymbol{\Omega}}{\partial t}\times\mathbf{r} + (\boldsymbol{\Omega}.\mathbf{r})\boldsymbol{\Omega} - (\boldsymbol{\Omega}.\boldsymbol{\Omega})\mathbf{r}, \tag{4}$$

where $\partial/\partial t$ is the time deviative in the rotating frame. In equation (4) second-order quantitives in $\mathbf{u}$ have been ignored. In spherical co-ordinates, (4) becomes

$$\left.\begin{aligned}
\frac{d^2u_r}{dt^2} &= \frac{\partial^2 u_r}{\partial t^2} - 2\omega\sin\theta\frac{\partial u_\phi}{\partial t} - \frac{2}{3}\omega\sigma\varepsilon r P_2{}^1(\cos\theta)\cos(\sigma t - \phi) \\
&\qquad - \frac{\partial}{\partial r}\left(W_c + \frac{\sigma+\omega}{\omega}W_T\right), \\
\frac{d^2u_\theta}{dt^2} &= \frac{\partial^2 u_\theta}{\partial t^2} - 2\omega\cos\theta\frac{\partial u_\phi}{\partial t} + \frac{8}{5}\omega\sigma\varepsilon r\frac{P_1{}^1(\cos\theta)}{\sin\theta}\cos(\sigma t - \phi) \\
&\qquad - \frac{4}{15}\omega\sigma\varepsilon r\frac{P_3{}^1(\cos\theta)}{\sin\theta}\cos(\sigma t-\phi) - \frac{1}{r}\frac{\partial}{\partial\theta}\left(W_c + \frac{\sigma+\omega}{\omega}W_T\right), \\
\frac{d^2u_\phi}{dt^2} &= \frac{\partial^2 u_\phi}{\partial t^2} + 2\omega\sin\theta\frac{\partial u_r}{\partial t} + 2\omega\cos\theta\frac{\partial u_\theta}{\partial t} - \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\left(W_c + \frac{\sigma+\omega}{\omega}W_T\right).
\end{aligned}\right\} \tag{5}$$

The various terms used in (5) are defined as follows:

$P_n{}^m(\cos\theta)$ is the associated Legendre function of degree n and order m. For $m \geqslant 0$, $P_n{}^m(u)$ is defined as

$$P_n{}^m(u) = \frac{(1-u^2)^{m/2}}{2^n n!}\frac{d^{n+m}}{du^{n+m}}(u^2-1)^n,\ (-1 \leqslant u \leqslant 1).$$

For $-$ve m it is defined as $P_n{}^{|m|}(\cos\theta)$.

W_c is the centrifugal potential, given by

$$W_c = \tfrac{1}{2}\omega^2 r^2 \sin^2\theta. \tag{6}$$

W_T is the tesseral potential arising from the variation of latitude caused by motion of the rotation axis and is given by

$$W_T = W_t(r)\, P_2{}^1(\cos\theta)\cos(\sigma t - \phi), \tag{7}$$

where

$$W_t(r) = -\tfrac{1}{3}\varepsilon\omega^2 r^2. \tag{8}$$

In order to determine W and T_{ij}, we make the following assumptions:

(i) The Earth is initially in hydrostatic equilibrium.

(ii) In the initial state, the equipotential surfaces coincide with the surface of equal density, compressibility and rigidity etc.

(iii) The dynamic stress–strain relation is that for a perfectly elastic and isotropic earth.

We define the following terms: $e(r)$ is the ellipticity of a surface of equal density; $g(r)$ is the gravitational acceleration; $\rho_0(r)$ is the initial mass density; W_m is the initial gravitational potential due to $\rho_0(r)$; W_0 is the total potential; $W_r(r)$ is a function of r only. The following relations hold (Jeffreys 1959, p. 145):

$$\nabla^2 W_m = -4\pi G\rho_0, \tag{9}$$

$$W_0 = W_m + W_c, \tag{10}$$

$$g(r) = -W_0' \doteqdot -\frac{d}{dr} W_0(r), \tag{11}$$

$$W_0 = W_r(r) + b(r)\sin^2\theta, \tag{12}$$

$$b(r) = e(r)\, g(r)\, r, \tag{13}$$

$$\nabla P = \rho_0 \nabla W_0. \tag{14}$$

The prime over W_0 in equation (11) indicates the derivative of W_0 along the external normal to the equipotential surface. This convention will be followed hereafter.

Assumption (ii) enables us to write

$$\left.\begin{aligned} \frac{\partial \rho_0}{\partial x_i} &= \frac{\rho_0'}{W_0'}\frac{\partial W_0}{\partial x_i}, \\ \frac{\partial \lambda}{\partial x_i} &= \frac{\lambda'}{W_0'}\frac{\partial W_0}{\partial x_i}, \\ \frac{\partial \mu}{\partial x_i} &= \frac{\mu'}{W_0'}\frac{\partial W_0}{\partial x_i}, \end{aligned}\right\} \tag{15}$$

where λ and μ are Lame's constants.

Furthermore, we can also write (Dahlen 1968)

$$\left.\begin{aligned} \rho_0 &= \rho_s(r) + \frac{\rho_s'}{W_0'} b(r)\sin^2\theta, \\ \lambda &= \lambda_s(r) + \frac{\lambda_s'}{W_0'} b(r)\sin^2\theta, \\ \mu &= \mu_s(r) + \frac{\mu_s'}{W_0'} b(r)\sin^2\theta, \end{aligned}\right\} \tag{16}$$

where ρ_s, λ_s and μ_s are the values of ρ_0, λ, and μ respectively for an equivalent spherical earth.

With assumption (iii) the additional stress τ_{ij} due to the deformation is given by

$$\tau_{ij} = \lambda\,\Delta\,\delta_{ij} + \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right), \tag{17}$$

where

$$\Delta = \operatorname{div} \mathbf{u} \tag{18}$$

is the dilatation, and δ_{ij} the Kronecker delta.

Due to deformation, there is a variation in volume density. The equation of continuity gives

$$\rho-\rho_0 = \mathrm{div}\,(\rho\mathbf{u}). \tag{19}$$

Using (15), we get

$$\rho-\rho_0 = -\rho_0\Delta-\frac{\rho_0'}{W_0'}\eta. \tag{20}$$

The work done by the deformation is η and is given by

$$\eta = u_r\frac{\partial W_0}{\partial r}+u_\theta\frac{1}{r}\frac{\partial W_0}{\partial\theta}+u_\phi\frac{1}{r\sin\theta}\frac{\partial W_0}{\partial\phi}. \tag{21}$$

The variation in volume density gives rise to a change in gravitational potential, W_a, which satisfies

$$\nabla^2 W_a = -4\pi G(\rho-\rho_0) = 4\pi G\left(\rho_0\Delta+\frac{\rho_0'}{W_0'}\eta\right). \tag{22}$$

The total potential of self-gravitation is then given by

$$W = W_m+W_a.$$

Since $W_m = W_0-W_c$, we get

$$W = W_0+W_a-W_c. \tag{23}$$

The stress T_{ij} consists of the initial hydrostatic stress T_{0ij} and the additional stress τ_{ij}. Since the initial hydrostatic pressure at a point $\mathbf{r}$ in the deformed state is the hydrostatic pressure at the point originally at $(\mathbf{r}-\mathbf{u})$, we find

$$\begin{aligned} T_{0ij} &= -P(\mathbf{r}-\mathbf{u})\delta_{ij} \\ &= -\big(P(\mathbf{r})-\mathbf{u}.\nabla P\big)\delta_{ij}. \end{aligned}$$

Using (14) and (21) in the above equation, we get

$$T_{0ij} = -(P-\rho_0\eta)\delta_{ij}. \tag{24}$$

Combining (24) and (17), we find that

$$T_{ij} = (-P+\rho_0\eta+\lambda\Delta)\delta_{ij}+\mu\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right). \tag{25}$$

Using (5), (23) and (25) in (3), we obtain, with the help of (15), the equation of motion for an elliptical rotating earth:

$$\begin{aligned} &\frac{\partial^2 u_r}{\partial t^2}-2\omega\sin\theta\frac{\partial u_\phi}{\partial t}-2/3\,\omega\sigma\varepsilon r\,P_2{}^1(\cos\theta)\cos(\sigma t-\phi) \\ &= F_r+\frac{\partial}{\partial r}\left(W_a+\frac{\sigma+\omega}{\omega}W_T+\eta+\frac{\lambda\Delta}{\rho_0}\right)-\beta(r)\,\Delta\frac{\partial W_0}{\partial r} \\ &+\frac{1}{\rho_0}\left\{\nabla\mu.\left(\nabla u_r+\frac{D\mathbf{u}}{\partial r}\right)+\mu\left(\nabla^2 u_r+\frac{\partial\Delta}{\partial r}\right)\right. \\ &\left.+\frac{2\mu}{r}\left(\frac{\partial u_r}{\partial r}+\frac{u_r}{r}-\Delta\right)\right\}, \end{aligned} \tag{26}$$

$$\frac{\partial^2 u_\theta}{\partial t^2} - 2\omega\cos\theta\frac{\partial u_\phi}{\partial t} + \frac{8}{5}\omega\sigma\varepsilon r\frac{P_1{}^1(\cos\theta)}{\sin\theta}\cos(\sigma t-\phi) - \frac{4}{15}\omega\sigma\varepsilon r\frac{P_1{}^3(\cos\theta)}{\sin\theta}\cos(\sigma t-\phi)$$

$$= F_\theta + \frac{1}{r}\frac{\partial}{\partial\theta}\left(W_a + \frac{\sigma+\omega}{\omega}W_T + \eta + \frac{\lambda\Delta}{\rho_0}\right) - \beta(r)\frac{\Delta}{r}\frac{\partial W_0}{\partial\theta}$$

$$+ \frac{1}{\rho_0}\left\{\nabla\mu.\left(r\nabla\left(\frac{u_\theta}{r}\right) + \frac{D\mathbf{u}}{r\partial\theta}\right) + \mu\left(\nabla^2 u_\theta + \frac{\partial\Delta}{r\partial\theta}\right)\right.$$

$$\left. + \frac{2}{r^2}\frac{\partial}{\partial\theta}(\mu u_r) - \frac{\mu}{r^2\sin^2\theta}\left(u_\theta + 2\cos\theta\frac{\partial u_\phi}{\partial\phi}\right)\right\}, \tag{27}$$

$$\frac{\partial^2 u_\phi}{\partial t^2} + 2\omega\sin\theta\frac{\partial u_r}{\partial t} + 2\omega\cos\theta\frac{\partial u_\theta}{\partial t}$$

