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Oscillations and Concentrations in Weak Solutions of the Incompressible Fluid Equations

Ronald J. DiPerna[1],* and Andrew J. Majda[2],**

[1] Department of Mathematics, University of California, Berkeley, CA 94720, USA
[2] Department of Mathematics and Program for Applied and Computational Mathematics, Princeton University, Princeton, NJ 08544, USA

Abstract. The authors introduce a new concept of measure-valued solution for the 3-D incompressible Euler equations in order to incorporate the complex phenomena present in limits of approximate solutions of these equations. One application of the concepts developed here is the following important result: a sequence of Leray-Hopf weak solutions of the Navier-Stokes equations converges in the high Reynolds number limit to a measure-valued solution of 3-D Euler defined for all positive times. The authors present several explicit examples of solution sequences for 3-D incompressible Euler with uniformly bounded local kinetic energy which exhibit complex phenomena involving both persistence of oscillations and development of concentrations. An extensions of the concept of Young measure is developed to incorporate these complex phenomena in the measure-valued solutions constructed here.

Introduction

The Euler equations for an incompressible homogeneous fluid in n-space dimensions are given by

$$\frac{Dv}{Dt} = -\nabla p, \quad x \in R^n, \quad t>0, \quad \operatorname{div} v = 0, \quad v(x,0) = v_0(x), \tag{1.1}$$

while the Navier-Stokes equations $\left(\text{with Reynolds number } \frac{1}{\varepsilon}\right)$ are given by

$$\frac{Dv^\varepsilon}{Dt} = -\nabla p^\varepsilon + \varepsilon \Delta v^\varepsilon, \quad x \in R^n, \quad t>0, \quad \operatorname{div} v^\varepsilon = 0, \quad v^\varepsilon(x,0) = v_0(x). \tag{1.2}$$

Here $v = {}^t(v_1, \ldots, v_n)$ is the fluid velocity, $\frac{Dv}{Dt} = \frac{\partial v}{\partial t} + v \cdot \nabla v$, is the convective derivative and p is the scalar pressure. The structure of solutions of the Euler

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equations and the behavior of sequences of solutions of the Navier-Stokes equations as $\varepsilon \to 0$ are problems of wide current interest. The same comment applies to other regularizations of the Euler equations through computational vortex methods. Such problems are motivated by attempts to understand turbulent structures in fluid flows. Numerical calculations [2–4] with smooth solutions of the Euler equations in three space dimensions and with vortex sheet initial data in two space dimensions [1, 13, 14] reveal increasingly complex behavior as the numerical regularization parameters tend to zero.

In the context of the Navier-Stokes limit the importance of the concept of measure-valued solution is illustrated by the following theorem proved in Sect. 5.

Theorem. *Consider a smooth divergence-free velocity field v_0 in $L^2(R^3)$ and let v_ε be any (Leray-Hopf) weak solution of the Navier-Stokes equations* (1.2) *with initial data v_0. Then as ε vanishes a subsequence has a limit which defines a measure-valued solution of the* 3-*D incompressible Euler equations.*

The motivation for the theorem is partly the following. For smooth initial data there exists a fixed interval of time $[0, T]$ on which the Navier-Stokes solutions v_ε converge strongly in L^2 as ε vanishes; the limiting field v is necessarily a conventional solution of the Euler equations. During this time interval the associated measure-valued solution of the Euler equations reduces to a Dirac mass at the point $v(x,t)$. However, numerical evidence [2–4] indicates that the complexity of the flow increases rapidly as time evolves. This suggests that beyond some critical time the Navier-Stokes solutions v_ε converge weakly (but not strongly) in L^2 due to the development of oscillations and/or concentrations. At the level of the measure-valued solution the appearance of oscillations and concentrations corresponds to the bursting of the Dirac mass into a measure with more complex structure. In this paper, we introduce a framework to study this possibility.

In this paper first we identify through rigorous examples some new phenomena that occur in solution sequences of the Euler equations including the persistence of oscillations and the development of concentrations. Within the context of sequences with uniformly bounded locally finite kinetic energy, the persistence of oscillations in three space dimensions is discussed in Sect. 2 and the development of concentrations in two space dimensions is discussed in Sect. 3.

With these examples as motivation, we develop a natural generalization of the classical Young measure which provides a representation for both oscillations and concentrations in arbitrary L^p function sequences. Then we introduce a notion of measure-valued solution for the Euler equations under the single assumption of locally bounded kinetic energy, see (1.13) and (1.14) below. This concept of measured-valued solution is used in the theorem mentioned above. We recall that the kinetic energy is the only positive definite conserved functional for fluid flows in three space dimensions, so a theory with this generality is needed. The remainder of the introduction is a summary of the developments presented below in this paper.

Tartar [27, 28] first recognized the importance of the Young measure for the problem of representing and analyzing oscillations. Some of the first applications of the Young measure to evolution equations dealt with L^∞-bounded solution sequences to scalar conservation laws and systems of conservation laws in one space dimension [5, 10, 27, 28]. In the context of L^∞, DiPerna [6] introduced the concept of measure-valued solution to a system of conservation laws. In order to study both oscillations and concentrations, an extension to L^p is required. The remainder of the introduction provides a discussion of some of the new phenomena and concepts developed in this paper along with applications to the Euler equations. Detailed proofs are given in the subsequent sections.

First, we recall the classical concept of the Young measure. Below, if G is a locally compact Hausdorff space, $M(G)$ denotes the Radon measures on G with finite total mass, $M^+(G)$ the subspace of nonnegative measures and $\operatorname{Prob} M(G)$ the subset of nonnegative measures with unit mass. The notation $w_\varepsilon \rightharpoonup w$ denotes weak convergence in M or L^p while $w_\varepsilon \to w$ denotes strong convergence. Consider an arbitrary sequence of vector fields $(x,t) \to v_\varepsilon(x,t)$ from $\Omega \subset R^n \times R$ to R^n satisfying the uniform L^∞ bound, $|v_\varepsilon(x,t)| \leqq C$, and converging weakly in L^p, $1 \leqq p < \infty$ to v. The parametrized Young measure theorem yields a fixed subsequence which we ignore in our notation and the existence of a Lebesgue measurable mapping

$$(x,t) \to \bar{v}_{(x,t)} \varepsilon \operatorname{Prob} M(R^n)$$

with $\operatorname{supp} \bar{v}_{(x,t)} \subset \{v : |v| \leqq C\}$, such that

$$g(v_\varepsilon) \rightharpoonup \langle v_{(x,t)}, g \rangle$$

for all continuous functions $g: R^n \to R$. Thus,

$$\lim \iint \phi g(v_\varepsilon) dx dt = \iint \phi \langle \bar{v}_{(x,t)}, g \rangle dx dt \quad \text{for all } \phi \text{ in } C_0(\Omega). \tag{1.3}$$

Here the expected value of g with respect to $\{\bar{v}_{(x,t)}\}$ is denoted by brackets:

$$\langle \bar{v}_{(x,t)}, g \rangle = \int_{R^n} g(v) d\bar{v}_{(x,t)}. \tag{1.4}$$

Furthermore, if $p < \infty$ then

$$\begin{gathered} v_\varepsilon \to v \text{ in } L^p \text{ iff } \bar{v} \text{ reduces to a Dirac mass:} \\ \bar{v}_{(x,t)} = \delta_{v(x,t)}. \end{gathered} \tag{1.5}$$

Thus the Young measure, $\bar{v} \equiv \{\bar{v}_{(x,t)}\}$, represents all composite weak limits of an L^∞-bounded sequence and non-Dirac structure reflects the persistence of oscillations in the limit process.

A conventional pointwise weak solution of the Euler equations is derived from the conservative form of (1.1), namely

$$\frac{\partial}{\partial t} v + \operatorname{div} v \otimes v + \nabla p = 0, \qquad \operatorname{div} v = 0,$$

through multiplication by suitable test functions and integration by parts. A divergence-free velocity field $v \in L^2(\Omega)$ is a weak solution of the Euler equations on a space-time region Ω if $\operatorname{div} v = 0$ and

$$\iint_\Omega (\phi_t \cdot v + \nabla \phi : v \otimes v) dx dt = 0, \tag{1.6}$$

for all test functions ϕ in $C_0^\infty(\Omega)$ with $\operatorname{div}\phi=0$. Of course, the divergence-free condition on v is also understood in the sense of distributions. Here $v\otimes v$ is the matrix $(v_i v_j)$, $\nabla\phi=\left(\frac{\partial\phi_i}{\partial x_j}\right)$ and $A:B$ is the matrix product $\sum_{i,j} a_{ij}b_{ij}$.

