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OSCILLATIONS IN A DELAY-LOGISTIC EQUATION*

By

K. GOPALSAMY

Flinders University of South Australia

Summary. Sufficient conditions are derived for all nonconstant nonnegative solutions of the equations of the form

$$\frac{dx(t)}{dt} = x(t)\left\{a - \sum_{j=1}^{n} b_j x(t - \tau_j)\right\}$$

and

$$\frac{dx(t)}{dt} = x(t)\left\{a - b\int_{-\infty}^{t} k(t-s)x(s)\,ds\right\}$$

to be oscillatory about their respective positive steady states. The results are complementary to those in [15].

1. Introduction. Consider a nonlinear (delay-logistic) equation of the form

$$\frac{dx(t)}{dt} = x(t)\left\{a - \sum_{j=1}^{n} b_j x(t - \tau_j)\right\}, \quad t > 0, \tag{1.1}$$

where a, b_j, τ_j $(j = 1, 2, \ldots, n)$ are positive constants. Equation (1.1) denotes a generalization of the equation

$$\frac{dN(t)}{dt} = rN(t)[K - N(t-\tau)]/K, \tag{1.2}$$

in which r, K, τ are positive constants. By a change of variables (1.2) can be put in the form (1.1) with $n = 1$. It has been suggested by Hutchinson [22] and Wangersky and Cunningham [42] that (1.2) can represent the dynamics of a single-species population system growing with a constant reproduction rate r toward a saturation level K, the term $[K - N(t)]/K$ denoting a feedback mechanism which takes τ units of time to respond to changes in the population size. Cunningham [5] indicated that (1.2) can be used to

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describe certain control systems and suggested that similar equations can also be used in economic studies of business cycles. A number of authors (May [30], Maynard Smith [31], Pielou [35]) have discussed (1.2) with respect to its potential application in mathematical ecology, especially concerning the dynamics of single-species population systems. By means of a change of variables (1.2) can be brought to an equation of the form

$$\frac{dy(s)}{ds} = -\alpha y(s-1)[1 + y(s)] \tag{1.3}$$

(for some constant $\alpha > 0$), which has also been studied by several authors, notably Jones [23, 24], Wright [43], and Kakutani and Markus [25].

Autonomous ordinary differential equations with delayed arguments have been used in modeling epidemics (Waltman [41]), fish populations (Walter [40]), blowfly populations (Taylor and Sokol [39], Perez et al. [36]), survival of red blood cells (Chow [3]), neurophysiology (Hadeler and Tomiuk [18], an der Heiden [20]), respiratory and hematopoietic disorders (Mackey and Glass [29]), physiology of breathing (Grodins et al. [17]), supply and demand in economics (Francis et al. [12]), biological immune response (Dibrov et al. [8]), and heat exchangers (Fowler [11] and Friedly and Krishnan [13]).

Recently some authors (Claeyssen [4], Stech [38], Nussbaum [34], Braddock and Van den Driessche [2], Hale [19]) have considered delay differential equations with two delays. The existing literature on scalar equations with two delays is mainly concerned with the derivation of conditions for the loss of linear stability of a steady state leading to a Hopf-type bifurcation to oscillations. While a study of (1.1) for an arbitrary positive integer n is of mathematical interest, there is some evidence of a need to study (1.1) with at least $n = 3$; for instance, Kitching [26] has indicated that a model of the dynamics of the Australian blowfly *Eucilia cuprina* must have three delays. We note that even in the case of linear scalar equations with three or more delays, few global characteristics of the equations are known which are valid for all possible values of the delay parameters due to the complex transcendental nature of the related characteristic equation. The situation in nonscalar systems is quite different, depending on how the delays appear in the equations (Gopalsamy [15, 16]).

It is intuitively clear that if all the delays in (1.1) are sufficiently small then the asymptotic behaviour as $t \to \infty$ of positive solutions of (1.1) is similar to that of an equation without delays; this aspect has been investigated by the author [15] and sufficient conditions have been derived for the global asymptotic stability of the positive steady state of (1.1) along with conditions for (1.1) to be nonoscillatory. It is our principal concern in the following to derive sufficient conditions for all the realistic solutions (i.e., nonconstant and nonnegative solutions) of (1.1) to be oscillatory about a steady state. Since fluctuating populations are susceptible to extinction due to sudden and unforeseen environmental disturbances, a knowledge of the conditions under which populations fluctuate indefinitely will be of some use in planning and designing control as well as management procedures.

