# OSCILLATION RESULTS OF HIGHER ORDER LINEAR DIFFERENTIAL EQUATION 

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#### Abstract

We study higher order linear differential equation $y^{(k)}+A_{1}(z) y=0$ with $k \geq 2$, where $A_{1}=A+h, A$ is a transcendental entire function of finite order with $\frac{1}{2} \leq \mu(A)<1$ and $h \neq 0$ is an entire function with $\rho(h)<\mu(A)$. Then it is shown that, if $f^{(k)}+A(z) f=0$ has a solution $f$ with $\lambda(f)<\mu(A)$ then exponent of convergence of zeros of any non trivial solutions of $y^{(k)}+A_{1}(z) y=0$ is infinite.


## 1. Introduction

For the understanding of this paper, we must know the basic facts of Nevanlinna's value distribution theory. For a meromorphic function $f, n(r, f), N(r, f), m(r, f)$ and $T(r, f)$ denote un-integrated counting function, integrated counting function, proximity function and characteristic function respectively. We also use first main theorem of Nevanlinna for a meromorphic function $f$, see [6, 9, 15]. Now we present elementary definitions of order of growth $\rho(f)$, lower order of growth $\mu(f)$ exponent of convergence of zeros $\lambda(f)$ for a meromorphic function $f$ to make the paper self contained.

$$
\begin{gathered}
\rho(f)=\limsup _{r \rightarrow \infty} \frac{\log T(r, f)}{\log r}, \\
\mu(A)=\liminf _{r \rightarrow \infty} \frac{\log T(r, f)}{\log r}, \\
\lambda(f)=\limsup _{r \rightarrow \infty} \frac{\log n\left(r, \frac{1}{f}\right)}{\log r}=\limsup _{r \rightarrow \infty} \frac{\log N\left(r, \frac{1}{f}\right)}{\log r},
\end{gathered}
$$

where

$$
N\left(r, \frac{1}{f}\right)=\int_{0}^{r} \frac{n\left(t, \frac{1}{f}\right)-n\left(0, \frac{1}{f}\right)}{t} d t+n\left(0, \frac{1}{f}\right) \log r .
$$

A meromorphic function $g(z)$ is a small function of $f(z)$ if $T(r, g)=S(r, f)$ and vice versa, here $S(r, f)$ denote such quantities which are of growth $o(T(r, f))$ as $r \rightarrow \infty$, outside of a possible exceptional set of finite linear measure.
Suppose $A$ be an entire function and $k \geq 2$ is an integer. The complex differential equation

$$
\begin{equation*}
f^{(k)}+A(z) f=0 \tag{1}
\end{equation*}
$$

[^0]has entire solutions $f_{j}(j=1,2, \ldots, k)$, as proved by Hille 7 that any solutions of (11) are entire functions whenever $A$ is an entire function. In this note, we are concerned with the zero distribution of solutions of linear differential equation of $k^{\text {th }}$ degree with perturbed coefficients. The research in this direction is called complex oscillation theory and to classify the oscillations of solutions of (1) has been a long standing problem since 1980s. And it is basically depends on finding the conditions on $A(z)$ so that the existing solution $f$ of (1) have $\lambda(f)=0$ or $\lambda(f) \geq \rho(A)$ or $\lambda(f)=\infty$. However, rest cases like $0<\lambda(f)<\rho(A)$ or $\lambda(f)=\rho(A)=\infty$ or $\rho(A)<\lambda(f)<\infty$ are also possible but they seems to be quite exceptional.
In this field, many mathematicians have done research by relating the exponent of convergence of zeros $\lambda\left(f_{j}\right)(j=1,2, \ldots, k)$ and order of growth $\rho$ of coefficient $A$, for e.g [1-3, 10, 12]. More results regarding complex oscillation of solutions of linear differential equation can be found in [4, 5, [1] and references therein.

In 2005 , A. Alotaibi [1] proved the following theorem, which shows that by doing small perturbation of equation (1) we get exponent of convergence of zeros of solution is at least the order of growth of coefficient $A$.

Theorem 1. [1] Suppose that $A$ is a transcendental entire function with $\rho(A)<\frac{1}{2}$, $k \geq 2$ and (1) has a solution $f$ with $\lambda(f)<\rho(A)$. Let

$$
\begin{equation*}
A_{1}=A+h, \tag{2}
\end{equation*}
$$

where $h \not \equiv 0$ is an entire function with $\rho(h)<\rho(A)$. Then exponent of convergence of zeros of any non trivial solution of

$$
\begin{equation*}
g^{(k)}+A_{1} g=0 \tag{3}
\end{equation*}
$$

does not have a solution $g$ with $\lambda(g)<\rho(A)$.
So, it seems interesting to find the condition on $A(z)$ so that exponent of convergence of any nontrivial solution of equation (3) is infinite, and hence in 2020, by using similar idea in [1] J. Long and Y. Li proved the following theorem by considering lower order of growth of $A$ along with the small perturbation of such equation.