$$= F_\phi + \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\left(W_a + \frac{\sigma+\omega}{\omega}W_T + \eta + \frac{\lambda\Delta}{\rho_0}\right) - \beta(r)\frac{\Delta}{r\sin\theta}\frac{\partial W_0}{\partial\phi}$$

$$+ \frac{1}{\rho_0}\left\{\nabla\mu.\left(r\sin\theta\nabla\left(\frac{u_\phi}{r\sin\theta}\right) + \frac{1}{r\sin\theta}\frac{D\mathbf{u}}{\partial\phi}\right) + \mu\left(\nabla^2 u_\phi + \frac{1}{r\sin\theta}\frac{\partial\Delta}{\partial\phi}\right)\right.$$

$$\left. + \frac{2}{r^2\sin\theta}\frac{\partial\mu}{\partial\phi}(\cot\theta\, u_\theta + u_r) + \frac{\mu}{r^2\sin\theta}\left(2\frac{\partial u_r}{\partial\phi} + 2\cot\theta\frac{\partial u_\theta}{\partial\phi} - \frac{u_\phi}{\sin\theta}\right)\right\}, \tag{28}$$

where

$$\left.\begin{aligned} \beta(r) &= -\frac{\lambda\rho_0'}{\rho_0{}^2 W_0'} + 1, \\ \frac{D\mathbf{u}}{\partial r} &= \hat{r}\frac{\partial u_r}{\partial r} + \hat{\theta}r\frac{\partial}{\partial r}\left(\frac{u_\theta}{r}\right) + \hat{\phi}r\frac{\partial}{\partial r}\left(\frac{u_\phi}{r}\right), \\ \frac{D\mathbf{u}}{r\partial\theta} &= \hat{r}\frac{\partial u_r}{r\partial\theta} + \hat{\theta}\frac{\partial u_\theta}{r\partial\theta} + \hat{\phi}\sin\theta\frac{\partial}{r\partial\theta}\left(\frac{u_\phi}{\sin\theta}\right), \end{aligned}\right\} \tag{29}$$

and

$$\frac{1}{r\sin\theta}\frac{D\mathbf{u}}{\partial\phi} = \hat{r}\frac{1}{r\sin\theta}\frac{\partial u_r}{\partial\phi} + \hat{\theta}\frac{1}{r\sin\theta}\frac{\partial u_\theta}{\partial\phi} + \hat{\phi}\frac{1}{r\sin\theta}\frac{\partial u_\phi}{\partial\phi}.$$

Equations (26), (27) and (28) are equivalent to the equations of motion derived by Molodensky (1961).

3. Expansion in spherical harmonics

The displacement vector $\mathbf{u} = (u_r, u_\theta, u_\phi)$ for a harmonic oscillation of the Earth of angular frequency σ can be expanded in spheroidal and toroidal fields. The displacement fields are generally given in complex form which includes spatial and temporal phase information. However, for the purpose at hand phase is irrelevant and the real form may be used. Then the components of a spheroidal field $\mathbf{S}_n^m(r, \theta, \phi)$ of degree n and order m are given by

$$\left.\begin{aligned}
(\mathbf{S}_n^m)_r &= U_n^m(r)\, P_n^m(\cos\theta)\cos(\sigma t - m\phi),\\
(\mathbf{S}_n^m)_\theta &= V_n^m(r)\frac{\partial}{\partial\theta} P_n^m(\cos\theta)\cos(\sigma t - m\phi),\\
(\mathbf{S}_n^m)_\phi &= m\, V_n^m(r)\frac{1}{\sin\theta} P_n^m(\cos\theta)\sin(\sigma t - m\phi),
\end{aligned}\right\} \tag{30}$$

and the components of a toroidal field of degree n and order m by

$$\left.\begin{aligned}
(\mathbf{T}_n^m)_r &= 0,\\
(\mathbf{T}_n^m)_\theta &= -m\, T_n^m(r)\frac{1}{\sin\theta} P_n^m(\cos\theta)\cos(\sigma t - m\phi),\\
(\mathbf{T}_n^m)_\phi &= -T_n^m(r)\frac{\partial}{\partial\theta} P_n^m(\cos\theta)\sin(\sigma t - m\phi).
\end{aligned}\right\} \tag{31}$$

The factor $(\sigma t - m\phi)$ determines the sense of motion as prograde or retrograde. For $m \geqslant 0$ a positive σ gives a prograde motion and a negative σ gives a retrograde motion. In the present work, we follow the convention adopted by Molodensky (1961) and consider σ as negative.

The main effect of ellipticity and rotation of the Earth is to bring about a coupling between spheroidal and toroidal fields of the same order. Thus an eigenfunction will have a displacement of the form

$$\mathbf{u} = \mathbf{S}_m^m + \mathbf{T}_{m+1}^m + \mathbf{S}_{m+2}^m + \mathbf{T}_{m+3}^m + \dots,$$

or

$$\mathbf{u} = \mathbf{T}_m^m + \mathbf{S}_{m+1}^m + \mathbf{T}_{m+2}^m + \mathbf{S}_{m+3}^m + \dots,$$

(Dahlen 1968; Smith 1974; Crossley 1975; Shen 1975). In the outer core, due to the vanishing of rigidity, the coupling between spheroidal and toroidal fields is strong. Crossley (1975), in treating the $\mathbf{S}_2^2$ core oscillations for a stable core, considered the effects of coupling from $\mathbf{S}_n^2$ and $\mathbf{T}_n^2$ up to $\mathbf{S}_{52}^2$. The results indicate that due to the rotation of the Earth, there exist infinite number of critical periods by which the free periods are divided into allowed and forbidden zones. The first allowed zone is found to be bounded above by a decreasing period which eventually reaches 12 sidereal hours. In view of Crossley's analysis, it is obvious that large errors will be introduced when a severe truncation of the coupling chain is made. However, until a better approach to the problem is found, numerical solutions are possible only for simple approximations. As a preliminary work, here we consider a displacement for the liquid outer core of the form

$$\mathbf{u} = \mathbf{T}_{n-1}^m + \mathbf{S}_n^m + \mathbf{T}_{n+1}^m. \tag{32}$$

Here $\mathbf{T}_{n-1}^m$ does not exist if n is equal to or smaller than m. We note that (32) is of the same form considered by Smith (1974) and Crossley (1975). With (32) as displacement for the outer core, the effects of rotation and ellipticity can be neglected in the solid Earth. This simplification is permitted by the boundary conditions (see equations (42)) and its validity has been demonstrated numerically by Crossley (1975). Thus corresponding to (32), the displacement in the solid earth assumes the simple form of $\mathbf{S}_n^m$.

The equations of motion for the liquid outer core may be expanded in spherical harmonics using (32) as the displacement. The resulting finite set of ordinary differential equations over radius is of the fourth order. However, the exact expansion is possible and requires about the same amount of work. We therefore work out the infinite set of ordinary differential equations which may prove useful in future when numerical methods can be improved upon.

The general displacement for an elliptical, rotating Earth under harmonic oscillation of angular frequency σ are given by

$$\mathbf{u} = \textstyle\sum_n \mathbf{S}_n^m + \sum_l \mathbf{T}_l^m \tag{32a}$$

where the displacement fields $\mathbf{S}_n^m$ and $\mathbf{T}_n^m$ are given by (30) and (31) respectively. The summations in (32a) are over

$$n = |m|,\ |m|+2,\ |m|+4,\ \ldots$$

$$\text{and } l = |m|+1,\ |m|+3,\ |m|+5,\ \ldots,$$

or over

$$n = |m|+1,\ |m|+3,\ |m|+5,\ \ldots,$$

$$\text{and } l = |m|,\ |m|+2,\ |m|+4,\ \ldots .$$

A summation over m is not necessary at present since displacement fields with different m are separable. For the same reason, from now on, we can drop the superscript m without any danger of confusion.

The equations of equilibrium can be reduced to an infinite set of ordinary differential equations. Our notation differs from the conventional y notations introduced by Alterman *et al.* (1959). However, the y notation is convenient only for treatment of a spherical Earth. In the present study we choose to follow a different but more descriptive notation.

With the displacement given by (32a), the following expressions are general:

$$W_a = \sum_n H_n(r)\, P_n^m(\cos\theta)\cos(\sigma t - m\phi), \tag{33.1}$$

$$\eta = \sum_n \eta_n(r)\, P_n^m(\cos\theta)\cos(\sigma t - m\phi), \tag{33.2}$$

$$\Delta = \sum_n \Delta_n(r)\, P_n^m(\cos\theta)\cos(\sigma t - m\phi), \tag{33.3}$$

$$\frac{\lambda\Delta}{\rho_0} = \sum X_n(r)\, P_n^m(\cos\theta)\cos(\sigma t - m\phi), \tag{33.4}$$

$$F_r = \sum_n {}_rF_n(r)\, P_n^m(\cos)\theta\, \cos(\sigma t - m\phi), \tag{33.5}$$

$$F_\theta = \sum_n {}_\theta F_n(r)\frac{\partial}{\partial\theta} P_n^m(\cos\theta)\cos(\sigma t - m\phi), \tag{33.6}$$

$$F_\phi = m\sum_n {}_\phi F_n(r)\frac{1}{\sin\theta} P_n^m(\cos\theta)\sin(\sigma t - m\phi). \tag{33.7}$$

The summations over n in (33) are the same as in (32a).

Using (32a) in (21), and equating the resulting equation to (33.2) we find

$$\eta_n(r) = -\frac{(n-1-m)(n-m)}{(2n-3)(2n-1)}\left(b(r)\,U_{n-2} - (n-2)\frac{2b}{r}V_{n-2}\right)$$

$$+m\left(-\frac{n-m}{2n-1}\right)\frac{2b}{r}T_{n-1} + \left(-\frac{n(n+1)-3m^2}{(2n-1)(2n+3)}\right)\frac{2b}{r}V_n$$

$$+\left(\frac{2(n^2+n-1+m^2)}{(2n-1)(2n+3)}\dot{b}-g\right)U_n+m\left(-\frac{n+1+m}{2n+3}\right)\frac{2b}{r}T_{n+1}$$

$$+\left(-\frac{(n+1+m)(n+2+m)}{(2n+3)(2n+5)}\right)\left(\dot{b}\,U_{n+2}+(n+3)\frac{2b}{r}V_{n+2}\right). \tag{34}$$

The dot over b in (34) indicates the derivative witth respect to the radius. This convention will be followed hereafter.

