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OSCILLATION OF LINEAR RETARDED DIFFERENTIAL EQUATION

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We shall study the oscillatory behaviour of solutions of the linear differential equation with retarded argument

$$(1) \qquad y^{(n)}(t) + p(t)\, y(g(t)) = 0\,, \quad n \geqq 2\,,$$

where $p(t)$ and $g(t)$ are continuous functions on $[a, \infty)$, $a \geqq 0$, $p(t) > 0$, $g(t) > 0$, $g(t) \leqq t$ and $\lim_{t\to\infty} g(t) = \infty$.

Our aim is to give new sufficient conditions that guarantee the oscillatory character of equation (1) as well as of the corresponding ordinary differential equation.

We shall use the following definitions.

A solution $y(t)$ of equation (1) is called *oscillatory* if it has arbitrarily large zeros, and it is called *nonoscillatory* otherwise.

Equation (1) is said *to have property A* if every solution of this equation is oscillatory if n is even, and every solution is either oscillatory or $\lim_{t\to\infty} y^{(i)}(t) = 0$, $i = 0, \ldots, n-1$ if n is odd.

We introduce the notation:

$$\gamma(t) = \sup\{s \geqq a : g(s) \leqq t\} \quad \text{for} \quad t \geqq a\,.$$

The function $\gamma(t)$ has the following properties that can be found in [10]: $t \leqq \gamma(t)$, $g[\gamma(t)] = t$, $t \geqq a$.

Lemma 1 (Kiguradze [3]). *Let $y(t)$ be a nonoscillatory solution of equation* (1) *and let*

$$y(t)\, y^{(n)}(t) \leqq 0 \quad \textit{for} \quad t \in [t_0, \infty)\,, \quad t_0 \geqq a\,.$$

Then there exist $t_1 \in [t_0, \infty)$ and an integer $l \in \{0, 1, \ldots, n-1\}$ such that $l + n$ is odd and

$$(2) \qquad y(t)\, y^{(i)}(t) > 0 \quad \text{for} \quad t \in [t_1, \infty) \quad (i = 0, \ldots, l)\,,$$

$$(-1)^{n+i}\, y(t)\, y^{(i)}(t) < 0 \quad \text{for} \quad t \in [t_1, \infty) \quad (i = l+1, l+2, \ldots, n)\,,$$

$$(3) \qquad (t - t_1)\, |y^{(l-i)}(t)| \leqq (1 + i)\, |y^{(l-i-1)}(t)| \quad \text{for} \quad t \in [t_1, \infty)$$

$$(i = 0, \ldots, l-1)\,, \quad 1 \leqq l \leqq n-1\,.$$

Theorem 1. *Let the following condition be fulfilled:*

(4)
$$\limsup_{t\to\infty}\left[\frac{1}{t}\int_T^t s[g(s)]^{n-1}\,p(s)\,\mathrm{d}s + t\int_t^\infty \frac{[g(s)]^{n-1}}{s}\,p(s)\,\mathrm{d}s\right] > (n-1)!\,,\quad T\in[a,\infty)\,.$$

Then equation (1) *has property A.*

Proof. Without loss of generality we assume that $y(t)$ is a nonoscillatory solution of equation (1) such that $y(g(t)) > 0$ for $t\in[t_0,\infty)$, $t_0 \geqq a$. Then with regard to Lemma 1 there exist $t_1\in[t_0,\infty)$ and $l\in\{0,1,\dots,n-1\}$ such that $n+l$ is odd and (2), (3) hold.

Let $l\in\{1,\dots,n-1\}$. According to (3), for sufficiently large $t_2\in[t_1,\infty)$ we obtain

$$y(g(t)) \geqq \frac{[g(t)-t_1]^{l-1}}{l!}\,y^{(l-1)}(g(t))\,,\quad t\geqq t_2\,. \tag{5}$$

