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OSCILLATION OF SOLUTIONS OF FORCED NEUTRAL DIFFERENTIAL EQUATIONS OF n-th ORDER

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1.

A great deal of work has been done in recent years in oscillation theory of neutral differential equations. Most of this work is concerned with linear homogeneous equations (For example, see [3, 4, 12, 13] and the references there in). Some authors have studied oscillatory behaviour of solutions and the problem of existence of a nonoscillatory solution of nonlinear homogeneous equations of neutral type. (See [1, 2, 6, 7, 14, 15, 18, 20]). The oscillation theory of forced ordinary and delay-differential equations has developed, to some extent, satisfactorily during last few years. However, it seems that very little work has been done on forced neutral differential equations (see [16,19]).

In this paper we are concerned with oscillatory behaviour of solutions of a class of forced neutral differential equations of nth order ($n \geqslant 1$) of the form

$$[x(t) + p(t)x(r(t))]^{(n)} + q(t)h(x(g(t))) = f(t) \tag{NH}$$

and the associated homogeneous equation

$$[x(t) + p(t)x(r(t))]^{(n)} + q(t)h(x(g(t))) = 0, \tag{H}$$

where the following assumptions hold:

i) p and $q \in C([\sigma, \infty), \mathbb{R})$,

ii) $h \in C(\mathbb{R}, \mathbb{R})$ such that $uh(u) > 0$ for $u \neq 0$,

iii) $f \in C([\sigma, \infty), \mathbb{R})$ and there exists $F \in C^{(n)}([\sigma, \infty), \mathbb{R})$ such that $F^{(n)}(t) = f(t)$,

iv) g and $r \in C([\sigma, \infty), \mathbb{R})$ such that $\lim\limits_{t\to\infty} g(t) = \infty$ and $\lim\limits_{t\to\infty} r(t) = \infty$.

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By a solution of (NH)/(H) we mean a real-valued continuous function x on $[T_x, \infty)$ for some $T_x \geqslant \sigma$ such that $\{x(t) + p(t)x(r(t))\}$ is n-times continuously differentiable and (NH)/(H) is satisfied for $t \in [T_X, \infty)$. Such a solution is said to be oscillatory if it has arbitrarily large zeros; otherwise, it is called nonoscillatory. Following assumptions are made for the use in the sequel:

(A_1) $-1 < -p_2 \leqslant p(t) \leqslant 0$,

(A_2) $0 \leqslant p(t) \leqslant p_1 < 1$,

(A_3) $p(t)p(r(t)) \geqslant 0$ and $-1 < -p_2 \leqslant p(t) \leqslant p_1 < 1$, where p_1 and p_2 are positive constants,

(A_4) $q(t) \geqslant 0$ and $\int_\sigma^\infty (g(t))^{n-1} q(t)\,\mathrm{d}t = \infty$,

(A_5) $q(t) \geqslant 0$ and $\int_\sigma^\infty q(t)\,\mathrm{d}t = \infty$,

(A_6) $h(u)$ is bounded away from zero if u is bounded away from zero, that is, $|u| > \delta$ implies that $h(u)| > \eta$, where $\eta > 0$ and $\delta > 0$,

(A_7) $h'(u) \geqslant 0$ and $h(u)$ is superlinear, that is, $h(u)$ satisfies

$$\int_c^\infty \frac{\mathrm{d}u}{h(u)} < \infty \quad \text{and} \quad \int_{-c}^{-\infty} \frac{\mathrm{d}u}{h(u)} < \infty$$

for every $c > \sigma$,

(A_8) $g(t) \leqslant t$ and $g'(t) \geqslant 0$,

(A_9) $r(t) \leqslant t$,

(A_{10}) $q(t) \leqslant 0$ and $\int_\sigma^\infty t^{n-1} q(t)\,\mathrm{d}t = -\infty$.

In the second section we consider oscillatory behaviour of solutions of (NH) with $q(t) \geqslant 0$ and the third section deals with the same problem for (NH) with $q(t) \leqslant 0$.

2.

In this section we study oscillatory behaviour of solutions of (H) and (NH) with $q(t) \geqslant 0$.

Theorem 2.1. *Suppose that the conditions* (A_2), (A_4), (A_7), (A_8) *and* (A_9) *hold. If n is even, then every solution of* (H) *is oscillatory. If n is odd, then every solution of* (H) *is oscillatory or tends to zero as $t \to \infty$.*

P r o o f. Let $x(t)$ be a nonoscillatory solution of (H) such that $x(t) > 0$ for $t \geqslant t_0 > \max\{\sigma, 0, T_x\}$. Hence there exists a $t_1 > t_0$ such that $x(r(t)) > 0$ and $x(g(t)) > 0$ for $t \geqslant t_1$. Setting, for $t \geqslant t_1$,

$$z(t) = x(t) + p(t)x(r(t)) > 0, \tag{2.1}$$

we obtain from (H) that

$$z^{(n)}(t) = -q(t)h(x(g(t))) \leqslant 0. \tag{2.2}$$

From a lemma due to Kiguradze (see [9, 11]) it follows that there exists an integer ℓ, $0 \leqslant \ell \leqslant n-1$, which is odd if n is even and even if n is odd, such that, for $t \geqslant t_1$,

$$z^{(k)}(t) > 0 \quad \text{for} \quad k = 0, 1, \ldots, \ell, (-1)^{\ell+k} z^{(k)}(t) \geqslant 0 \tag{2.3}$$

for $k = \ell+1, \ldots, n$ and

$$z^{(\ell)}(t) \leqslant k!(t-t_1)^{-k} z^{(\ell-k)}(t), k = 1, 2, \ldots, \ell. \tag{2.4}$$

For $t \geqslant 2t_1$, we get from (2.4) that

$$z^{(\ell)}(t) \leqslant k! 2^k t^{-k} z^{(l-k)}(t), k = 1, 2, \ldots, \ell.$$

There exists a $t_2 > 2t_1$ such that $g(t) > 2t_1$ for $t \geqslant t_2$. Consequently, for $t \geqslant t_2$,

$$z^{(\ell)}(g(t)) \leqslant (\ell-1)!\, 2^{\ell-1} (g(t))^{-\ell+1} z'(g(t)). \tag{2.5}$$

If n is even, then from (2.3) it follows that $z'(t) > 0$ for $t \geqslant t_2$. There exists a $t_3 > t_2$ such that $r(t) > t_2$ for $t \geqslant t_3$. Thus from (2.1) we obtain, using $r(t) \leqslant t$,

$$\begin{aligned} 0 < (1-p_1)z(t) &\leqslant z(t) - p(t)z(r(t)) \\ &\leqslant x(t) - p(t)p(r(t))x(r(r(t))) \\ &< x(t), \end{aligned}$$

for $t \geqslant t_3$. Consequently, for $t \geqslant t_4 > t_3$, we have

$$0 < (1-p_1)z(g(t)) < x(g(t)).$$

Multiplying (2.2) through by $(g(t))^{n-1}/h(x(g(t)))$ and integrating the resulting identity by parts from t_4 to t, we obtain