Using (32) in (18), and equating the resulting equation to (33.3), we find

$$\Delta_n=\dot{U}_n+\frac{2}{r}U_n-\frac{n(n+1)}{r}V_n. \tag{35}$$

Using (16) in $\lambda\Delta/\rho_0$, we find

$$X_n(r)=\frac{\lambda_s}{\rho_s}\Delta_n+\frac{\lambda_s}{\rho_s}\left(\frac{\dot{\lambda}_s}{\lambda_s}-\frac{\dot{\rho}_s}{\rho_s}\right)\frac{b(r)}{W_0'}\left(-\frac{(n-1-m)(n-m)}{(2n-3)(2n-1)}\Delta_{n-2}\right.$$

$$\left.+\frac{2(n^2+n-1+m^2)}{(2n-1)(2n+3)}\Delta_n-\frac{(n+1+m)(n+2+m)}{(2n+3)(2n+5)}\Delta_{n+2}\right). \tag{36}$$

Using (32) and (33) in (26), (27), and (28), we obtain

$$-\sigma^2U_n+2\omega\sigma\left(\frac{(n-1)(n-m)}{2n-1}T_{n-1}-m\,V_n\right.$$

$$\left.-\frac{(n+2)(n+1+m)}{2n+3}T_{n+1}\right)-2/3\,\omega\sigma\varepsilon r\,\delta_n{}^2\delta_m{}^1$$

$$=\frac{d}{dr}\left(H_n+\frac{\omega+\sigma}{\omega}W_t\delta_n{}^2\delta_m{}^1+\eta_n+X_n\right)+\beta g\,\Delta_n+{}_rF_n$$

$$+\left(-\frac{(n-1-m)(n-m)}{(2n-3)(2n-1)}\Delta_{n-2}+\frac{2(n^2+n-1+m^2)}{(2n-1)(2n+3)}\Delta_n\right.$$

$$\left.-\frac{(n+1+m)(n+2+m)}{(2n+3)(2n+5)}\Delta_{n+2}\right)\beta\dot{b}, \tag{37}$$

$$\frac{(n-3)(n-2-m)(n-1-m)}{(2n-5)(2n-3)}2\omega\sigma\,T_{n-3}+\left(-\frac{n-1-m}{2n-3}\right)\left(2m\omega\sigma+(n-2)\sigma^2\right)V_{n-2}$$

$$+\left(-\frac{n(n-1)-3m^2}{(2n-3)(2n+1)}2\omega\sigma+m\sigma^2\right)T_{n-1}+\left(-\frac{n+m}{2n+1}\right)(2m\omega\sigma-(n+1)\sigma^2)V_n$$

$$+\left(-\frac{(n+2)(n+m)(n+1+m)}{(2n+1)(2n+3)}\right)2\omega\sigma\,T_{n+1}$$

$$+\tfrac{8}{5}\,\omega\sigma\varepsilon r\,\delta_n{}^2\delta_m{}^1-\tfrac{4}{15}\,\omega\sigma\varepsilon r\,\delta_n{}^4\delta_m{}^1$$

$$\frown\ \frac{n-2)(n-1-m)}{2n-3}\Big)\frac{1}{r}\left(H_{n-2}+\frac{\omega+\sigma}{\omega}W_t\delta_n{}^4\delta_m{}^1+\eta_{n-2}+X_{n-2}+r_\theta F_{n-2}\right)$$

$$+\left(-\frac{(n+1)(n+m)}{2n+1}\right)\frac{1}{r}\left(H_n+\frac{\omega+\sigma}{\omega}W_t\delta_n^2\delta_m^1+\eta_n+X_n+r_\theta F_n\right)$$

$$+\left(\frac{(n-3-m)(n-2-m)(n-1-m)}{(2n-7)(2n-5)(2n-3)}\right)\frac{2\beta b}{r}\Delta_{n-4}$$

$$-\left(+\frac{(n+m)}{(2n-1)(2n+1)}\right)\left(\frac{2(n^2+n-1+m^2)}{2n+3}-\frac{(n-1-m)(n-1+m)}{2n-3}\right)$$

$$\times\frac{2\beta b}{r}\Delta_n-\left(-\frac{n-1-m}{(2n-3)(2n-1)}\right)\left(\frac{(n-m)(n+m)}{2n+1}\right.$$

$$\left.-\frac{2(n^2-3n+1+m^2)}{2n-5}\right)\frac{2\beta b}{r}\Delta_{n-2}$$

$$-\left(-\frac{(n+m)(n+1+m)(n+2+m)}{(2n+1)(2n+3)(2n+5)}\right)\frac{2\beta b}{r}\Delta_{n+2}, \tag{38}$$

and

$$\left(-\frac{(n-2)(n-1-m)(n-m)}{(2n-3)(2n-1)}\right)2\omega\sigma V_{n-2}+\left(\frac{n-m}{2n-1}\right)(2m\omega\sigma+(n-1)\sigma^2)T_{n-1}$$

$$+\left(\frac{n(n+1)-3m^2}{(2n-1)(2n+3)}2\omega\sigma-m\sigma^2\right)V_n+\left(\frac{n+1+m}{2n+3}\right)(2m\omega\sigma-(n+2)\sigma^2)T_{n+1}$$

$$+\left(\frac{(n+3)(n+1+m)(n+2+m)}{(2n+3)(2n+5)}\right)2\omega\sigma V_{n+2}+\left(\frac{(n-1-m)(n-m)}{(2n-3)(2n-1)}\right)2\omega\sigma U_{n-2}$$

$$+\left(-\frac{2(n^2+n-1+m^2)}{(2n-1)(2n+3)}\right)2\omega\sigma U_n+\left(\frac{(n+1+m)(n+2+m)}{(2n+3)(2n+5)}\right)2\omega\sigma U_{n+2}$$

$$=\frac{m}{r}\left(H_n+\frac{\omega+\sigma}{\omega}W_t\delta_n^2\delta_m^1+\eta_n+X_n+r_\phi F_n\right). \tag{39}$$

Finally, using (33.2) and (33.4) in (22), we get

$$\ddot{H}_n+\frac{2}{r}\dot{H}_n-\frac{n(n+1)}{r^2}H_n=+4\pi G\rho_s\beta\Delta_n+\frac{4\pi G\rho_s'}{W_0'}(\eta_n+X_n). \tag{40}$$

Equations (34)–(40) with $n = m, m+2, m+4, \ldots$, or $n = m+1, m+3, m+5, \ldots$ form the infinite set of ordinary differential equations that govern the small oscillations in the outer core.

Equations (34)–(40) for a spherical earth (e.g. Smylie 1974; Crossley 1975) are generally given in the y notations introduced by Alterman *et al.* (1959). The introduction of the present notations stems from the desire to retain explicitly the stability parameter β and to use the functions η_n and X_n in the equations. Notice that η_n and X_n correspond to $-gU_n$ and $\lambda\Delta_n/\rho$ respectively in the spherical earth, and $-\rho\,(\eta_n+X_n)$ is the nth degree harmonic change in hydrostatic pressure. For readers who are familiar with the y notations, we give the following identification:

$$y_1^n=U_n,\ y_2^n=\lambda_s\Delta_n,\ y_3^n=V_n,\ y_4^n=0,\ y_5^n=H_n,$$

and $y_6{}^n = \dot{H}_n - 4\pi G\rho_s U_n$. In Appendix A, the simplified set of hydrodynamic equations for $S_2{}^1$ oscillations is given in y notations. For the mantle and inner core equations which follow, $y_2{}^n = Z_n$, $y_4{}^n = Y_n$, and $y_6{}^n = Q_n$.

In the mantle and inner core, each spheroidal oscillation is governed by a set of sixth order ordinary differential equations. These equations are

$$\left.\begin{aligned}
\dot{U}_n &= -\frac{2\lambda}{\lambda+2\mu}\frac{1}{r}U_n + \frac{1}{\lambda+2\mu}Z_n + \frac{n(n+1)\lambda}{\lambda+2\mu}\frac{1}{r}V_n, \\
\dot{Z}_n &= \left(-\rho_0\sigma^2 - \frac{4}{r}\rho_0 g + 4\mu\frac{3\lambda+2\mu}{\lambda+2\mu}\frac{1}{r^2}\right)U_n \\
&\quad - \frac{4\mu}{\lambda+2\mu}\frac{1}{r}Z_n - \rho_0\left(Q_n + \frac{\sigma+\omega}{\omega}\frac{dW_t}{dr}\delta_n{}^2\delta_m{}^1\right) - \rho_{0\,r}F_n \\
&\quad + \left(\frac{n(n+1)}{r}\rho_0 g - \frac{2n(n+1)\mu}{\lambda+2\mu}\frac{(3\lambda+2\mu)}{r^2}\right)V_n + \frac{n(n+1)}{r}Y_n, \\
\dot{V}_n &= -\frac{1}{r}U_n + \frac{1}{r}V_n + \frac{1}{\mu}Y_n, \\
\dot{Y}_n &= \left(\frac{1}{r}\rho_0 g - \frac{2\mu(3\lambda+2\mu)}{(\lambda+2\mu)r^2}\right)U_n - \frac{\lambda}{\lambda+2\mu}\frac{1}{r}Z_n \\
&\quad - \frac{\rho_0}{r}\left(H_n + \frac{\omega+\sigma}{\omega}W_t\delta_n{}^2\delta_m{}^1\right) - \rho_{0\,\theta}F_n + \left\{-\rho_0\sigma^2\right. \\
&\quad \left. + \frac{2\mu}{\lambda+2\mu}\left((2n^2+2n-1)\lambda + 2(n^2+n-1)\mu\right)\frac{1}{r^2}\right\}V_n - \frac{3}{r}Y_n, \\
\dot{H}_n &= 4\pi G\rho_0 U_n + Q_n, \\
\dot{Q}_n &= -4\pi G\rho_0\frac{n(n+1)}{r}V_n + \frac{n(n+1)}{r^2}H_n - \frac{2}{r}Q_n,
\end{aligned}\right\} \quad (41)$$

where U_n is the radial displacement, Z_n the change in normal stress, V_n the transverse displacement, Y_n the change in transverse stress, H_n the change in gravitational potential, and Q_n the change in radial gravitational flux density.

The set of equations (41) can be solved numerically for the inner core and mantle respectively. The inner core solution must satisfy the regularity conditions at the origin while the mantle solution is subject to the conditions at the free surface which require that stress across the boundary vanishes and that the change in gravitational potential is harmonic. These are

$$Z_n(r_e) = Y_n(r_e) = 0,$$

and

$$H_n(r_e) + r_e Q_n(r_e)/(n+1) = 0,$$

where r_e is the radius of the Earth.