Theorem 2. [13] Suppose that $A$ is a transcendental entire function of finite order with $\mu(A)<\frac{1}{2}, k \geq 2$ and (11) has a solution $f$ with $\lambda(f)<\mu(A)$. Let $A_{1}$ satisfies (2) where $h \not \equiv 0$ is an entire function with $\rho(h)<\mu(A)$. Then exponent of convergence of zeros of any non trivial solution of (3) is infinite.

Motivated by above results, It is natural to ask what we can say about the coefficient $A(z)$ of $\rho(A) \geq \frac{1}{2}$ or $\mu(A) \geq \frac{1}{2}$. So we consider the case $\frac{1}{2} \leq \mu(A)<1$ and prove the following result.
Theorem 3. Suppose that $A$ is a transcendental entire function of finite order with $\frac{1}{2} \leq \mu(A)<1, k \geq 2$ and (11) has a solution $f$ with $\lambda(f)<\mu(A)$. Let $A_{1}$ satisfies(2) and $h \not \equiv 0$ is an entire function with $\rho(h)<\mu(A)$. Then exponent of convergence of zeros of any non trivial solution of

$$
\begin{equation*}
y^{(k)}+A_{1} y=0 \tag{4}
\end{equation*}
$$

is infinite.
By the proof of theorem 3, we can easily prove the following result.

Corollary 1. Let $A$ is a transcendental entire function of finite order with $\frac{1}{2} \leq$ $\mu(A)<1, k \geq 2$ and (1) has a solution $f$ with finitely many zeros, and let $A_{1}$ satisfies(2) and $h \not \equiv 0$ is an entire function with $\rho(h)<\mu(A)$. Then (4) does not have a non trivial solution with finitely many zeros.

In section 2 we state some lemmas and results. In section 3, we prove Theorem 3.

## 2. Auxiliary results

In this section we present some lemmas and definition which will be helpful in proving our main theorem. For a set $I \subset(0, \infty)$, the linear measure is defined by $m(I)=\int_{I} d t$. For a set $J \subset(1, \infty)$, the logarithmic measure is defined by $m_{l}(J)=\int_{J} \frac{1}{t} d t$ and the upper and lower logarithmic density of $J \subset[1, \infty)$ by

$$
\overline{\operatorname{logdens}} J=\underset{r \rightarrow \infty}{\limsup } \frac{m_{l}(J \cap[1, r])}{\log }
$$

and

$$
\underline{\text { logdens }} J=\liminf _{r \rightarrow \infty} \frac{m_{l}(J \cap[1, r])}{\log } .
$$

The logarithmic density actually gives an idea how big the set J is.
Definition 1. [9] Let $B\left(z_{n}, r_{n}\right)=\left\{z:\left|z-z_{n}\right|<r_{n}\right\}$ be the open disc in the complex plane. Countable union $\bigcup_{n=1}^{\infty} B\left(z_{n}, r_{n}\right)$ is said to be $R$-set if $z_{n} \rightarrow \infty$ and $\Sigma r_{n}$ is finite.

Now we state a very well known result regarding rational function, which we use in our main theorem's proof.

Lemma 1. [9] A meromorphic function $f$ is a rational function iff $T(r, f)=O(\log r)$.
Next we define a well known representation for higher order logarithmic derivatives through which we can easily show the possibility of the existence of a solution $f$ of (11) with no zeroes by taking $F=\frac{f^{\prime}}{f}, f=e^{P}$ where $P$ is an entire function.

Lemma 2. (Hayman's Lemma) [6] Let $f(z)$ be an analytic function, and let $F=\frac{f^{\prime}}{f}$. Then for $k \in \mathbb{N}$, we have

$$
\frac{f^{(k)}}{f}=F^{k}+\frac{k(k-1)}{2} F^{k-2} F^{\prime}+P_{k-2}(F),
$$

where $P_{k-2}$ is a differential polynomial with constant coefficients, which vanishes identically for $k \leq 2$ and has degree of $(k-2)$ when $k>2$.

Lemma 3. 14] Let $B(z)$ is an entire function with $\mu(B) \in\left[\frac{1}{2}, \infty\right)$ then there exists a sector $\Omega(\alpha, \beta), \beta-\alpha \geq \frac{\pi}{\mu(B)}$, such that

$$
\varlimsup_{r \rightarrow \infty} \frac{\log \log \left|B\left(r e^{\iota \theta}\right)\right|}{\log r} \geq \mu(B)
$$

$\forall \theta \in \Omega(\alpha, \beta)$, where $0 \leq \alpha<\beta<2 \pi$.