By extending the definition in [6] in a straightforward fashion and using (1.6) we say that a Young measure $\bar{v}$ is a *measure-valued solution* of the Euler equations if

$$\frac{\partial}{\partial t}\langle\bar{v},v\rangle+\operatorname{div}\langle\bar{v},v\otimes v\rangle+\nabla p=0 \qquad \operatorname{div}\langle\bar{v},v\rangle=0$$

for some distribution in p, i.e. if

$$\iint \phi_t\langle\bar{v}_{(x,t)},v\rangle+\nabla\phi:\langle\bar{v}_{(x,t)},v\otimes v\rangle dx\,dt=0$$

for all divergence-free vector fields ϕ in C_0^∞ and also if

$$\iint \nabla\psi\cdot\langle\bar{v}_{(x,t)},v\rangle dx\,dt=0$$

for all scalar fields ψ in $C_0^\infty(\Omega)$.

The Young measure associated with a sequence of conventional Euler solutions provides an example of a measure-valued solution.

Proposition 1. *If v_ε is a sequence of solutions of the Euler equation* (1.1) *such that*

$$|v_\varepsilon|_{L^\infty(\Omega)}\leqq C,$$

then the Young measure $\bar{v}$ constructed from this sequence defines a measure-valued solution of the Euler equations.

Other procedures for generating measure-valued solutions are discussed in Sect. 5.

In several of the applications to hyperbolic conservation laws [5, 10], the Young measure has been used as a tool in conjunction with the compensated compactness theory of Tartar and Murat [27, 28] to show that oscillations do not persist in nondegenerate systems in one space dimension, i.e. $\bar{v}_{(x,t)}=\delta$ if $t>0$ provided that $\bar{v}_{(x,0)}=\delta$. The absence of sustained oscillations has also been established in the context of the steady 2-D transonic equations by Morawetz [21] under the assumption that the solutions in question avoid the vacuum and stagnation states. The work of Lax and Levermore [16], Venakides [30, 31], and Flaschka, Forest and McLaughlin [12, 20] on the zero dispersion limit of the KdV equation provides the first nontrivial examples where oscillations persist. In these cases the Young measure has a simple structure related to modulated families of N-phase waves in different regions of space-time [6, 20]. The recent work of Roytburd and Slemrod [24, 25] and Rascle and Serre [23, 26] deals with very special non-Dirac structure in degenerate hyperbolic systems of conservation laws in one space dimension.

In Sect. 2, we present an example of an L^∞-bounded sequence of smooth solutions of the 3-D Euler equations which exhibits persistence of oscillations; the associated Young measure is not a Dirac mass. From Proposition 1, this Young measure provides a nontrivial example of a measure-valued solution of the 3-D Euler equations representing oscillations. The tacit assumption that v_ε is uniformly bounded in L^∞ used in constructing the classical Young measure is too restrictive for applications to incompressible fluid flow where the natural physical assumption requires only a locally uniform bound on the kinetic energy, i.e.

$$\iint_{\Omega\cap B_R} |v_\varepsilon(x,t)|^2 dxdt \leqq C_R \tag{1.8}$$

with $B_r=\{x: |x|<R\}$. In Sect. 3, we present several examples of solution sequences of the Euler equations which satisfy (1.8). These examples illustrate that the classical Young measure (1.3), (1.4) is not adequate to describe the behavior of quadratic functions like $v_\varepsilon \otimes v_\varepsilon$.

In view of the structure of the Euler equations, the smallest class of functions relevant to the study of composite limits consists of functions g with the form

$$g(v)=g_0(v)(1+|v|^2)+g_H\left(\frac{v}{|v|}\right)|v|^2, \tag{1.9}$$

where g_0 lies in the space $C_0(R^n)$ of continuous functions vanishing at infinity and g_H lies in the space $C(S^{n-1})$ of continuous functions on the unit sphere. The coefficients $(1+|v|^2)$ and $|v|^2$ are introduced for convenience. Clearly, the defining functions v and $v\otimes v$ of the Euler equations belong to this class with $g_0=v/(1+|v|^2)$ and $g_H=\theta\otimes\theta$.

In Sect. 3, we build several explicit solution sequences v_ε of the 2-D Euler equations which develop concentrations. Intuitively, the energy in the limit concentrates on a small set of measure zero in physical space while the mass of the associated probability escapes to infinity in the state space. In more precise terms the sequences v_ε have the following two properties. The locally uniform energy bound (1.8) holds and for every g of the form (1.9) the composite weak limit is given by

$$\lim \iint \phi g(v_\varepsilon)dxdt = \Gamma \int \phi(0,t)dt \int_{S^1} g_H \frac{d\theta}{2\pi} + \left\{\iint \phi dxdt\right\} g_0(0). \tag{1.10}$$

Here Γ is a fixed constant and ϕ lies in C_0^∞. Thus the composite limit consists of two terms which are singular with respect to Lebesgue measure. In contrast to (1.3) the first takes the form

$$\Gamma\delta(x)\langle v_{(x,t)}, g_H\rangle,$$

where $\delta(x)$ is the Dirac mass on the line $x=0$ and $(x,t)\to v_{(x,t)}$ is a $\delta(x)$-measurable map from $R^2\times R$ to Prob$M(S^1)$ given by uniform measure on S^1:

$$\langle v_{(x,t)}, g_H\rangle = \int_{S^1} g_H(\theta)\frac{d\theta}{2\pi}.$$

The second term takes the form $dxdt\otimes\delta_0$ and is a classical Young measure. Thus, a generalization of the Young measure is needed which incorporates this structure. In Sect. 4, we develop such a generalization which includes both oscillations and concentrations and we apply it to the problem of representing weak limits of exact and approximate solution sequences of the Euler equations under the single assumption of locally uniform boundedness of the kinetic energy (1.8).

The theory builds a generalized Young measure as a vector-valued measure v with special structure, see Theorems 4.1 and 4.2 below. The action of v on functions in the class (1.9) and in other separable spaces admits a simple form described in Theorems 1 and 2 below.

Theorem 1. *Generalized Young Measure. If $\{v_\varepsilon\}$ is an arbitrary family of functions whose L^2 norm on a set $\Omega \subset R^n \times R$ is uniformly bounded, then $\{v_\varepsilon\}$ contains a sequence with the following properties. There exists a measure μ in $M(\Omega)$ such that*

$$|v_\varepsilon|^2 \rightharpoonup \mu \quad in \quad M(\Omega),$$

and a μ-measurable map

$$(x,t) \to \{v^1_{(x,t)}, v^2_{(x,t)}\}$$

from Ω to $M^+(R^n) \oplus \operatorname{Prob} M(S^{n-1})$ such that for all g in (1.9),

$$g(v_\varepsilon) \rightharpoonup \langle v^1, g_0 \rangle (1+f) dxdt + \langle v^2, g_H \rangle d\mu, \tag{1.11}$$

where f denotes the Radon-Nikodym derivative of μ with respect to $dxdt$, i.e.

$$\lim_{\varepsilon \to 0} \iint \phi g(v_\varepsilon) dxdt = \iint \phi \langle v^1_{(x,t)}, g_0 \rangle (1+f) dxdt + \iint \phi \langle v^2_{(x,t)}, g_H \rangle d\mu$$

for all ϕ in C_0^∞.

Remark 1. If $\mu = \mu_s + f dxdt$ denotes the Lebesgue decomposition of μ into its singular and absolutely continuous parts, then $v^1_{(x,t)}$ vanishes on the set where μ_s is concentrated.

Remark 2. The triple (μ, v^1, v^2) represents a canonical decomposition of a vector-valued measure v which we shall call the generalized Young measure. It is naturally defined on a class of functions which is somewhat broader than (1.9) and yields

$$g(v_\varepsilon) \rightharpoonup \langle v, g \rangle$$

for all g in that class, cf. Sect. 4. The restriction of v to the class (1.9) gives

$$v = v^1 (dxdt + d\mu) + v^2 d\mu. \tag{1.12}$$

Definition. A generalized Young measure v is called a measured-valued solution of the incompressible Euler equations if

$$\iint \phi_t \cdot \langle v, v \rangle + \nabla \phi : \langle v, v \otimes v \rangle = 0 \tag{1.13}$$

for all divergence-free vector fields ϕ in C_0^∞ and

$$\iint \nabla \psi \cdot \langle v, v \rangle = 0 \tag{1.14}$$

for all scalar fields ψ in C_0^∞.

