Oscillations of complex networks

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A complex network processing information or physical flows is usually characterized by a number of macroscopic quantities such as the diameter and the betweenness centrality. An issue of significant theoretical and practical interest is how such quantities respond to sudden changes caused by attacks or disturbances in *recoverable networks*, i.e., functions of the affected nodes are only temporarily disabled or partially limited. By introducing a model to address this issue, we find that, for a finite-capacity network, perturbations can cause the network to *oscillate* persistently in the sense that the characterizing quantities vary periodically or randomly with time. We provide a theoretical estimate of the critical capacity-parameter value for the onset of the network oscillation. The finding is expected to have broad implications as it suggests that complex networks may be structurally highly dynamic.

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The response of a complex network to sudden changes such as intentional attacks, random failures, or abnormal load increase, has been of great interest [1–10] since the discoveries of the small-world [11] and the scale-free [12] topologies. The issue is particularly relevant for scale-free networks that are characterized by a power-law degree distribution. For such a network, generically there exists a small set of nodes with degrees significantly higher than those of the rest of the nodes. A scale-free network is thus robust against random failures, but it is vulnerable to intentional attacks [1]. This is particularly so when dynamics on the network is taken into account, which can lead to catastrophic breakdown of the network via the cascading process [5,8] even when the attack is on a single node. A basic assumption underlying the phenomenon of cascading breakdown is that a node fails if the load exceeds its capacity. As a result, the load of the failed node has to be transferred to other nodes, which causes more nodes to fail, and so on, leading to a cascade of failures that can eventually disintegrate the network.

There are situations in complex networks where overload does not necessarily lead to failures. For instance, in the Internet, when the number of information-carrying packets arriving at a node exceeds what it can handle, traffic congestion occurs. That is, overload of a node can lead to the waiting of packets but not to the failure of the node. As a result of the congestion, traffic detour becomes necessary in the sense that any optimal routes for new packets on the network try to avoid the congested nodes. This is equivalent to a change in the "weights" (to be defined more precisely below) of the congested nodes and, consequently, to changes in the macroscopic characterizing quantities of the network. This situation usually does not occur when the network is in a normal operational state, but it becomes likely when sudden disturbances, such as an attack or an abrupt large load increase, occur. A question is then whether the network can recover after a finite amount of time, in the sense that its characterizing quantities restore to their original values.

In this paper, we study a class of weighted scale-free networks, incorporating a feasible traffic-flow protocol, to address the above question. In the absence of any perturbations, the network is assumed to operate in its "normal" state so that its macroscopic characterizing quantities are constants. We find that, after a large perturbation, the network can indeed recover but only for large node capacities. When the node capacities are not significantly higher than their loads in the normal state, a surprising phenomenon arises: The macroscopic quantities of the network are never able to return to their unperturbed values but, instead, they exhibit persistent oscillations. In this sense we say the *network oscillates*. More remarkably, as the node capacities are decreased, both periodic and random oscillations can occur. The striking feature is that the oscillation phenomena, periodic or random, are caused solely by the interplay between the complex network topology and the traffic-flow protocol, regardless of the network parameters such as the degree distribution and the overall load fluctuations. For fixed network parameters, the oscillations exist regardless of the explicit form of the local node dynamics, requiring only the minimal rule that it simply causes the traffic to wait when overloaded. Our finding may have implications to many network-traffic problems. For instance, it can provide an alternative explanation, from the dynamical point of view, for the recently observed random oscillations in real Internet traffic flow [13,14] and provide some insights into the self-similar oscillations of the traffic flux observed in the worldwide web [15,16]. In a broader sense, that a complex can never recover and instead its basic characteristics exhibit persistent oscillations in regions far away from their original steady state can have an enormous impact on the function and role of the network, regardless of the context (e.g., whether physical, biological, or social).