The solution for the outer core is related to the solutions for mantle and inner core through conditions at the outer core boundaries. These conditions are the continuity of: the normal displacement, the change in normal stress, the change in transverse stress, the change in gravitational potential, and the change in normal gravitational flux density. Let a be the radial distance to the outer core–mantle boundary. Then we have

$$\left.\begin{aligned} \eta_n(a-) &= W_0'(a)\, U_n(a+), \\ \lambda(a-)\,\Delta_n\,(a-) &= Z_n(a+), \\ 0 &= Y_n(a+), \\ H_n(a-) &= H_n(a+), \\ H_n'(a-)-4\pi G\rho_0(a-)\frac{\eta_n(a-)}{W_0'} &= Q_n(a+). \end{aligned}\right\} \quad (42)$$

A similar set of conditions obtains at the inner core—outer core boundary. The boundary conditions (42) involve errors of the order of the ellipticity of the boundary. A complete formulation for the boundary conditions has been given by Smith (1974). However, for our present purpose, such formulation is unnecessary because errors of at least the order of ellipticity exist in the core solution.

4. Spheroidal oscillations of the Earth of degree 2 and order 1—free core oscillations and diurnal earth tides

The importance of this class of spheroidal oscillations lies in the fact that the axis of rotation of the earth is disturbed. We derive first the simplified set of differential equations using the approximation (32). Substituting $n = 2$ and $m = 1$ we have

$$\eta_2 = -\frac{2b}{r}\left(\frac{1}{3}T_1+\frac{1}{7}V_2+\frac{4}{7}T_3\right)+\left(\frac{4}{7}b-g\right)U_2, \quad (43.1)$$

$$\Delta_2 = \dot{U}_2+\frac{2}{r}U_2-\frac{6}{r}V_2, \quad (43.2)$$

$$X_2 = \frac{\lambda_s}{\rho_s}\Delta_2+\frac{4}{7}\frac{\lambda_s}{\rho_s}\left(\frac{\dot{\lambda}_s}{\lambda_s}-\frac{\dot{\rho}_s}{\rho_s}\right)\frac{b(r)}{W_0'}\Delta_2, \quad (43.3)$$

$$(\omega\sigma+\sigma^2)\,T_1-\frac{9}{5}\omega\sigma\,V_2 = \frac{3}{5}\omega\sigma\,U_2-\frac{3}{5}\frac{\beta b}{r}\Delta_2-\omega\sigma\varepsilon r, \quad (43.4)$$

$$\frac{8}{15}\omega\sigma\,V_2-\left(\frac{1}{6}\omega\sigma+\sigma^2\right)T_3 = \frac{4}{15}\omega\sigma\,U_2+\frac{1}{15}\frac{\beta b}{r}\Delta_2, \quad (43.5)$$

$$r\,B_2 = \eta_2+X_2+H_2, \quad (43.6)$$

$$\frac{d}{dr}(rB_2) = -\sigma^2 U_2-\beta\left(g-\frac{4}{7}b\right)\Delta_2 + \omega\sigma\left(\frac{2}{3}T_1-2V_2-\frac{32}{7}T_3\right)+\frac{2}{3}\omega\sigma\varepsilon r, \quad (43.7)$$

$$\ddot{H}_2+\frac{2}{r}\dot{H}_2+\left(\frac{4\pi G\rho_s'}{W_0'}-\frac{6}{r^2}\right)H_2 = +4\pi G\rho_s\,\beta\Delta_2+\frac{4\pi G\rho_s'}{W_0'}rB_2, \quad (43.8)$$

where

$$B_2 = \left(\frac{2}{3}\omega\sigma+\frac{1}{3}\sigma^2\right)T_1+\left(\frac{2}{7}\omega\sigma-\sigma^2\right)V_2 + \left(\frac{8}{7}\omega\sigma-\frac{16}{7}\sigma^2\right)T_3-\frac{8}{7}\omega\sigma\,U_2. \quad (44)$$

In (43), the function H_2 has been modified to include the term $(\omega+\sigma)W_t/\omega$, and the external force density $\mathbf{F}$ has been set to zero.

The equations (43) are arranged to facilitate numerical computations. For example, (43.4), (43.5), and (43.6) can be solved algebraically for T_1, V_2 and T_3 in terms of U_2, Δ_2, H_2, and ε. Thus the set of equations (43) can be conveniently rearranged as a fourth-order ordinary differential equation in U_2, Δ_2, and H_2. The constant ε is related to the motion of the axis of rotation of the Earth and must be determined from Euler's equation for the angular momentum.

The coefficients $(\omega\sigma+\sigma^2)$ for T_1 and $(\omega\sigma+6\sigma^2)$ for T_3 in (43.4) and (43.5) are important. When $(\omega\sigma+\sigma^2)\simeq 0$, T_1 approaches infinity. The result is an inertial oscillation of the Earth with a period of 23·883 hr. Similarly inertial oscillation takes place when $(\omega\sigma+6\sigma^2)\simeq 0$ and T_3 approaches infinity.

Equations (43.4), (43.5) and (43.6) show that except for inertial oscillations of the outer core, the functions T_1, U_2, V_2 and T_3 are of the same order of magnitude when σ is comparable to ω. This suggests that the neglected fields $\mathbf{S}_n{}^1$ and $\mathbf{T}_n{}^1$ may also be of the same order of magnitude. Therefore for such core oscillations, the approximation (32) is incomplete as has been emphasized earlier. On the other hand, for the inertial oscillation with a period of 23·883 hr, $\mathbf{T}_1{}^1$ is the dominating displacement field in the outer core. The ellipticity couples a small $\mathbf{S}_2{}^1$ to $\mathbf{T}_1{}^1$. But $\mathbf{S}_4{}^1$, $\mathbf{T}_5{}^1$, and so on can be expected to be negligible. Thus in this case, (32) is a good approximation. However, this conclusion must await numerical confirmation.

In vector notation, Euler's equation is (Munk & Macdonald 1960)

$$\frac{\partial \mathbf{M}}{\partial t}+\mathbf{\Omega}\times\mathbf{M}=\mathbf{L}, \tag{45}$$

where $\mathbf{M}$ is the angular momentum,

$\frac{\partial \mathbf{M}}{\partial t}$ is the time derivative of $\mathbf{M}$ in the rotating frame,

$\mathbf{L}$ is the external torque,

$$\mathbf{\Omega}=(\omega\varepsilon\cos\sigma t,\ \omega\varepsilon\sin\sigma t,\ \omega). \tag{46}$$

The components of $\mathbf{M}$, to first order in ε, in the cartesian system, are

$$\left.\begin{aligned} M_x &= \omega(I_{xx}\,\varepsilon\cos\sigma t-I_{xz})+\Delta M_x,\\ M_y &= \omega(I_{yy}\,\varepsilon\sin\sigma t-I_{yz})+\Delta M_y,\\ M_z &= \omega I_{zz}+\Delta M_z. \end{aligned}\right\} \tag{47}$$

Here I_{ij} are components of the inertia tensor and $\Delta\mathbf{M}$ is the change in angular momentum due to motion relative to the rotating frame. The rather simple form of (47) is due to proper choice of the rotating earth fixed system.

We have

$$\Delta\mathbf{M}=\int_{\text{earth}}\rho_0\mathbf{r}\times\frac{\partial \mathbf{u}}{\partial t}\,d\tau. \tag{48}$$

Using (32) in (48) we obtain

$$\Delta M_x=\sigma\zeta\cos\sigma t, \tag{49}$$

$$\Delta M_y=\sigma\zeta\sin\sigma t,$$

correct to first order in the ellipticity of the Earth with

$$\zeta = \frac{8\pi}{3} \int_{\text{outer core}} \rho r^3 T_1 \, dr. \tag{50}$$

In the present problem, toroidal fields $\mathbf{T}_1{}^1$ is neglected in the mantle and inner core so that the integration in (50) is over the outer core only.

The products of inertia I_{xz} and I_{yz} are due to the redistributions of volume density

$$\rho - \rho_0 = -\rho_0 \Delta - \frac{\rho_0'}{W_0'} \eta,$$

and the surface density $\rho\eta/W_0'$ at every surface of discontinuity.

$$\left.\begin{aligned} I_{xz} &= \int_\tau (\rho-\rho_0)\, xz \, d\tau + \sum_s \int_s \frac{\rho\eta}{W_0'} xz \, ds, \\ I_{yz} &= \int_\tau (\rho-\rho_0)\, yz \, d\tau + \sum_s \int_s \frac{\rho\eta}{W_0'} yz \, ds \end{aligned}\right\} \tag{51}$$

Using (22) it can be shown that

$$\left.\begin{aligned} I_{xz} &= f \cos \sigma t, \\ I_{yz} &= f \sin \sigma t, \end{aligned}\right\} \tag{52}$$

where

$$f = \frac{1}{5G} d^3 \left(2H_2(d) - d\, Q_2(d)\right) \delta_m{}^1, \tag{53}$$

where d is the radius of the Earth, and H_2 and Q_2 are defined in (41).

Using (49), (51) and (47) in (45) and letting the external torque $\mathbf{L} = \mathbf{0}$ we get, for free oscillation,

$$\varepsilon - \frac{\sigma+\omega}{\omega^2} \frac{1}{C} \left(\omega(A\varepsilon - f) + \sigma\zeta\right) = 0, \tag{54}$$

where $C = I_{zz}$ is the polar moment of inertia, and $A = I_{xx} = I_{yy}$ the equatorial moment of inertia.

The equations (54), (41) and (43) with the help of conditions at the origin, the surface, and the outer core boundaries completely determine the solution.

The equations developed above can be conveniently applied to the problem of diurnal earth tides and nutations by the inclusion of the forcing terms. The diurnal tidal potentials are of the form

$$W_2{}^1(r, \theta, \phi) = A_m r^2 \, P_2{}^1 (\cos\theta) \cos(\sigma t - \phi) \tag{55}$$

where A_m is the amplitude.

Inclusion of the tidal potential is effected by replacing the function H_2 in (43) with the function $H_2 + A_m r^2$.

The torque exerted on the equatorial bulge by the tidal force is given by

$$\mathbf{L} = \int \rho_0 \mathbf{r} \times \nabla W_2{}^1 \, d\tau. \tag{56}$$

In Cartesian co-ordinates

$$\left.\begin{aligned} L_x &= -\mathscr{L} \sin \sigma t, \\ L_y &= \mathscr{L} \cos \sigma t, \\ L_z &= 0, \end{aligned}\right\} \tag{57}$$

where $\mathscr{L} = A_m (C - A)$.

The inclusion of the torque (56) is effected by replacing the right-hand side of (54) by $-(\mathscr{L}/\omega^2 C)$.