Lemma 4. [9] Suppose that $f(z)$ is a meromorphic function of finite order. Then there exists a positive integer $N$ such that

$$
\frac{f^{\prime}(z)}{f(z)}=O\left(|z|^{N}\right)
$$

holds for large $z$ outside of an $R$-set.
Lemma 5. (Langley's theorem) [10] Let $A(z)$ be a transcendental entire function of finite order, and let $J_{1}$ be a subset of $[1, \infty)$ of infinite logarithmic measure and with the following property. For each $r \in J_{1}$, there exists an arc

$$
a_{r}=\left\{r e^{i t}: 0 \leq \alpha_{r} \leq t \leq \beta_{r} \leq 2 \pi\right\}
$$

of the circle $S(0, r)=\{z:|z|=r\}$ such that

$$
\lim _{r \rightarrow \infty, r \in J_{1}} \frac{\min \left\{\log |A(z)|: z \in a_{r}\right\}}{\log r}=+\infty .
$$

Let $k \geq 2$ and let $f$ be a solution of 1 with $\lambda(f)<\infty$. Then there exists a subset $J_{2} \subset[1, \infty)$ of finite measure, such that for large $r \in J_{0}=J_{1} \backslash J_{2}$, we have

$$
\frac{f^{\prime}}{f}=c_{r} A(z)^{1 / k}-\frac{k-1}{2 k} \frac{A^{\prime}(z)}{A(z)}+O\left(r^{-2}\right)
$$

holds for all $z \in a_{r}$, where the constant $c_{r}$ satisfies $c_{r}^{k}=-1$ and may depend on $r$, for a given $r \in J_{0}$ but not depend on $z$, and the branch of $A(z)^{1 / k}$ is analytic on $a_{r}$ (included in the case where $a_{r}$ is the whole circle $S(0, r)$ ).

## 3. Proof of main theorem

Proof. Given $A_{1}=A+h, \rho(h)<\mu(A) \leq \rho(A)$ implies $\rho\left(A_{1}\right)=\rho(A)$. Let equation (1) has a solution $f$ with $\lambda(f)<\mu(A) \leq \rho(A)$ and let us assume (4) has a solution $y$ with $\lambda(y)<\infty$. So we can take

$$
f=P e^{U}
$$

and

$$
y=Q e^{V}
$$

where $U, V, P, Q$ are entire functions of finite order [11.
Now $f=P e^{U}$ implies $\lambda(f)=\rho(P)<\infty[9]$ as $e^{U} \neq 0 \& \rho(P)=\lambda(P)$.
Similarly,

$$
\lambda(y)=\rho(Q) .
$$

Let

$$
\begin{align*}
& F=\frac{f^{\prime}}{f}  \tag{5}\\
& Y=\frac{y^{\prime}}{y} \tag{6}
\end{align*}
$$

Using $f=P e^{U} \& Y=Q e^{V}$, we get

$$
F=\frac{P e^{U} U^{\prime}+e^{U} P^{\prime}}{P e^{U}}
$$

$$
\begin{equation*}
=\frac{P^{\prime}}{P}+U^{\prime} \tag{7}
\end{equation*}
$$

Similarly,

$$
\begin{equation*}
Y=\frac{Q^{\prime}}{Q}+V^{\prime} \tag{8}
\end{equation*}
$$

Applying Hayman's Lemma 2, we get

$$
\begin{equation*}
\frac{f^{(k)}}{f}=F^{k}+\frac{k(k-1)}{2} F^{k-2} F^{\prime}+P_{k-2}(F) \tag{9}
\end{equation*}
$$

and

$$
\begin{equation*}
\frac{y^{(k)}}{y}=Y^{k}+\frac{k(k-1)}{2} Y^{k-2} Y^{\prime}+P_{k-2}(Y) \tag{10}
\end{equation*}
$$

where $P_{k-2}$ is a differential polynomial with constant coefficients, which vanishes identically for $k \leq 2$ and has degree of $k-2$, when $k>2$. Choose

$$
\begin{equation*}
\max \{\lambda(f), \lambda(y), \rho(h)\}<\beta<\gamma<\mu<1 \tag{11}
\end{equation*}
$$

Using lemma 3 for $A(z)$, there exists a sector $\Omega(\alpha, \beta)$ such that following inequality holds $\forall \theta \in \Omega(\alpha, \beta)$.

$$
\varlimsup_{r \rightarrow \infty} \frac{\log \log \left|A\left(r e^{\iota \theta}\right)\right|}{\log r} \geq \mu(A)
$$

