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Oscillation of Solutions of Nonlinear Differential Delay Equations of Arbitrary Order

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1. Introduction.

This paper is concerned with the oscillatory behavior of solutions of n-th order nonlinear differential delay equations of the form

$$(*) \qquad x^{(n)}(t)+F(t, x(\delta_0(t)), x'(\delta_1(t)), \cdots, x^{(n-1)}(\delta_{n-1}(t)))=0 ,$$

where

$$\delta_j(t) \leqq t, \qquad \lim_{t\to\infty} \delta_j(t)=\infty, \qquad j=0, 1, \cdots, n-1 .$$

All functions considered are assumed to be continuous on their domains of definition, and such that they guarantee the existence of solutions of (*) which are indefinitely extendable to the right. In what follows, we deal exclusively with such solutions which are nontrivial for all large t. A solution of (*) is called oscillatory if it has arbitrarily large zeros and nonoscillatory if it is eventually of constant sign.

The purpose of this paper is to present some criteria for all solutions of several variants of (*) to be oscillatory in the case n is even, and oscillatory or strongly monotone in the case n is odd.

In Section 2 we establish an oscillation criterion (Theorem 1) for the simplest equation

$$(A) \qquad x^{(n)}(t)+p(t)f(x(\delta(t)))=0 ,$$

which generalizes to arbitrary $n \geqq 2$ recent results of Odarič and Ševelo [8, Theorem 5; 11, Thoerem 2] for the second order delay equation of the form (A). The proof of Theorem 1 is based on combining the arguments of Odarič and Ševelo [8 ,11] with those of Ryder and Wend [10].

In Section 3 we give two oscillation theorems (Theorems 2 and 3) for the equation

$$(B) \qquad x^{(n)}(t)+p(t)g(x(\delta_0(t)), x'(\delta_1(t)), \cdots, x^{(n-1)}(\delta_{n-1}(t)))=0 .$$

Theorem 2 is a generalization of a theorem of Ševelo and Odarič [11, Theorem 5], while Theorem 3 generalizes to equation (B) results of Kartsatos

[4, 5] for the differential equation $x^{(n)}+p(t)g(x, x', \cdots, x^{(n-1)})=0$. The latter has points of contact with results of one of the authors [9].

In Section 4, we prove two oscillation theorems for the equation

$$x^{(n)}(t)+h(t, x(\delta(t)))=0\,, \tag{C}$$

under the assumption that $\delta(t)$ is merely continuous but the delay $\tau(t)=t-\delta(t)$ is bounded from above. These theorems generalize, on the one hand, some of the recent results of Bradley [2], and Gollwitzer [3] pertaining to second order differential delay equations, and, on the other, the oscillation criteria obtained by Ličko and Svec̆ [7], Ryder and Wend [10] for ordinary differential equations of the form $x^{(n)}+h(t, x)=0$.

We note that related results have been obtained by Bourkowski [1], Ladas [6] and Staikos and Petsaulus [12].

2. Oscillation of solutions of (A).

With regard to equation (A) assume that the following conditions are satisfied:

(a) $p(t)$ is continuous and eventually positive on $[a, \infty)$;

(b) $\delta(t)$ is a continuously differentiable nondecreasing function on $[a, \infty)$ such that $\delta(t)\leqq t$ and $\lim_{t\to\infty}\delta(t)=\infty$;

(c) $f(u)$ is a continuous nondecreasing function on $R=(-\infty, \infty)$ such that $uf(u)>0$ for $u\neq 0$, and

(d) $\liminf_{|u|\to\infty}\dfrac{|f(u)|}{|u|^{\alpha}}>0$ for some positive $\alpha\neq 1$.

THEOREM 1. *Let equation* (A) *satisfy* (a)—(d) *and in addition*

$$\int^{\infty}[\delta(t)]^{\alpha^*(n-1)}p(t)\,dt=\infty, \quad \alpha^*=\min(\alpha, 1)\,. \tag{1}$$

Then if n is even, every solution of (A) *is oscillatory, while if n is odd, every solution of* (A) *is oscillatory or tends monotonically to zero as $t\to\infty$ together with all its first $n-1$ derivatives.*

The proof of this theorem is carried out by contradiction and requires the following two lemmas which describe the possible behavior for large t of a nonoscillatory solution; see [10].