$$\begin{aligned} &\int_{t_4}^{t} (g(s))^{n-1} q(s)\,\mathrm{d}s \\ &\qquad = -\int_{t_4}^{t} \frac{(g(s))^{n-1} z^{(n)}(s)}{h(x(g(s)))}\,\mathrm{d}s \\ &\qquad \leqslant -\int_{t_4}^{t} \frac{(g(s))^{n-1} z^{(n)}(s)}{h((1-p_1)z(g(s)))}\,\mathrm{d}s \\ &\qquad \leqslant \beta_1 + (n-1)\int_{t_4}^{t} \frac{(g(s))^{n-2} g'(s) z^{(n-1)}(s)}{h((1-p_1)z(g(s)))}\,\mathrm{d}s \\ &\qquad\quad + \int_{t_4}^{t} (g(s))^{n-1} z^{(n-1)}(s) \frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{1}{h((1-p_1)z(g(s)))}\right)\mathrm{d}s, \end{aligned}$$

where $\beta_1 = \frac{(g(t_4))^{n-1} z^{(n-1)}(t_4)}{h((1-p_1)z(g(t_4)))} > 0$. As

$$\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{1}{h((1-p_1)z(g(s)))}\right) = -\frac{(1-p_1)h'((1-p_1)z(g(s)))z'(g(s))g'(s)}{h^2((1-p_1)z(g(s)))} \leqslant 0,$$

we have, for $t \geqslant t_4$,

$$\int_{t_4}^{t} (g(s))^{n-1} q(s)\,\mathrm{d}s \leqslant \beta_1 + (n-1)\int_{t_4}^{t} \frac{(g(s))^{n-2} g'(s) z^{(n-1)}(g(s))}{h((1-p_1)z(g(s)))}\,\mathrm{d}s.$$

Proceeding as above, we obtain with the help of (2.5)

$$\begin{aligned}
&\int_{t_4}^{t} (g(s))^{n-1} q(s)\,\mathrm{d}s \\
&\quad\leqslant \sum_{i=1}^{n-\ell} \beta_i + (n-1)(n-2)\dots\ell \int_{t_4}^{t} \frac{(g(s))^{\ell-1} g'(s) z^{(\ell)}(g(s))}{h((1-p_1)z(g(s)))}\,\mathrm{d}s \\
&\quad\leqslant \sum_{i=1}^{n-\ell} \beta_i + (n-1)!2^{\ell-1} \int_{t_4}^{t} \frac{g'(s)z'(g(s))}{h((1-p_1)z(g(s)))}\,\mathrm{d}s \\
&\quad\leqslant \sum_{i=1}^{n-\ell} \beta_i + \frac{(n-1)!2^{\ell-1}}{(1-p_1)} \int_{(1-p_1)z(g(t_4))}^{(1-p_1)z(g(t))} \frac{\mathrm{d}u}{h(u)} \\
&\quad\leqslant \sum_{i=1}^{n-\ell} \beta_i + \frac{(n-1)!2^{\ell-1}}{(1-p_1)} \int_{(1-p_1)z(g(t_4))}^{\infty} \frac{\mathrm{d}u}{h(u)},
\end{aligned}$$

where

$$\beta_i = (-1)^{i-1} \frac{(n-1)!}{(n-i)!} \frac{(g(t_4))^{n-i} z^{(n-i)}(g(t_4))}{h((1-p_1)z(g(t_4)))}$$

$i = 2, \dots, n-\ell$. This in turn implies that

$$\int_{t_4}^{\infty} (g(s))^{n-1} q(s)\,\mathrm{d}s < \infty,$$

a contradiction.

Suppose that n is odd. So ℓ is even. If $\ell > 0$, then we proceed as above to arrive at a contradiction. If $\ell = 0$, then $(-1)^k z^{(k)}(t) \geqslant 0$ for $k = 1, 2, \dots, n$. Multiplying (2.2) through by t^{n-1} and integrating the resulting identity from t_2 to t, we obtain

$$\int_{t_2}^{t} s^{n-1} q(s) h(x(g(s)))\,\mathrm{d}s = -\int_{t_2}^{t} s^{n-1} z^{(n)}(s)\,\mathrm{d}s \leqslant \sum_{i=1}^{n} \alpha_i,$$

where $\alpha_i = (-1)^{i+1}\frac{(n-1)!}{(n-i)!}t_2^{n-i}z^{(n-i)}(t_2) > 0, i = 1,\dots,n$. As $g(t) \leqslant t$ and $\lim\limits_{t\to\infty} z(t)$ exists, we have

$$\int_{t_2}^{\infty} (g(s))^{n-1}q(s)h(x(g(s)))\,\mathrm{d}s < \infty,$$

which, in view of (A_4), implies that $\lim\limits_{t\to\infty}\inf x(t) = 0$. Thus

$$\begin{aligned}\overline{\lim_{t\to\infty}}\, x(t) \leqslant \overline{\lim_{t\to\infty}}\, z(t) &= \underline{\lim_{t\to\infty}}\, z(t)\\ &\leqslant \underline{\lim_{t\to\infty}}\,[x(t) + p_1 x(r(t))]\\ &\leqslant \underline{\lim_{t\to\infty}}\, x(t) + p_1 \overline{\lim_{t\to\infty}}\, x(r(t))\\ &\leqslant p_1 \overline{\lim_{t\to\infty}}\, x(t)\end{aligned}$$

yields that $0 \leqslant (1-p_1)\overline{\lim\limits_{t\to\infty}}\, x(t) \leqslant 0$ and hence $\lim\limits_{t\to\infty} x(t) = 0$.

If $x(t) < 0$ for $t \geqslant t_0$, then we set $y(t) = -x(t)$ and hence (H) takes the form

$$[y(t) + p(t)y(r(t))]^{(n)} + q(t)h^*(y(g(t))) = 0 \tag{2.6}$$

where $h^*(u) = -h(-u)$. Proceeding as above one may obtain a contradiction when n is even. In case n is odd, one obtains a contradiction when

$$\ell > 0 \quad \text{and} \quad \lim_{t\to\infty} y(t) = 0 \quad \text{when} \quad \ell = 0.$$

Hence the theorem is proved. □

Remark 1. In [10] (See Theorem 2), Kusano and Onose have obtained conclusion of Theorem 2.1 for equations (H) with $p(t) \equiv 0$. Thus our theorem may be viewed as a generalization to neutral delay equations.

The following theorem asserts that the presence of a forcing term which is small in some sense does not affect substantially the oscillatory character of the associated unforced equations.

Theorem 2.2. *Let the conditions* (A_3), (A_4), (A_7), (A_8) *and* (A_9) *hold. Suppose that* $\lim\limits_{t\to\infty} F(t) = 0$. *Then every solution of* (NH) *is oscillatory or tends to zero as* $t\to\infty$ *if* (i) n *is even or* (ii) n *is odd and* $p_1 + p_2 < 1$.

Proof. Suppose that $x(t)$ is a nonoscillatory solution of (NH) such that $x(t) > 0$ for $t \geqslant t_0 > \max\{\sigma, 0, T_x\}$. Hence there exists a $t_1 > t_0$ such that $x(r(t)) > 0$ and $x(g(t)) > 0$ for $t \geqslant t_1$. Setting, for $t \geqslant t_1$,

$$z(t) = x(t) + p(t)x(r(t)) - F(t) \tag{2.7}$$

we obtain from (NH) that

$$z^{(n)}(t) = -q(t)h(x(g(t))) \leqslant 0. \tag{2.8}$$

Thus $z(t) < 0$ or > 0 for $t \geqslant t_2 > t_1$. If $z(t) < 0$ for $t \geqslant t_2$, then from (2.7) it follows that

$$x(t) < F(t) + p_2 x(r(t)).$$

Hence

$$\varlimsup_{t\to\infty} x(t) \leqslant \varlimsup_{t\to\infty} F(t) + p_2 \varlimsup_{t\to\infty} x(r(t)) \leqslant p_2 \varlimsup_{t\to\infty} x(t)$$

implies that $\lim_{t\to\infty} x(t) = 0$, desired conclusion. Next suppose that $z(t) > 0$ for $t \geqslant t_2$.