This gives

$$
\log \left|A\left(r e^{\iota \theta}\right)\right| \geq r^{\gamma}
$$

where $\gamma=\mu(A)-\epsilon$. Set

$$
J_{1}:=\left\{z=r e^{\iota \theta}:|z|=r>r_{0}, \alpha_{0}<\theta<\beta_{0}\right\},
$$

where $\alpha<\alpha_{0}<\beta_{0}<\beta$ and $r_{0}$ is a fixed number, satisfying

$$
\begin{equation*}
\inf _{|z|=r \in J_{1}} \log |A(z)| \geq r^{\gamma} \tag{12}
\end{equation*}
$$

where $J_{1}$ has a positive upper logarithmic density [8].
Using lemma 4, there exist a set $J_{2} \subset[1, \infty)$ which is a subset of finite measure, for some $t_{1} \in \mathbb{N}$,

$$
\begin{equation*}
\left|\frac{A^{\prime}(z)}{A(z)}\right|+\left|\frac{P^{\prime}(z)}{P(z)}\right|+\left|\frac{Q^{\prime}(z)}{Q(z)}\right| \leq r^{\tau} \tag{13}
\end{equation*}
$$

holds for $|z|=r \geq 1, r \notin J_{2}$. Now for large $r \in J_{1}$ and $\rho(h)<\rho(A)$, we have $A_{1}=A+h$. This gives

$$
\begin{aligned}
\log \left(A_{1}\right) & =\log (A+h) \\
& =\log A+o(1) \\
& \geq r^{\gamma}+o(1) .
\end{aligned}
$$

Next we calculate $\frac{f^{\prime}}{f}$ and $\frac{y^{\prime}}{y}$ in terms of $A(z)$. For applying lemma 5, take an arc $a_{r}:=\left\{z=r_{1} e^{\iota \theta}: \theta \in\left(\alpha_{1}, \beta_{1}\right)\right\}$ in $J_{1}$ for some fixed $r_{1}$, where $\alpha_{0}<\alpha_{1}<\beta_{1}<\beta_{0}$. Now for some $\theta_{1} \in\left(\alpha_{1}, \beta_{1}\right)$, we get

$$
\min \left\{\log |A(z)|: z \in a_{r}\right\}=\log \left|A\left(r e^{\iota \theta_{1}}\right)\right|
$$

Next,

$$
\frac{\log A\left(r e^{\iota \theta_{1}}\right)}{\log r} \geq \frac{r^{\gamma}}{\log r}
$$

As $\gamma<\mu<1$, so

$$
\lim _{r \rightarrow \infty} \frac{r^{\gamma}}{\log r} \rightarrow \infty
$$

Hence

$$
\lim _{r \rightarrow \infty} \frac{\min \left\{\log |A(z)|: z \in a_{r}\right\}}{\log r}=\infty
$$

Next, on applying lemma 5 in equations (1) and (4). The following equalities hold for large $r \in J_{0}$.

$$
\begin{gather*}
\frac{f^{\prime}}{f}=c A(z)^{1 / k}-\frac{k-1}{2 k} \frac{A^{\prime}(z)}{A(z)}+O\left(r^{-2}\right), z \in \Omega, c^{k}=-1  \tag{14}\\
\frac{y^{\prime}}{y}=d A_{1}(z)^{1 / k}-\frac{k-1}{2 k} \frac{A_{1}^{\prime}(z)}{A(z)}+O\left(r^{-2}\right), z \in \Omega, d^{k}=-1 \tag{15}
\end{gather*}
$$

Now expanding $A_{1}(z)^{1 / k}$ and $\frac{A_{1}^{\prime}(z)}{A(z)}$ in terms of $A(z)^{1 / k}$ and $\frac{A^{\prime}(z)}{A(z)}$ with the help of binomial theorem, we have

$$
\begin{aligned}
& \limsup _{r \rightarrow \infty} \frac{\log \log M(r, h)}{\log r}=\rho(h), \\
\Longrightarrow & |h(z)| \leq e^{r^{\rho(h)+o(1)}},
\end{aligned}
$$

where $M(r, h)$ is the maximum term.
For $|z|=r \rightarrow \infty, r \in J_{0}$, using above equation and equation (12), we get

$$
\left|\frac{h(z)}{A(z)}\right| \leq \frac{e^{r^{\rho(h)+0(1)}}}{e^{r \gamma}}=o(1)
$$

Similarly we can find

$$
\left|\frac{h^{\prime}(z)}{A(z)}\right| \leq \frac{e^{r \rho(h)+o(1)}}{e^{r \gamma}}=o(1) .
$$

