Czechoslovak Mathematical Journal

Ján Ohriska

Oscillation of second order delay and ordinary differential equation

Czechoslovak Mathematical Journal, Vol. 34 (1984), No. 1, 107–112

Persistent URL: http://dml.cz/dmlcz/101929

Czechoslovak Mathematical Journal, 34 (109) 1984, Praha

OSCILLATION OF SECOND ORDER DELAY AND ORDINARY DIFFERENTIAL EQUATION

JÁN OHRISKA, Košice

(Received September 27, 1982)

Let us consider the delay differential equation

$$u''(t) + p(t)\,u(\tau(t)) = 0 \tag{1}$$

and the ordinary differential equation

$$u''(t) + p(t)\,u(t) = 0\,, \tag{2}$$

where $p(t)$, $\tau(t)$ are real-valued and continuous on $[t_0, \infty)$. The following conditions are assumed to hold throughout the paper:

(i) $p(t) \geqq 0$, $p(t)$ is not identically zero in any neighborhood of infinity,

(ii) $\tau(t) \leqq t$, $\lim\limits_{t\to\infty} \tau(t) = \infty$.

We restrict our attention to those solutions of (1) which exist on some ray $[b, \infty)$ where $b \geqq t_0$ and which are non-trivial in any neighborhood of infinity. Such a solution is called *oscillatory* if it has arbitrarily large zeros. Otherwise the solution is caled *nonoscillatory*. An equation is called *oscillatory* if all its solutions are oscillatory.

The purpose of this paper is to present conditions which guarantee that the equation (1) or (2) is oscillatory.

It is clear that with a solution $u(t)$ of (1) or (2) also $-u(t)$ is its solution. This enables us to consider e.g. only positive nonoscillatory solutions of (1) or (2). Further, by (ii), if $u(t)$ is a nonoscillatory solution of (1) such that $u(t) > 0$ for $t \geqq t_1$ then there exists $t_2 \geqq t_1$ such that $u(\tau(t)) > 0$ for $t \geqq t_2$ and we see from (1) and (i) that $u''(t) \leqq$ $\leqq 0$ for $t \geqq t_2$. It can be seen that $u'(t) > 0$ for $t \geqq t_2$. Likewise for equation (2) we obtain $u'(t) > 0$ for $t \geqq t_1$ if $u(t) \geqq 0$ for $t \geqq t_1$.

We begin with two lemmas that will be useful in the proof of our main results.

Lemma 1. *Let $u(t) \in C^2_{[T,\infty)}$ and let*

$$u(t) > 0\,, \quad u'(t) > 0\,, \quad u''(t) \leqq 0 \quad \textit{for} \quad t \geqq T\,.$$

Then for each $k_1 \in (0, 1)$ *there is a* $T_{k_1} \geqq T$ *such that*

$$u(\tau(t)) \geqq k_1 \frac{\tau(t)}{t} u(t), \quad t \geqq T_{k_1}.$$

Proof of Lemma 1 may be found in [2].

Lemma 2. *Let* $u(t) \in C^2_{[T,\infty)}$ *and let*

$$u(t) > 0, \quad u'(t) > 0, \quad u''(t) \leqq 0 \quad \textit{for} \quad t \geqq T.$$

Then for each $k_2 \in (0, 1)$ *there is a* $T_{k_2} \geqq T$ *such that*

$$u(t) \geqq k_2 t\, u'(t), \quad t \geqq T_{k_2}.$$

Proof. Suppose that $t > T$. Then by the well-known Lagrange's theorem we have

$$u(t) - u(T) = u'(s)(t - T) \quad \text{for some} \quad s \in (T, t).$$

From this identity, according to the assumptions of Lemma 2 we obtain

$$u(t) \geqq u'(t)(t - T). \tag{3}$$

Now for any $k_2 \in (0, 1)$, $K = 1/(1 - k_2) > 1$, and for $t \geqq KT$ we have $T \leqq t/K$. Then

$$t - T \geqq t - t/K = k_2 t \quad \text{for} \quad t \geqq KT$$

and from (3) we have

$$u(t) \geqq k_2 t\, u'(t) \quad \text{for} \quad t \geqq T_{k_2},$$

where $T_{k_2} = KT$. The proof is complete.