Remark 3. Since the action of v in (1.13) is limited to the special class (1.9) we may appeal to the decomposition (1.12) and express (1.13) in the following concrete form

$$\iint \phi_t \cdot \left\langle v^1_{(x,t)}, \frac{v}{1+|v|^2} \right\rangle (1+f) dxdt + \nabla \phi : \quad v^2_{(x,t)}, \frac{v \otimes v}{|v|^2} \quad d\mu = 0.$$

The general abstract structure is discussed further in Sect. 4

The following proposition is an immediate consequence of Theorem 1 and the definitions above.

Proposition 2. *The generalized Young measure associated with any sequence of conventional solutions of the Euler equations with uniformly bounded local kinetic energy defines a measure-valued solution.*

Examples. Every classical weak solution v with locally finite kinetic energy defines a measure-valued solution through the following identifications:

$$d\mu = |v|^2 dx dt,$$
$$v^1_{(x,t)} = \delta_{v(x,t)}, \quad v^2_{(x,t)} = \delta_{\theta(x,t)} \varepsilon M(S^{n-1}),$$

where $\theta(x,t) = v(x,t)/|v(x,t)|$.

The explicit 2-*D* examples mentioned above and constructed in Sect. 3 have the form

$$\mu = \Gamma \delta(x), \quad v^1_{(x,t)} = \delta_0, \quad v^2_{(x,t)} = \text{uniform measure on } S^1.$$

This triple defines a stationary measure-valued solution $v = (\mu, v^1, v^2)$ which is highly concentrated and exhibits non-Dirac behavior on homogeneous functions.

In Sect. 4, several properties of the generalized Young measure are established which indicate its role in representing oscillations and concentrations. As a corollary of that discussion we obtain the following theorem.

Theorem 2. *Let $v = (\mu, v^1, v^2)$ be a generalized Young measure associated with an L^2 sequence $v_\varepsilon \rightharpoonup v$.*

A) *If $|v_\varepsilon|_\infty \leqq C$, then $\mu_s = 0$ and the classical Young measure $\bar{v}$ is recovered from the formula*

$$\bar{v}_{(x,t)} = (1 + |v|^2)^{-1} v^1_{(x,t)} (1 + f).$$

B) *Weak continuity of the type*

$$g(v_\varepsilon) \rightharpoonup g(v)$$

holds for all g of the form $g = g_0(1 + |v|^2)$ with $g_0 \in C_0(R^n)$ if and only if v^1 reduces to a weighted Dirac mass,

$$v^1_{(x,t)} = \alpha(x,t) \delta_{v(x,t)},$$

where $0 \leqq \alpha \leqq 1$ and satisfies $(1 + |v|^2) = \alpha(1 + f)$.

C) *Strong continuity holds, i.e.*

$$v_\varepsilon \to v \quad in \quad L^2$$

if and only if $\mu_s = 0$ and $\alpha = 1$ almost everywhere.

In two forthcoming papers [7, 8] we study the behavior of exact and approximate solution sequences of the 2-*D* incompressible Euler equations including regularizations of vortex sheet initial data. Under natural additional hypotheses on the initial data it is possible to rule out oscillations so that only concentrations arise. In this situation, statement B of Theorem 2 automatically holds.

In the context of the 2-*D* Euler equations the theory of concentration compactness developed by Lions [17, 18] could be invoked under the additional hypothesis of the form $|\nabla v_\varepsilon|_{L^1} \leqq C$. With this additional uniform bound it follows that $\alpha \equiv 1$ and that the singular part μ_s of μ is an at most countable number of point masses. However, in the context of two space dimensions the natural estimate from the dynamical point of view is an L^1 estimate on the vorticity, i.e. $|\operatorname{curl} v_\varepsilon|_{L^1} \leqq C$ (see [7]). In this situation the theory of concentration compactness does not apply; indeed several examples involving 2-*D* vortex sheets are presented in [7] for which $\alpha < 1$ on a set of positive measure. Although the theory of concentration compactness is not applicable it motivates a number of the theoretical constructions in this paper and in [7, 8]. More examples from fluids with concentrations are presented in [7].

In Sect. 4, we construct generalized Young measures in the form of vector-valued measures acting on arbitrary continuous functions with at most quadratic growth. The restriction of these measures to separable subspaces such as those of the form (1.9) admits explicit representations as in Theorem 1.1.

In Sect. 5, we discuss the zero viscosity limit for Navier-Stokes in connection with measure-valued solutions of the Euler equations and make several additional comments. The concept of measure-valued solutions introduced here applies to many other nonlinear equations of mathematical physics with natural L^p (energy bounds) and nonlinearities that are bounded in L^p such as the Yang-Mills equations, Vlasov-Maxwell, and Vlasov-Poisson equations. This general theory will be developed elsewhere by the authors (see [9]).

Persistence of Oscillations in Sequences of Smooth Solutions for Incompressible Flow

To build explicit examples, we begin with the following remark. If $\hat{v} = \{\hat{v}_1(x_1, x_2 t), \hat{v}_2(x_1, x_2, t)\}$ is the velocity field for a solution of the 2-*D* Navier-Stokes equations, then $v = \{\hat{v}_1(x_1, x_2, t),\ \hat{v}_2(x_1, x_2, t),\ v_3(x_1, x_2, t)\}$ is a solution of the 3-*D* Navier-Stokes equations provided that $v_3(x_1, x_2, t)$ satisfies the scalar diffusion equation with known coefficients $\hat{v}$ given by

$$\frac{\partial v_3}{\partial t} + \hat{v}_1 \frac{\partial v_3}{\partial x_1} + \hat{v}_2 \frac{\partial v_3}{\partial x_2} = \varepsilon \Delta v_3 . \tag{2.1}$$

In our explicit examples, we ignore viscosity and build oscillating sequences of solutions for the 3-*D* incompressible Euler equation.

First we consider a family of exact steady 2-*D* solutions given by the shear layers,

$$\hat{v}^\varepsilon = \left(v\left(\frac{x_2}{\varepsilon}, x_2\right), 0\right), \tag{2.2}$$

where $v(y, x_2)$ is a given smooth bounded function with period 1 in y and mean zero, i.e.,

$$\int_0^1 v(y, x_2)\, dy = 0 . \tag{2.3}$$

The assumption (2.3) is made for convenience only. We let $w(x_1, x_2, y)$ be any smooth bounded function with period 1 in y. By following the principle in the first paragraph of this section, we build exact smooth solutions of the 3-*D* Euler equations with the explicit form,

$$v = {}^t\left(v\left(\frac{x_2}{\varepsilon}, x_2\right), 0, w\left(x_1 - v\left(\frac{x_2}{\varepsilon}, x_2\right) t, x_2, \frac{x_2}{\varepsilon}\right)\right), \tag{2.4}$$

where $v_3^\varepsilon \equiv w\left(x_1 - v\left(\frac{x_2}{\varepsilon}, x_2\right)t, x_2, \frac{x_2}{\varepsilon}\right)$ is the explicit solution of the advection equation,

$$\frac{\partial v_3^\varepsilon}{\partial t} + v\left(\frac{x_2}{\varepsilon}, x_2\right)\frac{\partial v_3^\varepsilon}{\partial x_1} = 0, \qquad v_3^\varepsilon|_{t=0} = w\left(x_1, x_2, \frac{x_2}{\varepsilon}\right).$$

Clearly, v^ε satisfies the uniform L^∞-bound

$$|v^\varepsilon|_{L^\infty} \leqq C. \tag{2.5}$$

Below, we explicitly compute the Young measure, $\bar{v}_{x,t}$, defined in (1.3)–(1.5) and display this explicit measure as a smooth measure-valued solution of the Euler equations in (1.7) which is *not* a classical pointwise solution of these equations as required in (1.1) or (1.6).