In the following we show that $x(t)$ is bounded. For $n = 1$, we get $z'(t) \leqslant 0$ for $t \geqslant t_2$ from (2.8). Thus $z(t)$ is bounded. If $z(t) \leqslant L$ for $t \geqslant t_2$, where $L > 0$ is a constant, then proceeding as above we obtain

$$0 \leqslant (1 - p_2) \varlimsup_{t\to\infty} x(t) \leqslant L$$

and hence $x(t)$ is bounded. We claim that $x(t)$ is bounded for $n \geqslant 2$. If not, $x(t)$ is unbounded. Then $z(t)$ is unbounded because bounded $z(t)$ yields bounded $x(t)$ as above. This together with (2.8) implies that $z'(t) > 0$ for $t \geqslant t_3 > t_2$. Consequently, $\lim_{t\to\infty} z(t) = \infty$. It is possible to find $t_4 > t_3$ such that for $t \geqslant t_4$, we have

$$\begin{aligned}(1 - p_1)z(t) &\leqslant z(t) - p(t)z(r(t))\\ &\leqslant x(t) - p(t)p(r(t))x(r(r(t))) - F(t) + p(t)F(r(t))\\ &< x(t) + |F(t)| + |F(r(t))|\\ &< x(t) + \varepsilon,\end{aligned}$$

where $\varepsilon > 0$. Setting $y(t) = (1 - p_1)z(t) - \varepsilon$, we see that $y(t) < x(t)$ for $t \geqslant t_4$, $\lim_{t\to\infty} y(t) = \infty$ and (2.8) may be written as

$$y^{(n)}(t) + q_1(t)h(y(g(t))) = 0, \tag{2.9}$$

where

$$q_1(t) = (1 - p_1)q(t)h(x(g(t)))/h((1 - p_1)z(g(t)) - \varepsilon).$$

There exists $t_5 > t_4$ such that for $t \geqslant t_5$, $q_1(t) \geqslant (1 - p_1)q(t)$ and hence from (A_4) it follows that

$$\int_{t_5}^{\infty} (g(s))^{n-1} q_1(s)\,\mathrm{d}s = \infty.$$

Consequently, from Theorem 2.1 (with $p(t) \equiv 0$) we obtain that every solution of (2.9) is oscillatory if n is even and is oscillatory or tends to zero as $t \to \infty$ if n is odd. This is a contradiction in view of the observation that $\lim_{t\to\infty} y(t) = \infty$.

Thus $x(t)$ is bounded and hence $z(t)$ is bounded. From (2.8) it follows that

$$(-1)^{n+k} z^{(k)}(t) < 0 \qquad \text{for } k = 1, 2, \ldots, n-1. \tag{2.10}$$

for large t.

Let n be even. Hence (2.10) yields $z'(t) > 0$ for $t \geqslant t_3 > t_2$. As $z(t) > 0$ for $t \geqslant t_3$, then $\lim_{t\to\infty} z(t) = a > 0$ exists. Choosing $o < \varepsilon < (1-p_1)a$, setting $w(t) = (1-p_1)z(t) - \varepsilon$ and proceeding as above, we obtain $w(t) < x(t)$ and

$$w^{(n)}(t) + q_2(t)h(w(g(t))) = 0 \tag{2.11}$$

for $t \geqslant t_4 > t_3$, where

$$q_2(t) = (1-p_1)q(t)h(x(g(t)))/h((1-p_1)z(t) - \varepsilon).$$

From the given hypotheses it is clear that $q_2(t) \geqslant (1-p_1)q(t)$ for $t \geqslant t_5 > t_4$ and hence

$$\int_{t_5}^{\infty} (g(s))^{n-1} q_2(s)\,\mathrm{d}s = \infty.$$

This in turn implies, by Theorem 2.1, that $w(t)$ is oscillatory, a contradiction to the fact that $\lim_{t\to\infty} \omega(t) = (1-p_1)a - \varepsilon > 0$.

Suppose that n is odd and $p_1 + p_2 < 1$. Multiplying (2.8) through by t^{n-1} and integrating the resulting identity from t_3 to t we obtain, by using (2.10),

$$\int_{t_3}^{t} s^{n-1} q(s) h(x(g(s)))\,\mathrm{d}s = -\int_{t_3}^{t} s^{n-1} z^{(n)}(s)\,\mathrm{d}s < \sum_{i=1}^{n} \alpha_i,$$

where

$$\alpha_i = (-1)^{i+1} \frac{(n-1)!}{(n-i)!} t_3^{n-i} z^{(n-i)}(t_3) > 0, i = 1, \ldots, n. \tag{2.12}$$

This in turn implies that

$$\int_{t_3}^{\infty} (g(s))^{n-1} q(s) h(x(g(s)))\,\mathrm{d}s < \infty.$$

Consequently, in view of (A_4) we have $\liminf\limits_{t\to\infty} x(t) = 0$. Clearly, (2.7) yields

$$\begin{aligned}
\limsup_{t\to\infty} z(t) &\geqslant \limsup_{t\to\infty}\, [x(t) - p_2 x(r(t)) - F(t)] \\
&\geqslant \limsup_{t\to\infty}\, [x(t) - p_2 x(r(t))] + \liminf_{t\to\infty}\, (-F(t)) \\
&\geqslant \limsup_{t\to\infty}\, x(t) + \liminf_{t\to\infty}\, [-p_2 x(r(t))] \\
&\geqslant \limsup_{t\to\infty}\, x(t) - p_2 \limsup_{t\to\infty}\, x(r(t)) \\
&\geqslant (1 - p_2) \limsup_{t\to\infty}\, x(t).
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\liminf_{t\to\infty} z(t) &\leqslant \liminf_{t\to\infty}\, [x(t) + p_1 x(r(t)) - F(t)] \\
&\leqslant \liminf_{t\to\infty}\, [x(t) + p_1 x(r(t))] + \limsup_{t\to\infty}\, (-F(t)) \\
&\leqslant \liminf_{t\to\infty}\, x(t) + p_1 \limsup_{t\to\infty}\, x(r(t)) \\
&\leqslant p_1 \limsup_{t\to\infty}\, x(t).
\end{aligned}$$

From (2.10) we get $z'(t) < 0$ for $t \geqslant t_4 > t_3$ and hence $\lim\limits_{t\to\infty} z(t)$ exists. Thus

$$(1 - p_2) \limsup_{t\to\infty} x(t) \leqslant p_1 \limsup_{t\to\infty} x(t),$$

that is,

$$0 \leqslant (1 - p_1 - p_2) \limsup_{t\to\infty} x(t) \leqslant 0$$

and hence $\lim\limits_{t\to\infty} x(t) = 0$.

If $x(t) < 0$ for $t \geqslant t_0$, then we set $y(t) = -x(t)$ in (NH) to obtain

$$[y(t) + p(t)y(r(t))]^{(n)} + q(t)h^*(y(g(t))) = f^*(t),$$

where $h^*(u) = -h(-u)$ and $f^*(t) = -f(t)$. Proceeding as above we obtain necessary conclusions.

