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M. Gunaydin, Cihan Saclioglu

Institutions: CERN, University of Bonn

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OSCILLATOR-LIKE UNITARY REPRESENTATIONS OF NON-COMPACT
GROUPS WITH A JORDAN STRUCTURE AND THE NON-COMPACT
GROUPS OF SUPERGRAVITY

M. Günaydin

CERN -- Geneva

and

C. Saçlıoğlu *)

Physikalisches Institut der Universität Bonn
Bonn, F.R.G.

"Dedicated to Feza Gürsey on the occasion of his 60th birthday"

ABSTRACT

We give a general bosonic construction of oscillator-like unitary irreducible representations (UIR) of non-compact groups whose coset spaces with respect to their maximal compact subgroups are Hermitean symmetric. With the exception of $E_{7(7)}$, they include all the non-compact invariance groups of extended supergravity theories in four dimensions. These representations have the remarkable property that each UIR is uniquely determined by an irreducible representation of the maximal compact subgroup. We study the connection between our construction, the Hermitean symmetric spaces and the Tits-Koecher construction of the Lie algebras of corresponding groups. We then give the bosonic construction of the Lie algebra of $E_{7(7)}$ in $SU(8)$, $SO(8)$ and $U(7)$ bases and study its properties. Application of our method to $E_{7(7)}$ leads to reducible unitary representations.

*) Alexander von Humboldt Fellow, on leave from Physics Dept., Boğaziçi University, Istanbul/Turkey: work supported in part by TBTAk, The National Science and Technology Council of Turkey.

1. INTRODUCTION

Recently, Cremmer and Julia¹⁾ have discovered a set of non-compact invariance groups in the bosonic sectors of $N = 5, 6, 8$ extended supergravity theories in four dimensions, thereby generalizing the non-compact invariance group of the $N = 4$ theory found by Cremmer, Ferrara and Scherk²⁾. The vector field strengths in these theories and their duals get transformed into each other under the action of the non-compact group G and form a linear representation, whereas the scalar fields transform non-linearly as the coset space G/H where H is the maximal compact subgroup of G . The full invariance has the form $G_{\text{global}} \otimes H_{\text{local}}$ as in the two-dimensional generalized σ models³⁾.

Julia and Cremmer conjectured that the composite gauge fields associated with H_{local} may become dynamical at the quantum level just as in the two-dimensional CP^N models¹⁾. Ellis, Gaillard and Zumino (EGZ) have extended this idea and postulated that in $N = 8$ supergravity in addition to the vector bound states other bound states (fermionic as well as bosonic) form whose effective interactions at low energies correspond to a spontaneously broken grand unified theory based on $SU(5)$ with three families of quarks and leptons^{4),5)}. Again in analogy with CP^N models⁶⁾ it was suggested that the bound states in extended supergravity theories may fall into linear representations of G_{global} ⁷⁾. Since the global invariance for $N = 4, 5, 6, 8$ supergravity theories are all non-compact their unitary representations are infinite dimensional. In fact, an infinite set of bound states seems to be needed for giving superheavy masses to the unwanted helicity states in the EGZ program⁵⁾ or the extensions thereof⁸⁾.

In a previous publication we have given a construction of a class of oscillator-like unitary representations of some non-compact groups including those appearing in extended supergravity theories⁹⁾. Our purpose in this paper is to present an extension of our method for constructing unitary irreducible representations (UIR) and point out its connection to other mathematical structures; in particular to Jordan triple systems¹⁰⁾ and Hermitean symmetric spaces¹¹⁾. The plan of the paper is as follows: in Section 2 we give the bosonic construction of the Lie algebras of Ref. 9) in a generalized form which allows one to construct larger classes of UIRs. Specifically this section contains a construction of the Lie algebras of $SP(2n, \mathbb{R})$, $SO(2n)^*$, $SU(m, n)$ and $SO(m, n)$ in terms of boson annihilation and creation operators, some of which are well known in the literature^{12),13)}. We then present, in a generalized form, the extension of the standard construction which yields only the Lie algebras of the non-compact groups of supergravity. This extension uses boson operators transforming exactly like the vector fields in the corresponding supergravity theories. (The construction of $E_{7(7)}$ is deferred to Section 5.) In Section 3 we point out that with