The exact solution sequence in (2.4) defines a modulated one-phase wave with phase function $\frac{x_2}{\varepsilon}$. If $h(z, y)$ is a smooth bounded function of $(z, y) \in R^L \times R^1$ which is one-periodic in y, then the Riemann-Lebesgue lemma implies that $h^\varepsilon(z) = h\left(z, \frac{z}{\varepsilon}\right)$ has a weak limit $\bar{h}(z)$ given by averaging the fast scale, i.e. with $\bar{h}(z) = \int_0^1 h(z, s)ds$,

$$\lim_{\varepsilon\to 0} \int_{R^L} \phi(z)h^\varepsilon(z)dz = \int_{R^L} \phi(z)\bar{h}(z)dz \tag{2.6}$$

for all $\phi \in C_0(R^L)$. We apply the well-known principle in (2.6) to the exact solution sequence from (2.4). First, from (2.6) we observe that v^ε from (2.4) satisfies,

$$v^\varepsilon \rightharpoonup \bar{v}(x_2, x_2, t) = {}^t\left(0, 0, \int_0^1 w(x_1 - v(s, x_2)t, x_2, s)ds\right) \quad \text{as} \quad \varepsilon \to 0. \tag{2.7}$$

We remark that even though $\bar{v}(x_1, x_2, t)$ is a smooth velocity field, $\bar{v}$ is not generally a smooth solution of the Euler equations. With

$$\bar{v}_3 = \int_0^1 w(x_1 - v(s, x_2)t, x_2, s)ds \tag{2.8}$$

and $\bar{v}$ given in (2.7), $\bar{v}$ is a smooth solution of the Euler equations if and only if $\frac{\partial \bar{v}_3}{\partial t} = 0$. This condition is never satisfied unless either v vanishes or $w(x_1, x_2, y)$ has no x_1-dependence. For $g(v) = g(v_1, v_2, v_3)$ an arbitrary continuous function of v, we use (2.4) and (2.6) to compute that

$$\lim_{\varepsilon\to 0} \iint \phi(x, t)g(v^\varepsilon)dxdt = \iint \phi(x, t)\bar{g}dxdt \tag{2.9}$$

with $\bar{g}$ given by

$$\bar{g}(x_1, x_2, t) = \int_0^1 g(v(s, x_2), 0, w(x_1 - v(s, x_2)t, x_2, s))ds. \tag{2.10}$$

With (2.9) and (2.10) it is easy to compute the Young measure as defined in (1.3), (1.4). For each point (x_1, x_2, t) we consider the curve $V_{x,t}$: $[0, 1] \to R^3$ given by

$$V_{x,t}(s) = \{v(s, x_2), 0, w(x_1 - v(s, x_2)t, x_2, s)\}. \tag{2.11}$$

In a canonical fashion, the smooth map $V_{x,t}$ induces a probability measure on R^3, $\bar{v}_{x,t}$, via the formula

$$\bar{v}_{x,t}(E)=m(V_{x,t}^{-1}(E)) \tag{2.12}$$

for any Borel set E where m is Lebesgue measure on $[0,1]$. The reader can check that the measure $\bar{v}_{x,t}$ defined in (2.12) is the Young measure with all the properties in (1.3), (1.4). In particular, from (2.10),

$$\overline{v_1 v_3}(x_1,x_2,t)=\int_0^1 v(s,x_2)w(x_1-v(s,x_2)t,x_2,s)ds\,, \tag{2.13}$$

and with $\bar{v}_3$ from (2.8), we compute explicitly that

$$\frac{\partial \bar{v}_3}{\partial t}+\frac{\partial}{\partial x_1}\overline{v_1 v_3}=0\,. \tag{2.14}$$

With $\bar{v}$ given in (2.7), we see that (2.14) is the only non-trivial equation to check to guarantee that $\bar{v}$ defines a non-trivial measure-valued weak solution of the incompressible fluid equations as defined in (1.7)

Development of Concentrations in Smooth Solutions of the Euler Equations with Uniformly Bounded Local Kinetic Energy

Our first examples of sequences of exact solutions of the incompressible fluid equations exhibiting the phenomenon of concentration as described in (1.10) will be constructed from steady rotating eddies for two-dimensional incompressible flow.

It is well-known (see [15, 19]) that any smooth radial distribution of vorticity, $\omega(r)$, with $r=(x_1^2+x_1^2)^{1/2}$ defines an exact steady solution of the 2-D incompressible fluid equations via the formula

$$\begin{pmatrix} v_1(x) \\ v_2(x) \end{pmatrix}=\begin{pmatrix} -x_2 \\ x_1 \end{pmatrix} r^{-2}\int_0^r s\omega(s)ds\,. \tag{3.1}$$

Associated with the function $\omega(r)$, we define $W(r)$ and Γ by the formulae,

$$W(r)=\left(\int_0^r s\omega(s)ds\right)^2, \qquad \Gamma=2\pi\left(\int_0^\infty s\omega(s)ds\right)^2, \tag{3.2}$$

where we assume $s\omega(s)\in L^1([0,\infty])$ so that $W(r)$ is absolutely continuous and uniformly bounded. The remaining assumptions on $\omega(r)$ needed for the validity of our examples are summarized below. We consider the sequence of velocity fields $v^\varepsilon(x)$ defined by

$$v^\varepsilon(x)=\log\left(\frac{1}{\varepsilon}\right)^{-1/2}\varepsilon^{-1}\begin{pmatrix} v_1\left(\frac{x}{\varepsilon}\right) \\ v_2\left(\frac{x}{\varepsilon}\right) \end{pmatrix} \tag{3.3}$$

with v_1, v_2 defined in (3.1).

First, we verify the local boundedness of kinetic energy for the sequence v^ε as well as the explicit limit in (1.10) for functions g which are exactly homogeneous of degree two, i.e.

$$g(v)=g_H\left(\frac{v}{|v|}\right)|v|^2. \tag{3.4}$$

Since the solution sequence in (3.3) is time independent, we only need to verify (1.10) for test functions $\phi(x)\in C_0^\infty(R^2)$. By using polar coordinates, we compute

$$\lim_{\varepsilon\to 0}\int \phi(x)g_H\left(\frac{v^\varepsilon}{|v^\varepsilon|}\right)|v^\varepsilon|^2dx=\lim_{\varepsilon\to 0}\int \log\left(\frac{1}{\varepsilon}\right)^{-1}W\left(\frac{r}{\varepsilon}\right)\overline{g_H\phi}(r)d\log(r) \tag{3.5}$$

with $\overline{g_H\phi}$ defined by

$$\overline{g_H\phi}(r)=\int_0^{2\pi} g_H(-\sin\theta,\cos\theta)\phi(r\cos\theta, r\sin\theta)d\theta. \tag{3.6}$$

Integrating by parts in (3.5) we obtain

$$\begin{aligned}&\lim_{\varepsilon\to 0}\int \phi(x)g_H\left(\frac{v^\varepsilon}{|v^\varepsilon|}\right)|v^\varepsilon|^2dx\\ &=\left\{\lim_{\varepsilon\to 0}-\int_0^\infty \log\left(\frac{1}{\varepsilon}\right)^{-1}W'\left(\frac{r}{\varepsilon}\right)\log(r)\overline{g_H\phi}(r)\frac{dr}{\varepsilon}\right\}\\ &\quad+\left\{\lim_{\varepsilon\to 0}-\int_0^\infty \log\left(\frac{1}{\varepsilon}\right)^{-1}W\left(\frac{r}{\varepsilon}\right)\log(r)\overline{\tilde{g}_H\phi}(r)dr\right\}=\{1\}+\{2\}.\end{aligned} \tag{3.7}$$

Since $W(r)$ is uniformly bounded and ϕ has compact support, it is obvious that term $\{2\}$ satisfies the estimate,

$$|\{2\}|\leqq C\log\left(\frac{1}{\varepsilon}\right)^{-1}.$$

We rescale term $\{1\}$ by changing variables $r\to\frac{r}{\varepsilon}$, to get

$$\begin{aligned}\{1\}=\lim_{\varepsilon\to 0}&\int_0^\infty W'(r)\overline{g_H\phi}(\varepsilon r)dr\\ &-\int_0^\infty \log\left(\frac{1}{\varepsilon}\right)^{-1}W'(r)\log(r)\overline{g_H\phi}(\varepsilon r)dr=\{1\}^A+\{1\}^B.\end{aligned} \tag{3.8}$$

Under the mild additional assumption on $\omega(r)$ that

$$s\omega(s)\log(s)\in L^1([0,\infty)), \tag{3.9}$$

we trivially obtain the estimate,

$$|\{1\}^B|\leqq C\log\left(\frac{1}{\varepsilon}\right)^{-1}. \tag{3.10}$$

By recalling the definition of $\overline{g_H\phi}(r)$ from (3.6), we see that

$$\{1\}^A=\Gamma\phi(0)\int_0^{2\pi} g_H(-\sin\theta,\cos\theta)\frac{d\theta}{2\pi}. \tag{3.11}$$