LEMMA 1. *If $x(t)\in C^m[b, \infty)$, $x(t)\geqq 0$ and $x^{(m)}(t)$ is monotone on $[b, \infty)$, then either*

(i) $\lim_{t\to\infty}x^{(m)}(t)=0$, *or*

(ii) $\lim_{t\to\infty} x^{(m)}(t)>0$ *and* $x(t), x'(t), \cdots, x^{(m-1)}(t)$ *tend to* ∞ *as* $t\to\infty$.

LEMMA 2. *Suppose* $x(t)\in C^n[b,\infty)$, $x(t)\geqq 0$ *and* $x^{(n)}(t)\leqq 0$ *on* $[b,\infty)$. *Then exactly one of the following cases occurs:*

(I) $x'(t), \cdots, x^{(n-1)}(t)$ *tend monotonically to zero as* $t\to\infty$;

(II) *there is an odd integer* k, $1\leqq k\leqq n-1$, *such that* $\lim_{t\to\infty} x^{(n-j)}(t)=0$ *for* $1\leqq j\leqq k-1$, $\lim_{t\to\infty} x^{(n-k)}(t)\geqq 0$, $\lim_{t\to\infty} x^{(n-k-1)}(t)>0$ *and* $x(t), x'(t), \cdots, x^{(n-k-2)}(t)$ *tend to* ∞ *as* $t\to\infty$.

Analogous statement can be made if $x(t)\leqq 0$ and $x^{(n)}(t)\geqq 0$ on $[b,\infty)$.

PROOF OF THEOREM 1. Let $x(t)$ be a nonoscillatory solution of (A). We may assume that $x(t)$ is eventually positive. A parallel argument holds if $x(t)$ is eventually negative. Form (A)

$$x^{(n)}(t)=-p(t)f(x(\delta(t)))<0 \tag{2}$$

for large t, so, by Lemma 1, $x^{(n-1)}(t)$ decreases to a nonnegative limit as $t\to\infty$. Integration of (2) from t to infinity yields

$$x^{(n-1)}(t)\geqq\int_t^\infty p(u)f(x(\delta(u)))du$$

and a fortiori

$$x^{(n-1)}(\delta(t))\geqq\int_t^\infty p(u)f(x(\delta(u)))du \tag{3}$$

for large t, say $t\geqq T$.

Suppose Case I of Lemma 2 holds. Multiply both sides of (3) by $\delta'(t)$ and integrate the resulting inequality from t to s, $T<t<s$,

$$x^{(n-2)}(\delta(s))-x^{(n-2)}(\delta(t))\geqq\int_t^s[\delta(u)-\delta(t)]p(u)f(x(\delta(u)))du + [\delta(s)-\delta(t)]\int_s^\infty p(u)f(x(\delta(u)))du\,. \tag{4}$$

Since $x^{(n-2)}(\delta(t))$ increases to zero as $t\to\infty$, letting s tend to infinity in (4) we have

$$-x^{(n-2)}(\delta(t))\geqq\int_t^\infty[\delta(u)-\delta(t)]p(u)f(x(\delta(u)))du\,.$$

Proceeding in this fashion we see that

$$(-1)^n x'(\delta(t))\geqq\int_t^\infty\frac{[\delta(u)-\delta(t)]^{n-2}}{(n-2)\,!}p(u)f(x(\delta(u)))du\,. \tag{5}$$

If n is even, integrating (5) multiplied by $\delta'(t)$ from T to t, $T<t$,

$$x(\delta(t)) \geqq \int_T^t \frac{[\delta(u)-\delta(T)]^{n-1}}{(n-1)!} p(u)f(x(\delta(u)))du + \frac{[\delta(t)-\delta(T)]^{n-1}}{(n-1)!}\int_t^\infty p(u)f(x(\delta(u)))du . \tag{6}$$

Let $\alpha>1$. It follows from (6) that

$$[x(\delta(t))]^{-\alpha} \leqq \left\{\int_T^t \frac{[\delta(u)-\delta(T)]^{n-1}}{(n-1)!} p(u)f(x(\delta(u)))du\right\}^{-\alpha} . \tag{7}$$