Now expanding with the help of above two equations, we have

$$
\begin{aligned}
A_{1}^{\frac{1}{k}}(z) & =(A+h)^{\frac{1}{k}} \\
& =A^{\frac{1}{k}}\left(1+\frac{h}{A}\right)^{\frac{1}{k}} \\
& =A^{\frac{1}{k}}\left(1+O\left(\frac{|h|}{|A|}\right)\right),
\end{aligned}
$$

for $|z|=r \in J_{0}$. Similarly,

$$
\begin{aligned}
\frac{A_{1}^{\prime}(z)}{A(z)} & =\left(\frac{A^{\prime}+h}{A+h}\right) \\
& =\frac{A^{\prime}+h}{A\left(1+\frac{h}{A}\right)} \\
& =\frac{A^{\prime}+h^{\prime}}{A}\left(1-\frac{h}{A}+\frac{h^{2}}{A^{2}} \cdots\right) \\
& =\left(\frac{A^{\prime}}{A}+\frac{h^{\prime}}{A}\right)\left(1+O\left(\frac{|h|}{|A|}\right)\right) \\
& =\frac{A^{\prime}}{A}\left(1+O\left(\frac{|h|}{|A|}\right)\right),
\end{aligned}
$$

for $|z|=r \in J_{0}$. So we get equation

$$
\begin{equation*}
\frac{y^{\prime}}{y}=d A(z)^{1 / k}-\frac{k-1}{2 k} \frac{A^{\prime}(z)}{A(z)}+O\left(r^{-2}\right), d^{k}=-1 . \tag{16}
\end{equation*}
$$

Now we will prove $c=d$ for large $r \in J_{0}$. Let $d / c=\omega$, where $\omega^{k}=1$ and using equation 11, we get

$$
\begin{equation*}
\frac{y^{\prime}}{y}=c \omega A(z)^{1 / k}-\frac{k-1}{2 k} \frac{A^{\prime}(z)}{A(z)}+O\left(r^{-2}\right), w^{k}=1 . \tag{17}
\end{equation*}
$$

Multiply equation 9 with $\omega$,

$$
\omega \frac{f^{\prime}}{f}=\omega c A(z)^{1 / k}-\omega \frac{k-1}{2 k} \frac{A^{\prime}(z)}{A(z)}+O\left(r^{-2}\right), c^{k}=-1 .
$$

Now subtract equation 17 from above equation, we get

$$
\begin{gathered}
\omega \frac{f^{\prime}}{f}-\frac{y^{\prime}}{y}=\omega c A(z)^{1 / k}-\omega \frac{k-1}{2 k} \frac{A^{\prime}(z)}{A(z)}+O\left(r^{-2}\right)-\omega c\left(A(z)^{1 / k}\right)-\frac{k-1}{2 k} \frac{A^{\prime}(z)}{A(z)}+O\left(r^{-2}\right) \\
\omega\left(\frac{f^{\prime}}{f}-\frac{k-1}{2 k} \frac{A^{\prime}(z)}{A(z)}\right)=\frac{y^{\prime}}{y}+\frac{k-1}{2 k} \frac{A^{\prime}(z)}{A(z)}+O\left(r^{-2}\right)
\end{gathered}
$$

Now by using Argument principle, integrate above equation around $\left|z_{n}\right|=r_{n} \in J_{0}$, and $r_{n} \rightarrow \infty$ as $n \rightarrow \infty$, we get

$$
\begin{equation*}
\omega\left[2 \pi \iota n\left(r_{n}, \frac{1}{f}\right)+\frac{k-1}{2 k} 2 \pi \iota n\left(r_{n}, \frac{1}{A}\right)\right]+o(1)=2 \pi \iota n\left(r_{n}, \frac{1}{y}\right)+\frac{k-1}{2 k} 2 \pi \iota n\left(r_{n}, \frac{1}{A}\right) . \tag{18}
\end{equation*}
$$

In this equation R.H.S must be a positive integer as $n\left(r_{n}, \frac{1}{y}\right) \geq 0$ and $n\left(r_{n}, \frac{1}{A}\right)>0$.
Let us suppose that $n\left(r_{n}, \frac{1}{A}\right)=0$, if so then it implies $N\left(r_{n}, \frac{1}{A}\right)=0$.
Now since $\log |A(z)| \geq r^{\gamma}$ for $r>r_{0}$ i.e $\inf _{|z|=r_{n} \in J_{0}} \log |A(z)|$ is very large for $r_{n} \rightarrow \infty$, thus we get

$$
\begin{aligned}
m\left(r_{n}, \frac{1}{A}\right) & =\frac{1}{2 \pi} \int_{0}^{2 \pi} \log \left|\frac{1}{A\left(r e^{\ell \theta}\right)}\right| d \theta \\
& =0 .
\end{aligned}
$$