Clearly, if we choose $g_H \equiv 1$ and $\phi(r) \geqq 0$ so that $\phi(r) \equiv 1$ for $r \leqq R$ and ϕ vanishes for $r \geqq 2R$, we obtain the uniform kinetic energy bound,

$$\int_{|x| \leqq R} |v^\varepsilon|^2 \leqq C_R \tag{3.12}$$

while (3.5)–(3.11) imply that

$$\lim_{\varepsilon \to 0} \int \phi(x) g_H\left(\frac{v^\varepsilon}{|v^\varepsilon|}\right) |v^\varepsilon|^2 dx = \Gamma \left\langle \delta_0, \phi \int_{S^1} g_H d\theta \right\rangle \tag{3.13}$$

with Γ defined in (3.2). To verify the predicted limit in (1.10) including the additional effect on nonlinear functions with less than quadratic growth at infinity with the form $g_0(v)(1 + |v|^2)$ with $g_0(v) \in C_0(R^2)$ is much easier than (3.6)–(3.11) and is left as an exercise for the reader. To summarize, we have the following

Proposition 3.1. *Assume $\omega(r)$ is a radial vorticity distribution with*

$$r\omega(r) \log(r) \in L^1([0, \infty))$$

and $\int_0^\infty r\omega(r) dr \neq 0$, then v^ε defined from ω via (3.3) *is a sequence of exact solutions of the incompressible fluid equations which exhibit the concentration phenomena described in* (1.10) *of the introduction.*

Remark. A detailed study of the phenomena of concentration for 2-*D* incompressible flows is contained in the author's two companion papers [7, 8]. Here we use these examples to show that both oscillations and concentrations occur in general sequences for 3-*D* fluid flows.

Examples with Time-Dependent Concentration

Next, we give explicit examples with extremely rapid temporal oscillation which exhibit dynamic concentration like the phenomenon which we have just observed in steady exact solutions. These exact solution sequences will be space-time rescalings of the uniformly rotating Kirchhoff elliptical vortex (see [15]) which we describe next.

We consider a constant patch of vorticity $\omega(x_1, x_2)$ with an elliptical shape:

$$\omega(x_1, x_2) = \begin{cases} \tilde{\omega}, & \dfrac{x_1^2}{a^2} + \dfrac{x_2^2}{b^2} \leqq 1 \\ 0, & \dfrac{x_1^2}{a^2} + \dfrac{x_2^2}{b^2} > 1, \end{cases} \tag{3.14}$$

where $\tilde{\omega}$ is constant. The velocity field corresponding to the vorticity distribution ω is constructed from a stream function ψ by the formulae,

$$v = \begin{pmatrix} -\psi_{x_2} \\ \psi_{x_1} \end{pmatrix}, \qquad \Delta\psi = -\omega. \tag{3.15}$$

If we denote the constant Γ by $\Gamma = \frac{1}{2\pi}\tilde{\omega}x$ area of ellipse, it is easy to determine the behavior of v for $|x| \to \infty$ and in fact,

$$\left| v(x) - \begin{pmatrix} -x_2 \\ x_1 \end{pmatrix} \frac{\Gamma}{1+|x|^2} \right| \leqq C(1+|x|)^{-(1+\eta)} \tag{3.16}$$

for some constants C and $\eta > 0$. The velocity field v does not define a steady solution of the Euler equations. However if we introduce the rotation matrix $\sigma(\theta)$ with

$$\sigma(\theta) = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}, \tag{3.17}$$

then there exists a time period L (positive or negative) depending on $\tilde{\omega}$ and the ellipse ratio, b/a so that $\tilde{v}(x,t)$ given by

$$\tilde{v}(x,t) = \sigma(2\pi L^{-1}t)v(\sigma^T(2\pi L^{-1}t)x) \tag{3.18}$$

is an exact time periodic solution of the 2-D incompressible fluid equations with

$$\tilde{v}(x,0) = v_0(x). \tag{3.19}$$

This solution is the well-known Kirchoff rotating elliptical vortex.

We generate an exact solution sequence for the inviscid Euler equations by rescaling the Kirchoff elliptical vortex solution from (3.18). In general, if $v(x,t)$ solves the Euler equations, then $\beta v(\lambda x, \lambda\beta t)$ also solves the Euler equations for arbitrary constants β, λ. Motivated by the rescaled sequence of steady solutions from (3.3), we consider the exact solution sequence for 2-D Euler given by

$$v_\varepsilon(x,t) = \left(\log\left(\frac{1}{\varepsilon}\right)\right)^{-1/2} \varepsilon^{-1}\tilde{v}\left(\frac{x}{\varepsilon}, \log\left(\frac{1}{\varepsilon}\right)^{-1/2} \varepsilon^{-2}t\right) \tag{3.20}$$

with $\tilde{v}$ any Kirchoff elliptical vortex solution as defined in (3.18). Since v_ε is a function of the fast time scale $\tau = \left(\log\frac{1}{\varepsilon}\right)^{-1/2} \varepsilon^{-2}t$, these solutions exhibit extremely rapid temporal oscillation as $\varepsilon \to 0$. However, we shall show below that these rapid temporal oscillations time-average in an extremely simple fashion due to the development of concentrations.

As in (3.5)–(3.13), we explicitly evaluate

$$\lim_{\varepsilon\to 0} \iint \phi_1(t)\phi_2(x) g_H\left(\frac{v_\varepsilon}{|v_\varepsilon|}\right) |v_\varepsilon|^2 dx dt \tag{3.21}$$

with $\phi_1 \in C_0^\infty(R^1)$, $\phi_2 \in C_0^\infty(R^2)$. We use the dense class of sums of products as test functions to clarify the discussion below. Next, we choose a radial vortex $\hat{\omega}(r) \in C_0^\infty(R^2)$ so that $\int_0^\infty r\hat{\omega}(r)dr = \Gamma$. The corresponding velocity $\hat{v}$ is given by (3.1),

$$\hat{v} = \begin{pmatrix} -x_2 \\ x_1 \end{pmatrix} r^{-2} \int_0^r s\hat{\omega}(s)ds. \tag{3.22}$$

From (3.16) and (3.22), we have the estimate

$$|v(x) - \hat{v}(x)| \leqq C(1+|x|)^{-(1+\eta)}, \qquad \eta > 0. \tag{3.24}$$

The following lemma is easily proved by straightforward estimates adding and subtracting terms and using (3.24).

Lemma. *Set* $\hat{v}(x,t)=\sigma(2\pi L^{-1}t)\hat{v}(\sigma^T(2\pi L^{-1}t)x)$ *and define*

$$\hat{v}_\varepsilon=\left(\log\frac{1}{\varepsilon}\right)^{-1/2}\varepsilon^{-1}\hat{v}\left(\frac{x}{\varepsilon},\left(\log\frac{1}{\varepsilon}\right)^{-1/2}\varepsilon^{-2}t\right),$$

then

$$\max_{-\infty\leqq t<\infty}\left|\int_{R^2}\phi_2(x)\left(g_H\left(\frac{v_\varepsilon}{|v_\varepsilon|}\right)|v_\varepsilon|^2-g_H\left(\frac{\hat{v}_\varepsilon}{|\hat{v}_\varepsilon|}\right)|\hat{v}_\varepsilon|^2\right)dx\right|\leqq C\left(\log\frac{1}{\varepsilon}\right)^{-1}. \tag{3.25}$$

With the above lemma, we only need to compute,

$$\lim_{\varepsilon\to 0}\int_0^\infty\log\left(\frac{1}{\varepsilon}\right)^{-1}W\left(\frac{r}{\varepsilon}\right)\overline{g_H\phi_1\phi_2}(r)d\log(r) \tag{3.26}$$

in order to evaluate the limit in (3.21). Here $\overline{g_H\phi_1\phi_2}(r)$ is analogous to formula in (3.6) but also includes a time integral over rapid time oscillations, i.e.

$$\overline{g_H\phi_1\phi_2}(r)=\lim_{\varepsilon\to 0}\int_{-\infty}^{\infty}\int_0^{2\pi}\phi_1(t)\phi_2(r\cos\theta,r\sin\theta)g_H^\varepsilon(\theta,t)d\theta dt, \tag{3.27}$$

with

$$g_H^\varepsilon(\theta,t)=g_H\left(-\sin\left(\theta+2\pi L^{-1}\left(\log\frac{1}{\varepsilon}\right)^{-1/2}\varepsilon^{-2}t\right),\right.$$
$$\left.\times\cos\left(\theta+2\pi L^{-1}\left(\log\frac{1}{\varepsilon}\right)^{-1/2}\varepsilon^{-2}t\right)\right). \tag{3.28}$$