It has been argued in the literature on mathematical ecology that continuously distributed delays are more appropriate than discrete delays as in (1.1) or (1.2) (see, for instance, May [30], Cushing [6]). Accordingly, we will also consider in the following the oscillatory nature of a scalar integrodifferential equation of the form

$$\frac{dx(t)}{dt} = x(t)\left\{a - b\int_{-\infty}^{t} k(t-s)x(s)\,ds\right\}, \quad t > 0, \tag{1.4}$$

where a, b are positive constants and k corresponds to a "delayed weighting kernel" representing the manner in which the past history of the species influences its present growth rate. Under suitable assumptions on k, the local stability of a positive steady state and the existence of periodic solutions arising from the loss of such stability have been discussed by Cushing [7] and Stech [37]. While (1.4) may be biologically more realistic, there is considerable difficulty in choosing or determining suitable delay kernels in (1.4). However, due to the ensuing analytical convenience, kernels of the type

$$k(t) = \frac{\sigma^{m+1}t^m}{m!}\exp\{-\sigma t\}, \quad t \geqslant 0,\ m = 0,1,2,\dots, \tag{1.5}$$

where σ is a positive constant, have been extensively used in integrodifferential equations. It is possible to convert (1.4) with (1.5) into a vector system of ordinary differential equations by means of a linear "chain trick" (see MacDonald [28]) originally due to Fargue [10]. Since (1.1) denotes a generalization of equations of the form of (1.2), it is of some interest to study (1.1) for its own sake, and we will examine the oscillatory nature of (1.3) below. There are not many results in the literature on the oscillation of integrodifferential equations (or equations with unbounded delays) except for some partial results due to the author [14], Levin [27], and Myschkis [33].

2. An oscillatory delay-differential equation. We begin with a note that if the delays are absent in (1.1) and (1.2) then no solution of (1.1) and (1.2) corresponding to positive initial conditions will be oscillatory. The following preparation will be useful for our discussion of oscillations of (1.1). Let $\tau = \max_{1\leqslant j\leqslant n}\tau_j$, $\tau_* = \min_{1\leqslant j\leqslant n}\tau_j$ and consider (1.1) together with initial conditions of the form

$$x(s) = \varphi(s) \geqslant 0, \qquad \varphi \text{ is continuous on } [-\tau, 0], \qquad \varphi(0) > 0. \tag{2.1}$$

Since we will assume that $\tau_j > 0$ $(j = 1,2,\dots,n)$, the method of steps (Bellman and Cooke [1], El'sgol'ts and Norkin [9]) is applicable for (1.1) and (2.1), with which one can show that solutions of (1.1) and (2.1) exist on intervals of the form $[m\tau_*, (m+1)\tau_*]$ $(m = 0,1,2,\dots)$. Thus it will follow that solutions of (1.1) and (2.1) exist for all $t \geqslant 0$ and remain bounded on $[-\tau, \infty)$. Also, since any solution of (1.1) and (2.1) satisfies a relation of the form

$$x(t) = \varphi(0)\exp\left[\int_0^t \left\{a - \sum_{j=1}^{n} b_j x(s-\tau_j)\right\} ds\right], \quad t \geqslant 0,$$

we have that $x(t) > 0$ for $t \geqslant 0$. The system (1.1) has a positive steady state $x^* = a/(\sum_{j=1}^{n} b_j)$. We can introduce a change of variables by the formula

$$X(t) \equiv \log[x(t)/x^*], \quad t \geqslant 0, \tag{2.2}$$

and derive that

$$\frac{dX(t)}{dt} = -x^*\sum_{j=1}^{n} b_j\left[\exp\{X(t-\tau_j)\} - 1\right], \quad t \geqslant 0. \tag{2.3}$$

We will use the following:

DEFINITION. The system (1.1) is said to be oscillatory about its steady state x^* if and only if $x(t) - x^*$ has at least one zero for t in every interval of the form $[\alpha, \infty)$ for arbitrary positive α, where $x(t)$ is a solution of (1.1) corresponding to any initial condition of the type in (2.1).

We can now formulate our principal result as follows.

THEOREM 2.1. Assume the following:

(i) The constants a, b_j, τ_j $(j = 1, 2, \ldots, n)$ are positive.

(ii)

$$ex^* \sum_{j=1}^{n} b_j \tau_j > 1, \text{ where } x^* = a / \sum_{j=1}^{n} b_j. \tag{2.4}$$

Then the system (1.1) corresponding to initial conditions of the type in (2.1) is oscillatory about x^*.

Proof. It is clear from our preparation that it is sufficient to show that nonconstant solutions of (2.3) are oscillatory about zero. Suppose there exists a nonconstant solution of (2.3) defined on $[-\tau, \infty)$ which is not oscillatory about zero. Then there exists a positive number $t^* > \tau$ and a nonconstant solution $X^*(t)$ of (2.3) such that $X^*(t)$ is bounded for $t \geqslant -\tau$ and either

$$X^*(t) > 0 \quad \text{for } t \geqslant t^* \qquad \text{or} \qquad X^*(t) < 0 \quad \text{for } t \geqslant t^*. \tag{2.5}$$