the exception of $SO(m,n)$ ($m \neq 2$ and $n \neq 2$) all the Lie algebras of Section 2 decompose as $L = L^+ \oplus L^0 \oplus L^-$ where L^0 is the Lie algebra of the maximal compact subgroup H that contains an Abelian $U(1)$ factor. L^+ and L^- spaces are conjugate to each other and carry opposite $U(1)$ charges. This decomposition shows that the coset space G/H is a Hermitean symmetric space and the Lie algebra L can be constructed from a so-called Hermitian Jordan triple system. This Jordan structure is discussed and the Tits-Koecher construction of Lie algebras from Jordan triple systems is given. In Section 4 we formulate our general method for constructing UIRs for non-compact groups with a Jordan structure in the Fock space of the corresponding boson operators. Section 5 contains the construction of the Lie algebra of $E_{7(7)}$ which does not have a Jordan structure with respect to its maximal compact subgroup $SU(8)$. Rewriting the Lie algebra of $E_{7(7)}$ in the $SO(8)$ basis we show its triality properties. We then give the $U(7)$ basis of $E_{7(7)}$ and indicate its connection to the Kantor construction of the Lie algebras of the E series in terms of antisymmetric tensors of rank three^{14),15)}. This suggests a possible link between their emergence in $N = 8$ extended supergravity theories in various dimensions and the Kantor construction.

In the last section we show how applying our methods to the case of $E_{7(7)}$ leads to infinitely reducible unitary representations, which may still be of relevance to supergravity¹⁶⁾. We then mention a method due to Gell-Mann for constructing UIRs of $E_{7(7)}$ on certain coset spaces of its maximal compact subgroup $SU(8)$ ^{17),18)}. We conclude with the suggestion that in addition to the unitary representations constructed by using boson operators transforming like the vector fields one can construct further classes of unitary representations using boson operators transforming like the scalar fields in supergravity theories via the operator methods developed by Gürsey and his collaborators¹⁹⁾.

2. BOSONIC CONSTRUCTION OF THE LIE ALGEBRAS OF A CLASS OF NON-COMPACT GROUPS

In this section we give a bosonic construction of the Lie algebras of the non-compact groups $Sp(2n,R)$, $SO(2n)^*$, $SU(n,m)$ and $SO(n,m)$ in a more general form than the one considered in Ref. 9). This generalization is trivial on the Lie algebra level in the sense that it corresponds to taking direct sums but as we shall see later it leads to the construction of a much larger class of UIRs by our methods. The Lie algebras $Sp(2n,R)$, $SO(2n)^*$, $SU(n,m)$ and $SO(n,2)$ have a Jordan structure with respect to their maximal compact subalgebra as explained in the next section. The non-compact groups that come up in extended supergravity theories in four dimensions all have a Jordan structure with respect to

their maximal compact subgroups. The only exception is the non-compact group $E_{7(7)}$ of $N = 8$ supergravity¹⁾ which does not have a Jordan structure with respect to its maximal compact subgroup $SU(8)$. We treat the bosonic construction of $E_{7(7)}$ separately in a later section.

Consider N pairs of boson annihilation and creation operators $a_i(K)$, $b_i(K)$ and $a_i^\dagger(K)$, $b_i^\dagger(K)$ where $i = 1, \dots, n$ denotes a $U(n)$ index and $K = 1, \dots, N$ labels the different pairs which can be infinitely many in certain cases of physical interest as is shown in the Appendix. We shall denote the creation operators by upper indices; thus $a_i(K)^\dagger = a^i(K)$, $b_i(K)^\dagger = b^i(K)$. They obey the canonical commutation relations

$$\begin{aligned} [a_i(K), a^j(L)] &= \delta_i^j \delta^{KL} \\ [b_i(K), b^j(L)] &= \delta_i^j \delta^{KL} \\ [a_i(K), a_j(L)] &= 0 = [b_i(K), b_j(L)] \end{aligned} \quad (2.1)$$