The critical fact from (3.27) is that the dependence of $\overline{g_H\phi_1\phi_2}$ on r occurs only through the test function ϕ_2 so that the argument in (3.5)–(3.11) can be repeated without changes to yield

$$\lim_{\varepsilon\to 0}\iint\phi_1\phi_2 g_H\left(\frac{v_\varepsilon}{|v_\varepsilon|}\right)|v_\varepsilon|^2dxdt=\Gamma\phi_2(0)\lim_{\varepsilon\to 0}\int_0^{2\pi}\int_{-\infty}^{\infty}\phi_1(t)g_H^\varepsilon(\theta,t)\frac{d\theta}{2\pi}, \tag{3.29}$$

with Γ the constant from (3.2). We consider the function $G\left(\left(\log\frac{1}{\varepsilon}\right)^{-1/2}\varepsilon^{-2}t\right)$ defined by

$$G=\int_0^{2\pi}g_H^\varepsilon(\theta,t)d\theta.$$

The function $G(\tau)$ is a periodic function with period L; according to the averaging lemma we have already used in (2.6),

$$\begin{aligned}&\lim_{\varepsilon\to 0}\int_0^{2\pi}\int_{-\infty}^{\infty}\phi_1(t)g_H^\varepsilon(\theta,t)d\theta dt\\&=\lim_{\varepsilon\to 0}\int_{-\infty}^{\infty}\phi_1(t)G\left(\left(\log\frac{1}{\varepsilon}\right)^{-1/2}\varepsilon^{-2}t\right)dt=\int_{-\infty}^{\infty}\phi_1(t)\left(\frac{1}{L}\int_0^L G(\tau)d\tau\right)dt\\&=\int_{-\infty}^{\infty}\phi_1(t)\int_0^{2\pi}g_H(-\sin(\theta),\cos\theta)\frac{d\theta}{2\pi}.\end{aligned} \tag{3.30}$$

With (3.29) and (3.30) we have verified all of the properties in (1.10) needed to construct the explicit sequence v^ε defined from (3.18)–(3.20) with both concentration and rapid temporal oscillation.

4. The Generalized Young Measure in L^p

In this section we develop an extension of the concept of Young Measure as described in (1.3), (1.4) to sequences u_ε satisfying a uniform L^p bound of the form

$$\int_\Omega |u_\varepsilon(y)|^p dy \leqq C, \tag{4.1}$$

where Ω is a bounded subset of R^m and $p \geqq 1$. The functions u_ε take values in R^n. In the special case where the general theory is applied with $p=2$ to the algebra of functions of the form (1.9), we obtain a generalized Young measure representing oscillations and concentrations and having the properties listed in Theorems 1 and 2.

These theorems are connected with the concept of measure-valued solution for the incompressible Euler equations as defined in the introduction. The construction of measure-valued solutions of the 3-D Euler equations as limits of Leray-Hopf solutions of the 3-D Navier-Stokes equations is discussed in detail in Sect. 5.

The analysis splits into two parts related to the construction and refined structure of the generalized Young measure. Theorems 1 and 2 follow from a series of remarks and corollaries in the development presented next. The main tools are elementary and include the Riesz representation theorem, the Radon-Nikodym theorem and compactifications of R^n associated with subalgebras of the space of bounded continuous function on R^n. The theorems quoted below are found in Chaps. 3, 4, and 7 of the elementary text by Folland [11]. In particular the required properties of compactifications and regular algebras are included in pp. 137–140 and the associated exercises.

Construction of the Generalized Young Measure

Given the uniform L^p bound in (4.1), the largest class of nonlinear functions where we expect to compute the weak limit

$$\lim_{\varepsilon \to 0} \int_\Omega \phi(y) g\{u_\varepsilon(y)\} dy$$

with $\phi \in C_0(\Omega)$ consists of functions of the form

$$g(u) = \tilde{g}(u)(1+|u|^p) \tag{4.2}$$

with $\tilde{g}$ in the space $BC(R^n)$ of bounded continuous functions on R^n. A canonical norm on the space of functions in (4.2) is defined by

$$|g|_\infty \equiv \max_{u \in R^n} |\tilde{g}(u)| = |\tilde{g}|_\infty . \tag{4.3}$$

After passing to a subsequence the following general representation theorem holds for functions in $BC(R^n)$.

Theorem 4.1. *There exists a scalar measure σ in $M^+(\Omega)$ and a bounded linear transformation*

$$T: BC(R^n) \mapsto L^\infty(\sigma)$$

such that

$$\{1+|u_\varepsilon(y)|^p\}dy \rightharpoonup \sigma \tag{4.4}$$

and

$$g\{u_\varepsilon(y)\}dy \rightharpoonup T(\tilde{g})d\sigma\,. \tag{4.5}$$

The mode of convergence in (4.5), (4.6) is weak-star, i.e.

$$\lim \int \phi\{1+|u_\varepsilon|^p\}dy = \int \phi d\sigma$$
$$\lim \int \phi g\{u_\varepsilon\}dy = \int \phi T(\tilde{g})d\sigma\,,$$

for all ϕ in $C_0(\Omega)$ so that σ and $T(\tilde{g})$ represent the composite weak limit.

Before proving this theorem we record a corollary that interprets (4.4), (4.5) in terms of vector-valued measures. We recall that weak-star measurable vector-valued measure v on Ω associates to each Borel set E in Ω an element $v(E)$ of the dual space X^* of a Banach space X in such a way that $v(E)$ is weak-star countable additive: if E is a countable disjoint union of sets E_j, then

$$v(E) = w^* - \lim_{N\to\infty} \sum_{j=1}^{N} v(E_j) \tag{4.6A}$$

in the sense that

$$\langle v(E), g\rangle = \lim \sum_{j=1}^{N} \langle v(E_j), g\rangle \tag{4.6B}$$

for each g in X.

If we denote this space of vector-valued measures by $M(\Omega, X^*)$ and if we set $X = BC(R^n)$ we identify the limit in (4.5) as an element of $M(\Omega, X^*)$. Combining this with definition (4.6) yields the following consequence of Theorem 4.1.

Corollary 4.1. *The scalar measure σ and the linear transformation T from Theorem 4.1 define a vector-valued measure v in $M(\Omega, BC(R^n)^*)$ through the formula*

$$\langle v(E), g\rangle = \int_E T(\tilde{g})d\sigma\,, \tag{4.7}$$

where E is a Borel set.

Thus if $\tilde{g} \in BC(R^n)$, then all composite weak limits of the sequence u_ε are represented by the vector-valued measure v.

Proof of Theorem 4.1. The space $BC(R^n)$ is isometrically isomorphic to the space of continuous functions on the Stone-Cech compactification of R^n denoted by βR^n. Here and below we shall ignore canonical isomorphisms between spaces of continuous functions and say for example that $\tilde{g} \in C(\beta R^n)$ if $\tilde{g} \in BC(R^n)$. We recall that the set of all linear combinations of the form

$$\phi(x)\tilde{g}(u)\,, \quad \phi \in C_0(\Omega)\,, \quad \tilde{g} \in C(\beta R^n)$$

is dense in $C_0(\Omega \times \beta R^n)$ and that βR^n is a compact Hausdorff space. We shall prove Theorem 4.1 by utilizing the Riesz representation theorem to express the dual of $C_0(\Omega \times \beta R^n)$ in terms of the Radon measures on $\Omega \times \beta R^n$,

$$C_0(\Omega \times \beta R^n)^* = M(\Omega \times \beta R^n).$$

For each fixed $\varepsilon > 0$, the function u_ε induces a unique element v_ε of $M^+(\Omega \times \beta R^n)$ via the formula

$$\langle v_\varepsilon, \phi\tilde{g}\rangle \equiv \int_\Omega \phi g(u_\varepsilon) dx = \int_\Omega \phi \tilde{g}(u_\varepsilon)(1+|u_\varepsilon|^p) dy,$$

for $\phi \in C_0(\Omega)$, $\tilde{g} \in C(\beta R^n)$. The uniform L^p bound in (4.1) implies a uniform bound on the total mass of v_ε,

$$|v_\varepsilon(\Omega \times \beta R^n)| \leqq C. \tag{4.7}$$

Thus, weak-star compactness in $M^+(\Omega \times \beta R^n)$ and the Riesz representation theorem lead to the following assertion for an appropriate subsequence. There exists an element v of $M^+(\Omega \times \beta R^n)$ such that

$$v_\varepsilon \rightharpoonup v \quad \text{in} \quad M^+(\Omega \times \beta R^n) \tag{4.8A}$$

and, in particular,

$$\lim \int_\Omega \phi g(u_\varepsilon) dy = \langle v, \phi\tilde{g}\rangle \tag{4.8B}$$

for $\phi \in C_0(\Omega)$, $\tilde{g} \in BC(R^n)$. The measure v yields the vector-valued measure in Corollary 4.1.