Our strategy of proof is to show that both these possibilities lead to contradictions. Consider the case $X^*(t) > 0$ for $t \geqslant t^*$. It will follow from (2.3) that

$$\frac{dX^*(t)}{dt} = -x^* \sum_{j=1}^{n} b_j \left[\exp\left\{ X^*(t - \tau_j)\right\} - 1\right], \quad t > t^* + \tau, \tag{2.6}$$

which implies that $dX^*(t)/dt < 0$ for $t > t^* + \tau$; thus

$$\lim_{t \to \infty} X^*(t) \text{ exists and we let } c = \lim_{t \to \infty} X^*(t) \tag{2.7}$$

for a constant $c \geqslant 0$. One can see from (2.6) that $c = 0$ and that

$$\frac{dX^*(t)}{dt} \leqslant -x^* \sum_{j=1}^{n} b_j X^*(t - \tau_j), \quad t > t^* + \tau, \tag{2.8}$$

which implies on integration that

$$X^*(t) \geqslant x^* \sum_{j=1}^{n} b_j \int_t^{\infty} X^*(s - \tau_j)\, ds, \quad t > t^* + \tau. \tag{2.9}$$

We now show that the existence of a bounded positive function $X^*(t)$ for $t \in [t^*, \infty)$ satisfying (2.9) and $\lim_{t \to \infty} X^*(t) = 0$ implies the existence of a positive solution of the delay-differential system

$$\frac{dy^*(t)}{dt} = -x^* \sum_{j=1}^{n} b_j y^*(t - \tau_j), \quad t > t^* + \tau. \tag{2.10}$$

Define a set S as follows:

$$S = \{x \in C([t^*, \infty), \mathbb{R}) \mid 0 \leqslant x(t) \leqslant X^*(t);\ t \geqslant t^*\}$$

and for each $x \in S$ we can define $\hat{x}$: $[t^*, \infty) \to [0, \infty)$ by the following:

$$\hat{x}(t) = \begin{cases} x(t), & t \geqslant t^* + \tau, \\ x(t^* + \tau) + X^*(t) - X^*(t^* + \tau), & t \in [t^*, t^* + \tau). \end{cases} \tag{2.11}$$

It is easy to see

$$0 \leqslant \hat{x}(t) \leqslant X^*(t) \quad \text{for } t \geqslant t^*.$$

We consider a map T defined on S by (as in the case of $\hat{x}$)

$$(Tx)(t) = \begin{cases} x^* \sum_{j=1}^{n} b_j \int_t^\infty x(s - \tau_j)\, ds, & t > t^* + \tau, \\ x^* \sum_{j=1}^{n} b_j \int_{t+\tau}^\infty x(s - \tau_j)\, ds + X^*(t) - X^*(t^* + \tau), & t \in [t^*, t^* + \tau], \end{cases} \tag{2.12}$$

and note

$$0 \leqslant (Tx)(t) \leqslant x^* \sum_{j=1}^{n} b_j \int_t^\infty X^*(s - \tau_j)\, ds \leqslant X^*(t), \quad t \geqslant t^* + \tau. \tag{2.13}$$

Define a sequence of functions $\{y_n\}$ $(n = 0, 1, 2, \ldots)$ such that

$$y_0 = X^*, \qquad y_n = Ty_{n-1}, \quad n = 1, 2, 3, \ldots. \tag{2.14}$$

It follows from the definition of T and the nature of X^* that

$$0 \leqslant y_n(t) \leqslant y_{n-1}(t) \leqslant X^*(t) \quad \text{for } t > t^*,\ n = 1, 2, 3, \ldots,$$

and hence the following limit exists in a pointwise sense:

$$\lim_{n \to \infty} y_n(t) = y^*(t), \quad t > t^*. \tag{2.15}$$

By Lebesgue's convergence theorem, y^* is a solution of

$$0 \leqslant y^*(t) = \begin{cases} x^* \sum_{j=1}^{n} b_j \int_t^\infty y^*(s - \tau_j)\, ds, & t > t^* + \tau, \\ x^* \sum_{j=1}^{n} b_j \int_{t^*+\tau}^\infty y(s - \tau_j)\, ds + X^*(t) - X^*(t^* + \tau), & t \in [t^*, t^* + \tau], \end{cases}$$

which implies that y^* is a positive solution of (2.10) since $X^*(t) - X^*(t^* + \tau) > 0$ for $t \in [t^*, t^* + \tau)$. But by Lemma 1 of the appendix all bounded solutions of (2.10) are oscillatory about zero when (2.14) holds. Thus the existence of $X^*(t) > 0$ for $t > t^*$ leads to a contradiction.

The second possibility in (2.5) is treated by showing that the existence of an eventually negative solution X^* of (2.6) will lead to an inequality of the form

$$\frac{dx^*(t)}{dt} \geqslant -x^*\alpha \sum_{j=1}^{n} b_j X^*(t - \tau_j), \quad t > t^* + \tau, \tag{2.16}$$

where α is a positive constant which can be chosen such that

$$ex^*\alpha \sum_{j=1}^{n} b_j\tau_j > 1. \tag{2.17}$$

The author is indebted to Professor G. Ladas for the following arguments in the derivation of (2.16)–(2.17).

Let us suppose that $X^*(t) < 0$ for $t > t^*$ and note that for such X^*, $dX^*(t)/dt > 0$, eventually implying that $\lim_{t\to\infty} X^*(t)$ exists and such a limit is in fact zero; thus the convergence in $\lim_{t\to\infty} X^*(t) = 0$ is monotonic in t eventually.