Hence the theorem is proved. □

Corollary 2.3. (a) *Suppose that the conditions of Theorem* 2.2 *are satisfied. Then all unbounded solutions of* (NH) *are oscillatory if* (i) n *is even or* (ii) n *is odd and* $p_1 + p_2 < 1$.

(b) *Let the conditions of Theorem* 2.2 *be satisfied. Then all nonoscillatory solutions of* (NH) *tend to zero as* $t \to \infty$ *if* (i) n *is even or* (ii) n *is odd and* $p_1 + p_2 < 1$.

Remark 2. (i) We may note that the condition $p_1 + p_2 < 1$ is satisfied if (A_1) or (A_2) is assumed.

(ii) If $r(t) = t - \tau, \tau > 0$ and $p(t)$ is τ-periodic, then $p(t)p(r(t)) \geqslant 0$ holds.

Theorem 2.4. *Suppose that the assumptions* (A_2), (A_4), (A_7), (A_8) *and* (A_9) *hold. Let $F(t)$ be oscillatory such that $\lim_{t\to\infty} F(t) = 0$. Then every solution of* (NH) *is oscillatory if n is even and is oscillatory or tends to zero as $t \to \infty$ if n is odd.*

Proof. Proceeding as in Theorem 2.2 we obtain $z(t) < 0$ or > 0 for $t \geqslant t_2 > t_1$. But $z(t) < 0$ for $t \geqslant t_2$ yields $0 < x(t) < F(t)$, $t \geqslant t_2$, a contradiction because $F(t)$ is assumed to be oscillatory. The rest of the proof is similar to that of Theorem 2.2.

In the following theorem we obtain results similar to those in Theorem 2.2, when the assumption (A_4) is replaced by the stronger assumption (A_5) and without the superlinearity condition on h. For $n = 1$, (A_4) $\equiv$ (A_5). □

Theorem 2.5. *Let the conditions* (A_3), (A_5), (A_6) *and* (A_9) *be satisfied and $\lim_{t\to\infty} F(t) = 0$. Then the conclusions of Theorem 2.2 hold.*

Proof. We proceed as in Theorem 2.2 to arrive at $z(t) < 0$ or > 0 for $t \geqslant t_2 > t_1$. Clearly $z(t) < 0$ for $t \geqslant t_2$ implies that $\lim_{t\to\infty} x(t) = 0$. Next let $z(t) > 0$ for $t \geqslant t_2$. If $n = 1$, then from (2.8) we obtain $z(t)$ is bounded, say, $z(t) \leqslant L$ for $t \geqslant t_2$, where $L > 0$ is a constant. Hence from (2.7) we get

$$0 \leqslant (1 - p_2) \varlimsup_{t\to\infty} x(t) \leqslant L$$

Consequently, $(x(t)$ is bounded. For $n \geqslant 2$, we claim that $x(t)$ is bounded. If not, then $z(t)$ is unbounded and hence from (2.8) it follows that $z'(t) > 0$ for large t. Thus $\lim_{t\to\infty} z(t) = \infty$. Proceeding as in Theorem 2.2 we obtain

$$(1 - p_1)z(t) - \varepsilon < x(t) \tag{2.13}$$

for large t, where $\varepsilon > 0$ is arbitrary. There exists a $t_3 > t_2$ such that $x(g(t)) > \delta > 0$ for $t \geqslant t_3$ and hence $h(x(g(t))) > \eta$ for $t \geqslant t_3$. Clearly $z^{(n-1)}(t) > 0$ for $t \geqslant t_4 > t_3$. Otherwise, $z(t) < 0$ for large t. Integrating (2.8) from t_4 to t, we obtain

$$\eta \int_{t_4}^{t} q(s)\,\mathrm{d}s \leqslant \int_{t_4}^{t} q(s)h(x(g(s)))\,\mathrm{d}s < z^{(n-1)}(t_4),$$

that is,

$$\int_{t_4}^{\infty} q(t)\,\mathrm{d}t < \infty,$$

a contradiction. Hence our claim holds. Thus $z(t)$ is bounded and (2.10) holds.

If n is even, then $z'(t) > 0$ for large t. Hence $\lim_{t\to\infty} z(t) = a > 0$ exists. For $0 < b < a$, there exists $t_5 > t_4$ such that $z(t) > a - b$ for $t \geqslant t_5$. Choosing $o < \varepsilon < (1-p_1)(a-b)$, we obtain from (2.13) that $x(g(t)) > (1-p_1)(a-b) - \varepsilon > 0$ for $t \geqslant t_6 > t_5$. Integrating (2.8) from t_6 to t and using (A_6) we contradict (A_5). Hence $z(t) > 0$ for large t is not possible when n is even. Suppose that n is odd and $p_1 + p_2 < 1$. Proceeding as in Theorem 2.2 we obtain

$$t_4^{n-1}\int_{t_4}^{t} q(s)h(x(g(s)))\,\mathrm{d}s \leqslant \int_{t_4}^{t} s^{n-1}q(s)h(x(g(s)))\,\mathrm{d}s < \sum_{i=1}^{n}\alpha_i,$$

where α_i is given by (2.12) with t_3 replaced by t_4. This in turn implies that $\varliminf_{t\to\infty} x(t) = 0$. One may proceed as in Theorem 2.2 to obtain $\varlimsup_{t\to\infty} x(t) = 0$ and hence $\lim_{t\to\infty} x(t) = 0$.

The case $x(t) < 0$ for large t may be treated as in Theorem 2.2.

This completes the proof of the theorem. □

Remark 3. We may note that Theorem 2.5 holds for advanced neutral equations and generalizes the results in [1, 3, 4, 12, 13].

Theorem 2.6. *Let the conditions* (A_2), (A_5), (A_6) *and* (A_9) *be satisfied. Let* $F(t)$ *be oscillatory such that* $\lim_{t\to\infty} F(t) = 0$. *Then the conclusions of Theorem* 2.4 *hold.*

The proof is similar to that of Theorem 2.5 and hence is omitted.

We may note that if $f(t) \equiv 0$ and $p(t)$ satisfies (A_2), then $z(t)$ cannot be < 0 for large t, where $z(t)$ is given by (2.7). Hence we have the following result for (H).

Theorem 2.7. *Suppose that* (A_2), (A_5), (A_6) *and* (A_9) *hold. Then every solution of* (H) *is oscillatory if n is even and every solution of* (H) *is oscillatory or tends to zero as $t \to \infty$ if n is odd.*

Following examples illustrate above theorems.

Example 1. Consider

$$\begin{aligned}[x(t) + (1+2\sin t)\mathrm{e}^{-2\pi}x(t-2\pi)]''' &+ \mathrm{e}^{-\sigma/3}tx^{1/3}(t-\sigma)\\ &= 2\mathrm{e}^{-t}(2\sin t + 2\cos t - 1) + t\mathrm{e}^{-t/3}\end{aligned} \tag{2.14}$$

$t \geqslant \max\{2\pi, \sigma\}$, where $\sigma > 0$. Clearly, $-1 < -\mathrm{e}^{-2\pi} \leqslant p(t) \leqslant 3\mathrm{e}^{-2\pi} < 1$, where $p(t) = (1+2\sin t)\mathrm{e}^{-2\pi}$ and $F(t) = 2\mathrm{e}^{-t}(1+\sin t) - 27(t+9)\mathrm{e}^{-t/3} \to 0$ as $t \to \infty$. From theorem 2.5, it follows that every solution of (2.14) is oscillatory or tends to zero as $t \to \infty$. In particular, $x(t) = \mathrm{e}^{-t}$ is a solution of (2.14) which $\to 0$ as $t \to \infty$. As the condition (A_7) fails to hold here, Theorem 2.2 cannot be applied to this example.