Setting $\tilde{g} \equiv 1$ in (4.8B) shows that the scalar measure σ corresponds to the projection of v onto R^n, i.e. (4.4) holds and

$$\sigma(E) = v(E \times \beta R^n)$$

for all Borel sets E in Ω. It follows from (4.7) and (4.8) that the scalar measure $v_{\tilde{g}}$ defined on Ω by

$$\langle v_{\tilde{g}}, \phi\rangle \equiv \langle v, \phi\tilde{g}\rangle$$

satisfies the inequality

$$|\langle v_{\tilde{g}}, \phi\rangle| \leqq C|\tilde{g}|_\infty \int |\phi| d\sigma, \tag{4.9}$$

which implies that $|v_{\tilde{g}}(E)| \leqq C|\tilde{g}|_\infty \sigma(E)$ for all E, and therefore that $v_{\tilde{g}}$ is absolutely continuous with respect to σ. It follows from the Radon-Nikodym theorem that for each $\tilde{g}$ the measure $v_{\tilde{g}}$ admits a representation of the form $v_{\tilde{g}}(E) = \int_E f d\sigma$, where $f \in L^1(\sigma)$. The integrand f depends linearly on g and defines the transformation T in the statement of Theorem 4.1. The boundedness of T follows from (4.9) and the duality between L^1 and L^∞. The proof of Theorem 4.1 is complete.

A similar argument applies to continuous functions of the form

$$g(u) = g_H\left(\frac{u}{|u|}\right)|u|^p, \tag{4.10}$$

where $g_H \in C(S^{n-1})$ and yields a scalar measure μ with

$$|u|^p \rightharpoonup \mu \quad \text{in} \quad M(\Omega).$$

For functions of the form (4.10) the properties in Theorem 4.1 and Corollary 4.1 hold with μ replacing σ. From the definitions we have $\sigma=\mu+dy$. Thus, there exists a bounded linear transformation

$$\mathcal{S}: C(S^{n-1}) \mapsto L^\infty(\mu)$$

such that

$$\lim \int \phi g_H(u_\varepsilon)\, |u_\varepsilon|^p dy = \int \phi \mathcal{S}(g_H) d\mu \tag{4.11}$$

for all g_H in $C(S^{n-1})$ and ϕ in $C_0(\Omega)$. For convenience in the study of composite weak limits associated with nonlinear functions g of the form (1.9) we shall use μ as the basic reference measure.

Concrete Representations on Separable Subspaces

We begin with a simple construction which produces the family of measures $\nu^2_{(x,t)} \in \operatorname{Prob} M(S^{n-1})$ in Theorem 1 of the introduction. We use the version of Theorem 4.1 dealing with homogeneous functions of the form (4.10) as described in (4.11).

Theorem 4.2 (Young measure for homogeneous functions). *There exists a μ-measurable map*

$$y \mapsto \nu_y$$

from Ω to $\operatorname{Prob} M(S^{n-1})$ *such that*

$$\lim_{\varepsilon\to 0} \int \phi g(u_\varepsilon) dy = \int \phi \langle \nu_y, g_H\rangle d\mu$$

for all g of the form $g=g_H|u|^p$ and all ϕ in $C_0(\Omega)$. Here the bracket denotes expected value

$$\langle \nu_y, g_H\rangle = \int_{S^{n-1}} g_H d\nu_y .$$

Proof. Choose a countable dense set of functions h_j in the separable space $C(S^{n-1})$. From (4.11) there exists a set N with $\mu(N)=0$ so that

$$|\mathcal{S}\circ h_j(y)| \leqq C|h_j|_\infty \tag{4.12}$$

if $y\in\Omega\backslash N$. The uniform boundedness of the functional which evaluates the integrand $\mathcal{S}(h_j)$ at a specified point y as in (4.12) is a direct consequence of the boundedness of S as a map from $C(S^{n-1})$ to $L^\infty(\mu)$. It is necessary to avoid only a countable number of null sets in order to achieve (4.12) on a single null set N.

By the Riesz representation theorem there exists a unique element ν_y of $M^+(S^{n-1})$ such that

$$\langle \nu_y, h_j\rangle = \mathcal{S}\circ h_j(y),$$

if $y\in\Omega\backslash N$. This yields the desired μ-measurable map from Ω to $\operatorname{Prob} M(S^{n-1})$. We see that ν_g is a probability measure by choosing $h\equiv 1$. This argument follows the proof of the classical Young measure theorem as given by Tartar in [28].

The same proof yields a more concrete representation of the functions $T\tilde{g}$ of Theorem 4.1 if $\tilde{g}$ lies in a *separable completely regular* subalgebra of $BC(R^n)$. We recall that a subalgebra F is completely regular if it is closed in the maximum norm, contains constants, and separates points and closed sets, cf. [11]. Examples are

$F_\infty = BC(R^n)$ and $F_0 = C_0(R^n) \cup \{\text{constants}\}$. Here $C_0(R^n)$ denotes the space of continuous functions which vanish at infinity. Notice that F_0 is separable.

Associated with any completely regular subalgebra F of $BC(R^n)$ is a compactification $\beta_F R^n$ of R^n. We recall that $\beta_F R^n$ is a compact completely regular Hausdorff space and that the subalgebra F is isomorphic to $C(\beta_F R^n)$. Thus, each continuous linear functional on F is canonically identified with an element of $M(\beta_F R^n)$. The special case $\beta_{F_0} R^n$ coincides with the one-point compactification of R^n and hence

$$M(\beta_{F_0} R^n) = M(R^n) \oplus \{\alpha \delta_{np}\}, \tag{4.13}$$

where

$$\langle \delta_{np}, 1 \rangle = 1 \quad \text{and} \quad \langle \delta_{np}, g_0 \rangle = 0$$

for all g_0 in $C_0(R^n)$. Thus the measures on the one-point compactification of R^n admit a decomposition into a measure on R^n and a Dirac mass at the "north pole." In general the following ordering principle holds:

$$F_1 \subset F_2 \Rightarrow \beta_{F_1} R^n \subset \beta_{F_2} R^n.$$

Theorem 4.3 [Concrete Representation for Separable Completely Regular Subalgebras F of $BC(R^n)$]. *There exists a σ-measurable map*

$$y \mapsto \nu_y$$

from Ω to $\operatorname{Prob} M(\beta_F R^n)$ *such that for every $\tilde{g}$ in F*

$$\lim_{\varepsilon \to 0} \int \phi g(u_\varepsilon) dy = \int \phi \langle \nu_y, \tilde{g} \rangle d\sigma \tag{4.14}$$

for all ϕ in $C_0(\Omega)$. Here

$$\langle \nu_y, \tilde{g} \rangle \equiv \int_{\beta_F R^n} \tilde{g} d\nu_y. \tag{4.15}$$

The proof of Theorem 4.3 follows the same argument of Tartar in [28] that was sketched for Theorem 4.2 and is omitted here.

We complete the proof of the generalized Young measure Theorem 1 by using F_0 in Theorem 4.3. In view of the identification expressed by (4.13) and the defining properties (4.14) and (4.15), we note that the piece $\nu^1_{(x,t)}$ of the Young measure in Theorem 1 is generally not a probability measure since some of the mass can leak to infinity in the limit. This feature is made explicit in the following corollary of Theorem 4.3 which is motivated by the fact that every ν in $\operatorname{Prob} M(\beta_{F_0} R^n)$ has the form

$$\nu = \nu^1 + (1 - \alpha) \delta_{np},$$

where $\nu^1 \in M^+(R^n)$ and $0 \leqq \alpha \equiv \nu^1(R^n) \leqq 1$.