We will now show the existence of functions $\xi_j = \xi_j(t)$, $j = 1, 2, \ldots, n$, such that for all large enough t,

$$e^{X^*(t-\tau_j)} - 1 = X(t-\tau_j)e^{X^*(\xi_j(t))}, \quad j = 1, 2, \ldots, n,\ t > t^* + \tau. \tag{2.18}$$

For $t > t^* + \tau$ and $t_1 > t^* + \tau$, we have

$$e^{X^*(t-\tau_j)} - e^{X^*(t_1)} = \left[X^*(t-\tau_j) - X(t_1)\right]e^{X^*(\theta_j)},\ j = 1, 2, \ldots, n, \tag{2.19}$$

where $X^*(\theta_j)$ lies between $X^*(t-\tau_j)$ and $X^*(t_1)$. Considering the limiting case of (2.19) as $t_1 \to \infty$,

$$e^{X^*(t-\tau_j)} - 1 = X^*(t-\tau_j)e^{X^*(\xi_j(t))}, \quad t > t^* + \tau,\ j = 1, 2, \ldots, n, \tag{2.20}$$

for some functions $\xi_j = \xi_j(t)$, $j = 1, 2, \ldots, n$, $t > t^* + \tau$, such that $\xi_j(t) \to \infty$ as $t \to \infty$. Now by the monotonic nature of $\xi_j(t) \to \infty$ for $t > t^* + \tau$ and since $X^*(\xi_j(t)) \to 0$ monotonically as $t \to \infty$, it will follow from (2.20) that for sufficiently large $t > t^{**}$ (say)

$$\frac{dX^*(t)}{dt} = -x^* \sum_{j=1}^{n} b_j X^*(t-\tau_j)e^{X^*(\xi_j(t))}, \quad t > t^{**}, \tag{2.21}$$

$$> -x^* \sum_{j=1}^{n} b_j X^*(t-\tau_j)e^{X^*(\xi_j(t^{**}))} \tag{2.22}$$

where t^{**} can be chosen such that

$$x^*e \sum_{j=1}^{n} b_j\tau_j e^{X^*(\xi_j(t^{**}))} > 1. \tag{2.23}$$

The possibility of choosing such t^{**} is a consequence of $X^*(t) \to 0$, $\xi_j(t) \to \infty$ as $t \to \infty$ and our hypothesis $ex^*\sum_{j=1}^{n} b_j\tau_j > 1$. Now by the first part of our proof, the existence of an eventually negative solution X^* of (2.22) and (2.23) is impossible, and this completes the proof.

3. Logistic equation with unbounded delays. We will now consider a nonlinear integrodifferential equation of the form

$$\frac{dx(t)}{dt} = x(t)\left[a - b\int_{-\infty}^{t} k(t-s)x(s)\,ds\right], \quad t \geqslant t_0 > -\infty, \tag{3.1}$$

together with initial conditions φ on $(-\infty, t_0]$ of the type

$$x(s) = \varphi(s) \geqslant 0, \qquad s \in (-\infty, t_0], \qquad \varphi(t_0) > 0, \tag{3.2}$$

where φ is assumed to be bounded and piecewise (locally) continuous on $(-\infty, t_0]$. Precisely, we will establish the following for (3.1) and (3.2).

THEOREM 3.1. Suppose the following hold:

(i) a, b are positive constants and k: $[0, \infty) \to [0, \infty)$ is piecewise (locally) continuous on $[0, \infty)$ such that

$$\int_0^\infty k(s)\,ds = 1, \qquad \int_0^\infty sk(s)\,ds < \infty, \qquad \int_0^\infty k(s)e^{\mu s}\,ds < \infty, \tag{3.3}$$

for some positive number μ [note that the third of (3.3) will also imply the first two of (3.3)].

(ii)

$$bex^*\int_0^\infty sk(s)\,ds > 1 \quad \text{where } x^* = a/b. \tag{3.4}$$

Then all nonoscillatory solutions of (3.1) corresponding to initial conditions of the type in (3.2) are such that

$$\lim_{t\to\infty}\left[x(t) - x^*\right] = 0, \tag{3.5}$$

and if $\lim_{s\to-\infty} x(s) \neq x^*$, with $\lim_{t\to\infty} x(t) = x^*$, then the convergence in (3.5) cannot be monotonic; in any case all solutions of (3.1) and (3.2) are oscillatory on $[0, \infty)$.

Proof. The local existence of solutions of (3.1)–(3.4) on an interval of the form $[t_0, t_0 + T)$ for some possibly small $T > 0$ will follow from the elements of integrodifferential equations (Miller [32]). The form of (3.1) and the nonnegativity of φ on $(-\infty, t_0]$ together with $\varphi(t_0) > 0$ will imply that any solution of (3.1) and (3.2) will remain nonnegative for those $t > t_0$ for which such a solution exists. Since solutions of (3.1)–(3.3) satisfy

$$\begin{aligned} x(t) &= \varphi(t_0)\exp\left[\int_{t_0}^t \left\{a - \int_{-\infty}^s k(s+\eta)x(\eta)\,d\eta\right\}ds\right] \\ &\leqslant \varphi(t_0)e^{a(t-t_0)}, \quad t \geqslant t_0, \end{aligned}$$