Multiplying both sides of (7) by $\frac{[\delta(t)-\delta(T)]^{n-1}}{(n-1)!} p(t)f(x(\delta(t)))$ and integrating from t_1 to t_2, $T<t_1<t_2$,

$$\int_{t_1}^{t_2} \frac{[\delta(t)-\delta(T)]^{n-1}}{(n-1)!} p(t)f(x(\delta(t)))[x(\delta(t))]^{-\alpha}dt \leqq \frac{1}{1-\alpha}\left\{\int_T^t \frac{[\delta(u)-\delta(T)]^{n-1}}{(n-1)!} p(u)f(x(\delta(u)))du\right\}^{1-\alpha}\Bigg|_{t_1}^{t_2} . \tag{8}$$

As t_2 tends to infinity the right side remains finite, hence the integral on the left converges.

If $x(\delta(t))$ increases to a finite limit c as $t\to\infty$, then there exists a constant $\tau>T$ such that $c/2\leqq x(\delta(t))<c$ for $t\geqq\tau$. Then from (8)

$$\frac{f(c/2)}{c^\alpha}\int_\tau^\infty [\delta(t)-\delta(T)]^{n-1}p(t)dt \leqq \int_\tau^\infty [\delta(t)-\delta(T)]^{n-1}p(t)f(x(\delta(t)))[x(\delta(t))]^{-\alpha}dt$$

which contradicts (1).

If $x(\delta(t))$ increases to infinity as $t\to\infty$, then, in view of (d), there exists a positive constant M such that

$$f(x(\delta(t)))[x(\delta(t))]^{-\alpha}\geqq M \qquad \text{for} \quad t\geqq t_1 ,$$

provided that t_1 is sufficiently large, hence from (8)

$$M\int_\tau^\infty [\delta(t)-\delta(T)]^{n-1}p(t)dt \leqq \int_\tau^\infty [\delta(t)-\delta(T)]^{n-1}p(t)f(x(\delta(t)))[x(\delta(t))]^{-\alpha}dt$$

which again contradicts (1).

Let $0<\alpha<1$. It follows from (6) with the first integral on the right side removed that

$$(9)\qquad [x(\delta(t))]^{-\alpha}[\delta(t)-\delta(T)]^{\alpha(n-1)} \leqq \left\{\int_t^\infty \frac{p(u)f(x(\delta(u)))}{(n-1)!}du\right\}^{-\alpha}.$$

Multiplying both sides of (9) by $p(t)f(x(\delta(t)))/(n-1)!$ and integrating from t_1 to t_2, $T<t_1<t_2$,

$$\int_{t_1}^{t_2} \frac{[\delta(t)-\delta(T)]^{\alpha(n-1)}}{(n-1)!} p(t)f(x(\delta(t)))[x(\delta(t))]^{-\alpha}dt$$
$$\leqq -\frac{1}{1-\alpha}\left\{\int_t^\infty \frac{p(u)f(x(\delta(u)))}{(n-1)!}du\right\}^{1-\alpha}\Bigg|_{t_1}^{t_2},$$

from which the desired contradiction

$$\int^\infty [\delta(t)]^{\alpha(n-1)}p(t)dt<\infty$$

can be derived exactly as in the case $\alpha>1$.

If n is odd, then

$$(5')\qquad -x'(\delta(t)) \geqq \int_t^\infty \frac{[\delta(u)-\delta(t)]^{n-2}}{(n-2)!} p(u)f(x(\delta(u)))du,$$

which implies that $x'(t)\leqq 0$ for all large t. Hence $x(t)$ decreases to a limit $h\geqq 0$. Suppose $h>0$. Then, by integration of (5′) multiplied by $\delta'(t)$ from T to t,

$$x(\delta(T))-x(\delta(t)) \geqq \int_T^t \frac{[\delta(u)-\delta(T)]^{n-1}}{(n-1)!} p(u)f(x(\delta(u)))du$$
$$+\frac{[\delta(t)-\delta(T)]^{n-1}}{(n-1)!}\int_t^\infty p(u)f(x(\delta(u)))du.$$