From the identity

$$z^{(j)}(t) = \sum_{i=j}^{k-1}(-1)^{i-j}\frac{(s-t)^{i-j}}{(i-j)!}\,z^{(i)}(s) + \tag{6}$$
$$+\frac{(-1)^{k-j}}{(k-j-1)!}\int_t^s (u-t)^{k-j-1}\,z^{(k)}(u)\,\mathrm{d}u\,,\quad s\geqq t\geqq t_2\,,$$

where $1\leqq k\leqq n$, $z\in C_n([a,\infty),R)$, we obtain for $k=n-l+1$, $j=1$

$$z'(t) = \sum_{i=1}^{n-l}(-1)^{i-1}\frac{(s-t)^{i-1}}{(i-1)!}\,z^{(i)}(s) + \tag{7}$$
$$+\frac{(-1)^{n-l}}{(n-l-1)!}\int_t^s (u-t)^{n-l-1}\,z^{(n-l+1)}(u)\,\mathrm{d}u\,.$$

Choose $z(t) = y^{(l-1)}(t)$. Then by (2) we have $(-1)^{i-1}\,z^{(i)}(s) > 0$, $i = 1,\dots,n-l$ and (7) implies

$$z'(t)\geqq \frac{(-1)^{n-l}}{(n-l-1)!}\int_t^\infty (u-t)^{n-l-1}\,z^{(n-l+1)}(u)\,\mathrm{d}u\,.$$

Using (5) and equation (1) in the last inequality we obtain

$$z'(t)\geqq \frac{1}{(n-l-1)!\,l!}\int_t^\infty (u-t)^{n-l-1}\,[g(u)-t_1]^{l-1}\,p(u)\,z(g(u))\,\mathrm{d}u\,. \tag{8}$$

Integrating (8) from T to t, $t > T\geqq t_2$, we get

$$z(t)\geqq \frac{1}{(n-l)!\,l!}\int_T^t (u-T)^{n-l}\,[g(u)-t_1]^{l-1}\,p(u)\,z(g(u))\,\mathrm{d}u +$$
$$+\frac{1}{(n-l-1)!\,l!}\int_t^\infty [g(u)-t_1]^{l-1}\,p(u)\,z(g(u))\int_T^t (u-s)^{n-l-1}\,\mathrm{d}s\,\mathrm{d}u\,,$$

$$z(t) \geqq \frac{1}{(n-l)!\, l!}\left[\int_T^t (u-T)^{n-l}\,[g(u)-t_1]^{l-1}\, p(u)\, z(g(u))\, du + \right.$$

$$\left. + (t-T)\int_t^\infty (u-T)^{n-l-1}\,[g(u)-t_1]^{l-1}\, p(u)\, z(g(u))\, du\right].$$

We choose $T_1 \geqq T$ such that $g(u) \geqq T$ for $u \geqq T_1$. Then for $t \geqq T_1$ we have

$$z(t) \geqq \frac{1}{(n-l)!\, l!}\left[\int_{T_1}^t (u-T)^{n-l}\,[g(u)-T]^{l-1}\, p(u)\, z(g(u))\, du + \right. \tag{9}$$

$$\left. + (t-T)\int_t^\infty (u-T)^{n-l-1}\,[g(u)-T]^{l-1}\, p(u)\, z(g(u))\, du\right].$$

It follows from (3) that the function $z(t)/(t-T)$, $t > T$, is nonincreasing. So we have

$$z(g(u)) \geqq \frac{g(u)-T}{t-T}\, z(t)\,, \quad t \geqq u > T_1\,, \quad \text{and} \quad z(g(u)) \geqq \frac{g(u)-T}{u-T}\, z(u)\,,$$

since $g(u) \leqq u$.

Then (9) yields

$$(n-l)!\, l! \geqq \frac{1}{t-T}\int_{T_1}^t (u-T)^{n-l}\,[g(u)-T]^l\, p(u)\, du +$$

$$+ (t-T)\int_t^\infty (u-T)^{n-l-2}\,[g(u)-T]^l\, p(u)\, du\,.$$

The last inequality for $l \in \{1, \ldots, n-1\}$ implies

$$(n-1)! \geqq (n-l)!\, l! \geqq \frac{1}{t-T}\int_{T_1}^t (u-T)^{n-l}\,[g(u)-T]^l\, p(u)\, du +$$

$$+ (t-T)\int_t^\infty (u-T)^{n-l-2}\,[g(u)-T]^l\, p(u)\, du\,.$$

Since $(u-T)^{n-l}\,[g(u)-T]^l \geqq (u-T)\,[g(u)-T]^{n-1}$ for $1 \leqq l \leqq n-1$, $u \geqq T_1$, we have

$$(n-1)! \geqq \frac{1}{t-T}\int_{T_1}^t (u-T)\,[g(u)-T]^{n-1}\, p(u)\, du +$$

$$+ (t-T)\int_t^\infty \frac{[g(u)-T]^{n-1}}{u-T}\, p(u)\, du\,,$$

which contradicts condition (4).