it will follow that solutions of (3.1)–(3.3) are defined for all $t \geqslant 0$ (at least by continuation) and remain nonnegative on $[t_0, \infty)$. As before we let

$$X(t) \equiv \log\left[x(t)/x^*\right], \quad t \geqslant t_0,$$

and derive that

$$\frac{dX(t)}{dt} = -bx^*\int_{-\infty}^t k(t-s)\left[\exp\{X(s)\} - 1\right]ds, \quad t > t_0, \tag{3.6}$$

and it is enough to show that all nonoscillatory solutions of (3.6) are such that

$$\lim_{t\to\infty} X(t) = 0 \tag{3.7}$$

and if $\lim_{s\to-\infty} X(s) \neq 0$ then the convergence in (3.7) cannot be monotonic.

Suppose now that a solution of (3.6) has at most a finite number of zeros on $[0, \infty)$ (i.e., nonoscillatory); then there exists a $t^* > 0$ such that $X(t) \neq 0$ for $t \geqslant t^*$. We now rewrite (3.6) in the form

$$\begin{aligned}\frac{dX(t+t^*)}{dt} &= -bx^*\int_{-\infty}^{t+t^*} k(t+t^*-s)[\exp\{X(s)\}-1]\,ds\\ &= -bx^*\int_{-\infty}^{t^*} k(t+t^*-s)\{\exp[X(s)]-1\}\,ds\\ &\quad -bx^*\int_{t^*}^{t^*+t} k(t+t^*-s)[\exp\{X(s)\}-1]\,ds\\ &= -bx^*\int_0^t k(t-\eta)[\exp\{X(t^*+\eta)\}-1]\,d\eta - f(t),\end{aligned} \tag{3.8}$$

where

$$f(t) = bx^*\int_{-\infty}^{t^*} k(t+t^*-s)[\exp\{X(s)\}-1]\,ds. \tag{3.9}$$

It is not difficult to verify that

$$|f(t)| \leqslant b\Big(\sup_{s\leqslant t^*}|x(s)-x^*|\Big)\int_t^\infty k(\eta)\,d\eta \to 0 \quad \text{as } t\to\infty, \tag{3.10}$$

and furthermore if we let

$$Q(t) = \int_t^\infty f(s)\,ds,$$

then

$$|Q(t)| \leqslant b\Big(\sup_{s\leqslant t^*}|x(s)-x^*|\Big)\int_t^\infty \eta k(\eta)\,d\eta \to 0 \quad \text{as } t\to\infty. \tag{3.11}$$

If we now define

$$Z(t) \equiv X(t+t^*), \quad t \geqslant 0, \tag{3.12}$$

then we have from (3.8)

$$\frac{dZ(t)}{dt} = -bx^*\int_0^t k(t-\eta)[e^{Z(\eta)}-1]\,d\eta - f(t), \tag{3.13}$$

with $Z(t) \neq 0$ on $[0,\infty)$ [since $X(t+t^*) \neq 0$ for $t \geqslant 0$]. There are now two possibilities: either $Z(t) > 0$ on $[0,\infty)$ or $Z(t) < 0$ on $[0,\infty)$. Suppose $Z(t) > 0$ on $[0,\infty)$ and let

$$Y(t) = Z(t) - Q(t). \tag{3.14}$$

Then

$$\frac{dY(t)}{dt} = -bx^*\int_0^t k(t-\eta)[\exp\{Y(\eta)+Q(\eta)\}-1]\,d\eta, \quad t > 0. \tag{3.15}$$

Since $Y(t) + Q(t) > 0$ for $t \geqslant 0$, it will follow from (3.15) that for some constant (say) c,

$$\lim_{t\to\infty}[Z(t) - Q(t)] = c. \tag{3.16}$$

If $c < 0$, it will then follow that eventually $Z(t)$ becomes negative for large t since $Q(t) \to 0$ as $t \to \infty$. Thus $c \geqslant 0$, and if $c > 0$, it will follow from (3.15) that

$$\lim_{t\to\infty} \frac{dY(t)}{dt} = -bx^*[e^c - 1]\int_0^\infty k(s)\,ds < 0, \tag{3.17}$$

and (3.17) will imply that $Y(t) \to -\infty$ and $t \to \infty$. Thus $c = 0$ and hence we have $\lim_{t\to\infty} Z(t) = \lim_{t\to\infty} X(t + t^*) = 0$.