Letting $t\to\infty$, we arrive at the following contradiction:

$$x(\delta(T))>x(\delta(T))-h \geqq \int_T^\infty \frac{[\delta(u)-\delta(T)]^{n-1}}{(n-1)!} p(u)f(x(\delta(u)))du$$
$$\geqq \frac{f(h)}{(n-1)!}\int_T^\infty [\delta(u)-\delta(T)]^{n-1}p(u)du.$$

Suppose now Case II of Lemma 2 holds. Observe that there exists $t_0\geqq T$ such that $x^{(j)}(t)>0$ for $t\geqq t_0$, $j=0, 1, \cdots, n-k-1$. Proceeding as in Case I,

$$x^{(n-k)}(\delta(t)) \geqq \int_t^\infty \frac{[\delta(u)-\delta(t)]^{k-1}}{(k-1)!} p(u)f(x(\delta(u)))du.$$

Integrating the above inequality multiplied by $\delta'(t)$ from t_0 to t, $t_0<t$,

(10) $$x^{(n-k-1)}(\delta(t)) \geqq \frac{[\delta(t)-\delta(t_0)]^k}{k!}\int_t^\infty p(u)f(x(\delta(u)))du .$$

Multiplication of both sides of (10) by $\delta'(t)$ and integration from t_0 to t yield

$$x^{(n-k-2)}(\delta(t)) \geqq \frac{[\delta(t)-\delta(t_0)]^{k+1}}{(k+1)!}\int_t^\infty p(u)f(x(\delta(u)))du .$$

Repeating this procedure,

$$x'(\delta(t)) \geqq \frac{[\delta(t)-\delta(t_0)]^{n-2}}{(n-2)!}\int_t^\infty p(u)f(x(\delta(u)))du ,$$

which leads directly to the following inequality analogous to (6):

$$x(\delta(t)) \geqq \int_{t_0}^t \frac{[\delta(u)-\delta(t_0)]^{n-1}}{(n-1)!} p(u)f(x(\delta(u)))du + \frac{[\delta(t)-\delta(t_0)]^{n-1}}{(n-1)!}\int_t^\infty p(u)f(x(\delta(u)))du .$$

The proof now proceeds exactly as in Case 1.

This completes the proof of Theorem 1.

COROLLARY. *Consider the equation*

(D) $$x^{(n)}(t)+p(t)[x(\delta(t))]^\gamma=0,$$

where $p(t)$ *and* $\delta(t)$ *satisfy, respectively, conditions* (a) *and* (b) *of Theorem* 1, *and* γ *is the ratio of odd positive integers relatively prime. Let any one of the following hold*:

(i) $\gamma>1$, $\int^\infty [\delta(t)]^{n-1}p(t)dt=\infty$;

(ii) $\gamma=1$, $\int^\infty [\delta(t)]^{n-1-\varepsilon}p(t)dt=\infty$ *for some* ε *with* $0<\varepsilon<1$;

(iii) $\gamma<1$, $\int^\infty [\delta(t)]^{\gamma(n-1)}p(t)dt=\infty$.

Then every solution of equation (D) *is oscillatory when* n *is even, and every solution of* (D) *is oscillatory or tends to zero together with its first* $n-1$ *derivatives as* $t\to\infty$ *when* n *is odd.*

PROOF. It is only necessary to consider equation (D) with $\gamma=1$. Since this equation satisfies condition (d) for any α less than 1, it follows from the proof of Theorem 1 that

$$\int^\infty [\delta(t)]^{\alpha(n-1)}p(t)dt<\infty$$

for any α with $0<\alpha<1$, in particular, for $\alpha=1-\varepsilon/(n-1)$. But this is not consistent with our hypothesis.

3. Oscillation of solutions of (B).

The following theorem is a straightforward extension of Theorem 1.