Example 2. Consider

$$(2.15)\qquad \begin{aligned}&[x(t) - (1 + 2\cos t)e^{-2\pi}x(t - 2\pi)]'' + e^{2t-3\pi}x^3(t - \pi)\\ &\quad = e^{-t}(4\cos 2t + 3\sin 2t - \sin^3 t), \quad t \geqslant 2\pi.\end{aligned}$$

Clearly, $-1 < -3e^{-2\pi} \leqslant p(t) \leqslant e^{-2\pi} < 1$, where $p(t) = -e^{-2\pi}(1 + 2\cos t)$ and

$$F(t) = -e^{-t}\Big[\sin 2t + \frac{3}{8}\cos t - \frac{1}{200}(3\cos 3t - 4\sin 3t)\Big].$$

Either from Theorem 2.2 or from Theorem 2.5 it follows that every solution of (2.15) is oscillatory or tends to zero as $t \to \infty$. In particular, we see that $x(t) = e^{-t}\sin t$ is an oscillatory solution of (2.15) which tends to zero as $t \to \infty$.

Example 3. Consider

$$(2.16)\qquad \begin{aligned}&\Big[x(t) + \frac{1 + 2\sin t}{6}x(t - 2\pi)\Big]''' + (t - \sigma)^{-3/2}x^5(t - \sigma)\\ &\quad = -\frac{6}{t^4} - \frac{\cos t}{3(t - 2\pi)} + \frac{\sin t}{(t - 2\pi)^2} + \frac{2\cos t}{(t - 2\pi)^3}\\ &\qquad - \frac{(1 + 2\sin t)}{(t - 2\pi)^4} + \frac{1}{(t - \sigma)^{13/2}},\end{aligned}$$

$t > \sigma > 0$. Clearly, all the conditions of Theorem 2.2 are satisfied with $p(t) = \frac{1+2\sin t}{4}$ satisfying $-1 < -\frac{1}{6} \leqslant p(t) \leqslant \frac{1}{2} < 1$ and $F(t) = \frac{1}{t} + \frac{1+2\sin t}{6}\cdot\frac{1}{t-2\pi} - \frac{8}{693}\frac{1}{(t-\sigma)^{7/2}} \to 0$ as $t \to \infty$. Thus every solution of (2.16) is socillatory or $\to 0$ as $t \to \infty$. In particular, $x(t) = \frac{1}{t}$ is such a solution of (2.16). We may note that (A_5) fails to hold here and hence Theorem 2.5 cannot be applied to this example.

Example 4. Consider

$$(2.17)\qquad \begin{aligned}&\Big[x(t) + \frac{1}{t}x(t - \pi)\Big]'' + x(t - 2\pi)\\ &\quad = \Big(\frac{1}{t} - \frac{2}{t^3}\Big)\sin t + \frac{2\cos t}{t^2}, \quad t > 1\end{aligned}$$

Here $F(t) = -\frac{1}{t}\sin t$. From Theorem 2.6 it follows that every solution of (2.17) is oscillatory. We may see that $x(t) = \sin t$ is an oscillatory solution of the equation.

In the following an attempt has been made to obtain a result similar to above theorems when $r(t) = t - \tau, \tau > 0$ and $F(t)$ is τ-periodic.

Theorem 2.8. *Suppose that the conditions* (A_2), (A_4), (A_7) *and* (A_8) *are satisfied. Let* $r(t) = t - \tau$ *and* $F(t)$ *be* τ*-periodic. Then every solution of* (NH) *is* *oscillatory if n is even and is either oscillatory or $\lim\limits_{t\to\infty} |x(t)| = 0$ and*

$$0 \leqslant \overline{\lim_{t\to\infty}}\, |x(t)| \leqslant \frac{b_2 - b_1}{1 - p_1}$$

if n is odd, where b_1 and b_2 are lower and upper bounds of $F(t)$ respectively.

Proof. Let $x(t)$ be a nonoscillatory solution of (NH) such that $x(t) > 0$ for $t > t_0$. Hence there exists a $t_1 > t_0 + \tau$ such that $x(g(t)) > 0$ for $t \geqslant t_1$. Setting, for $t \geqslant t_1$,

$$z(t) = x(t) + p(t)x(t-\tau) - F(t) \tag{2.18}$$

we obtain

$$z(t) + F(t) > 0 \tag{2.19}$$

and

$$z^{(n)}(t) = -q(t)h\big(x(g(t))\big) \leqslant 0. \tag{2.20}$$

Thus $z(t) < 0$ or > 0 for $t \geqslant t_2 \geqslant t_1$.

Clearly, $z(t) < 0$ for $t \geqslant t_2$ implies that $x(t) < F(t) - p(t)x(t-\tau) < b_2$, that is, $x(t)$ is bounded. Let $z(t) > 0$ for $t \geqslant t_2$. If $n = 1$, then $z'(t) \leqslant 0$ for $t \geqslant t_2$ and hence $z(t)$ is bounded. Thus $x(t)$ is bounded. Let $n \geqslant 2$. We claim that $x(t)$ is bounded. If not, $x(t)$ is unbounded. Hence $z(t)$ is unbounded. Consequently, $z'(t) > 0$ for $t \geqslant t_3 > t_2$. Thus $\lim\limits_{t\to\infty} z(t) = \infty$. From (2.18) we obtain, for $t \geqslant t_3$,

$$\begin{aligned} z(t) - p(t)z(t-\tau) &\leqslant x(t) - F(t) + p(t)F(t-\tau) \\ &\leqslant x(t) - b_1 + b, \end{aligned}$$

where $b = \max\{|b_1|, |b_2|\}$, that is

$$(1 - p_1)z(t) \leqslant z(t) - p(t)z(t-\tau) \leqslant x(t) - b_1 + b.$$

Setting $y(t) = (1 - p_1)z(t) + b_1 - b$, for $t \geqslant t_3$, we have $y(t) \leqslant x(t)$, $\lim\limits_{t\to\infty} y(t) = \infty$ and $y(t)$ is a solution of the equation

$$y^{(n)}(t) + q_1(t)h\big(y(g(t))\big) = 0 \tag{2.21}$$

for $t \geqslant t_4 > t_3$, where

$$q_1(t) = \frac{(1-p_1)q(t)h\bigl(x(g(t))\bigr)}{h\bigl((1-p_1)z\bigl(g(t)\bigr) + b_1 - b\bigr)} \geqslant (1-p_1)q(t).$$

Consequently,

$$\int_{t_4}^{\infty} \bigl(g(t)\bigr)^{n-1} q_1(t)\,\mathrm{d}t = \infty.$$

Hence from Theorem 2.1 with $p(t) \equiv 0$ it follows that every solution of (2.21) is oscillatory if n is even and is oscillatory or $\to 0$ as $t \to \infty$ if n is odd. This contradicts the fact that $\lim\limits_{t\to\infty} y(t) = \infty$ and hence our claim that $x(t)$ is bounded holds. Thus $z(t)$ is bounded and

$$(-1)^{n+k} z^{(k)}(t) < 0, \quad k = 1, 2, \ldots, n-1 \tag{2.22}$$

for large t.