One can show by similar arguments that if $Z(t) < 0$ on $[0, \infty)$ then $\lim_{t\to\infty} X(t + t^*) = \lim_{t\to\infty} Z(t) = 0$. We will turn to the mode of convergence; let us first suppose that

$$\lim_{s\to-\infty} Z(s) > 0, \quad Z(t) > 0 \text{ for } t > -\infty \text{ and } Z(t) \to 0 \text{ as } t \to \infty. \tag{3.18}$$

We will show that (3.18) will lead to a contradiction of the result of Lemma 2 of the appendix. When (3.18) holds we have from (3.13) that there exists a positive solution $Z(t)$ of the linear integrodifferential inequality

$$\frac{dZ(t)}{dt} \leqslant -bx^*\int_0^t k(t-\eta)Z(\eta)\,d\eta - f(t), \quad t > 0, \tag{3.19}$$

where $f(t) > 0$ for $t > 0$. Define a sequence $\{W_n(t);\ n = 0, 1, 2, 3, \ldots;\ t \geqslant 0\}$ by the following:

$$W_0(t) = Z(t), \quad t \geqslant 0, \tag{3.20}$$

$$W_{n+1}(t) = bx^*\int_t^\infty \left\{\int_0^s k(s-\eta)W_n(\eta)\,d\eta\right\} ds + \int_t^\infty f(s)\,ds, \quad t \geqslant 0,\ n = 0, 1, 2, 3, \ldots. \tag{3.21}$$

Integration of (3.19) leads to

$$Z(\infty) - Z(t) \leqslant -bx^*\int_t^\infty \left\{\int_0^s k(s-\eta)Z(\eta)\,d\eta\right\} ds - \int_t^\infty f(s)\,ds, \quad t \geqslant 0,$$

which on using $Z(\infty) = 0$ implies that

$$Z(t) \geqslant bx^*\int_t^\infty \left\{\int_0^s k(s-\eta)Z(\eta)\,d\eta\right\} ds + \int_t^\infty f(s)\,ds, \quad t \geqslant 0. \tag{3.22}$$

It will follow from (3.20)–(3.22) and $f(t) \geqslant 0$ that

$$0 < \cdots W_{n+1}(t) \leqslant W_n(t) \leqslant \cdots \leqslant W_2(t) \leqslant W_1(t) \leqslant W_0(t), \quad t \geqslant 0,\ n \geqslant 1, 2, 3, \ldots. \tag{3.23}$$

It is easy to see from (3.23) that the limit of the sequence $\{W_n(t)\}$ as $n \to \infty$ exists in a pointwise sense and we let

$$\lim_{n\to\infty} W_n(t) = W^*(t), \quad t \geqslant 0. \tag{3.24}$$

By Levi's theorem [21] on integration it will then follow from (3.21) that

$$W^*(t) = bx^*\int_t^\infty \left\{\int_0^s k(s-\eta)W^*(\eta)\,d\eta\right\} ds + \int_t^\infty f(s)\,ds, \quad t \geqslant 0, \tag{3.25}$$

in which $W^*(t)$ cannot be zero for any finite $t \geqslant 0$ due to the positivity of the integrands in (3.25). But (3.25) implies that $W^*(t)$ is a positive solution of the integrodifferential equation

$$\frac{dU(t)}{dt} = -bx^* \int_0^t k(t-\eta)U(\eta)\,d\eta - f(t); \tag{3.26}$$

but this is impossible since by Lemma 2 of the Appendix all solutions of (3.26) are oscillatory under the hypotheses of the theorem.

If in (3.18) we have $Z(t) < 0$ for $t > -\infty$, then we will have from (3.13), on using $Z(t) \to 0$ as $t \to \infty$,

$$\frac{dZ(t)}{dt} = -bx^* \int_0^t k(t-\eta)\left\{\frac{e^{Z(\eta)} - 1}{Z(\eta)}\right\} Z(\eta)\,d\eta - f(t) \tag{3.27}$$

$$\geqslant \left\{\lim_{t\to\infty} \inf\left[\frac{e^{Z(t)} - 1}{Z(t)}\right]\right\}(-bx^*)\int_0^t k(t-\eta)Z(\eta)\,d\eta - f(t)$$

$$\geqslant -bx^* \int_0^t k(t-\eta)Z(\eta)\,d\eta - f(t). \tag{3.28}$$

We note that in (3.28) $f(t) < 0$ for $t \geqslant 0$, and repeat a procedure similar to that in the above and derive that the existence of a negative solution of the inequality leads to the existence of a negative solution of (3.16), which is again a contradiction of the result of Lemma 2 of the Appendix.

The last assertion is proved as follows. We first note that all solutions of (3.1) and (3.2) existing on $[0, \infty)$ are bounded as $t \to \infty$. For instance, suppose a solution $x(t)$ of (3.1) and (3.2) is not bounded for $t \to \infty$. Then there will exist a sequence $\{t_n;\ n = 1, 2, 3, \ldots\}$, $t_n \to \infty$ as $n \to \infty$, such that $x(t_n) \to \infty$ as $n \to \infty$ and $dx(t_n)/dt > 0$, $n = 1, 2, 3, \ldots$. But we have from (3.1) and (3.2),

$$\frac{dx(t_n)}{dt} = x(t_n)\left[a - b\int_0^{t_n} k(t_n - s)x(s)\,ds - b\int_{-\infty}^0 k(t_n - s)\varphi(s)\,ds\right]$$

$$= x(t_n)\left[a - bx(\theta t_n)\int_0^{t_n} k(\eta)\,d\eta - b\int_{-\infty}^0 k(t_n - s)\varphi(s)\,ds\right], \quad 0 < \theta < 1,$$

$$< 0 \quad \text{if } n \text{ is sufficiently large,}$$

and this contradiction shows that $\lim_{t\to\infty} \sup x(t) < \infty$. Now as before the existence of a nonoscillatory bounded solution of (3.1) and (3.2) leads to the existence of a nonoscillatory bounded solution of (3.26), which by Lemma 2 of the Appendix is not possible, and this completes the proof.