Hence

$$
T\left(r_{n}, \frac{1}{A}\right)=m\left(r_{n}, \frac{1}{A}\right)+n\left(r_{n}, \frac{1}{A}\right)=0 .
$$

Now with the help of Nevanlinna's first fundamental theorem $T\left(r_{n}, A\right)=T\left(r_{n}, \frac{1}{A}\right)+$ $O(1)$, we get

$$
T\left(r_{n}, A\right)=O(1)
$$

which is a contradiction with the fact that A is transcendental. So our supposition is wrong and hence $n\left(r_{n} \cdot \frac{1}{A}\right)>0$. So $n\left(r_{n}, \frac{1}{A}\right)+n\left(r_{n}, \frac{1}{f}\right)$ is a positive integer. Now as $n\left(r_{n}, \frac{1}{A}\right) \geq 1$ implies $\frac{k-1}{2 k} n\left(r_{n}, \frac{1}{A}\right)+n\left(r_{n}, \frac{1}{f}\right) \geq \frac{k-1}{2 k}$. Taking modulus of imaginary part of both sides after multiplying with $\omega$, we get

$$
\left|\operatorname{Im}\left[\omega\left(n\left(r_{n}, \frac{1}{f}\right)+\frac{k-1}{2 k} n\left(r_{n}, \frac{1}{A}\right)\right)\right]\right| \geq \frac{k-1}{2 k}|\operatorname{Im}(\omega)|
$$

and taking imaginary part of both sides of 18, we get

$$
\operatorname{Im}\left[\omega\left(n\left(r_{n}, \frac{1}{f}\right)+\frac{k-1}{2 k} n\left(r_{n}, \frac{1}{A}\right)\right)\right]+\operatorname{Im}(o(1))=0
$$

so

$$
\begin{aligned}
|\operatorname{Imo}(1)|= & \left|-\operatorname{Im}\left[\omega\left(n\left(r_{n}, \frac{1}{f}\right)+\frac{k-1}{2 k} n\left(r_{n}, \frac{1}{A}\right)\right)\right]\right| \\
& \geq \frac{k-1}{2 k}|\operatorname{Im}(\omega)| \\
& \geq \frac{k-1}{2 k} \Delta,
\end{aligned}
$$

where $\Delta:=\inf \left\{|\operatorname{Im}(\omega)|: \omega^{k}=1, \operatorname{Im}(\omega) \neq 0\right\}$. It is obvious $\Delta>0$. Now for sufficiently large $r_{n} \in J_{0}$ and $|\operatorname{Im}(o(1))|<\Delta \frac{k-1}{2 k}$, we get $\operatorname{Im}(\omega)=0$.
As $\omega^{k}=1$ with imaginary part zero implies either $\omega=-1$ or $\omega=1$.
But $\omega=-1$ contradicts with (18), so $\omega=1$ and hence $c=d$. By using it we can write equation (16) as

$$
\begin{equation*}
Y(z)=\frac{y^{\prime}}{y}=c A(z)^{1 / k}-\frac{k-1}{2 k} \frac{A^{\prime}(z)}{A(z)}+O\left(r^{-2}\right), c^{k}=-1, \tag{19}
\end{equation*}
$$

for $|z|=r \in J_{0}$. Subtracting equation (17) from equation (14), we get

$$
\frac{f^{\prime}(z)}{f(z)}=\frac{y^{\prime}(z)}{y(z)}+O(1),|z|=r .
$$

Now as poles of $\frac{f^{\prime}}{f}=$ zeroes of $f$, hence for large $r \in J_{0}$, we have

$$
n\left(r, \frac{1}{f}\right)=n\left(r, \frac{1}{y}\right) .
$$

With the help of equations (6), (77), (8) and (9) we get,

$$
\frac{P^{\prime}}{P}+U^{\prime}=\frac{Q^{\prime}}{Q}+V^{\prime}+O(1)
$$

Using above equation and (13), we get

$$
\begin{aligned}
\left|U^{\prime}-V^{\prime}\right| & \leq\left|\frac{P^{\prime}}{P}\right|+\left|\frac{Q^{\prime}}{Q}\right| \\
& \leq 2 r^{\tau}
\end{aligned}
$$

which holds for all $|z|=r$ with large $r \in J_{0} \backslash J_{2}$. Next

$$
\log M\left(r, U^{\prime}-V^{\prime}\right) \leq \log \left(2 r^{\tau}\right)
$$

where $M\left(r, U^{\prime}-V^{\prime}\right)$ is the maximum term, this gives

$$
\frac{T\left(r, U^{\prime}-V^{\prime}\right)}{\log r} \leq \tau
$$

As $U^{\prime}-V^{\prime}$ is entire function so

$$
M\left(r, U^{\prime}-V^{\prime}\right)=T\left(r, U^{\prime}-V^{\prime}\right)
$$