We begin by constructing a scale-free network of N nodes using the standard growth and preferential-attachment mechanism [12]. We next define the node capacity by using the model in Ref. [8],

$$C_i = (1+\alpha)L_i(0), \quad (1)$$

where $L_i(0)$ is the initial load on node i, which is approximately the load in a normal operational state (free of traffic congestion), and $\alpha > 0$ is the *capacity* parameter. The load L_i can be conveniently chosen to be the betweenness [17,18], which is the total number of optimal paths [19] between all pairs of nodes passing through node i. To define an optimal path at time t, say at this time the weights associated with node i and with node j are $w_i(t)$ and $w_j(t)$ (to be defined below according to the degree of traffic congestion), respectively, where there is a direct link l_{ij} between the two nodes. Given a pair of nodes, one packet generating and another receiving, the optimal path is the one that minimizes the sum of all weights w_i of nodes that constitute the path. Finally, we define a traffic protocol on the network by assuming that, at each time step, one packet is to be communicated between any pair of nodes. There are thus $N(N-1)/2$ packets to be transported across the whole network at any time. When a packet is generated, its destination and the optimal path that the packet is going to travel toward it are determined.

In a computer or a communication network, a meaningful quantity to characterize a link is the time required to transfer a data packet through this link. When the traffic flow is free, it takes one time unit for a node to transport a packet. When congestion occurs, it may take a substantially longer time for a packet to pass through a node. For instance, suppose at time t there are $J_i(t)$ packets at node i, where $J_i(t) > C_i$. Since the node can process C_i packets at any time, the waiting time for a packet at the end of the queue is $1+\mathrm{int}[J_i(t)/C_i]$, where $\mathrm{int}[\cdot]$ is the integer part of the fraction in the square bracket. These considerations lead to the following definition of *instantaneous* weight for node i:

$$w_i(t) = 1 + \mathrm{int}\left[\frac{J_i(t)}{C_i}\right], \quad \text{for } i = 1, \ldots, N, \quad (2)$$

from which the instantaneous weights for any node in the network and hence a set of instantaneous optimal paths can be calculated accordingly. For free traffic flow on the network, we have $J_i(t) < C_i$ and hence $w_i(t)=1$ so that the network is nonweighted. In this case, the optimal path reduces to the shortest path.

The above model of traffic dynamics on a weighted network allows us to investigate the response of the network to perturbations in a systematic way. In particular, since the node capacities are the key to the occurrence of traffic congestion, it is meaningful to choose the capacity parameter α in Eq. (1) as a bifurcation parameter. To apply perturbation, we locate the node with the largest betweenness $B_{\max}$ in the network and generate a large number of packets, say ten times of $B_{\max}$, at time $t=0$. The network is then allowed to relax according to our model. Initially, because of the congestion at the largest-betweenness node caused by the perturbation, its weight assumes a large value. As a result, there is a high probability that the optimal paths originally passing through this node change routes. This can lead to a sudden increase in the network diameter, which is the average of all optimal-path length. As time goes by, the congestion will cascade to other nodes that adopt some detoured optimal paths. As a result, the diameter increases quickly and reaches its maximum at some time. During this process the congestion situation is lessened at the attacking node while it gets worse at the nodes that paths detour to. After this, the network begins to "absorb" the congestions due to the load tolerance and the recovery process starts, reducing the diameter. The same processes apply to other macroscopic characterizing quantities of the network. Due to the imbalance of load distribution and the high density of optimal paths, the final state where the system recovers to is difficult to predict [23]. Therefore, we are interested in whether these quantities can return to their "normal" or the steady-state values before the perturbation.