4. A brief discussion. The conditions of Theorems 2.1 and 3.1 are formulated as sufficient conditions only for the respective nonlinear systems to be oscillatory. It is not known whether the conditions (2.4) and (3.4) are necessary also for (1.1) and (3.1), respectively, to be oscillatory. However, a condition somewhat weaker than (2.4) of the type

$$ex^*\left(\sum_{j=1}^n b_j\right)\tau > 1, \quad \tau = \max(\tau_1, \ldots, \tau_n), \tag{4.1}$$

is known to be necessary for (1.1) to be oscillatory, and a detailed discussion of this aspect can be found in [15]. It is also shown in [15] that if

$$x^*\left(\sum_{j=1}^{n} b_j\tau_j\right) < 1, \tag{4.2}$$

then the steady state x^* of (1.1) is globally attractive of nonoscillatory solutions; as a consequence of this and our Theorem 2.1, it will follow that if

$$\tau e x^* \sum_{j=1}^{n} b_j > 1 \quad \text{and} x^* \sum_{j=1}^{n} b_j\tau_j < 1, \tag{4.3}$$

then no solution of (1.1) can be monotonic and the convergence of $x(t)$ to x^* as $t \to \infty$ is oscillatory. This is an extension of a single-delay result of Kakutani and Markus [25] to the multidelay logistic equation (1.1).

It is not difficult to generalize and extend the results of our Theorems 2.1 and 3.1 for systems of the form

$$\frac{dx(t)}{dt} = x(t)\left\{a - b_0x(t) - \sum_{j=1}^{n} b_jx(t-\tau_j)\right\} \tag{4.4}$$

and

$$\frac{dy(t)}{dt} = y(t)\left\{a - b_0y(t) - b\int_{-\infty}^{t} k(t-s)y(s)\,ds\right\}, \tag{4.5}$$

where a, b_0, b_j, τ_j ($j = 1, 2, \ldots, n$) are positive constants and k is a suitable delay kernel. The relevant lemmas needed for a discussion of oscillation of (4.4) and (4.5) can be obtained from Corollaries 2.1 and 2.2 of [14]. We conclude with the remark that it is worthwhile to obtain conditions under which nonscalar systems of the form

$$\frac{dx_i(t)}{dt} = x_i(t)\left\{b_i + \sum_{j=1}^{n} a_{ij}x_j(t-\tau_{ij})\right\} \tag{4.6}$$

will be oscillatory or nonoscillatory, where τ_{ij} are nonnegative constants and b_i, a_{ij} ($i, j = 1, 2, \ldots, n$) are arbitrary constants; systems of the form (4.6) are of fundamental theoretical interest in mathematical ecology and, as remarked in the introduction, identification of a system as either oscillatory or nonoscillatory is a necessary prerequisite for understanding and managing complex time-delayed systems.

5. Appendix. Detailed statements concerning the applicability of Laplace transform techniques for delay-differential and integrodifferential equations can be found in [14]. Proofs of the following lemmas are briefly outlined for the sake of completeness of our results.

LEMMA 1. Suppose x^*, b_j, τ_j ($j = 1, 2, \ldots, n$) are positive constants satisfying the first of (2.4). Then all bounded solutions of the linear system

$$\frac{du(t)}{dt} = -x^* \sum_{j=1}^{n} b_ju(t-\tau_j) \tag{5.1}$$

corresponding to continuous initial conditions $u(s) = \psi(s)$, $\psi \in [-\tau, 0]$ ($\tau = \max_{1 \leqslant j \leqslant n} \tau_j$), are oscillatory on $[0, \infty)$.

Proof. The characteristic equation associated with (5.1) is given by

$$\lambda = -x^* \sum_{j=1}^{n} b_j e^{-\lambda\tau_j}. \tag{5.2}$$

Any solution of (5.1) is of the form

$$u(t) = \sum_m p_m(t) e^{\lambda_m t}, \tag{5.3}$$

where the summation is over all the roots of (5.2); the convergence of the series in (5.3) has been discussed in Bellman and Cooke [1], the coefficients $p_m(t)$ being determined as follows:

$$p_m(t) = \text{residue of} \left\{ \frac{e^{\lambda t}\left[u(0) - x^* \sum_{j=1}^n b_j e^{-\lambda\tau_j} \int_{-\tau_j}^0 e^{-\lambda\eta} u(\eta)\, d\eta\right]}{\lambda + x^* \sum_{j=1}^n b_j e^{-\lambda\tau_j}} \right\} \tag{5.4}$$

at a root λ_m of (5.2).