If n is even, then from (2.22) it follows that $z'(t) > 0$ for large t. From (2.18) we obtain

$$z(t) - p(t)z(t-\tau) \leqslant x(t) - F(t) + p(t)F(t-\tau),$$

that is,

$$\bigl(1 - p(t)\bigr)\bigl(z(t) + F(t)\bigr) \leqslant x(t).$$

Using (2.19) we get

$$(1-p_1)\bigl(z(t) + F(t)\bigr) \leqslant x(t).$$

Hence

$$(1-p_1)\bigl(z(t) + b_1\bigr) \leqslant x(t)$$

for large t, say, $t \geqslant t_5$. Since $F(t)$ is continuous, and τ-periodic, there exists a $t' > t_5$ such that $F(t') = b_1$. Hence, for $t \geqslant t'$, $z(t) + b_1 = z(t) + F(t') \geqslant z(t') + F(t') > 0$. Setting $v(t) = (1-p_1)\bigl(z(t) + b_1\bigr)$ for $t \geqslant t'$, we have $0 < v(t) \leqslant x(t)$, $v'(t) > 0$ and $v(t)$ is bounded. Hence $\lim\limits_{t\to\infty} v(t) > 0$ exists and $v(t)$ is a solution of

$$v^{(n)}(t) + q_2(t)h\bigl(v(g(t))\bigr) = 0 \tag{2.23}$$

for $t \geqslant t_6 > t'$, where

$$q_2(t) = \frac{(1-p_1)q(t)h\bigl(x(g(t))\bigr)}{h\bigl((1-p_1)\bigl(z(g(t)) + b_1\bigr)\bigr)} \geqslant (1-p_1)q(t)$$

From Theorem 2.1 with $p(t) \equiv 0$ it follows that every solution of (2.23) is oscillatory if n is even and is oscillatory or $\to 0$ as $t \to \infty$ if n is odd, a contradiction to the fact that $\lim_{t\to\infty} v(t) > 0$.

Next suppose that n is odd. Hence for large t, $z'(t) \leqslant 0$ if $n = 1$ and $z'(t) < 0$ if $n \geqslant 3$. Thus $\lim_{t\to\infty} z(t)$ exists. Multiplying (2.20) through by t^{n-1} and integrating the resulting identity from t_7 to t, $t_7 > t_6$, we get

$$\begin{aligned}
&\int_{t_7}^{t} s^{n-1} q(s) h\big(x(g(s))\big)\,\mathrm{d}s \\
&\qquad = -\int_{t_7}^{t} s^{n-1} z^{(n)}(s)\,\mathrm{d}s \\
&\qquad \leqslant \begin{cases} \sum\limits_{i=1}^{n} \alpha_i, & \text{if} \quad z(t) > 0 \qquad \text{for} \quad t \geqslant t_7 \\ \sum\limits_{i=1}^{n-1} \alpha_i - (n-1)!\, z(t), & \text{if} \quad z(t) < 0 \quad \text{for} \quad t \geqslant t_7, \end{cases}
\end{aligned}$$

where $\alpha_i > 0$, $i = 1, \ldots, n$, is given by (2.12) with t_3 replaced by t_7. Hence

$$\int_{t_7}^{\infty} s^{n-1} q(s) h\big(x(g(s))\big)\,\mathrm{d}s < \infty. \tag{2.24}$$

This in turn implies that

$$\int_{t_7}^{\infty} \big(g(s)\big)^{n-1} q(s) h\big(x(g(s))\big)\,\mathrm{d}s < \infty.$$

Consequently, in view of (A_4), $\underline{\lim}_{t\to\infty} x(t) = 0$. Further, using (2.18), we get

$$\begin{aligned}
\overline{\lim}_{t\to\infty} \big(x(t) - F(t)\big) &\leqslant \overline{\lim}_{t\to\infty} [x(t) + p(t)x(t-\tau) - F(t)] \\
&\leqslant \lim_{t\to\infty} z(t) \\
&\leqslant \underline{\lim}_{t\to\infty} [x(t) + p(t)x(t-\tau) - F(t)] \\
&\leqslant \underline{\lim}_{t\to\infty} x(t) + \overline{\lim}_{t\to\infty} \big(p(t)x(t-\tau) - F(t)\big) \\
&\leqslant p_1 \overline{\lim}_{t\to\infty} x(t-\tau) - b_1 \\
&\leqslant p_1 \overline{\lim}_{t\to\infty} x(t) - b
\end{aligned}$$

and

$$\overline{\lim}_{t\to\infty} \big(x(t) - F(t)\big) \geqslant \overline{\lim}_{t\to\infty} x(t) - b_2.$$

Thus

$$\overline{\lim}_{t\to\infty} x(t) \leqslant \frac{b_2 - b_1}{1 - p_1}.$$

The case $x(t) < 0$ for large t may similarly be dealt with.

Hence the Theorem is proved. □

In the following theorem we replace the condition (A_4) by the stronger condition (A_5). However, it is possible to obtain results similar to those in Theorem 2.8 without the superlinearity condition (A_7) on h. Moreover, the following theorem holds when $g(t) > t$.

Theorem 2.9. *Let the conditions* (A_2), (A_5) *and* (A_6) *hold. Let* $r(t) = t - \tau$ *and* $F(t)$ *be* τ*-periodic. Then the conclusions of Theorem* 2.8 *hold.*

Proof. Proceeding as in Theorem 2.8, we obtain $y(t) \leqslant x(t)$ and $\lim_{t\to\infty} y(t) = \infty$. Consequently, $\lim_{t\to\infty} x(t) = \infty$. Thus, for $M > 0$, there exists a $T_1 > 0$ such that $x\big(g(t)\big) > M$ for $t \geqslant T_1$. Hence $h(x(g(t))) > M_1 > 0$ for $t > T_1$. Clearly, $z^{(n-1)}(t) > 0$ for large t because $z^{(n-1)}(t) < 0$ for large t implies that $z(t) < 0$ for large t, a contradiction. Integrating (2.20) from T_2 to t, $T_2 > T_1$, we obtain

$$\int_{T_2}^{\infty} q(s)\,\mathrm{d}s < \infty,$$

a contradiction. Hence $x(t)$ is bounded. Consequently, $z(t)$ is bounded and (2.22) holds.

If n is even, we proceed as in Theorem 2.8 to obtain $0 < v(t) \leqslant x(t)$ and $\lim_{t\to\infty} v(t) = \lambda$, $0 < \lambda < \infty$, where $v(t) = (1 - p_1)\big(z(t) + b_1\big)$, $t \geqslant t'$. Hence $x(g(t)) > \eta > 0$ for $t \geqslant T_3 > t'$. Thus $h\big(x(g(t))\big) > \delta > 0$ for $t \geqslant T_3$. Now integrating (2.20) from T_3 to t, we arrive at a contradiction as above.

If n is odd, then one proceeds as in Theorem 2.8 to obtain (2.24) which in turn yields

$$\int_{t_7}^{\infty} q(t)h\big(x(g(t))\big)\,\mathrm{d}t < \infty.$$

Hence $\varliminf_{t\to\infty} x(t) = 0$. The rest of the proof is similar to that of Theorem 2.8.