At this point, it is interesting to show that the theory of diurnal earth tides and nutations by Molodensky (1961) can be derived from our equations. Molodensky assumed that the Adams & Williamson condition (Adams & Williamson 1923) is satisfied in the liquid core so that

$$\beta(r) = -\frac{\lambda \rho_0'}{\rho_0^2 W_0'} + 1 = 0. \tag{58}$$

Next, we observe that for diurnal earth tides $(\omega+\sigma)/\omega \ll 1$. From the equations (43.4), (43.5) and (43.6) it can be seen that V_2, U_2 and T_3 are of the order $(\omega+\sigma)\, T_1/\omega$. Thus, equation (44) may be approximated by

$$B_2 = \left(\frac{2}{3}\omega\sigma + \frac{1}{3}\sigma^2\right) T_1. \tag{59}$$

Using (59) in (43.7), and neglecting small terms, we get

$$\left(\frac{2}{3}\omega\sigma + \frac{1}{3}\sigma^2\right)\frac{d}{dr}(r\, T_1) = \frac{2}{3}\omega\sigma T_1 + \frac{2}{3}\omega\sigma\varepsilon r.$$

The last equation means

$$T_1 = -\alpha r, \tag{60}$$

where α is the resonant parameter in Molodensky's theory. If we write

$$K(r) = H_2(r) - rB_2(r), \tag{61}$$

then (43.8) becomes

$$\ddot{K} + \frac{2}{r}\dot{K} + \left(\frac{4\pi G\rho_s'}{W_0'} - \frac{6}{r^2}\right) K = 0. \tag{62}$$

This is equation (30) in Molodensky's paper. Now, the function $b(r)$ given by (13) satisfies (62) so that we can write

$$2b(r) = K_1(r). \tag{63}$$

Next, we rewrite (43.4) as

$$9V_2 = -3U_2 + 5\left(\varepsilon - \frac{\sigma+\omega}{\omega}\alpha\right) r. \tag{64}$$

Substituting (64) in (43.2), we find

$$\Delta_2 = \frac{1}{r^4}\frac{d}{dr}(r^4 U_2) - \frac{10}{3}\left(\varepsilon - \frac{\sigma+\omega}{\omega}\alpha\right). \tag{65}$$

To relate U_2 to η_2, we neglect terms of the order $\sigma+\omega/\omega$ in (43.1), and get

$$\eta_2 = -\frac{2}{3}\cdot\frac{b}{r} T_1 - gU_2.$$

Upon using (60) and (63), the above equation becomes

$$U_2 = \frac{1}{W_0'}\left(\eta_2 + \frac{1}{3}\alpha K_1\right). \tag{66}$$

Now, using (61) and (58) in (43.6) and neglecting small terms, we get

$$\Delta_2 = -\frac{\rho_s'}{\rho_s W_0'}\left(\frac{1}{3}K+\eta_2\right). \tag{67}$$

Combining (65), (66) and (67), we find

$$\frac{d}{dr}\left(\frac{\rho_s r^4 \eta_2}{W_0'}+\frac{1}{3}\alpha\frac{\rho_s r^4 K_1}{W_0'}\right)+\frac{1}{3}\frac{\rho_s' r^4}{W_0'}(K-\alpha K_1) = \frac{10}{3}r^4\rho_s\left(\varepsilon-\frac{\sigma+\omega}{\omega}\alpha\right). \tag{68}$$

But from (62),

$$r^4\frac{\rho_s'}{W_0'}(K-\alpha K_1) = -\frac{1}{4\pi G}\frac{d}{dr}\left(r^6\frac{d}{dr}\left(\frac{K-\alpha K_1}{r^2}\right)\right). \tag{69}$$

Substituting (69) in (68) and integrating over the outer core, we finally get

$$\left\{\frac{3}{W_0'}\rho_s r^4\eta_2+\frac{\alpha}{W_0'}\rho_s r^4 K_1-\frac{r^6}{4\pi G}\frac{d}{dr}\left(\frac{K-\alpha K_1}{r^2}\right)\right\}_c^b = 5\nu\int_c^b \rho_s r^4\,dr. \tag{70}$$

This is equation (39) in Molodensky's paper, except that here the constant ν is given by

$$\nu = 2\left(-\frac{\omega+\sigma}{\omega}\alpha+\varepsilon\right), \tag{71}$$

while in Molodensky's theory it is given by

$$\nu = 2\left(\frac{\omega+\sigma}{\sigma}\alpha-\frac{\omega}{\sigma}\varepsilon\right). \tag{72}$$

However, this difference can be eliminated by replacing the parameter α and the function K_1 with $(\omega/\sigma)\alpha$ and $(\sigma/\omega)K_1$ respectively.

We note that in deriving Molodensky's equations, equation (43.5) has not been used. This means that Molodensky's theory is based on a displacement for the outer core of the form

$$\mathbf{u} = \mathbf{T}_1^{\,1}+\mathbf{S}_2^{\,1}, \tag{73}$$

as compared to our present form of

$$\mathbf{u} = \mathbf{T}_1^{\,1}+\mathbf{S}_2^{\,1}+\mathbf{T}_3^{\,1}. \tag{74}$$

5. Numerical calculation and results

The numerical computations were performed on earth models with different polytropic cores (Pekeris & Accad 1972). The interest in using these earth models stems from the fact that the function

$$\beta(r) = -\frac{\lambda\rho_0'}{\rho_0^2 W_0'}+1$$

determines the stability of the outer core. The restoring force, when a particle is displaced radially, is proportional to $\beta(r)$. The core is in neutral equilibrium when

Table 1

Earth models for different β

r (km)	C_p (km s^{-1})	C_s (km s^{-1})	M_3 ρ_0 (g cm^{-3})	$\beta = -0{\cdot}2$ ρ_0 (g cm^{-3})	$\beta = 0{\cdot}0$ ρ_0 (g cm^{-3})	$\beta = +0{\cdot}2$ ρ_0 (g cm^{-3})
6371	6·30	3·55	2·840			
6338	6·30	3·55	2·840			
6338	8·16	4·65	3·386			
6311	8·15	4·60	3·474			
6271	8·00	4·40	3·488			
6221	7·85	4·35	3·462			
6171	8·05	4·40	3·413			
6071	8·50	4·60	3·374			
5958	9·06	5·00	3·569			
5871	9·60	5·30	3·812			
5771	10·10	5·60	4·047			
5671	10·50	5·90	4·215			
5571	10·90	6·15	4·373			
5471	11·30	6·30	4·502			
5371	11·40	6·35	4·613			
5171	11·80	6·50	4·852			
4971	12·05	6·60	4·955			
4771	12·30	6·75	5·040			
4571	12·55	6·85	5·066			
4371	12·80	6·95	5·072			
4171	13·00	7·00	5·085			
3971	13·20	7·10	5·090			
3771	13·45	7·20	5·092			
3571	13·70	7·25	5·086			
3491	13·70	7·20	5·239			
3473	13·65	7·20	5·279			
3473	8·04		10·087	9·795	10·020	10·246
3123	8·44		10·637	10·449	10·573	10·693
2776	8·90		11·082	11·023	11·051	11·073
2429	9·31		11·478	11·517	11·457	11·392
2082	9·63		11·809	11·939	11·799	11·657
1735	9·88		12·079	12·293	12·084	11·876
1388	10·08		12·290	12·581	12·314	12·052
1318·6	10·11		12·321	12·630	12·354	12·082
1297·8	10·11		12·330	12·645	12·365	12·091
1283·9	10·17		12·337	12·654	12·373	12·097
1249·2	10·48		12·352	12·677	12·390	12·110
1249·2	10·48	3·16	12·352	12·677	12·390	12·110
1214·5	10·76	3·16	12·368	12·697	12·407	12·123
1179·8	10·93	3·16	12·382	12·717	12·422	12·135
1145·1	11·04	3·16	12·400	12·735	12·437	12·146
1110·4	11·09	3·16	12·412	12·753	12·451	12·156
1075·7	11·12	3·16	12·429	12·770	12·464	12·166
1041·0	11·13	3·16	12·443	12·786	12·477	12·176
867·5	11·15	3·16	12·501	12·860	12·536	12·221
694·0	11·17	3·16	12·551	12·921	12·584	12·257
520·5	11·17	3·16	12·590	12·968	12·621	12·285
347·0	11·16	3·16	12·614	13·003	12·648	12·306
173·5	11·15	3·16	12·629	13·023	12·665	12·318
0·0	11·15	3·16	12·635	13·030	12·670	12·322

The parameter β is defined in equation (29). The model is M_3 of Pekeris & Accad (1972) modified to allow for a solid inner core. For $\beta = +0{\cdot}2$ the density in the core has been slightly altered to correspond to the total mass of the Earth.

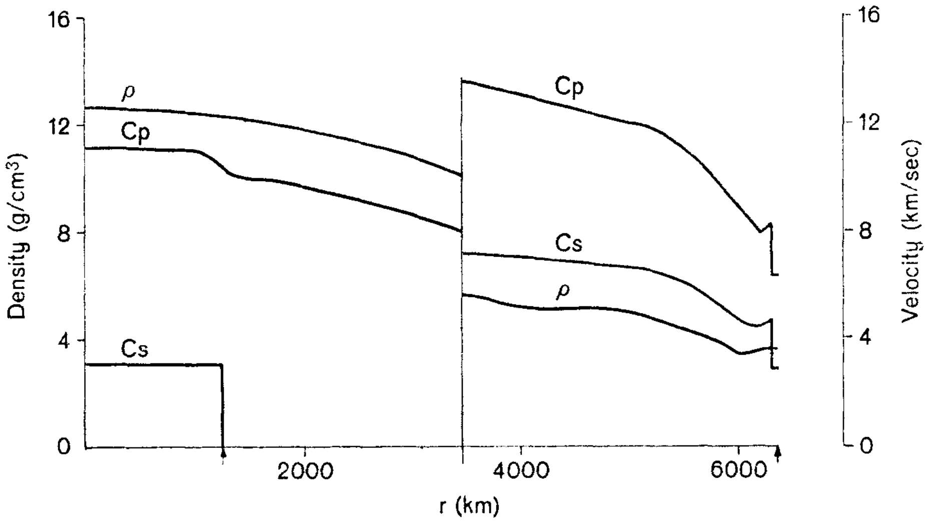

FIG. 2. Mechanical properties of the earth mode M_3 (Pekeris 1966).

$\beta = 0$ (Adams & Williamson 1923); for $\beta < 0$, the core is stable and for $\beta > 0$ the core is unstable.