Suppose now that (5.1) has a bounded nonoscillatory solution; then it will follow that (5.2) has at least one real nonpositive root, say $\lambda = -\mu$, for some $\mu \geqslant 0$; but $\mu = 0$ is not possible. Thus we have from

$$\mu = x^* \sum_{j=1}^{n} b_j e^{\mu\tau_j}$$

for some $\mu > 0$ that

$$1 = x^* \sum_{j=1}^{n} b_j \tau_j \left(e^{\mu\tau_j}/\mu\tau_j\right) \geqslant x^* \left(\sum_{j=1}^{n} b_j \tau_j\right) e, \tag{5.5}$$

which contradicts the first of (2.4). Thus (5.1) cannot have a bounded nonoscillatory solution and the proof of the lemma is complete.

Lemma 2. Assume that the conditions of Theorem 3.1 hold. Then all bounded solutions of the scalar linear integrodifferential equation

$$\frac{dV(t)}{dt} = -bx^* \int_0^t k(t-\eta) V(\eta)\, d\eta - f(t), \quad t > 0,\ V(0) \neq 0, \tag{5.6}$$

where f is any bounded continuous function on $[0, \infty)$ such that $f(t) \to 0$ as $t \to \infty$ and $Q(t) = \int_t^\infty f(s)\, ds \to 0$ as $t \to \infty$, have the following character:

(i) If V is of the same sign on $[0, \infty)$ then $\lim_{t \to \infty} V(t)$ exists and $\lim_{t \to \infty} V(t) = 0$.

(ii) If V is any bounded solution of (5.6) on $[0, \infty)$ then V is oscillatory (has zeros on intervals of the form $[\alpha, \infty)$ for arbitrary positive α) on $[0, \infty)$.

Proof. Suppose $V(t) > 0$ for $t > 0$. If we let $S(t) = V(t) - Q(t)$ then

$$\begin{aligned}\frac{dS(t)}{dt} &= -bx^* \int_0^t k(t-\eta)[S(\eta) + Q(\eta)]\, d\eta \\ &< 0 \quad \text{for } t > 0,\end{aligned} \tag{5.7}$$

implying that $\lim_{t\to\infty} S(t)$ exists. If $s_0 = \lim_{t\to\infty} S(t)$ then we have $s_0 = \lim_{t\to\infty} V(t)$ and it follows from our hypotheses on k and (5.6) that

$$0 = -bx^*\left(\int_0^\infty k(\eta)\,d\eta\right)s_0 \quad \text{since } f(t) \to 0 \text{ as } t \to \infty,$$

which implies $s_0 = 0$; if $V(t) < 0$ on $[0, \infty)$ then the arguments are similar.

Now let us suppose that (5.5) has a bounded solution on $[0, \infty)$ having at most a finite number of zeros on $[0, \infty)$. It will follow from our hypotheses on f that the Laplace transform of f exists and furthermore if

$$\tilde{F}(\lambda) = \int_0^\infty f(t)e^{-\lambda t}\,dt = P(\lambda)/Q(\lambda) \quad \text{(say)}$$

then P is analytic on the complex λ-plane and the zeros of Q or the poles of $\tilde{F}$ cannot have nonnegative real parts.

By means of Laplace transforms one can show that a solution of (5.6) can be represented in the form

$$V(t) = \sum_i p_i(t)e^{\lambda_i t} - \sum_j q_j(t)e^{\mu_j t}, \quad t > 0, \tag{5.8}$$

where

$$p_i(t) = \text{residue of } \frac{e^{\lambda t}[V(0) - \tilde{F}(\lambda)]}{\lambda + bx^*\int_0^\infty k(s)e^{-\lambda s}\,ds} \tag{5.9}$$

at a root $\lambda = \lambda_i$ of

$$\lambda + bx^*\int_0^\infty k(s)e^{-\lambda s}\,ds = 0, \tag{5.10}$$

$$q_j(t) = \text{residue of } e^{\lambda t}p(\lambda)\left[Q(\lambda)\left\{\lambda + bx^*\int_0^\infty k(s)e^{-s}\,ds\right\}\right]^{-1} \tag{5.11}$$

at a root $\mu = \mu_j$ of $Q(\mu) = 0$.

The convergence and the representation of the solution in the form (5.8) are established as in the case of discrete delays (see, for instance, Zubov [44]). Since the zeros of $Q(\lambda)$ cannot have nonnegative real parts, the bounded nonoscillatory nature of V and $V(0) \neq 0$ implies that there exists a real nonpositive root, say $\lambda = -\lambda^*$ (for some $\lambda^* \geqslant 0$), of (5.10). Since $\lambda^* = 0$ is not possible, $\lambda^* > 0$, and it will then follow that

$$\lambda^* = bx^*\int_0^\infty k(s)e^{\lambda_* s}\,ds,$$

leading to

$$\begin{aligned} 1 &= bx^*\int_0^\infty k(s)s\left[e^{\lambda^* s}/\lambda^* s\right]ds \\ &\geqslant bx^*e\int_0^\infty k(s)s\,ds, \end{aligned}$$

which contradicts (3.4). Thus (5.6) cannot have a nonoscillatory solution bounded on $[0, \infty)$ and the proof is complete.

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