THEOREM 2. *With regard to equation* (B) *assume that*

(a) *$p(t)$ is continuous and eventually positive on $[a, \infty)$;*

(b) *$\delta_j(t)$, $j=0, 1, \cdots, n-1$, are continuously differentiable nondecreasing functions on $[a, \infty)$ such that $\delta_j(t)\leqq t$ and $\lim_{t\to\infty}\delta_j(t)=\infty$;*

(c) *$g(u, u_1, \cdots, u_{n-1})$ is continuous on R^n and satisfies*

$$g(u, u_1, \cdots, u_{n-1})\leqq f_1(u)\phi_1(u_1, \cdots, u_{n-1}) \qquad \text{if} \quad u\geqq 0,$$

$$g(u, u_1, \cdots, u_{n-1})\geqq f_2(u)\phi_2(u_1, \cdots, u_{n-1}) \qquad \text{if} \quad u< 0,$$

where

(d) *$f_j(u)$, $j=1, 2$, are continuous nondecreasing functions on R such that $uf_j(u)>0$ for $u\neq 0$ and*

$$\liminf_{|u|\to\infty}\frac{|f_j(u)|}{|u|^{\alpha_j}}>0 \text{ for some positive } \alpha_j\neq 1, \text{ and}$$

(e) *$\phi_j(u_1, \cdots, u_{n-1})$, $j=1, 2$, are continuous on R^{n-1} and*

$$\inf_{(u_1,\cdots,u_{n-1})\in R^{n-1}}\phi_j(u_1, \cdots, u_{n-1})=\beta_j>0;$$

(f) $$\int^{\infty}[\delta_0(t)]^{\alpha_j^*(n-1)}p(t)dt=\infty, \quad \alpha_j^*=\min(\alpha_j, 1), \quad j=1, 2.$$

Then, every solution of (B) *is oscillatory if n is even, and every solution of* (B) *is oscillatory or tends to zero together with its first $n-1$ derivatives if n is odd.*

The proof of this theorem follows exactly the same procedure as the proof of Theorem 1. As before, the contradiction is obtained from the assumption that equation (B) has a nonoscillatory solution. The details are omitted.

LFMMA 3. *Let $\phi(t)$ be a function such that $\phi(t)\in C^n[b, \infty)$, $\phi(t)>0$ and $\phi^{(n)}(t)\leqq 0$ on $[b, \infty)$, and let $\delta_j(t)$, $j=0, 1, \cdots, n-1$, satisfy* (b) *of Theorem* 2 *and the additional condition*

(11) $$\lim_{t\to\infty}\delta_0'(t)=\delta>0\,.$$

Then it holds that

(12) $$\lim_{t\to\infty}\frac{\phi^{(j)}(\delta_j(t))}{\phi(\delta_0(t))}=0 \qquad \textit{for} \quad 1\leqq j\leqq n-1\,,$$

unless $\phi(t)$ and its first $n-1$ derivatives tend to zero as $t\to\infty$. The exceptional case may arise only when n is odd.

This lemma is a variant of a proposition given in [9, Lemma 2] and can be proved quite similarly.

Let us now consider equation (B) which satisfies, in addition to (a), (b) of Theorem 2 and (11), the following condition (G):

$g(u, u_1, \cdots, u_{n-1})$ is continuous on R^n, $ug(u, u_1, \cdots, u_{n-1})>0$ for every $(u, u_1, \cdots, u_{n-1})\in R^n$ with $u\neq 0$, and for every $(u, u_1, \cdots, u_{n-1})\in R^n$ and every $\lambda>0$, $g(-u, -u_1, \cdots, -u_{n-1})=-g(u, u_1, \cdots, u_{n-1})$, $g(\lambda u, \lambda u_1, \cdots, \lambda u_{n-1})=\lambda^\gamma g(u, u_1, \cdots, u_{n-1})$, where γ is the ratio of odd positive integers relatively prime.

We shall prove the following

THEOREM 3. *Suppose equation* (B) *satisfies, in addition to* (a), (b), (11) *and* (G), *any one of the following conditions*:

(i) $\gamma>1$, $\displaystyle\int^\infty[\delta_0(t)]^{n-1}p(t)dt=\infty$;

(ii) $\gamma=1$, $\displaystyle\int^\infty[\delta_0(t)]^{n-1-\varepsilon}p(t)dt=\infty$ for some $\varepsilon>0$ with $0<\varepsilon<1$;

(iii) $\gamma<1$, $\displaystyle\int^\infty[\delta_0(t)]^{\gamma(n-1)}p(t)dt=\infty$.