Thus the theorem is proved. □

Example 5. Consider

$$\left[x(t) + \frac{1}{2}x(t-2\pi)\right]'' + x^3(t-\pi) = -\frac{3}{2}\cos t - \cos^3 t, \quad t > 2\pi \tag{2.25}$$

Clearly, $F(t) = \frac{1}{9}\cos^3 t + \frac{13}{6}\cos t$ is 2π-periodic. From Theorem 2.9 it follows that all solutions of (2.25) are oscillatory. In particular, $x(t) = \cos t$ is a oscillatory solution of the equation.

3.

This section deals with oscillatory behaviour of solutions of (NH) with $q(t) \leqslant 0$.

Theorem 3.1. *Suppose that the conditions* (A_3), (A_6), (A_9) *and* (A_{10}) *hold. Let* $\lim_{t\to\infty} F(t) = 0$. *Then every bounded solution of* (NH) *is oscillatory or tends to zero as* $t \to \infty$ *if* (i) n *is odd or* (ii) n *is even and* $p_1 + p_2 < 1$.

Proof. Let $x(t)$ be a bounded nonoscillatory solution of (NH) such that $x(t) > 0$ for $t \geqslant t_0 > \max\{\sigma, T_x\}$. The case $x(t) < 0$ for $t \geqslant t_0$ may similarly be dealt with. Thus there exists a $t_1 > t_0$ such that $x\big(r(t)\big) > 0$ and $x\big(g(t)\big) > 0$ for $t \geqslant t_1$. Setting

$$z(t) = x(t) + p(t)x\big(r(t)\big) - F(t) \tag{3.1}$$

for $t \geqslant t_1$, we obtain $z(t)$ is bounded and

$$z^{(n)}(t) = -q(t)h\big(x(g(t))\big) \geqslant 0 \tag{3.2}$$

for $t \geqslant t_1$. Hence $z(t) < 0$ or > 0 for $t \geqslant t_2 > t_1$. However, $z(t) < 0$ for $t \geqslant t_2$ implies that

$$x(t) < -p(t)x\big(r(t)\big) + F(t) < p_2 x\big(r(t)\big) + F(t).$$

Thus $\overline{\lim}_{t\to\infty} x(t) \leqslant p_2 \overline{\lim}_{t\to\infty} x\big(r(t)\big) \leqslant p_2 \overline{\lim}_{t\to\infty} x(t)$ implies that $\lim_{t\to\infty} x(t) = 0$. Let $z(t) > 0$ for $t \geqslant t_2$. Let $n \geqslant 2$. As $z^{(n-1)}(t) > 0$ for large t implies that $\lim_{t\to\infty} z(t) = \infty$, a contradiction, we have $z^{(n-1)}(t) < 0$ for large t. From Kiguradze's lemma (see [9,11]) it follows that there exists an integer ℓ, $0 \leqslant \ell \leqslant n-2$, which is odd or even according to n is odd or even respectively, such that, for $t \geqslant t_3 > t_2$,

$$\begin{aligned} z^{(k)}(t) &> 0 \qquad \text{for } k = 0, 1, \ldots, \ell, \\ (-1)^{\ell+k} z^{(k)}(t) &> 0 \qquad \text{for } k = \ell+1, \ldots, n-1. \end{aligned} \tag{3.3}$$

Let n be odd such that $n > 2$. Then $\ell = 1$; otherwise, $\lim_{t\to\infty} z(t) = \infty$, a contradiction. Thus from (3.3) we get $z'(t) > 0$ for $t \geqslant t_3$ and hence $\lim_{t\to\infty} z(t)$ exists. Multiplying (3.2) through by t^{n-1} and integrating the resulting identity from t_3 to t, we get

$$\begin{aligned} \int_{t_3}^{t} s^{n-1} q(s) h\big(x(g(s))\big)\,\mathrm{d}s &= -\int_{t_3}^{t} s^{n-1} z^{(n)}(s)\,\mathrm{d}s \\ &> \sum_{i=1}^{n} \alpha_i - (n-1)! z(t) > \sum_{i=1}^{n} \alpha_i - (n-1)!\gamma, \end{aligned}$$

where

$$\alpha_i = (-1)^{i+1} \frac{(n-1)!}{(n-i)!} t_3^{n-i} z^{(n-i)}(t_3) < 0 \tag{3.4}$$

and

$$\gamma = \lim_{t\to\infty} z(t) > 0.$$

Hence

$$\int_{t_3}^{\infty} s^{n-1} q(s) h\big(x(g(s))\big)\, ds > -\infty.$$

This in turn implies, in view of (A_{10}), that $\underline{\lim}_{t\to\infty} x(t) = 0$. On the other hand, from (3.1) we get

$$\begin{aligned}(1-p_1)z(t) &< z(t) - p(t)z\big(r(t)\big) \\ &< x(t) - F(t) + p(t)F\big(r(t)\big).\end{aligned}$$

For $0 < \varepsilon < (1-p_1)\gamma$, there exists $t_4 > t_3$ such that

$$\begin{aligned}(1-p_1)z(t) &< x(t) + |F(t)| + \big|F\big(r(t)\big)\big| \\ &< x(t) + \varepsilon.\end{aligned}$$

Thus $\underline{\lim}_{t\to\infty} x(t) \geqslant (1-p_1)\gamma - \varepsilon > 0$, a contradiction. If $n = 1$, then (3.2) yields $z'(t) \geqslant 0$ for $t \geqslant t_1$. One proceeds as above to obtain necessary contradiction.

Suppose that n is even. Hence ℓ is even. If $\ell \geqslant 2$, then $\lim_{t\to\infty} z(t) = \infty$, a contradiction. Thus $\ell = 0$. Consequently, (3.3) yields

$$(-1)^k z^{(k)}(t) > 0, \quad k = 1, 2, \ldots, n-1.$$

Hence $\lim_{t\to\infty} z(t)$ exists. Multiplying (3.2) through by t^{n-1} and integrating the resulting identity from t_3 to t, we obtain

$$\int_{t_3}^{t} s^{n-1} q(s) h\big(x(g(s))\big)\, ds > \sum_{i=1}^{n} \alpha_i,$$

where α_i is given by (3.4). Hence using (A_{10}) we get $\underline{\lim}_{t\to\infty} x(t) = 0$. Further, (3.1) yields

$$\begin{aligned}\overline{\lim}_{t\to\infty} z(t) &\geqslant \overline{\lim}_{t\to\infty} \big[x(t) - p_2 x\big(r(t)\big) - F(t)\big] \\ &\geqslant \overline{\lim}_{t\to\infty} \big[x(t) - p_2 x\big(r(t)\big)\big] + \underline{\lim}_{t\to\infty} \big\{-F(t)\big\} \\ &\geqslant \overline{\lim}_{t\to\infty} x(t) - p_2 \overline{\lim}_{t\to\infty} x\big(r(t)\big) \\ &\geqslant (1-p_2) \overline{\lim}_{t\to\infty} x(t)\end{aligned}$$

and

$$
\begin{aligned}
\varliminf_{t\to\infty} z(t) &\leqslant \varliminf_{t\to\infty} \big[x(t) + p_1 x\big(r(t)\big) - F(t)\big] \\
&\leqslant \varliminf_{t\to\infty} \big[x(t) + p_1 x\big(r(t)\big)\big] + \varlimsup_{t\to\infty} \big\{ - F(t)\big\} \\
&\leqslant \varliminf_{t\to\infty} x(t) + p_1 \varlimsup_{t\to\infty} x\big(r(t)\big) \\
&\leqslant p_1 \varlimsup_{t\to\infty} x(t).
\end{aligned}
$$

As $\lim\limits_{t\to\infty} z(t)$ exists, we have

$$(1 - p_2) \varlimsup_{t\to\infty} x(t) \leqslant p_1 \varlimsup_{t\to\infty} x(t)$$

that is,

$$(1 - p_1 - p_2) \varlimsup_{t\to\infty} x(t) \leqslant 0.$$

Hence $\lim\limits_{t\to\infty} x(t) = 0$.