The following notation will be used:

$$\gamma(t) = \sup \{s \geqq t_0 \mid \tau(s) \leqq t\} \quad \text{for} \quad t \geqq t_0.$$

It is clear that $t \leqq \gamma(t)$ and $\tau(\gamma(t)) = t$.

Theorem 1. *Let*

$$\limsup_{t\to\infty} t \int_t^\infty p(x) \frac{\tau(x)}{x} \, dx > 1, \tag{4}$$

or

$$\limsup_{t\to\infty} t \int_{\gamma(t)}^\infty p(x) \, dx > 1. \tag{5}$$

Then the equation (1) *is oscillatory.*

Proof. If the conclusion is not true, then there exists a nonoscillatory solution $u(t)$ of (1), e.g. such that $u(t) > 0$ and also $u(\tau(t)) > 0$ for $t \geqq t_1 \geqq t_0$. Then $u''(t) \leqq 0$ and $u'(t) > 0$ for $t \geqq t_1$ and $\lim_{t\to\infty} u'(t) \geqq 0$.

Integrating the equation (1) from t to ∞ $(t \geqq t_1)$ we have

$$u'(t) \geqq \int_t^\infty p(x)\, u(\tau(x))\, \mathrm{d}x \,. \tag{6}$$

If $k_2 \in (0, 1)$, then according to Lemma 2 there exists a number $t_2 \geqq t_1$ such that $u(t) \geqq k_2 t\, u'(t)$ for $t \geqq t_2$. We may suppose without loss of generality that $t_2 > 0$ and then the inequality (6), by Lemma 2, yields

$$u(t) \geqq k_2 t \int_t^\infty p(x)\, u(\tau(x))\, \mathrm{d}x \,, \quad t \geqq t_2 \,. \tag{7}$$

Now we shall proceed in the proof of the conditions (4) and (5) separately.

a) Using Lemma 1 in (7) we obtain

$$u(t) \geqq k^2 t \int_t^\infty p(x) \frac{\tau(x)}{x}\, u(x)\, \mathrm{d}x \,, \quad t \geqq t_3 \geqq t_2 \,,$$

where $k = \min\{k_1, k_2\}$. Since the function $u(t)$ is positive and increasing, it follows from the above inequality that

$$1 \geqq k^2 t \int_t^\infty p(x) \frac{\tau(x)}{x}\, \mathrm{d}x \,, \quad t \geqq t_3 \,. \tag{8}$$

From (8) it follows that

$$\limsup_{t \to \infty} t \int_t^\infty p(x) \frac{\tau(x)}{x}\, \mathrm{d}x < \infty \,.$$

If we put

$$\limsup_{t \to \infty} t \int_t^\infty p(x) \frac{\tau(x)}{x}\, \mathrm{d}x = a$$

and suppose that (4) holds, then there exists a sequence of points $\{s_q\}$ such that $\lim_{q \to \infty} s_q = \infty$ and

$$\lim_{q \to \infty} s_q \int_{s_q}^\infty p(x) \frac{\tau(x)}{x}\, \mathrm{d}x = a > 1 \,.$$

So for $\varepsilon = \frac{1}{2}(a - 1) > 0$ there exists a number Q such that for every $q > Q$ we have

$$\frac{a+1}{2} = a - \frac{a-1}{2} < s_q \int_{s_q}^\infty p(x) \frac{\tau(x)}{x}\, \mathrm{d}x \,. \tag{9}$$