Then if n is even, every solution of (B) *is oscillatory, and if n is odd, every solution of* (B) *is oscillatory or tends to zero as $t\to\infty$ together with its first $n-1$ derivatives.*

PROOF. Let $x(t)$ be a nonoscillatory solution of (B). Since, by (G), $-x(t)$ is again a solution of (B), we may suppose that $x(t)$ is eventually positive. On account of (B), $x^{(n)}(t)\leqq 0$ for sufficiently large t, so it follows from Lemma 3 that either $x(t)$ tends to zero together with its first $n-1$ derivatives as $t\to\infty$, or $x(t)$ satisfies (12), remaining bounded away from zero for all large t, say $t\geqq t_0$.

Let the latter possibility hold. Then $x(t)$ must satisfy the equation

(13) $$y^{(n)}(t)+Q(t)[y(\delta_0(t))]^\gamma=0$$

for $t \in [t_0, \infty)$, where

$$Q(t)=p(t)\,g\Big(1, \frac{x'(\delta_1(t))}{x(\delta_0(t))}, \dots, \frac{x^{(n-1)}(\delta_{n-1}(t))}{x(\delta_0(t))}\Big).$$

In view of (12), there is a $t_1 \geqq t_0$ and $\varepsilon < g(1, 0, \dots, 0)$ such that

$$g(1, 0, \dots, 0)-\varepsilon < g\Big(1, \frac{x'(\delta_1(t_1))}{x(\delta_0(t))}, \dots, \frac{x^{(n-1)}(\delta_{n-1}(t))}{x(\delta_0(t))}\Big) < g(1, 0, \dots, 0)+\varepsilon$$

for all $t \geqq t_1$. Consequently, if (i) holds, then for equation (13) with $\gamma > 1$ we have

$$\int^{\infty} [\delta_0(t)]^{n-1} Q(t)\,dt \geqq [g(1, 0, \dots, 0)-\varepsilon] \int^{\infty} [\delta_0(t)]^{n-1} p(t)\,dt = \infty\,,$$

which implies by Corollary to Theorem 1 that every solution of (B) is oscillatory or tends monotonically to zero as $t \to \infty$. The contradiction thus obtained shows that the assertion of the theorem is true in the case (i). The cases (ii), (iii) can be discussed similarly by using the corresponding results of Corollary to Theorem 1.

REMARK. It is easily verified that, in the presence of condition (11), conditions (i), (ii), (iii) of Theorem 3 are equivalent to the following (i′), (ii′), (iii′), respectively:

(i′) $\gamma > 1$, $\int^{\infty} t^{n-1} p(t)\,dt = \infty$;

(ii′) $\gamma = 1$, $\int^{\infty} t^{n-1-\varepsilon} p(t)\,dt = \infty$ for some $\varepsilon > 0$ with $0 < \varepsilon < 1$;

(iii′) $\gamma < 1$, $\int^{\infty} t^{\gamma(n-1)} p(t)\,dt = \infty$.

4. Oscillation of solutions of (C).

Finally, we study the oscillatory properties of solutions of equation (C) under the assumption that $\delta(t)$ is merely continuous but the delay $\tau(t) = t - \delta(t)$ is bounded from above by a positive constant M.

Assume moreover that the following conditions are satisfied:

(a) $h(t, x)$ is continuous in $S = [a, \infty) \times R$;

(b) $h(t, x) \geqq a(t)\phi(x)$ if $x > 0$ and $h(t, x) \leqq b(t)\psi(x)$ if $x < 0$, $(t, x) \in S$, where

(c) $a(t)$ and $b(t)$ are nonnegative locally integrable on $[a, \infty)$ and neither $a(t)$ nor $b(t)$ is identically zero on any subinterval of $[a, \infty)$,

(d) $\phi(x)$ and $\psi(x)$ are nondecreasing, and $x\phi(x)>0$ and $x\psi(x)>0$ on R for $x\neq 0$, and

(e) for some $\varepsilon\geqq 0$,

$$\int_{\varepsilon}^{\infty}\frac{du}{\phi(u)}<\infty \quad\text{and}\quad \int_{-\varepsilon}^{-\infty}\frac{du}{\psi(u)}<\infty .$$

THEOREM 4. *Let the function $h(t, x)$ in* (C) *satisfy* (a)—(e) *and assume that*

$$\int^{\infty} t^{n-1}a(t)\,dt=\int^{\infty} t^{n-1}b(t)\,dt=\infty . \tag{15}$$