By above inequality, we get

$$
T\left(r, U^{\prime}-V^{\prime}\right)=O(\log r)
$$

This implies $U^{\prime}-V^{\prime}$ is a rational function but as $U$ and $V$ are entire so $P_{0}=U^{\prime}-V^{\prime}$ is a polynomial. From equation (8) and (9),

$$
F=\frac{P^{\prime}}{P}+U^{\prime}=\frac{P^{\prime}}{P}+P_{0}+V^{\prime}
$$

and

$$
Y=\frac{Q^{\prime}}{Q}+V^{\prime}
$$

This implies

$$
\begin{equation*}
F=\frac{P^{\prime}}{P}+P_{0}+Y-\frac{Q^{\prime}}{Q}=Y+M \tag{20}
\end{equation*}
$$

where $M=\frac{P^{\prime}}{P}-\frac{Q^{\prime}}{Q}+P_{0}$. Using (1) and (10), we get

$$
\begin{equation*}
\frac{f^{(k)}}{f}=F^{k}+\frac{k(k-1)}{2} F^{k-2} F^{\prime}+P_{k-2}(F)=-A \tag{21}
\end{equation*}
$$

where $P_{k-2}$ is a differential polynomial with constant coefficients, which vanishes identically for $k \leq 2$ and has degree of atmost $k-2$, when $k>2$.
By equations (2), (4) and (11), we get

$$
\begin{equation*}
\frac{y^{(k)}}{y}=Y^{k}+\frac{k(k-1)}{2} Y^{k-2} Y^{\prime}+P_{k-2}(G)=-A-h . \tag{22}
\end{equation*}
$$

Using equations (20) and (21), we get

$$
(Y+M)^{k}+\frac{k(k-1)}{2}(Y+M)^{k-2}\left(Y^{\prime}+M^{\prime}\right)+P_{k-2}(Y+M)=-A
$$

With the help of Binomial theorem, we can expand above equality and we obtain

$$
Y^{k}+M k Y^{k-1}+\frac{k(k-1)}{2} Y^{k-2} Y^{\prime}+B_{k-2}(Y, M)=-A
$$

where $B_{k-2}$ represent a polynomial in $\mathrm{M}, \mathrm{Y}$, and their derivatives with the total degree of atmost $k-2$. Now combining above equation with equation (22), we get

$$
\begin{equation*}
h=k M Y^{k-1}+R_{k-2}(Y, M) . \tag{23}
\end{equation*}
$$

Now we will claim $M \not \equiv 0$. On the contrary, let $M \equiv 0$. As $F=Y+M$ implies $F=Y$, then by equations (21) and (22), we get $h=0$, which contradicts with the hypothesis and hence claim is true.
Now divide equation (23) by $M Y^{k-2}$, we get

$$
\begin{equation*}
k Y+\frac{R_{k-2}(Y, M)}{M Y^{k-2}}=\frac{h}{M Y^{k-2}} . \tag{24}
\end{equation*}
$$

Assume that $|Y|>1$ and $\frac{R_{k-2}(Y, M)}{M Y^{k-2}}$ is a sum of the terms

$$
\frac{1}{M Y^{k-2}} M^{p_{0}}\left(M^{\prime}\right)^{p_{1}} \ldots\left(M^{(k)}\right)^{p_{k}} Y^{q_{0}}\left(Y^{\prime}\right)^{q_{1}} \ldots\left(Y^{(k)}\right)^{q_{k}}
$$

where $q_{0}+q_{1}+\ldots q_{k} \leq k-2$.
As $|Y|>1$ implies $\frac{1}{|Y|}<1$ and taking modulus of above equation, we get

$$
\begin{gather*}
|M|^{p_{0}+p_{1}+\ldots p_{k}-1}\left|\frac{M^{\prime}}{M}\right|^{p_{1}} \ldots \ldots . .\left|\frac{M^{(k)}}{M}\right|^{p_{k}}|Y|^{q_{0}+q_{1}+\ldots q_{k}-k+2}\left|\frac{Y^{\prime}}{Y}\right|^{q_{1}} \ldots\left|\frac{Y^{(k)}}{Y}\right|^{q_{k}} \\
\leq|M|^{p_{0}+p_{1}+\ldots p_{k}-1}\left|\frac{M^{\prime}}{M}\right|^{p_{1}} \ldots\left|\frac{M^{(k)}}{M}\right|^{p_{k}}\left|\frac{Y^{\prime}}{Y}\right|^{q_{1}} \ldots\left|\frac{Y^{(k)}}{Y}\right|^{q_{k}} \tag{25}
\end{gather*}
$$