Let $l = 0$. Then n is odd and condition (4) implies

$$\int^\infty t^{n-1}\, p(t)\, dt = \infty\,. \tag{10}$$

Otherwise, if

$$\int^{\infty} t^{n-1}\, p(t)\, \mathrm{d}t < \infty\,,$$

then we can choose $T \in [t_2, \infty)$ so large that

$$\int_T^{\infty} t^{n-1}\, p(t)\, \mathrm{d}t \leqq \tfrac{1}{2}(n-1)!\,,$$

and by (4) we have

$$(n-1)! < \limsup_{t\to\infty} \left[\int_T^t [g(s)]^{n-1}\, p(s)\, \mathrm{d}s + \int_t^{\infty} [g(s)]^{n-1}\, p(s)\, \mathrm{d}s\right] \leqq \tfrac{1}{2}(n-1)!\,,$$

which is a contradiction.

From (6) for $j = 0$, $k = n$, $z(t) = y(t)$, we find that

$$y(t) \geqq -\frac{1}{(n-1)!}\int_t^{\infty} (u-t)^{n-1}\, y^{(n)}(u)\, \mathrm{d}u\,, \quad t \geqq t_2\,,$$

and

$$y(t) \geqq \frac{1}{(n-1)!}\int_t^{\infty} (u-t)^{n-1}\, p(u)\, y(g(u))\, \mathrm{d}u\,.$$

If $y(t)$ is bounded below by a positive constant c, then

$$y(T) \geqq \frac{c}{(n-1)!}\int_T^{\infty} (u-T)^{n-1}\, p(u)\, \mathrm{d}u\,, \quad T \geqq t_2\,,$$

which is a contradiction with (10). So $\lim_{t\to\infty} y(t) = 0$. This completes the proof.

Corollary 1. *Consider the ordinary differential equation*

$$y^{(n)}(t) + p(t)\, y(t) = 0\,, \quad n \geqq 2\,. \tag{11}$$

Let the following condition be fulfilled:

$$\limsup_{t\to\infty} \left[\frac{1}{t}\int_T^t s^n\, p(s)\, \mathrm{d}s + t\int_t^{\infty} s^{n-2}\, p(s)\, \mathrm{d}s\right] > (n-1)!\,, \quad T \in [a, \infty).$$

Then equation (11) *has property A.*

Theorem 2. *Let $g(t)$ be continuously differentiable and suppose that there exists a function $\omega \in C_1([a, \infty), (0, \infty))$ bounded below by a positive constant and such that*

$$\int_T^{\infty} \frac{\mathrm{d}t}{t\omega(g(t))} < \infty \quad \textit{and} \quad \int_T^{\infty} \frac{[g(t)]^{n-1}\, p(t)}{\omega(g(t))}\, \mathrm{d}t = \infty\,, \quad T \in [a, \infty)\,. \tag{12}$$

Then equation (1) *has property A.*

Proof. We shall show that (12) implies

$$\limsup_{t\to\infty} \frac{1}{t} \int_T^t s[g(s)]^{n-1} p(s)\, ds = \infty . \tag{13}$$

Assume that

$$\frac{1}{t} \int_T^t s[g(s)]^{n-1} p(s)\, ds \leqq c \quad \text{for} \quad t \in [T, \infty) .$$