Then if n is even, every solution of (C) *is oscillatory, and if n is odd, every solution is either oscillatory or tends monotonically to zero together with its first $n-1$ derivatives.*

Theorem 4 is a natural extension of a theorem of Ryder and Wend [10, Theorem 1] for the differential equation $x^{(n)}+h(t, x)=0$. This theorem also extends to arbitrary $n\geqq 2$ results of Bradley [2, Theorem 1] and Gollwitzer [3, Theorem 1] for the delay equation $x''(t)+a(t)[x(\delta(t))]^{\gamma}=0$, γ being the ratio of odd integers.

PROOF. The proof is essentially that contained in Ryder and Wend [10], so we only sketch it.

Let $x(t)$ be a nonoscillatory solution of (C). We may assume that $x(t)>0$ for $t\geqq t_0>0$, since a parallel argument holds if $x(t)$ is eventually negative. In view of (b)-(d)

$$x^{(n)}(t)=-h(t, x(\delta(t)))\leqq -a(t)\phi(x(\delta(t)))\leqq 0 \tag{16}$$

for $t\geqq t_1=t_0+M$. Now we have to examine two cases: Cases I and II of Lemma 2.

Suppose Case I holds. Then an integration of (16) $n-1$ times gives

$$(-1)^n x'(t)\geqq\int_t^{\infty}\frac{(v-t)^{n-2}}{(n-2)\,!}a(v+M)\phi(x(v))\,dv\geqq 0, \qquad t\geqq t_1 . \tag{17}$$

If n is even, then integrating (17) from t_1 to t, we have

$$x(t)\geqq\int_{t_1}^{t}\frac{(v-t_1)^{n-1}}{(n-1)\,!}a(v+M)\phi(x(v))\,dv ,$$

from which, by using (d) and (e), we can derive the inequality

$$\int_{t_1}^{\infty}\frac{(v-t_1)^{n-1}}{(n-1)\,!}a(v+M)\,dv<\infty .$$

But this is impossible because of the hypothesis (15).

If n is odd, then

$$-x'(t) \geqq \int_t^\infty \frac{(v-t)^{n-2}}{(n-2)!} a(v+M)\phi(x(v))dv \geqq 0\,, \tag{17'}$$

so $x(t)$ decreases to a limit $L \geqq 0$ as $t \to \infty$. Let $L>0$. Then, integrating (17′) from t_1 to ∞,

$$x(t_1) > \phi(L) \int_{t_1}^\infty \frac{(v-t_1)^{n-1}}{(n-1)!} a(v+M)dv\,,$$

which again contradicts (15).

Suppose Case II holds. Proceeding exactly as in [10] it can be shown that there is a $\tau \geqq t_1$ such that the inequality

$$x'(t) \geqq \int_t^\infty \frac{(t-\tau)^{n-1}}{(n-2)!} a(v+M)\phi(x(v))dv \tag{18}$$

holds for $t \geqq \tau$, regardless of the parity of n. An integration of (18) from τ to t yields

$$x(t) \geqq \int_\tau^t \frac{(v-\tau)^{n-1}}{(n-1)!} a(v+M)\phi(x(v))dv\,,$$

and the required contradiction is obtained just as in Case I.

Our final result is contained in the following theorem which constitutes a generalization of results of Ryder and Wend [10, Theorem 2] and Gollwitzer [3, Theorem 2].

THEOREM 5. *Let the function* $h(t, x)$ *satisfy* (a)—(d) *and*

(f) *there exist positive constants* λ_0, M, N *and constants* α, β *with* $0 \leqq \alpha < 1$, $0 \leqq \beta < 1$ *such that for* $\lambda \geqq \lambda_0$

$$\phi(\lambda x) \geqq M\lambda^\alpha \phi(x), \qquad x>0,$$

$$\psi(\lambda x) \leqq N\lambda^\beta \psi(x), \qquad x<0,$$

and assume that

$$\int^\infty t^{\alpha(n-1)} a(t)dt = \int^\infty t^{\beta(n-1)} b(t)dt = \infty\,. \tag{19}$$

Then every solution of (C) *is oscillatory when* n *is even, while every solution is either oscillatory or tends monotonically to zero together with its* $n-1$ *derivatives when* n *is odd.*

PROOF. Let n be even and suppose there exists a nonoscillatory solution $x(t)$ such that $x(t)>0$ for $t \geqq t_0$. Since $x^{(n)}(t) \leqq 0$ for $t \geqq t_1 = t_0 + M$,

$x^{(n-1)}(t)$ is nonincreasing and positive on $[t_1, \infty)$; i.e., there exists a constant K such that $0 < x^{(n-1)}(t) < K$ for $t \geqq t_1$.