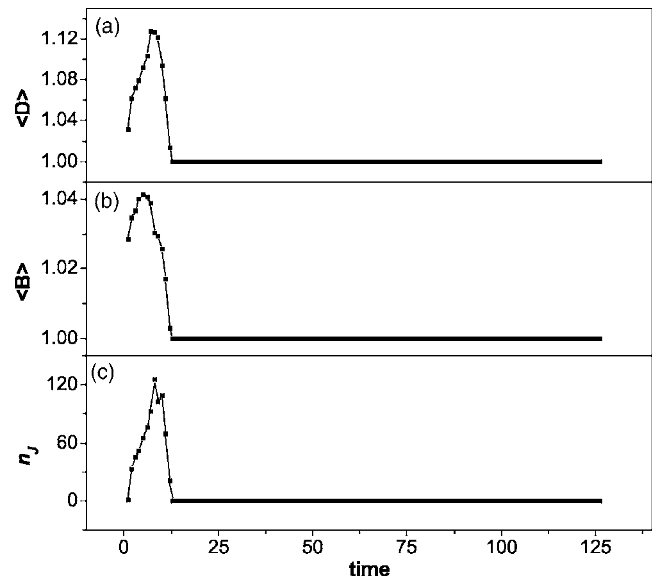

FIG. 1. For a scale-free network of 1000 nodes and for $\alpha=0.4$, time evolutions of three macroscopic quantities: (a) The normalized network diameter, (b) the normalized betweenness centrality, and (c) the number of jammed nodes. The quantities return to their respective steady-state values after a brief transient.

For relatively large value of α, the ability of the network to process and transport packets is strong, so we expect the network to be able to relax to its unperturbed state. This is exemplified in Figs. 1(a)–1(c), the time evolutions of three macroscopic quantities, the normalized diameter $\langle D\rangle$, the normalized betweenness centrality $\langle B\rangle$, and the number of jammed nodes n_J, respectively, of a scale-free network of 1000 nodes for $\alpha=0.4$. [For this network, the values of the diameter and of the betweenness centrality in the unperturbed state are $\langle D_0\rangle \approx 5.18$ and $\langle B_0\rangle \approx 2.35\times 10^6$. The plotted quantities in (a) and (b) are normalized with respect to these "static" values.] We see that, after about seven time steps, these quantities reach their maximum values and, after another about five steps, these quantities return to their respective unperturbed values. In this case, the large perturbation causes the network to oscillate but only for a transient time period. As the capacity parameter α is reduced, a remarkable phenomenon occurs: After an initial transient the network never returns to its steady state but, instead, it exhibits persistent oscillations. Figures 2(a)–2(c) show periodic oscillations for $\alpha=0.31$, where the legends are the same as