Three models with $\beta = 0{\cdot}2$, $0{\cdot}0$ and $-0{\cdot}2$ (equation (29) are listed in Table 1 and plotted in Figs 2, 3(a) and (b). The models are derived from earth model M3 (Pekeris 1966) by Pekeris & Accad (1972). We follow Smylie (1974) with the introduction of a solid inner core. However, the radius of the inner core is set at 1249·2 km instead of 1214·5 km (Smylie 1974). This may affect slightly the periods of undertones.

For $\beta = +0{\cdot}2$ the density in the core has been slightly increased as the density given by Pekeris & Accad leads to a deficiency in the total mass of the Earth.

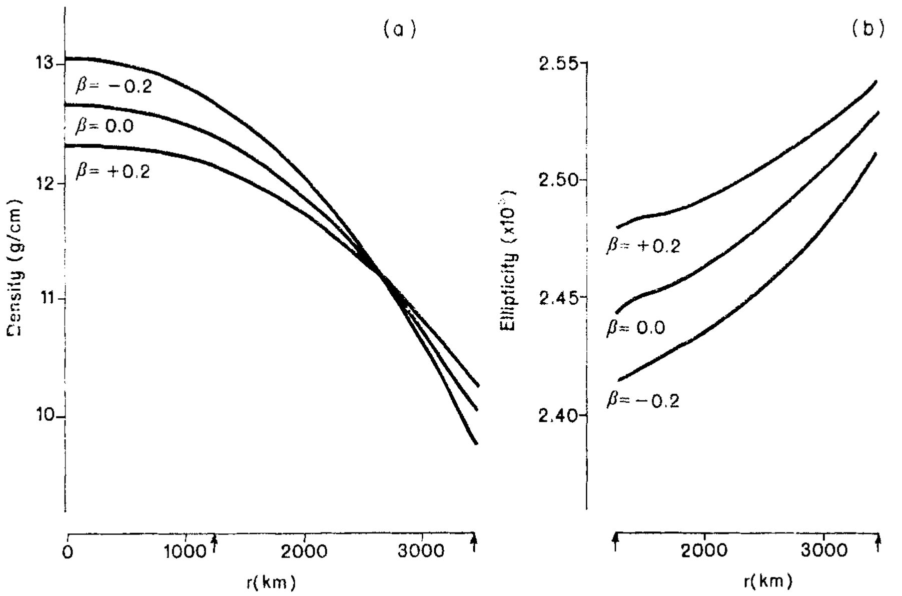

FIG. 3(a). Density distribution of the uniform polytropic cores given in Table 1. (b) Ellipticity in the outer core.

A neutral or unstable core is incapable of free oscillation. This and some other characteristic dynamic responses of the uniform polytropic cores have been demonstrated by Pekeris & Accad. Their conclusions are true for a spherical non-rotating earth. Our results show that in an elliptical rotating earth all three polytropic core models are capable of free oscillation.

Equations (41) and (43) were numerically integrated with spline interpolation and the Runge–Kutta scheme. The integration was initiated at the centre of the Earth and proceeded radially outwards. The step size was varied so that the ratio between the step size and the radius remains constant. It has been found that the integration is stable if this ratio is 0·004 or smaller.

The centre of the Earth is a singular point for equations (41). Therefore the numerical integration must be started at a finite distance (say r_1) from the centre. Thus initial solutions of (41) at r_1 must be found. One method is to assume r_1 is sufficiently small so that power series expansion of (41) can be applied. However, an equivalent approach is to assume that for $r \leqslant r_1$ the Earth is homogeneous. The exact solutions of (41) for $r \leqslant r_1$ are given by Love (1911). Since equation (41) constitutes a sixth order differential equation at r_1 there exist three independent solutions. Integration by Runge–Kutta method carry the free constants to the top of the inner core where one is eliminated by the vanishing of the transverse stress. At the bottom of the outer core, the constant ε is introduced. The new set of three constants is then propagated to the top of the outer core by numerical integration of (43). At the bottom of the mantle another free constant is introduced to account for the discontinuity in transverse displacement. The four constants are finally determined at the free surface by the three boundary conditions and equation (54).

5.1 *Free core oscillations and free wobbles*

The periods of free spheroidal oscillations of degree 2 and order 1 have been computed. The 'elastic' modes are ordinarily designated $S_2{}^1$ with fundamental mode as ${}_0S_2{}^1$. We follow Dahlen (1974) and Crossley (1975) by designating the νth undertone as ${}_{-\nu}S_2{}^1$. For convenience ${}_{-\nu}S_2{}^1$ is generally written as $S_2{}^1$ unless specifically called for. Two notations ($S_2{}^1C$ and $S_2{}^1\,T_1$), described later, are introduced to describe different classes of $S_2{}^1$. Table 2 lists periods of ${}_{-\nu}S_2{}^1$ and ${}_0S_2{}^1$ for different models. An upper bound for all possible periods has not yet been identified. Our computations have been arbitrarily limited to less than 28 hr.

Table 2

Periods, in hours, of free spheroidal core oscillations of degree 2 *and order* 1.

	$\beta = -0{\cdot}2$	$\beta = 0{\cdot}0$	$\beta = +0{\cdot}2$
${}_0S_2{}^1$	0·8894	0·8906	0·8906
$S_2{}^1C$	6·585	11·789	9·493
	8·992	14·508	10·664
	11·182	17·023	12·883
	12·477		13·664
	14·492		15·086
	17·417		15·727
	18·382		17·522
	19·462		20·211
	20·132		22·400
	25·332		27·931
	27·802		
$S_2{}^1\,T_1$	23·883	23·883	23·883

The parameter β is defined in equation (29)

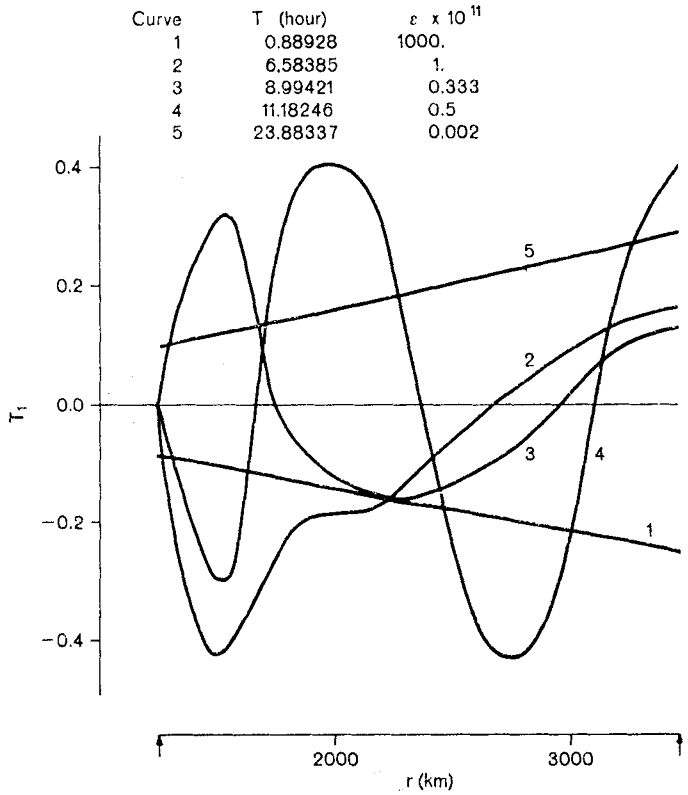

FIG. 4. The toroidal displacement T_1 in the outer core for free spheroidal oscillation for $n = 2$, $m = 1$ and $\beta = -0{\cdot}2$. Relative normalisation is indicated by the amplitude of the free wobble, ε, for Figs 4–7.

All three earth models are capable of free core oscillations in the case of a rotating, elliptical earth. Pekeris & Accad (1972) showed that in a spherical non-rotating earth, only stable cores are capable of free oscillations. Smylie (1974) and Crossley (1975) considered the effects of rotation alone on S_2^2 modes, and only for stable cores. Further, Smylie (1974), considered the effects of self-coupling and set 36 hr as the upper bound for free periods of S_2^2. Crossley (1975), set 12 hr as the upper bound for free periods of S_2^2. Crossley further inferred that core oscillations with periods greater than 12 hr are inertial oscillations. However, our results have shown no such critical barrier to the free period. We have found only one distinct inertial oscillation with a period of 23·883 hr. It is difficult to compare our results with those of Smylie & Crossley, since we are considering S_2^1 oscillations while they are considering S_2^2 oscillations. But it is possible that the disagreement is due to the effects of ellipticity.

The period of ${}_0S_2^1$ depends only slightly on β, the parameter determining core stability. But the spectrum of S_2^1 is strongly governed by the value of β. We designate a subclass of S_2^1 modes with strong dependence on β as S_2^1C.

The only core mode that is independent of β is the one with a period of 23·883 hr, 3 mins short of a sidereal day. This is designated $S_2^1\ T_1$ because of the existence of large T_1 toroidal motions in the liquid core.

The different characteristics of ${}_0S_2^1$, $S_2^1\ C$ and $S_2^1\ T_1$ prompts us to examine the dynamic behaviour of the Earth under these oscillations. In Figs 4, 5, 6 and 7, we plot the functions T_1, η_2, Z_2 and H_2 respectively for ${}_0S_2^1$, three $S_2^1\ C$ and $S_2^1\ T_1$.

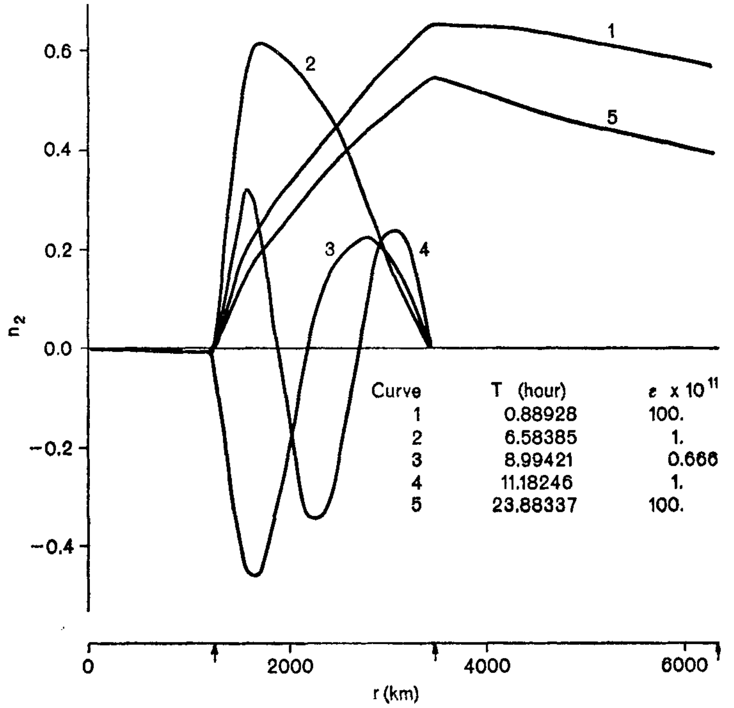

FIG. 5. The normal displacement η_2 for free spheroidal oscillations for $n = 2$, $m = 1$ and $\beta = -0{\cdot}2$.