Corollary 4.2. *There exists a σ-measurable function α with $0 \leqq \alpha \leqq 1$ in terms of which the measure ν_y of Theorem 4.3 takes the form*

$$\nu_y = \nu^1_y + \{1 - \alpha(y)\} \delta_{np},$$

where $\nu^1_y \in M^+(R^n)$ and $\nu^1_y(R^n) = \alpha(y)$. Thus, if

$$g(u) = g_0(u)(1 + |u|^p) \tag{4.16}$$

with g_0 *in* $C_0(R^n)$, *then*

$$\lim \int \phi g(u_\varepsilon) dy = \int \phi \langle v_y^1, g_0 \rangle d\sigma .$$

Refined Structure of the Generalized Young Measure

Next we establish the properties of the generalized Young measure which were stated in a special case in Theorem 1.2 as well as the absolute continuity of $v_{(x,t)}^1$ stated in Theorem 1. For this purpose we appeal to the Lebesgue decomposition of the measure μ in Theorem 4.3 $\mu = \mu_s + f dy$, where $\mu_s \perp dy$ and $f \in L^1(\Omega)$, $\sigma = dy + \mu$.

Proposition. *If* g *takes the form* (4.16) *with* g_0 *in* $C_0(R^n)$, *then*

$$\lim \int \phi g(u_\varepsilon) dy = \int \phi \langle v_y^1, g_0 \rangle (1+f) dy , \tag{4.17}$$

where v_y^1 *is defined in Corollary 4.2. In particular*

$$\int \phi \langle v_y^1, g_0 \rangle d\mu_s = 0 \tag{4.18}$$

for all g_0 *in* $C_0(R^n)$.

Proof. Taking into account the fact that (4.17) is trivially true if g_0 vanishes outside a ball of radius R, we consider the truncated sequence $u_\varepsilon^R = \gamma_\varepsilon^R u_\varepsilon$, where γ_ε^R is the characteristic function of the set where $|u_\varepsilon| \leqq R$. Thus, if g_0 vanishes for $|u| > R$, then

$$g_0(u_\varepsilon^R)(1+|u_\varepsilon^R|^p) = g_0(u_\varepsilon)(1+|u_\varepsilon|^p) ,$$

and we may appeal to the classical Young measure representation (1.3), (1.4) to deduce the existence of a family v_y^R in $\operatorname{Prob} M(R^n)$ such that

$$\lim \int \phi g(u_\varepsilon^R) dy = \int \phi \langle v_y^R, g \rangle dy = \int \phi \langle v_y^1, g_0 \rangle (1+f) dy + \int \phi \langle v_y^1, g_0 \rangle d\mu_s$$

for all g of the form $g = g_0(1+|u|^p)$. Thus

$$\int \phi \langle v_y^1, g_0 \rangle d\mu_s = 0 \tag{4.19}$$

for all continuous g_0 which vanish for $|u| > R$. The desired property (4.18) follows from the facts that (4.19) defines a continuous linear functional on $C_0(R^n)$ if ϕ is fixed and also that the class of g_0 vanishing outside a ball is dense in $C_0(R^n)$.

An immediate consequence of (4.19) is the following.

Corollary 4.3. *If* $|u_\varepsilon|_\infty \leqq C$, *then* $\mu_s = 0$ *and* $v_y = (1+f)(1+|u|^p)^{-1} v_y^1$ *is the classical Young measure defined in* (1.3), (1.4).

We conclude this section by verifying parts B and C of Theorem 2. Assume that

$$u_\varepsilon \rightharpoonup u \quad \text{in} \quad L^p(\Omega) \tag{4.20}$$

with $1 < p < \infty$ and that

$$g(u_\varepsilon) \rightharpoonup g(u) \quad \text{in} \quad M(\Omega) \tag{4.21}$$

for all g with $g = g_0(u)(1+|u|^p)$, $g_0 \in C_0(R^n)$. From (4.21) and Proposition 4.1 it follows that

$$\langle v_y^1, g_0 \rangle f(y) = \langle \delta_{u(y)}, g \rangle \tag{4.22}$$

for almost all y. The relation (4.22) implies that v_y^1 is a weighted Dirac mass a.e. with respect to $f dy$. Thus, Corollary 4.2 implies that there exists $\alpha(y)$ with $0 \leqq \alpha \leqq 1$ such that

$$(1+|u|^p)=\alpha(y) f(y) \text{ a.e.} \tag{4.23}$$

Finally, $u_\varepsilon \to u$ in L^p if and only if

$$|u_\varepsilon|^p \rightharpoonup |u|^p \quad \text{in} \quad M(\Omega) \tag{4.24}$$

by the Vitali-Hahn-Saks theorem. By applying Theorem 4.2 with $g_H \equiv 1$ it follows that (4.24) is valid if and only if

$$|u|^p dy = \mu_s + f dy, \tag{4.25}$$

i.e. $\mu_s = 0$ and $f = |u|^p$ a.e.

The discussion in (4.20)–(4.25) applied to the case $p=2$ yields parts B and C of Theorem 2 in the introduction.

5. Measure-Valued Solutions of the Euler Equations and the Zero Diffusion Limit of the Navier-Stokes Equation

First we formulate an elementary proposition with the following intuitive content: for the Euler equations, weak stability together with weak consistency implies convergence to a measure-valued solution.

Proposition 5.1. *Assume v_ε is a sequence of functions satisfying* $\operatorname{div} v_\varepsilon = 0$.
A) *Weak Stability: For any $\Omega \subset R^n \times R^+$ there exists a constant $C(\Omega)$ such that*

$$\iint_\Omega |v_\varepsilon(x,t)|^2 dx dt \leqq C(\Omega).$$

B) *Weak Consistency: For all divergence-free test functions ϕ in $C_0^\infty(\Omega)$,*

$$\lim_{\varepsilon \to 0} \iint \phi_t \cdot v_\varepsilon + \nabla \phi : v_\varepsilon \otimes v_\varepsilon dx dt = 0.$$

If $v = (\mu, v^1, v^2)$ is the associated generalized Young measure from Theorem 1, then v is a measure valued solution of the Euler equations on Ω.

Proof. It follows from the definitions that

$$\begin{aligned} 0 &= \lim_{\varepsilon \to 0} \iint \phi_t \cdot v_\varepsilon + \nabla \phi : v_\varepsilon \otimes v_\varepsilon dx dt \\ &= \iint \phi_t \cdot \left\langle v_{(x,t)}^1, \frac{v}{1+|v|^2} \right\rangle (1+f) dx dt + \nabla \phi : \langle v_{(x,t)}^2, \theta \otimes \theta \rangle d\mu. \end{aligned}$$

We prove the theorem stated in the introduction as an immediate corollary of Proposition 5.1. It is well-known [29] that for every $v_0(x) \in L^2(R^3)$ with $\operatorname{div} v_0 = 0$ in the sense of distributions, there is at least one Leray-Hopf weak solution $v_\varepsilon(x,t)$ to the Navier-Stokes equations satisfying the kinetic energy inequality

$$\max_{0 \leqq t < +\infty} \int_{R^3} |v_\varepsilon(x,t)|^2 dx \leqq \int_{R^3} |v_0(x)|^2 dx. \tag{5.2}$$

Thus, in the high Reynolds number limit, v_ε satisfies the weak stability estimate.

Multiplying the Navier-Stokes equations by the test function ϕ, we have

$$\iint \phi_t v_\varepsilon + \nabla\phi : v_\varepsilon \otimes v_\varepsilon) dx dt = \varepsilon \iint \Delta\phi v_e dx dt . \tag{5.3}$$

Since $|\iint \Delta\phi v_3| \leqq C|v_\varepsilon|_{L^2} \leqq CT|v_0|^2_{L^2}$, the identity in (5.3) implies that v^ε is weakly consistent with Euler. The proof of the theorem is completed through an application of Proposition 5.1 once we remark that the initial data $v_0(x)$ is assumed in an appropriate weak sense (see [9]).

We conclude with a few additional comments. It is also possible to construct measure-valued solutions of the 3-*D* Euler equations and verify Proposition 5.1 for a class of 3-*D* vortex algorithms. This is interesting because most of the detailed complexity in inviscid 3-*D* flows has been found computationally through these numerical methods [3, 4]. Our ideas also apply to other interesting equations for fluid flow like those with variable density or the Bousinesq approximation. The machinery in this paper also applies to approximating sequences of measure-valued solutions. In personal communication, J. Keller, W. Craig, and C. Foias have remarked that statistical solutions of Navier-Stokes induce measure-valued solutions of the Navier-Stokes equations in a canonical fashion. We plan to develop all of the above ideas and other connections with turbulence theory in a forthcoming publication (see [9]).

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