This completes the proof of the theorem. □

Theorem 3.2. *Suppose that* $q(t) \leqslant 0$, (A_3) *and* (A_9) *hold and* $F(t)$ *is bounded. Then every unbounded solution of* (NH) *is either oscillatory or tends to* $\pm\infty$ *as* $t \to \infty$.

Proof. Let $x(t)$ be an unbounded nonoscillatory solution of (NH) such that $x(t) > 0$ for $t \geqslant t_0 > \max\{\sigma, T_x\}$. Hence there exists a $t_1 > t_0$ such that $x\big(r(t)\big) > 0$ and $x\big(g(t)\big) > 0$ for $t \geqslant t_1$. Setting $z(t)$ as in (3.1), we get $z^{(n)}(t) \geqslant 0$ for $t \geqslant t_1$. Thus $z(t) < 0$ or > 0 for large t. Clearly, $z(t) < 0$ for $t \geqslant t_2 > t_1$ implies that $x(t) < M + p_2 x\big(r(t)\big)$, where $|F(t)| \leqslant M$. Consequently,

$$(1 - p_2) \varlimsup_{t\to\infty} x(t) \leqslant M,$$

a contradiction to the fact that $x(t)$ is unbounded. Thus $z(t) > 0$ for large t. Clearly, $z(t)$ is unbounded, because $z(t)$ is bounded implies that $x(t) \leqslant L + p_2 x\big(r(t)\big)$, where L is the upper bound of $z(t) + F(t)$. This in turn implies that $(1 - p_2) \varlimsup\limits_{t\to\infty} x(t) \leqslant L$, a contradiction. Thus $z'(t) > 0$ for large t if $n \geqslant 2$. If $n = 1$, then by (3.2) $z'(t) \geqslant 0$ for large t. Hence $\lim\limits_{t\to\infty} z(t) = \infty$. From (3.1) we obtain

$$
\begin{aligned}
(1 - p_1) z(t) &\leqslant z(t) - p(t) z\big(r(t)\big) \\
&\leqslant x(t) + p(t) F\big(r(t)\big) - F(t) \\
&\leqslant x(t) + 2M.
\end{aligned}
$$

This in turn implies that $\lim_{t\to\infty} x(t) = \infty$.

The case $x(t) < 0$ for large t may similarly be dealt with.

Hence the theorem is proved.

□

Corollary 3.3. *Suppose that all the conditions of Theorem 3.1 are satisfied. If $x(t)$ is a nonoscillatory solution of* (NH), *then either $\lim_{t\to\infty} x(t) = 0$ or $\lim_{t\to\infty} |x(t)| = \infty$ provided that* (i) *n is odd or* (ii) *n is even and $p_1 + p_2 < 1$.*

It follows from Theorems 3.1 *and* 3.2.

Example 6. Consider

$$\text{(3.5)} \qquad \big[x(t) + \mathrm{e}^{-2\pi}(2\sin t + 1)x(t-2\pi)\big]'' - x(t-\pi) = \mathrm{e}^{-t}(4\sin t - 4\cos 2t - 3\sin 2t) + \mathrm{e}^{-t+\pi}\cos t,$$

$t > 2\pi$. Clearly, $-1 < -\mathrm{e}^{-2\pi} \leqslant p(t) \leqslant 3\mathrm{e}^{-2\pi} < 1$, where $p(t) = \mathrm{e}^{-2\pi}(2\sin t + 1)$ and

$$F(t) = \mathrm{e}^{-t}\Big[2\cos t + \sin 2t - \frac{1}{2}\mathrm{e}^{\pi}\sin t\Big] \to 0 \quad \text{as} \quad t\to\infty.$$

From Theorem 3.1 it follows that every bounded solution of (3.5) is oscillatory or tends to zero as $t \to \infty$. In particular, $x(t) = \mathrm{e}^{-t}\cos t$ is an oscillatory solution of (3.5) which tends to zero as $t\to\infty$.

Theorem 3.4. *Let the conditions* (A_2), (A_6) *and* (A_{10}) *hold. Let $r(t) = t - \tau$ and $F(t)$ be τ-periodic. Then every bounded solution of* (NH) *is oscillatory if n is odd and is either oscillatory or $\underline{\lim}_{t\to\infty} |x(t)| = 0$ and*

$$0 \leqslant \overline{\lim_{t\to\infty}}\, |x(t)| \leqslant \frac{b_2 - b_1}{1 - p_1}$$

if n is even, where b_1 and b_2 are lower and upper bounds of $F(t)$ respectively.

P r o o f. Suppose that $x(t)$ is bounded nonoscillatory solution of (NH) such that $x(t) > 0$ for $t > t_0$. Hence there exists a $t_1 > t_0 + \tau$ such that $x\big(g(t)\big) > 0$ for $t \geqslant t_1$. Setting $z(t)$ as in (2.18) we obtain $z(t)$ is bounded, $z(t) + F(t) > 0, z^{(n)}(t) \geqslant 0$ for $t \geqslant t_1$ and

$$(-1)^{n+k} z^{(k)}(t) > 0, \quad k = 1, 2, \ldots, n-1$$

for large t.

Let n be odd. Thus $z'(t) > 0$ for $t \geqslant t_2 > t_1$. Then proceeding as in Theorem 2.8 (when n is even) we obtain $\lim_{t\to\infty} v(t) = \lambda$, $0 < \lambda < \infty$, where $v(t) = (1-p_1)\big(z(t)+b_1\big)$

such that $0 < v(t) \leqslant x(t)$. Hence, for $0 < \varepsilon < \lambda$, $x(g(t)) > \lambda - \varepsilon$ for $t \geqslant t_3 > t_2$. Consequently, in view of (A_6), we have $h(x(g(t))) > \eta > 0$. Hence integrating

$$t^{n-1}v^{(n)}(t) = -(1-p_1)q(t)t^{n-1}h(x(g(t)))$$

from t_3 to t yields

$$\begin{aligned}\eta \int_{t_3}^{t} s^{n-1}q(s)\,\mathrm{d}s &> \int_{t_3}^{t} q(s)s^{n-1}h(x(g(s)))\,\mathrm{d}s\\ &> -\frac{1}{1-p_1}\int_{t_3}^{t} s^{n-1}v^{(n)}(s)\,\mathrm{d}s\\ &> \gamma,\end{aligned}$$

where $-\infty < \gamma < 0$. Hence

$$\int_{t_3}^{\infty} s^{n-1}q(s)\,\mathrm{d}s > -\infty,$$

a contradiction.

If n is even, then one proceeds as in Theorem 2.8 (when n is odd) to obtain required results.

The case when $x(t) < 0$ for large t may be treated similarly.

This completes the proof of the theorem. □

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