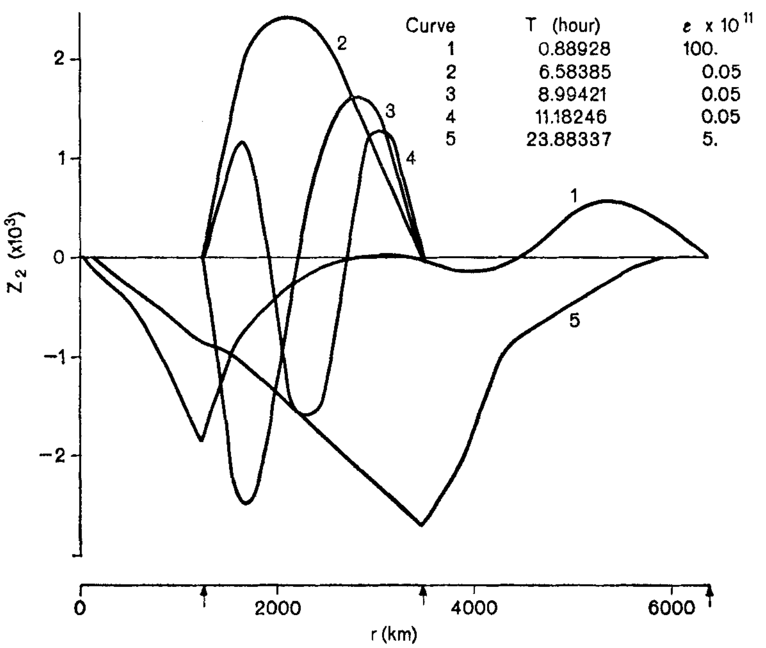

FIG. 6. The normal stress Z_2 for free spheroidal oscillations for $n = 2$, $m = 1$ and $\beta = -0{\cdot}2$.

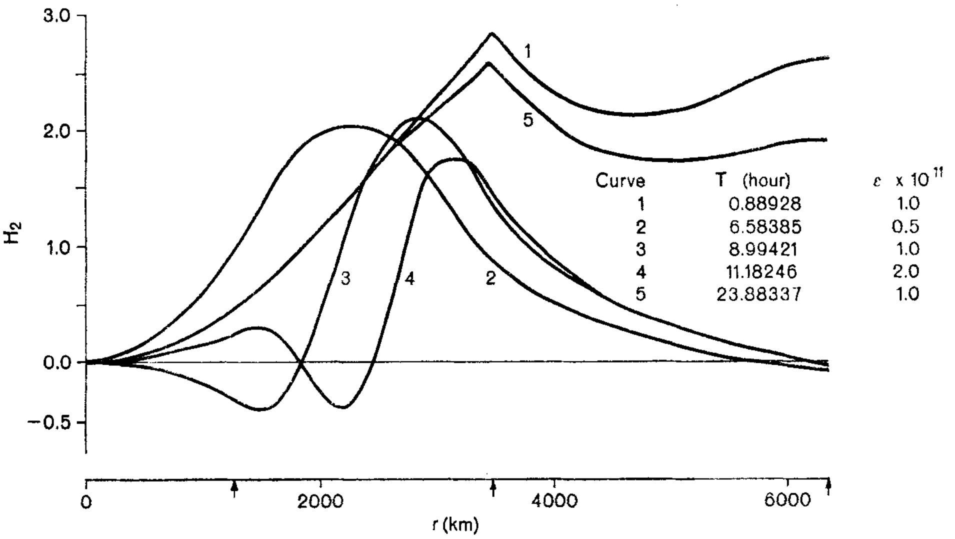

FIG. 7. The change in gravitational potential H_2 for free spheroidal oscillation for $n = 2$, $m = 1$ and $\beta = -0{\cdot}2$.

The earth model used is the one with stable core $\beta = -0{\cdot}2$. The functions are normalized with respect to the amplitude of the associated wobble. The normalization is relative and not absolute. For example, in Fig. 4, the normalization means that T_1 for curve 1 is magnified by 1000/0·0002 relative to T_1 for curve 5.

For ${}_0S_2{}^1$, the fundamental elastic mode, the function T_1 is non-zero but very small compared to η_2. This means that the motions in the outer core is predominantly spheroidal. Since T_1 is linear in radius, the outer core rotates as a rigid body relative to the mantle and inner core. But the amplitude of this relative rotation is small. The dependence of η_2, Z_2 and H_2 on radius shows that for this mode, the entire earth is deformed.

In the case of $S_2{}^1\,C$, the spheroidal core modes, the functions T_1 and η_2 are of the same order of magnitude in the outer core. A rough interpretation of this is that gravitational and inertial forces play about equal roles. The radial dependence of T_1 shows that the outer core does not rotate rigidly with respect to the mantle and inner core. This type of free oscillations has displacements and stresses mainly confined to the outer core. Only the change in gravitational potential has significant distribution in the mantle.

For $S_2{}^1\,T_1$, the inertial oscillation mode, the most significant feature is in the outer core, with $T_1 \gg \eta_2$. From a comparison of curve 5 on Fig. 4 with that on Fig. 5, it can be seen that $T_1 \sim 2{\cdot}5\times 10^4\,\eta_2$. Since T_1 is a linear function of radius, the dominant motion of the outer core is a rigid rotation relative to the mantle and inner core. The relative rotation depends primarily on the moment of inertia of the outer core and not the density stratification. This explains why the period is so insensitive to the stability of the outer core. One interesting feature of $S_2{}^1\,T_1$ is that apart from the large rigid rotation of the outer core relative to the mantle and inner core, the response of the Earth resembles that of ${}_0S_2{}^1$. However, the significant distribution of η_2 in the mantle is due to the existence of ellipticity at the core–mantle boundary. Without the ellipticity the large T_1 will not be able to contribute to η_2 in the mantle. Therefore it is incorrect to treat inertial oscillations of the Earth without considering the effects of ellipticity.

The fact that T_1 for $S_2{}^1\,T_1$, is large can be seen from equations (43.4), (43.5) and (43.6). Since for $S_2{}^1\,T_1$, $\omega\sigma+\sigma^2 \sim 0$, we expect T_1 to be of the order of

Table 3

The radial coefficients of the displacement and change of gravity at the surface for some $S_2{}^1$ free oscillations

	Period (hr)	Radial displacements (cm)	Transverse displacements (cm)	Change in gravity (mgal)
${}_0S_2{}^1$	0·8893	2710	60	−0·528
$S_2{}^1C$	6·584	−600	61	−1·048
$S_2{}^1C$	8·994	−372	43	−0·763
$S_2{}^1C$	11·182	−228	24	−0·419
$S_2{}^1\,T_1$	23·883	1881	9	−0·134

The amplitude of the related wobbles is assumed to be 1 arc sec

$(\omega\sigma/(\omega\sigma+\sigma^2))U_2$. Such consideration of the existence of $S_2{}^1\,T_1$ at an angular frequency $|\sigma| \sim \omega$ raises the possibility of a $S_2{}^1\,T_3$ at an angular frequency of $|\sigma| \sim \omega/6$. And if we use the exact equations (34)–(40) for the outer core, we would expect to have $S_2{}^1\,T_5$, $S_2{}^1\,T_7$ and so on. However, this conjecture must await numerical confirmation.

The wobble that is associated with $S_2{}^1\,T_1$ is generally referred to as the ' nearly diurnal wobble ' and has recently been discussed by Toomre (1974) and Rochester *et al.* (1974). Toomre (1974) has mentioned that the existence of the liquid outer core should lead to more wobbles of the Earth than just the nearly diurnal wobble. The present work confirms Toomre's conclusions. Any $S_2{}^1$ oscillation is associated with a wobble. Table 2 gives the spectrum of wobbles induced by the existence of the liquid outer core. However, the $S_2{}^1\,C$ wobbles are different in character as compared to the $S_2{}^1\,T_1$ wobble. We note here that due to our sign convention for the angular frequency σ the wobbles associated with $S_2{}^1$ are retrograde. Prograde wobbles are associated with $S_2{}^{-1}$ but the discussions will be deferred to another report.

The periods of $S_2{}^1\,C$ free oscillations depend strongly on the stability of the outer core. If $S_2{}^1\,C$ can be observed on the surface the density stratification of the outer core can be resolved. The excitation of these core oscillations by earthquakes is possible since displacement fields exist in the mantle. But the input of energy into these modes from earthquakes is unknown. And observation will be difficult because the energy is concentrated in the outer core as evident from Figs 4, 5, 6 and 7. We provide the kinematic response of the ${}_0S_2{}^1$, $S_2{}^1\,T_1$ and three $S_2{}^1\,C$ at the free surface in Table 3. The radial and transverse displacements, and the change in gravity at the surface of the Earth are given by arbitrarily assuming the amplitude of the wobble to be 1″00.

5.2 *Diurnal earth tides and nutations*

The principal components of diurnal tides and associated nutations are given in Table 4 (Melchior 1966). The theoretical amplitudes of the nutations are those for a hypothetical rigid earth. The discrepancies between these and the observed values are genuine and can only be removed by considering the relative rotation of the liquid core (Melchior 1971).