The $U(n)$ generators are then

$$I_n^m = \vec{a}^m \cdot \vec{a}_n + \vec{b}^m \cdot \vec{b}_n \quad (2.2)$$

where the dot product represents a sum over the generation index K , i.e.,

$$\vec{a}^m \cdot \vec{a}_n \equiv \sum_{K=1}^N a^m(K) a_n(K)$$

The $U(n)$ algebra can be extended to the Lie algebra of a non-compact group with a maximal compact subgroup $U(n)$:

a) $U(n) \rightarrow SP(2n, \mathbb{R})$:

The symmetric diboson operators

$$S_{ij} = \vec{a}_i \cdot \vec{b}_j + \vec{a}_j \cdot \vec{b}_i \quad ; \quad S^{ij} = \vec{a}^i \cdot \vec{b}^j + \vec{a}^j \cdot \vec{b}^i \quad (2.3)$$

together with the I_n^m obey the commutation relations

$$\begin{aligned} [S_{ij}, S^{kl}] &= \delta_j^l I_i^k + \delta_i^k I_j^l + \delta_j^k I_i^l + \delta_i^l I_j^k \\ [I_n^m, S_{ij}] &= -\delta_i^m S_{nj} - \delta_j^m S_{in} \\ [I_n^m, S^{ij}] &= \delta_i^m S^{nj} + \delta_n^j S^{im} \end{aligned} \quad (2.4)$$

which corresponds to the Lie algebra of $SP(2n, \mathbb{R})$ in a so-called split basis.

b) $U(n) \rightarrow SO(2n)^*$:

The antisymmetric diboson operators

$$\begin{aligned} A_{ij} &= \vec{a}_i \cdot \vec{b}_j - \vec{a}_j \cdot \vec{b}_i \\ A^{ij} &= \vec{a}^i \cdot \vec{b}^j - \vec{a}^j \cdot \vec{b}^i \end{aligned} \quad (2.5)$$

and the I_n^m satisfy

$$\begin{aligned} [A_{ij}, A^{kl}] &= \delta_j^l I_i^k + \delta_i^k I_j^l - \delta_j^k I_i^l - \delta_i^l I_j^k \\ [I_n^m, A_{ij}] &= -\delta_i^m A_{nj} - \delta_j^m A_{in} \\ [I_n^m, A^{ij}] &= \delta_n^i A^{mj} + \delta_n^j A^{im} \end{aligned} \quad (2.6)$$

This is the Lie algebra of $SO(2n)^*$ with maximal compact subgroup $U(n)$ in a split basis.

Instead of considering the particular combination I_n^m one can also take the operators

$$\begin{aligned} P_j^i &= \vec{a}^i \cdot \vec{a}_j - \frac{1}{n} \delta_j^i (\vec{a}^m \cdot \vec{a}_m) \\ R_j^i &= \vec{b}_j \cdot \vec{b}^i - \frac{1}{n} \delta_j^i (\vec{b}_m \cdot \vec{b}^m) \\ Q &= \vec{a}^m \cdot \vec{a}_m + \vec{b}_m \cdot \vec{b}^m \end{aligned} \quad (2.7)$$

which generate the Lie algebra of $S(U(n) \times U(n))$. These operators together with the non-symmetrized diboson operators

$$\begin{aligned} U_{ij} &= \vec{a}_i \cdot \vec{b}_j \\ U^{ij} &= \vec{a}^i \cdot \vec{b}^j \end{aligned} \quad (2.8)$$

give us the Lie algebra of $SU(n, n)$.

If the indices of the boson operators a and b run differently, i.e., for a_i , $i = 1, \dots, n$ and b_μ , $\mu = 1, \dots, m$ then the operators

$$\begin{aligned} P_j^i &= \vec{a}^i \cdot \vec{a}_j - \frac{1}{n} \delta_j^i (\vec{a}^m \cdot \vec{a}_m) \\ R_\nu^\mu &= \vec{b}_\nu \cdot \vec{b}^\mu - \frac{1}{m} \delta_\nu^\mu (\vec{b}_\lambda \cdot \vec{b}^\lambda) \\ N &= \frac{1}{n} \vec{a}^m \cdot \vec{a}_m + \frac{1}{m} \vec{b}_\lambda \cdot \vec{b}^\lambda \end{aligned} \quad (2.9)$$