Now if we choose $q > Q$ so that $s_q \geqq t_3$ and, moreover, numbers $k_1, k_2 \in (0, 1)$ such that $\sqrt{(2/(a+1))} < k < 1$, then (9) implies

$$k^2 s_q \int_{s_q}^\infty p(x) \frac{\tau(x)}{x}\, \mathrm{d}x > \frac{2}{a+1} \cdot \frac{a+1}{2} = 1$$

which contradicts (8).

b) Since $\gamma(t) \geqq t$, it follows from (7) that

$$u(t) \geqq k_2 t \int_{\gamma(t)}^{\infty} p(x)\, u(\tau(x))\, \mathrm{d}x\,, \quad t \geqq t_2\,.$$

Because the function $u(t)$ is increasing and $\tau(x) \geqq t$ for $x \geqq \gamma(t)$, the above inequality gives

$$u(t) \geqq k_2 t u(t) \int_{\gamma(t)}^{\infty} p(x)\, \mathrm{d}x\,, \quad t \geqq t_2$$

or

$$1 \geqq k_2 t \int_{\gamma(t)}^{\infty} p(x)\, \mathrm{d}x\,, \quad t \geqq t_2\,. \tag{10}$$

From (10) it follows that

$$b = \limsup_{t \to \infty} t \int_{\gamma(t)}^{\infty} p(x)\, \mathrm{d}x < \infty\,.$$

Suppose that (5) holds. Then similarly as above we again obtain a contradiction. This completes the proof.

Theorem 2. *Let*

$$\int^{\infty} \exp\left(-k \int^{s} \tau(x)\, p(x)\, \mathrm{d}x\right) \mathrm{d}s < \infty \quad \textit{for some} \quad k \in (0, 1)\,. \tag{11}$$

Then the equation (1) *is oscillatory.*

Proof. Let $u(t)$ be a nonoscillatory solution of (1), e.g. such that $u(t) > 0$, $u(\tau(t)) > 0$ for $t \geqq t_1 \geqq t_0$. Then $u'(t) > 0$ and $u''(t) \leqq 0$ for $t \geqq t_1$, and by Lemmas 1 and 2 we know that for any $k \in (0, 1)$ there is $t_2 \geqq t_1$ such that for $t \geqq t_2$ we have

$$u(\tau(t)) \geqq k\, \tau(t)\, u'(t)\,. \tag{12}$$

Now, if we estimate $u(\tau(t))$ in (1) by (12), we easily obtain that

$$\frac{u''(t)}{u'(t)} \leqq -k\, \tau(t)\, p(t)\,, \quad t \geqq t_2\,.$$

Integrating the above inequality from t_2 to t $(t \geqq t_2)$ we have

$$u'(t) \leqq u'(t_2) \exp\left(-k \int_{t_2}^{t} \tau(x)\, p(x)\, \mathrm{d}x\right).$$

Another integration from t_3 to t $(t \geqq t_3 \geqq t_2)$ yields

$$u(t) \leqq u(t_3) + u'(t_2) \int_{t_3}^{t} \exp\left(-k \int_{t_2}^{s} \tau(x)\, p(x)\, \mathrm{d}x\right) \mathrm{d}s\,.$$

Since the condition (11) holds we see from the last inequality that the nonoscillatory solution $u(t)$ of (1) is bounded.

On the other hand, if the condition (11) is satisfied then $\int^{\infty} \tau(x)\, p(x)\, \mathrm{d}x = \infty$ and this is a sufficient condition for the oscillation of all bounded solutions of (1) (see e.g. [2], p. 51). This contradiction proves the theorem.

Remark 1. In [2], L. Erbe showed that the equation (1) is oscillatory if

$$\liminf_{t\to\infty} t \int_t^{\infty} p(x) \frac{\tau(x)}{x}\, \mathrm{d}x > \frac{1}{4}.$$

It is obvious that this sufficient condition for oscillation of the equation (1) is better than our condition (4) in the case when there exists

$$\lim_{t\to\infty} t \int_t^{\infty} p(x) \frac{\tau(x)}{x}\, \mathrm{d}x,$$

however, in the opposite case it is not true in general.