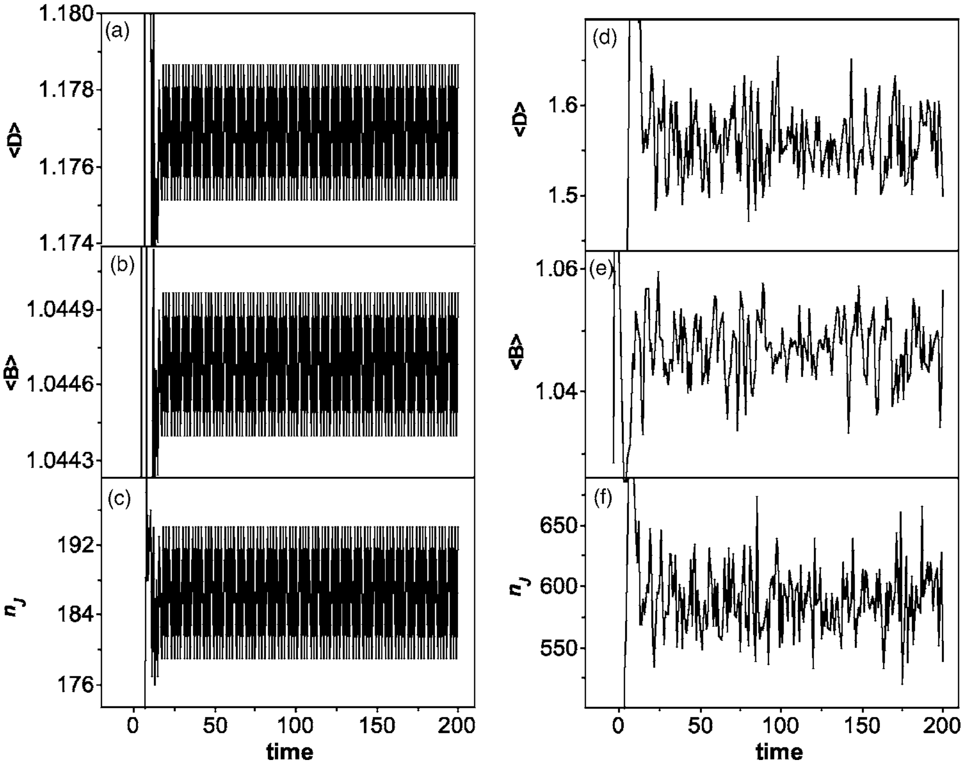

FIG. 2. (a)–(c) For the same scale-free network but for $\alpha = 0.31$, periodic oscillations of (a) the normalized diameter, (b) the normalized betweenness centrality, and (c) the number of jammed nodes. (d)–(f) Random oscillations of the same set of quantities for $\alpha = 0.2$.

for Figs. 1(a)–1(c), respectively. The oscillations are in fact period-2 in that each macroscopic quantity can assume two distinct values, neither being the steady-state value, and the quantity alternates between the two values. For smaller value of α, random oscillations [20] occur, as shown in Figs. 2(d)–2(f) for $\alpha = 0.2$.

The critical value α_c of the capacity parameter, below which persistent network oscillations can occur, can be estimated by noting that, for a given node j, the maximally possible increase in the load before traffic congestion occurs is $\alpha L_j(0)$. The weight-assignment rule in our traffic protocol, Eq. (2), stipulates that the most probable weight change be unity. Now regard α as a control parameter. For a *fixed* amount of change ΔL_j in the load, free flow of traffic is guaranteed if $\alpha L_j(0) > \Delta L_j$ but traffic congestion occurs if $\alpha L_j(0) < \Delta L_j$. The critical value α_c is then given by

$$\alpha_c = \Delta L_j / L_j(0), \qquad (3)$$

which is independent of the degree variable k [8]. Since the load distribution with respect to k is algebraic [18], this suggests that, in order for Eq. (3) to be meaningful, ΔL_j must follow an algebraic scaling law with the *same* exponent. Since the amount of possible weight change is approximately fixed, the resulting load change is also fixed. To give an example, we consider a weighted scale-free network of parameters $N=3000$ and $\langle k \rangle = 4$. Initially all nodes are assigned the same unit weight. The algebraic load distribution is shown in Fig. 3 (squares, the upper data set) on a logarithmic scale. The algebraic scaling exponent is about 1.5. Next we choose nodes of degree k and give them a sudden unit increase in the weight. A recent work shows that for weighted scale-free networks, a weight increase of a node typically causes its load to decrease [21]. The load change ΔL as a function of k is shown in Fig. 3 (circles, the lower data set). We see that on the logarithmic scale, ΔL versus k is parallel to the initial load-degree distribution curve, justifying the use of Eq. (3). Numerically we obtain $\alpha_c^* \approx 0.37$. Since in a realistic situation there are more nodes with weights above the uniform background value of unity and since the amount of

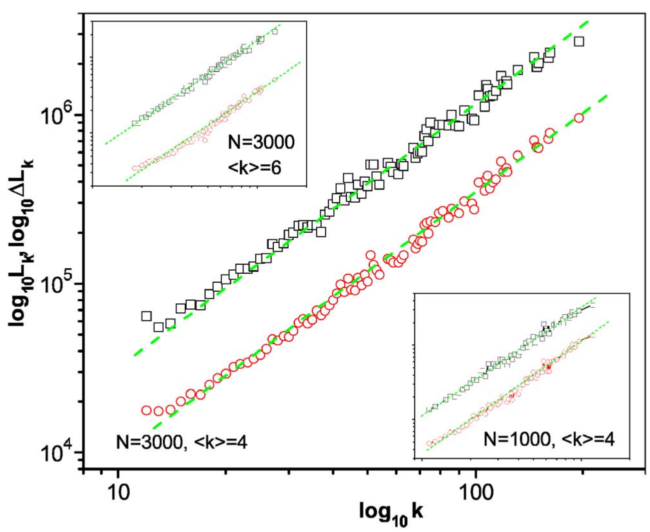

FIG. 3. (Color online) For a scale-free network of $N=3000$ and $\langle k \rangle = 4$, algebraic scaling of the initial load with the degree variable k (upper data set) and the scaling of the load change caused by unit weight change (lower data set). The parallelism of the two sets validates the use of Eq. (3). The insets show similar plots but for different network parameters. These results suggest that the critical value α_c for network oscillation is insensitive to the structural details of the network. The results are averaged over 50 realizations.