$$i, j, \dots = 1, \dots, n \quad ; \quad \mu, \nu, \lambda, \dots = 1, \dots, m$$

together with the diboson operators

$$\begin{aligned} U_{i\mu} &= \vec{a}_i \cdot \vec{b}_\mu \\ U^{i\mu} &= \vec{a}^i \cdot \vec{b}^\mu \end{aligned} \quad (2.10)$$

generate the Lie algebra of $SU(m, n)$

$$\begin{aligned} [U_{i\mu}, U^{j\nu}] &= \delta_i^j R_\mu^\nu + \delta_\mu^\nu P_i^j + \delta_\mu^\nu \delta_i^j N \\ [P_j^i, U_{k\mu}] &= -\delta_k^i U_{j\mu} + \frac{1}{n} \delta_j^i U_{k\mu} \\ [P_j^i, U^{k\mu}] &= \delta_j^k U^{i\mu} - \frac{1}{n} \delta_j^i U^{k\mu} \\ [P_j^i, P_\ell^k] &= \delta_j^k P_\ell^i - \delta_\ell^i P_j^k \end{aligned} \quad (2.11)$$

$$\begin{aligned}
 [R_\nu^\mu, U_{i\lambda}] &= -\delta_\lambda^\mu U_{i\nu} + \frac{1}{m} \delta_\nu^\mu U_{i\lambda} \\
 [R_\nu^\mu, U^{i\lambda}] &= \delta_\nu^\lambda U^{i\mu} - \frac{1}{m} \delta_\nu^\mu U^{i\lambda} \\
 [N, U_{i\mu}] &= -\left(\frac{1}{m} + \frac{1}{n}\right) U_{i\mu} \\
 [N, U^{i\mu}] &= \left(\frac{1}{m} + \frac{1}{n}\right) U^{i\mu} \\
 [R_\nu^\mu, R_\delta^\lambda] &= \delta_\nu^\lambda R_\delta^\mu - \delta_\delta^\mu R_\nu^\lambda
 \end{aligned} \tag{2.12}$$

The following subset of the above operators

$$i(P_j^i - P_i^j), \quad i(R_\nu^\mu - R_\mu^\nu), \quad (U_{i\mu} + U^{i\mu})$$

generate the $SO(n,m)$ subalgebra of $SU(n,m)$. Of the non-compact groups $SO(n,m)$ only those for which $n = 2$ or $m = 2$ have a Jordan structure as explained in the following section.

Now we repeat the above extension procedure by considering annihilation and creation operators transforming like the antisymmetric tensor representation of $U(n)$. This is of interest as one obtains exactly the Lie algebras of the non-compact groups occurring as global symmetries in extended supergravity theories and nothing else⁹⁾. We now have

$$\begin{aligned}
 [a_{ij}(\kappa), a_{kl}^{}(L)] &= \delta^{\kappa L} (\delta_i^k \delta_j^l - \delta_i^l \delta_j^k) \\
 [b_{ij}(\kappa), b_{kl}^{}(L)] &= \delta^{\kappa L} (\delta_i^k \delta_j^l - \delta_i^l \delta_j^k) \\
 [a_{ij}(\kappa), a_{kl}^{}(L)] &= 0 = [b_{ij}(\kappa), b_{kl}^{}(L)]
 \end{aligned} \tag{2.13}$$

where

$$\begin{aligned}
 a_{ij}(\kappa) &= -a_{ji}(\kappa) \quad ; \quad b_{ij}(\kappa) = -b_{ji}(\kappa) \\
 i, j, \dots &= 1, \dots, n \quad \quad \quad \kappa, L, \dots = 1, \dots, N
 \end{aligned}$$