Table 5 gives the amplitudes of nutations calculated from the present theory and the theory of Molodensky (1961). The close agreement between the two theories indicates that for the $S_2{}^1\,T_1$ nearly diurnal mode (73) is as good an approximation as (74). It appears that for this mode, higher order approximations for the displacement would still give the same results. The resonance effects at the frequency of $S_2{}^1\,T_1$ lead to the correction $(\varepsilon-\varepsilon_0)/\varepsilon_0$. Comparison of the results with the observed values given in Table 5 shows that the agreement is very good. Perfect agreement cannot be

Table 4

Principal diurnal Earth tides and astronomical nutations

Symbol	Diurnal tides: Doodson's Argument	Diurnal tides: Frequency in degrees per hour	North–South component of acceleration at the Equator (gals)	Period in sidereal days	Nutations: Amplitude in longitude	Nutations: Observed (O) Theoretical (T)* in obliquity
Q_1	135·655	13·3986609	+5·9428	9·157938		
	195·455	16·6834763	−0·2561			
O_1	145·555	13·9430356	+31·0391	13·698192	0″093 (O)	0″097 (O)
OO_1	185·555	16·1391017	−1·3366	Fortnightly	0″0876 (T)	0″0944 (T)
M_1	155·655	14·4966940	−2·4410	27·629992		
J_1	175·455	15·5854433	−2·4410			
π_1	162·556	14·9178647	+0·8474	122·082681		
	168·554	15·1642724	−0·0362			
P_1	163·555	14·9589314	+14·4815	183·121117	0″529 (O)	0″575 (O)
	167·555	15·1232059	−0·6226	semi-annual	0″5104 (T)	0″5558 (T)
S_1	164·556	15·0000020	−0·3484	366·259758		
ψ_1	166·554	15·0821353	−0·3484			
	165·575	15·0454814	−0·1268	3408·493577		
	165·545	15·0388622	+0·8647	6816·987155	6″848 (O)	9″203 (O)
	165·565	15·0432751	+5·9148	principal	6″8672 (T)	9″2232 (T)
K_1	165·555	15·0410686	+43·6898			

Reference: Melchior (1971)

* The theoretical amplitudes of nutations for a rigid earth are those given by Molodensky (1961)

Table 5

Theoretical amplitudes of nutation

Earth Model	Doodson's Argument	$\frac{\varepsilon-\varepsilon_0}{\varepsilon_0}\times 10^6$ Present theory	$\frac{\varepsilon-\varepsilon_0}{\varepsilon_0}\times 10^6$ Molodensky theory	Period in sidereal days	Nutations Amplitude Present theory (P) Molodensky (M) Longitude	Obliquity
	145·555	+28·4		13·698192	0″0899 (P)	0″0973 (P)
	185·555	+73·0				
$\beta = +0{\cdot}2$	163·555	+36·0		183·121117	0″5274 (P)	0″5768 (P)
	167·555	+89·1				
	165·545	+3·29		6816·987155	6″8330 (P)	9″1968 (P)
	165·565	−3·77				
	145·555	+28·4	+28·2	13·698192	0″0899 (P)	0″0973 (P)
	185·555	+76·9	+72·2		0″0899 (M)	0″0974 (M)
$\beta = 0{\cdot}0$	163·555	+35·9	+34·5	183·121117	0″5274 (P)	0″5768 (P)
	167·555	+88·3	+84·1		0″5273 (M)	0″5758 (M)
	165·545	+3·29	+3·17	6816·987155	6″8328 (P)	9″1966 (P)
	165·565	−3·78	−3·64		6″8342 (M)	9″1977 (M)
	145·555	+28·6		13·698192	0″0899 (P)	0″0973 (P)
	185·555	+76·2				
$\beta = -0{\cdot}2$	163·555	+35·8		183·121117	0″5274 (P)	0″5768 (P)
	167·555	+87·4				
	165·545	+3·30		6816·987155	6″8327 (P)	9″1965 (P)
	165·565	−3·79				

Table 6

Diurnal tidal Love numbers

		$\beta = -0\cdot2$			$\beta = 0\cdot0$ Present theory			$\beta = 0\cdot0$ Molodensky's theory			$\beta = +0\cdot2$		
Symbol	Doodson's argument	h	k		h	k	l	h	k	l	h	k	l
Q_1	135·655	0·6096	0·2994	0·08422	0·6101	0·3002	0·08431	0·6096	0·2995	0·09427	0·6103	0·3009	0·09441
0_1	145·555	0·6092	0·2991	0·08421	0·6093	0·2998	0·09434	0·6090	0·2993	0·09431	0·6096	0·3005	0·08444
M_1	155·655	0·6062	0·2978	0·08436	0·6069	0·2986	0·09442	0·6067	0·2983	0·09440	0·6072	0·2992	0·09452
π_1	162·556	0·5928	0·2911	0·09479	0·5932	0·2917	0·09487	0·5936	0·2918	0·08484	0·5935	0·2924	0·08496
P_1	163·555	0·5861	0·2877	0·08501	0·5865	0·2884	0·08509	0·5871	0·2886	0·08505	0·5868	0·2890	0·08517
S_1	164·556	0·5719	0·2806	0·08547	0·5723	0·2813	0·08555	0·5734	0·2817	0·0855	0·5726	0·2819	0·08563
	165·545	0·5272	0·2583	0·08693	0·5278	0·2590	0·28700	0·5302	0·2601	0·08690	0·5284	0·2597	0·08706
K_1	165·555	0·5214	0·2554	0·08712	0·5221	0·2561	0·08718	0·5246	0·2573	0·08709	0·5227	0·2569	0·08724
	165·565	0·5148	0·2521	0·08734	0·5155	0·2528	0·08739	0·5182	0·2541	0·08729	0·5162	0·2536	0·08745
	165·575	0·5071	0·2482	0·08759	0·5079	0·2490	0·08764	0·5108	0·2604	0·08754	0·5087	0·2498	0·08769
ψ_1*	166·554	0·9379	0·4633	0·07350	0·9486	0·4696	0·07333	0·9296	0·4600	0·07393	0·9598	0·4761	0·07314
ϕ_1	167·555	0·6696	0·3294	0·08228	0·6706	0·3304	0·08236	0·6680	0·3291	0·08243	0·6715	0·3315	0·08244
	168·544	0·6434	0·3163	0·08313	0·6441	0·3172	0·08322	0·6426	0·3164	0·08326	0·6447	0·3180	0·08330
J_1	175·455	0·6173	0·3032	0·08399	0·6177	0·3040	0·08408	0·6173	0·3038	0·08409	0·6183	0·3047	0·08415
00_1	185·555	0·6144	0·3018	0·08408	0·6148	0·3025	0·08418	0·6146	0·3026	0·08420	0·6131	0·3030	0·08449
	195·455	0·6136	0·3014	0·08412	0·6139	0·3021	0·08421	0·6138	0·3023	0·08424	0·6141	0·3028	0·08432

* Free oscillation of the Earth occurs at a frequency between tides with Doodson's arguments 165·575 and 166·554. This is true for all three polytropic models.

expected because we have not considered the effects of the core–mantle couplings, the viscosity of the Earth, and the effects of oceans.

The resonance effects of the diurnal tides due to the existence of $S_2{}^1\ T_1$ are also reflected in the diurnal tidal Love numbers. In Table 6, diurnal tidal Love numbers are given for the three uniform polytropic cores. The asymptotic behaviour of the Love numbers at the frequency of $S_2{}^1\ T_1$ is clearly observable. Observations (Melchior 1966, p. 383) have confirmed the general trend of frequency dependence of the Love numbers.

Acknowledgment

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Appendix A

The equations (43) in y notations

Let

$$\left.\begin{aligned} y_1^2 &= U_2, & y_2^2 &= \lambda\Delta_2,\\ y_3^2 &= V_2, & y_4^2 &= 0,\\ y_5^2 &= H_2, & y_6^2 &= \dot{H}_2 - 4\pi G\rho U_2,\\ y_7^1 &= T_1, & y_7^3 &= T_3, \end{aligned}\right\} \quad \text{(A1)}$$

then equations (43.2) and (43.4)–(43.8) becomes

$$\dot{y}_1^2 = -\frac{2}{r}y_1^2 + \frac{1}{\lambda}y_2^2 + \frac{6}{r}y_3^2, \quad \text{(A2)}$$

$$\begin{aligned} \dot{y}_2^2 = {} & \left(-\rho\sigma^2 - \frac{4\rho g}{r}\right)y_1^2 + \frac{6\rho g}{r}y_3^2 - \rho y_6^2 \\ & + 2\rho\omega\sigma\left(-y_3^2 + \frac{1}{3}y_7^1 - \frac{16}{7}y_7^3 - \frac{1}{3}\varepsilon r\right) \\ & + \frac{4}{7}\frac{\beta\rho\dot{b}}{\lambda}y_2^2 + \frac{d}{dr}\left(-\frac{4}{7}\dot{b}\,y_1^2 + \frac{4}{7}\frac{b}{g\lambda}\frac{d}{dr}\left(\frac{\lambda}{\rho}\right)y_2^2\right. \\ & \left. + \frac{2b}{r}\left(\frac{1}{7}y_3^2 + \frac{1}{3}y_7^1 + \frac{4}{7}y_7^3\right)\right). \end{aligned} \quad \text{(A3)}$$

$$\dot{y}_5^2 = 4\pi G\rho\, y_1^2 + y_6^2, \quad \text{(A4)}$$

$$\begin{aligned} \dot{y}_6^2 = {} & -\frac{24\pi G\rho}{r}y_3^2 + \frac{6}{r^2}y_5^2 - \frac{2}{r}y_6^2 \\ & + \frac{4\pi G}{g}\dot{\rho}\left(-\frac{4}{7}\dot{b}\,y_1^2 - \frac{4}{7}\frac{b}{\lambda}y_2^2 + \frac{2b}{r}\left(\frac{1}{7}y_3^2 + \frac{1}{3}y_7^1 + \frac{4}{7}y_7^3\right)\right). \end{aligned} \quad \text{(A5)}$$

$$\left(\frac{2}{7}\omega\sigma-\sigma^2+\frac{2}{7}\frac{b}{r^2}\right)y_3{}^2+\left(\frac{2}{3}\omega\sigma+\frac{1}{3}\sigma^2+\frac{2}{3}\frac{b}{r^2}\right)y_7{}^1$$

$$+\left(\frac{8}{7}\omega\sigma-\frac{16}{7}\sigma^2+\frac{8}{7}\frac{b}{r^2}\right)y_7{}^3=\left(\frac{8}{7}\omega\sigma+\frac{4}{7}\frac{\dot{b}}{r}-\frac{g}{r}\right)y_1{}^2$$

$$+\left(-\frac{4}{7}\frac{b}{rg\lambda}\frac{d}{dr}\left(\frac{\lambda}{\rho}\right)+\frac{1}{\rho r}\right)y_2{}^2+\frac{1}{r}y_5{}^2, \tag{A6}$$

$$11\,\omega\sigma\,y_3{}^2+(\omega\sigma+\sigma^2)\,y_7{}^1-4\,(\omega\sigma+6\sigma^2)\,y_7{}^3=7\,\omega\sigma\,y_1{}^2+\frac{\beta b}{r\lambda}y_2{}^2-\omega\sigma\varepsilon r, \tag{A7}$$

$$-16\,\omega\sigma\,y_3{}^2+5\,(\omega\sigma+6\sigma^2)\,y_7{}^3=-8\,\omega\sigma\,y_1{}^2-2\frac{\beta b}{r\lambda}y_2{}^2. \tag{A8}$$

In the above equations, the parameters ρ, λ, and μ are the ρ_s, λ_s and μ_s respectively defined in equation (16). The subscript s is dropped for convenience.