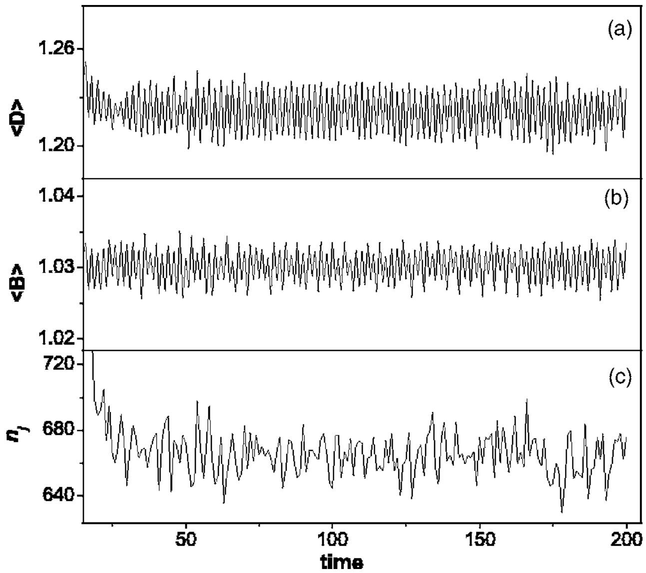

FIG. 4. For the Internet at the autonomous system level with capacity parameter $\alpha=0.2$, evolutions of (a) the normalized diameter, (b) the normalized betweenness centrality, and (c) the number of jammed nodes. Persistent oscillations of the Internet are observed.

weight change can be more than unity, this value of α_c is only approximate. Indeed, direct numerical computations give $\alpha_c \approx 0.32$. The two estimates are nonetheless consistent. An interesting observation is that, for the same degree distribution, the value of α_c is insensitive to network parameters like the network size and the average degree, as shown in the two insets in Fig. 3. In particular, for $N=1000$ and $\langle k \rangle = 4$ (inset in the lower-right corner), we have $\alpha_c \approx 0.39$, while for $N=3000$ and $\langle k \rangle = 6$ (upper-left corner), we obtain $\alpha_c \approx 0.40$. This phenomenon of network-parameter independency can be understood by noting that the load variation at a node caused by its weight change is mainly determined by the probability that optimal paths through this node appear or disappear, as a result of the weight change. This probability is independent of the network size and the average degree of the node [21]. The value of α_c, of course, depends on the degree distribution and the traffic protocol. Indeed, the simulations indicate that the value of α_c is slightly increased as the degree distribution becomes homogeneous and, for homogeneous random networks, the value of α_c is about $\alpha_c \approx 0.45$ [23].

Can oscillations be expected in realistic networks? To address this question, we test the stability of the Internet at the autonomous system level [22]. The network comprises 6474 nodes and 13 895 links, the average diameter is $\langle D(0) \rangle \approx 4.71$, the largest value of the degree is 1460, and the load of this node is $L_j \approx 1.97 \times 10^8$. By setting $\alpha=0.2$, we apply perturbation of strength $P=10 \times L_j$ at the largest-degree node (to mimic an attack) and let the Internet evolve according to our traffic protocol. The time evolutions of the normalized diameter $\langle D \rangle$, of the normalized betweenness $\langle B \rangle$, and of the number of congested nodes n_J are shown in Figs. 4(a)–4(c). Again, persistent oscillations are observed.