It is easy to see that the $n = 2$ and $n = 3$ cases revert to the standard construction in the form discussed under a) and b) above. New algebras are found only for $n = 4, 5, 6, 7, 8$ and the diboson operators that extend the $U(n)$ generators $\vec{a}^{ik} \cdot \vec{a}_{jk} + \vec{b}_{jk} \cdot \vec{b}^{ik}$ to the Lie algebra of a non-compact group all have the form

$$A_{i_{n-1} \dots i_n} \equiv \epsilon_{i_1 i_2 \dots i_n} \vec{a}^{i_1 i_2} \cdot \vec{b}^{i_3 i_4}$$

and

$$A_{i_{n-1} \dots i_n} \equiv \epsilon_{i_1 i_2 \dots i_n} \vec{a}_{i_1 i_2} \cdot \vec{b}_{i_3 i_4}$$

Thus for $n = 4$ we have the dibosons

$$\begin{aligned} Q^+ &= \frac{1}{4} \epsilon_{ijkl} \vec{a}^{ij} \cdot \vec{b}^{kl} \\ Q^- &= \frac{1}{4} \epsilon^{ijkl} \vec{a}_{ij} \cdot \vec{b}_{kl} \end{aligned} \quad (2.14)$$

which, together with $Q^0 = \frac{1}{4}(\vec{a}^{ij} \cdot \vec{a}_{ij} + \vec{b}_{ij} \cdot \vec{b}^{ij})$ representing the trace of $U(4)$ generate the $SU(1,1)$ algebra

$$\begin{aligned} [Q^+, Q^-] &= -2Q^0 \\ [Q^0, Q^+] &= Q^+ \\ [Q^0, Q^-] &= -Q^- \end{aligned} \quad (2.15)$$

With the remaining $SU(4)$ generators $T_j^i = \vec{a}^{in} \cdot \vec{a}_{jn} + \vec{b}_{jn} \cdot \vec{b}^{in} - \delta_j^i Q^0$ the resulting extension is $U(4) \rightarrow SU(4) \times SU(1,1)$.

For $n = 5$ one has the first simple Lie algebra of a non-compact group. Again we split the $U(5)$ generators into the trace

$$Q = \vec{a}^{ij} \cdot \vec{a}_{ij} + \vec{b}_{ij} \cdot \vec{b}^{ij} \quad (2.16)$$

and the $SU(5)$ part:

$$T_j^i = \vec{a}^{ik} \cdot \vec{a}_{jk} + \vec{b}_{jk} \cdot \vec{b}^{ik} - \frac{1}{5} \delta_j^i Q \quad (2.17)$$

and use the diboson operators

$$A_i = \frac{\sqrt{2}}{4} \epsilon_{ijklm} \vec{a}^k \cdot \vec{b}^{lm} \quad (2.18)$$

$$A^i = \frac{\sqrt{2}}{4} \epsilon^{ijklm} \vec{a}_{jk} \cdot \vec{b}_{lm}$$

to arrive at the algebra of $SU(5,1)$:

$$\begin{aligned} [A_i, A^j] &= T_i^j - \frac{3}{10} \delta_i^j Q \\ [T_i^k, A^j] &= \delta_i^j A^k - \frac{1}{5} \delta_i^k A^j \\ [T_i^k, A_j] &= -\delta_j^k A_i + \frac{1}{5} \delta_i^k A_j \\ [Q, A_i] &= 4 A_i \\ [Q, A^i] &= -4 A^i \end{aligned} \quad (2.19)$$

Under the action of the $SU(5,1)$ group the boson operators $a_{ij}(K)$ and $b^{ij}(K)$ transform into each other and form a 20-dimensional representation of $SU(5,1)$.

Finally, we treat $n = 6$: there are 15 diboson operators $\epsilon_{ijklmn} \vec{a}^{kl} \cdot \vec{b}^{mn}$ and their conjugates. With the 36 generators of $U(6)$ one thus has a total of 66 operators, suggesting $SO(12)^*$. However, the algebra does not close, because unlike the $U(5)$ case, $SO(12)^*$ does not have a 30 dimensional representation corresponding to $a_{ij}(K) \oplus b^{ij}(K)$. The simplest way to remedy the situation is to introduce two $SU(6)$ singlet boson operators $v(K)$ and $w^\dagger(K)$ in order to build up the 32 dimensional semi-spinor of $SO(12)^*$. Indeed this 32 dimensional representation of $SO(12)^*$ decomposes as $1 \oplus 15 \oplus \overline{15} \oplus 1$ with respect to $SU(6)$. The correct diboson generators now are