Then

$$\int_T^t \frac{[g(s)]^{n-1} p(s)}{\omega(g(s))}\, ds = \frac{1}{t\, \omega(g(t))} \int_T^t u[g(u)]^{n-1} p(u)\, du +$$

$$+ \int_T^t \frac{\omega(g(s)) + s[\omega(g(s))]'}{s^2\, \omega^2(g(s))} \int_T^s u[g(u)]^{n-1} p(u)\, du\, ds \leqq$$

$$\leqq \frac{c}{m} + \int_T^t \frac{\omega(g(s)) + s[\omega(g(s))]'}{s\, \omega^2(g(s))} \frac{1}{s} \int_T^s u[g(u)]^{n-1} p(u)\, du\, ds \leqq$$

$$\leqq c\left[\frac{1}{m} + \int_T^t \frac{ds}{s\, \omega(g(s))} + \int_T^t \frac{|[\omega(g(s))]'|}{\omega^2(g(s))}\, ds\right] < \infty ,$$

where $m = \inf\{\omega(g(t)) : t \in [T, \infty)\}$, which is a contradiction with (12). So condition (13) is fulfilled and we can apply Theorem 1.

Corollary 2. *Let there exist a function $\omega \in C_1([a, \infty), (0, \infty))$ bounded below by a positive constant and such that*

$$\int_T^\infty \frac{dt}{t\, \omega(t)} < \infty \quad \text{and} \quad \int_T^\infty \frac{t^{n-1} p(t)}{\omega(t)}\, dt = \infty , \quad T \in [a, \infty) . \tag{14}$$

Then equation (11) has property A.

Remark. According to (14), Mikusinski's condition

$$\int^\infty t^{n-1-\varepsilon} p(t)\, dt = \infty , \quad \varepsilon > 0 , \tag{15}$$

which guarantees the validity of the assertion of Corollary 2, implies the condition of Corollary 1.

Example. For the equation

$$y^{(4)}(t) + ct^{-4} y(t) = 0 , \quad c > 3! , \quad t > 0 ,$$

the condition of Corollary 1 is fulfilled and so every solution of this equation is oscillatory. However, condition (15) is not satisfied.

Theorem 3. *Suppose that*

(16)
$$\limsup_{t\to\infty}\left[\frac{1}{t}\int_T^t s[g(s)]^{n-1}\,p(s)\,ds + t\int_{\gamma(t)}^{\infty}[g(s)]^{n-2}\,p(s)\,ds\right] > (n-1)!\,,\quad T\in[a,\infty)\,.$$

Then equation (1) *has property A.*

Proof. Let $y(t)$ be a solution of equation (1) such that $y(g(t)) > 0$ for $t \in [t_0, \infty)$, $t_0 \geqq a$. With regard to Lemma 1 there exist $t_1 \in [t_0, \infty)$ and $l \in \{0, 1, \ldots, n-1\}$ such that $n + l$ is odd and (2), (3) hold.

Let $l \in \{1, \ldots, n-1\}$. We choose $T_1 \geqq T \geqq t_1$ such that $g(u) \geqq T$ for $u \geqq T_1$. From (9), in view of the properties of the function $\gamma(t)$, we obtain

$$z(t) \geqq \frac{1}{(n-l)!\,l!}\left[\int_{T_1}^t (u-T)^{n-l}\,[g(u)-T]^{l-1}\,p(u)\,z(g(u))\,du + \right.$$
$$\left. + (t-T)\int_{\gamma(t)}^{\infty}(u-T)^{n-l-1}\,[g(u)-T]^{l-1}\,p(u)\,z(g(u))\,du\right].$$

Using $z(g(u)) \geqq (g(u)-T)/(t-T)\,z(t)$, $t \geqq u > T_1$, and the properties of the function $\gamma(t)$ we conclude

$$(n-l)!\,l! \geqq \frac{1}{t-T}\int_{T_1}^t (u-T)^{n-l}\,[g(u)-T]^l\,p(u)\,du + \tag{17}$$
$$+ (t-T)\int_{\gamma(t)}^{\infty}(u-T)^{n-l-1}\,[g(u)-T]^{l-1}\,p(u)\,du\,.$$

For $l \in \{1, \ldots, n-1\}$, (17) implies that

$$(n-1)! \geqq \frac{1}{t-T}\int_{T_1}^t (u-T)\,[g(u)-T]^{n-1}\,p(u)\,du +$$
$$+ (t-T)\int_{\gamma(t)}^{\infty}[g(u)-T]^{n-2}\,p(u)\,du\,,$$

which contradicts condition (16).

Let $l = 0$. Using the same argument as in the proof of Theorem 1 we find that $\lim_{t\to\infty} y(t) = 0$. This completes the proof.

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