We attribute the oscillations to the dynamical interplay between the network topology and the traffic protocol, and regard the existence of loop structure in topology and the adoption of optimal path in protocol as the basic ingredients for the network to oscillate. The loop structure provides multiple options for the communication between two nodes, while the optimal-path protocol decides which option should be adopted at each time step. For most real systems, these two ingredients are naturally fulfilled and thus the oscillation phenomenon is expected to be generic. For clarity, we have used the standard scale-free network as a representative model to illustrate the phenomenon, but extensive numerical simulations have shown that this phenomenon is general for complex networks, regardless of system details such as the network type, the degree distribution, the average degree, the network size, the perturbation size and position, etc. However, while the oscillation phenomenon is generic for complex systems (i.e., whether network can oscillate), the oscillation details can be significantly affected by the system configurations (i.e., how network oscillates). For example, the numerical simulations suggest that oscillations are enhanced in homogeneous networks in the sense that both the value of α_c and the oscillation amplitude are increased and, for the given degree distribution, the avalanche size and the recovery time are closed related to the size and position of the perturbation [23]. The property of perturbation robustness suggests another advantage of scale-free networks over homogeneous random networks, which may stimulate a new direction for network study.

For simplicity, we have assumed a discrete version of the queuing protocol, i.e., the queuing time is unity when node is not overloaded and increases linearly with congestion when overloaded [see Eq. (2)]. To check if the network oscillation exists for other kinds of queuing protocols, we have replaced the discrete version with (i) a continuous version, i.e., the node weight is real value and Eq. (2) is replaced by $w_i(t)=\frac{J_i(t)}{C_i}$, and (ii) a variable version where node capacity is inversely proportional to the congestion situation, i.e., replacing Eq. (2) by $w_i(t)=1+\mathrm{int}\left[\frac{J_i(t)}{C_i(t)}\right]$ with $C_i(t)=C_i$ when $J_i(t)<C_i$ (not overloaded) and $C_i(t)=C_i \times \frac{C_i(t)}{J_i(t)}$ when $J_i(t)>C_i$ (overloaded). The oscillation phenomena are well confirmed in both cases. The only difference is that, for the same set of network parameters, the oscillation amplitude is decreased in the continuous capacity protocol while it is increased in the inverse capacity protocol [23]. Another interesting thing is that as the capacity parameter α decreases further from the value where the period-2 oscillation occurs, regular oscillations or higher period will arise. However, the transitions from low-periodic oscillations to high-periodic oscillations are nonsmooth. The final oscillation state is highly sensitive to the network parameters as well as the initial conditions, e.g., the perturbation size. Therefore the transition from the period oscillation to the chaotic oscillation is nonsmooth. Nevertheless, as α decreases, the trend from the low-period oscillations to the high-period ones and further to the random ones are still clear. A detailed study of this transition will be another issue of interest, since here chaos is generated from the topology complexity of networks instead of the nonlinear functions.

To emphasize the deterministic nature of the network oscillations, we have not allowed the network capacity and traffic load to have random fluctuations. Such fluctuations will generally enhance the oscillations, as represented by larger oscillation amplitudes [23]. The deterministic oscillations distinguish our work from others on network traffic oscillations [24,25], where the underlying models are stochastic. In addition, our model is different from the congestion control models where chaotic flux oscillations have been observed [13,14]. In particular, we have considered the competition among simple node dynamics on complex topologies while the congestion models focus on the competition of complicated node dynamics on simple topologies.

In summary, we have discovered that a complex network of finite capacity can oscillate in the sense that its macroscopic quantities exhibit persistent periodic or random oscillations in response to external perturbations. While the study has arisen as a problem of network security, our findings may have broad implications for general traffic networks which, when applying the evolutionary weighted model, need further specifications. Whereas there can be all sorts of dynamical processes on a complex network, our finding indicates that there can be physically meaningful situations where the network itself is never static but highly dynamic. As a primary model for the evolutionary weighted network, oscillations of macroscopic quantities are only one aspect originating from the interplay between the topology and the local dynamics. Other topics such as adaptivity where the network topology updates according to the local dynamics may warrant further studies.

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