$$A_{ij} = \frac{1}{4} \epsilon_{ijklmn} \vec{a}^{kl} \cdot \vec{b}^{mn} + \frac{1}{\sqrt{2}} (\vec{a}_{ij} \cdot \vec{v} + \vec{b}_{ij} \cdot \vec{w})$$

$$A^{ij} = \frac{1}{4} \epsilon^{ijklmn} \vec{a}_{kl} \cdot \vec{b}_{mn} + \frac{1}{\sqrt{2}} (\vec{a}^{ij} \cdot \vec{v}^\dagger + \vec{b}^{ij} \cdot \vec{w}^\dagger) \quad (2.20)$$

whereas the SU(6) and U(1) generators are given by

$$\begin{aligned} T_j^i &= \vec{a}^{in} \cdot \vec{a}_{jn} + \vec{b}^{in} \cdot \vec{b}_{jn} - \frac{1}{6} \delta_j^i (\vec{a}^{mn} \cdot \vec{a}_{mn} + \vec{b}^{mn} \cdot \vec{b}_{mn}) \\ Q &= \vec{a}^{mn} \cdot \vec{a}_{mn} + \vec{b}^{mn} \cdot \vec{b}_{mn} - 6 \vec{v}^{\dagger} \cdot \vec{v} - 6 \vec{w} \cdot \vec{w}^{\dagger} \end{aligned} \quad (2.21)$$

They satisfy the commutation relations of SO(12)*

$$\begin{aligned} [A_{ij}, A^{kl}] &= \frac{1}{2} (\delta_i^k T_j^l + \delta_j^l T_i^k - \delta_i^l T_j^k - \delta_j^k T_i^l) \\ &\quad - \frac{1}{12} (\delta_i^k \delta_j^l - \delta_j^k \delta_i^l) Q \\ [T_j^i, A_{kl}] &= -\delta_k^i A_{jl} - \delta_l^i A_{kj} + \frac{1}{3} \delta_j^i A_{kl} \\ [Q, A_{ij}] &= -4 A_{ij} \\ [T_j^i, A^{kl}] &= \delta_j^k A^{il} + \delta_j^l A^{ki} - \frac{1}{3} \delta_j^i A^{kl} \\ [Q, A^{ij}] &= 4 A^{ij} \end{aligned} \quad (2.22)$$

From the basis we have chosen for the Lie algebras above, it may not be obvious what non-compact form of the respective groups we are dealing with. Since we are interested in constructing unitary representations we assume implicitly that we are working in a Hermitean basis, i.e., all the generators of our group are Hermitian operators. In the above bases this is not the case. Therefore, we must take suitable linear combinations of the operators above to go to the Hermitean basis in which all the generators H_i of the group are Hermitian and the structure constants f_{ijk} defined by $[H_i, H_j] = if_{ijk} H_k$ are all pure real. Then the operator $U(g) = e^{iH_i w^i}$ representing a general group element is unitary (w^i are real group parameters). It is in this basis that we calculate the Killing metric so as to determine the form of non-compactness.

3. LIE ALGEBRAS WITH A JORDAN STRUCTURE AND HERMITEAN SYMMETRIC SPACES

We define a simple Lie algebra with a Jordan structure as a Lie algebra L that has a three-dimensional graded form, i.e.,

$$L = L^- \oplus L^0 \oplus L^+ \quad (3.1)$$

where L^0 contains the generator Q of an Abelian $U(1)$ factor such that $L^0 = H \oplus Q$ and $[Q, H] = 0$, $[L^+, L^+] = 0$, $[Q, L^\pm] = \pm L^\pm$. In addition we have a conjugation $+$ such that $(L^+)^+ \cong L^-$, $(L^0)^+ \cong L^0$. Because of the grading and simplicity of the algebra we have $[L^0, L^+] \cong L^+$, $[L^0, L^-] \cong L^-$, $[L^+, L^-] \cong L^0$.