Remark 2. Another sufficient condition for the oscillation of the equation (1) has been obtained by V. N. Ševelo and N. V. Varech in [4]. They proved that such condition is

$$\int^{\infty} [\tau(t)]^{1-\varepsilon} p(t)\, \mathrm{d}t = \infty, \quad 0 < \varepsilon \leqq 1.$$

We can show that Theorem 1 or 2 cannot be covered by this result. Namely, if we put $\tau(t) = t/2$ and $p(t) = 3/t^2$ then the conditions (4), (5) and (11) for $k > 2/3$ are satisfied but

$$\int^{\infty} [\tau(t)]^{1-\varepsilon} p(t)\, \mathrm{d}t < \infty$$

for every $\varepsilon > 0$.

It is easy to see that Theorem 1 or 2 holds also in the case $\tau(t) \equiv t$. Because $\gamma(t) \equiv \equiv t$ if $\tau(t) \equiv t$, so according to Theorems 1 and 2 we may formulate the following result.

Corollary 1. *Let*

(12) $$\limsup_{t\to\infty} t \int_t^{\infty} p(x)\, \mathrm{d}x > 1,$$

or

(13) $$\int^{\infty} \exp\left(-k \int^{s} x\, p(x)\, \mathrm{d}x\right) \mathrm{d}s < \infty \quad \text{for some} \quad k \in (0, 1).$$

Then the equation (2) *is oscillatory.*

Remark 3. In this remark we mention two well known sufficient conditions for the oscillation of the equation (2) and compare them with our result.

a) J. G. Mikusiński proved in [3] that such a condition is

$$(14) \qquad \int^{\infty} t^{1-\varepsilon} p(t)\,\mathrm{d}t = \infty\,, \quad \varepsilon > 0\,.$$

b) In [5], A. Vintner showed that the equation (2) is oscillatory provided

$$(15) \qquad \lim_{t\to\infty} \frac{1}{t} \int_{t_0}^{t} \int_{t_0}^{s} p(x)\,\mathrm{d}x\,\mathrm{d}s = \infty\,.$$

Now, e.g. if we put $p(t) = 3/t^2$, then the conditions (14) and (15) are not satisfied, but the conditions (12) and (13) are satisfied. Thus we see that Corollary 1 can be covered by none of the previous results.

Remark 4. In a recent paper [1] T. A. Čanturija has proved the following oscillation theorem.

Theorem (Theorem 2.3 in [1]). *If $p(t) \geqq 0$ and*

$$\limsup_{t\to\infty} t \int_t^{\infty} x^{n-2} p(x)\,\mathrm{d}x > (n-1)!\,,$$

then all solutions of the equation

$$(16) \qquad u^{(n)}(t) + p(t)\,u(t) = 0\,, \quad n \geqq 3$$

are oscillatory for n even, and every solution of (16) is either oscillatory or tends to zero as $t \to \infty$ for n odd.

It is evident that our result (Corollary 1) extends the above Čanturija's result for $n = 2$.

References

[1] *T. A. Čanturija:* Integral criteria of oscillation of solutions of linear differential equations of higher order (Russian). Differ urav., *XVI*, 3, 1980, 470—482.

[2] *L. Erbe:* Oscillation criteria for second order nonlinear delay equations. Canad. Math. Bull. *16* (1), 1973, 49—56.

[3] *J. G. Mikusiński:* On Fite's oscillation theorems. Colloq. Math. *II*, 1949, 34—39.

[4] *V. N. Ševelo, N. V. Varech:* On certain properties of solutions of differential equations with a delay (Ukrainian) UMŽ, *24*, 6, 1972, 807—813.

[5] *A. Vintner:* A criterion of oscillatory stability. Quart. Appl. Math. 7, 1, 1949, 115—117.

Author's address: 041 54 Košice, nám. Febr. víť. 9, ČSSR (PF UPJŠ).