Proceeding as in [10] we can deduce that

$$x^{(n)}(t) + A^{\alpha} B a(t)(t-M)^{\alpha(n-1)}[x^{(n-1)}(t)]^{\alpha} \leqq 0 \tag{20}$$

for $t \geqq t^* = 2^n t_1 + M$, where $A = 2^{-n^2/(n-1)!}$ and $B = M[\lambda_0/x(t^*)]^{\alpha} \times \phi(x(t^*)/\lambda_0)$.

Dividing (20) by $[x^{(n-1)}(t)]^{\alpha}$ and integrating from τ to T, $t^* < \tau < T$, we have

$$A^{\alpha} B \int_{\tau}^{T} (t-M)^{\alpha(n-1)} a(t)\, dt \leqq \int_{x^{(n-1)}(T)}^{x^{(n+1)}(\tau)} \frac{du}{u^{\alpha}}.$$

Since the latter integral is bounded from above by $\int_0^K du/u^{\alpha}$, we have the required contradiction by letting $T \to \infty$.

The case where $x(t) < 0$ for $t \geqq t_0$ can be treated similarly.

Let n be odd and let $x(t)$ be a nonoscillatory solution of (C). Then, arguing as in the case of even n, we are led to a contradiction unless $x(t)$ approaches zero as $t \to \infty$. Consequently, a nonoscillatory solution of (C), if it exists, does tend to zero together with its first $n-1$ derivatives when t increases to infinity.

References

[1] F. Bourkowski, Oscillation theorems for a second order non-linear functional differential equation, J. Math. Anal. Appl. **33** (1971), 258-262.

[2] J.S. Bradley, Oscillation theorems for a second-order delay equation, J. Differential Equations **8** (1970), 397-403.

[3] H.E. Gollwitzer, On nonlinear oscillation for a second order delay equation, J. Math. Anal. Appl. **26** (1969), 385-389; erratum, ibid. **29** (1970), 464.

[4] A.G. Kartsatos, Criteria for oscillation of solutions of differential equations of arbitrary order, Proc. Japan Acad, **44** (1968), 599-602.

[5] A.G. Kartsatos, On oscillation of solutions of even order nonlinear differential equations, J. Differential Equations **6** (1969), 232-237.

[6] G. Ladas, Oscillation and asymptotic behavior of solutions of differential equations with retarded argument, J. Differential Equations **10** (1971), 281-290.

[7] I. Ličko and M. Švec, Le caractère oscillatoire des solutions de l'équation $y^{(n)} + f(x)y^{\alpha} = 0$, $n > 1$, Czechoslovak Math. J. **13** (1963), 481-491.

[8] O.N. Odarič and V.N. Ševelo, On oscillatory properties of solutions of second order nonlinear differential equations with delay, Mathematical Physics **4** (1968, Kiev), 146-153. (Russian)

[9] H. Onose, Some oscillation criterifa for n-th order nonlinear delay-differential equations, Hiroshima Math. J. **1** (1971), 171-176.

[10] G.H. Ryder and D.V.V. Wend, Oscillation of solutions of certain ordinary differential equations of n-th order, Proc. Amer. Math. Soc. **25** (1970), 463-469.

[11] V.N. Ševelo and O.N. Odarič, Some problems of the theory of oscillation (nonoscillation) of solutions of second order differential equations with delay, Ukrain. Math. J. **23** (1971), 508-516. (Russian)

[12] V. A. Staikos and A. G. Petsaulus, Some oscillaticn criteria for second order nonlinear delay-differential equations, J. Math. Anal. Appl. **30** (1970), 695–701.

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