Taking proximity function on both sides of equation 24, we get

$$
\begin{array}{r}
m(r, k Y)=m\left(r,-\frac{R_{k-2}}{M Y^{k-2}}+\frac{h}{M Y^{k-2}}\right) \\
m(r, Y) \leq m\left(r,-\frac{R_{k-2}}{M Y^{k-2}}\right)+m\left(r, \frac{h}{M Y^{k-2}}\right)+\log 2
\end{array}
$$

Using equation (25), we get

$$
\begin{aligned}
m(r, Y) \leq & m\left(r,|M|^{p_{0}+p_{1}+\ldots p_{k}-1}\right)+m\left(r,\left|\frac{M^{\prime}}{M}\right|\right)+\ldots m\left(r,\left|\frac{M^{(k)}}{M}\right|\right) \\
& +m\left(r,\left|\frac{Y^{\prime}}{Y}\right|\right) \ldots m\left(r,\left|\frac{Y^{(k)}}{Y}\right|\right)+m(r, h)+m\left(r, \frac{1}{M}\right) \\
& +m\left(r, \frac{1}{Y^{k-2}}\right)+\log 4 .
\end{aligned}
$$

Now as M and Y are rational functions, so $m\left(r, \frac{M^{\prime}}{M}\right)=S(r, M)$ and $m\left(r, \frac{Y^{\prime}}{Y}\right)=$ $S(r, Y)$, hence we get

$$
m(, Y) \leq c_{0} m(r, M)+m\left(r, \frac{1}{M}\right)+m(r, h)+s(r, M)+S(r, Y)
$$

where $c_{0}=p_{0}+p_{1}+\ldots p_{k}-1$, positive constant. Now by adding and subtracting terms $c_{0} N(r, M), N\left(r, \frac{1}{M}\right), N(r, h)$ and using first fundamental theorem of Nevanlinna, we get

$$
\begin{align*}
m(r, Y) & \leq c_{0} T(r, M)+T\left(r, \frac{1}{M}\right)+T(r, h)+S(r, Y) \\
& =\left(c_{0}+1\right) T(r, M)+T(r, h)+S(r, Y) \tag{26}
\end{align*}
$$

Take proximity function on both sides of equation (22), we get

$$
\begin{aligned}
m(r, A) \leq & m\left(r, Y^{k}\right)+m\left(r, \frac{k(k-1)}{2} Y^{k-2} Y^{\prime}\right)+m\left(r, P_{k-2}(Y)\right)+m(r, h) \\
& \leq k m(r, Y)+m\left(r, \frac{k(k-1)}{2}\right)+m\left(r, Y^{k-2}\right)+m\left(r, \frac{Y^{\prime}}{Y}\right) \\
& +m(r, Y)+S(r, Y)+S(r, h)
\end{aligned}
$$

With some simple calculations, we get

$$
\begin{equation*}
m(r, A) \leq c_{2} m(r, Y)+O(\log r) \tag{27}
\end{equation*}
$$

Using equation (26) and (27), we get

$$
\begin{equation*}
T(r, A) \leq c_{3} T(r, M)+c_{2} T(r, h)+S(r, Y)+O(\log r) \tag{28}
\end{equation*}
$$

As

$$
\begin{aligned}
T(r, M) & =T\left(r, \frac{P^{\prime}}{P}-\frac{Q^{\prime}}{Q}+P_{0}\right) \\
& =m\left(r, \frac{P^{\prime}}{P}-\frac{Q^{\prime}}{Q}+P_{0}\right)+N\left(r, \frac{P^{\prime}}{P}-\frac{Q^{\prime}}{Q}+P_{0}\right) \\
& \leq S(r, P)+S(r, Q)+O(\log r)+N\left(r, \frac{P^{\prime}}{P}\right)+N\left(r, \frac{Q^{\prime}}{Q}\right)+N\left(r, P_{0}\right) \\
& =N\left(r, \frac{1}{P}\right)+N\left(r, \frac{1}{Q}\right)+S(r, W) \\
& \leq T(r, P)+T(r, Q)+S(r, W)
\end{aligned}
$$

where W is some entire function with order of growth $\beta$. Using equation (12) with $\rho(P)=\lambda(f)$ and $\rho(Q)=\lambda(y)$, we have

$$
T(r, M) \leq o\left(r^{\beta}\right)
$$

Now as $\rho(h)<\beta$, we get

$$
T(r, h) \leq o\left(r^{\beta}\right)
$$

Using these in equation (28), we get

$$
T(r, A) \leq o\left(r^{\beta}\right)
$$

for $r \in J_{0} \backslash J_{2}$. Hence contradiction arises. This completes the proof.

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