All simple Lie algebras L with a Jordan structure can be related to the so-called Jordan triple systems by the following simple method^{20), 21)}. Denote the elements of L that lie in the L^+ subspace by U_a and the elements of the L^- space by U_a^+ where a belongs to some vector space V . Further denote the commutator of U_a and U_b^+ as S_{ab}

$$S_{ab} \equiv [U_a, U_b^+] \quad (3.2)$$

Because of the grading $[U_a, U_b] = 0 = [U_a^+, U_b^+]$. Through the commutator of S_{ab} with U_c one defines a triple product (abc) in the vector space V :

$$[S_{ab}, U_c] = U_{(abc)} \quad (3.3)$$

Then all the commutation relations can be expressed in terms of the triple product (abc) by using the Jacobi identities.

$$\begin{aligned} [S_{ab}, U_c^+] &= -U_{(bac)}^+ \\ [S_{ab}, S_{cd}] &= S_{a(dcb)} - S_{(cda)b} \end{aligned} \quad (3.4)$$

Jacobi identities impose two conditions on the triple product:

$$(abc) = (cba) \quad (3.5a)$$

$$(ab(cd x)) - (cd(ab x)) - (a(dcb)x) + ((cda)bx) = 0 \quad (3.5b)$$

These conditions define a Jordan triple system¹⁰⁾. Therefore, given any Jordan triple system one can construct a Lie algebra with a Jordan structure as above. This construction of a Lie algebra with a Jordan structure is known as the Tits-Koecher construction^{20),21)}. It has also been extended to Lie superalgebras with a Jordan structure^{22),23)}.

A Jordan algebra with a symmetric product $a \cdot b = \frac{1}{2}(ab+ba)$ defines a Jordan triple system with the triple product

$$(abc) \equiv a \cdot (b \cdot c) - b \cdot (a \cdot c) + (a \cdot b) \cdot c \quad (3.6)$$

that satisfies the conditions (3.5a) and (3.5b).

Below we list the Lie algebras L and their respective subalgebras L^0 that can be constructed from various Jordan algebras using the Jordan triple product (3.6).

Jordan Algebra	L^0	L
$J_n^{\mathbb{R}}$	$U(n)$	$Sp(2n)$
$J_n^{\mathbb{C}}$	$SU(n) \times SU(n) \times U(1)$	$SU(2n)$
$J_n^{\mathbb{H}}$	$U(2n)$	$SO(4n)$
J_3^8	$E_6 \times U(1)$	E_7
$\Gamma(d)$	$SO(d+1) \times SO(2)$	$SO(d+3)$

where $J_n^{\mathbb{R}}$, $J_n^{\mathbb{C}}$, $J_n^{\mathbb{H}}$ denote the Jordan algebras of $n \times n$ real symmetric, complex Hermitean and quaternionic Hermitean matrices respectively. J_3^8 is the exceptional Jordan algebra of 3×3 Hermitean octonionic matrices. $\Gamma(d)$ denotes the Jordan algebra of γ matrices in d dimensions

$$\gamma_\mu \cdot \gamma_\nu \equiv \frac{1}{2} \{ \gamma_\mu, \gamma_\nu \} = \delta_{\mu\nu} 1 \quad ; \quad \mu, \nu = 1, \dots, d$$

Rectangular $n \times m$ matrices over the real numbers \mathbb{R} , complex numbers \mathbb{C} and quaternions \mathbb{H} define a Jordan triple system under the triple product²²⁾

$$(abc) \equiv a \bar{b}^T c + c \bar{b}^T a \quad (3.7)$$

where the bar denotes conjugation over the underlying division algebra \mathbb{R}, \mathbb{C} or \mathbb{H} and T is transposition. The corresponding Lie algebras L and L^0 are

Jordan Triple System	L^0	L
$K_{nm}^{\mathbb{R}}$	$SU(n) \times SU(m) \times U(1)$	$SU(n+m)$
$K_{nm}^{\mathbb{C}}$	$SU(n) \times SU(n) \times SU(m) \times SU(m) \times U(1)$	$SU(n+m) \times SU(n+m)$
$K_{nm}^{\mathbb{H}}$	$SU(2n) \times SU(2m) \times U(1)$	$SU(2n+2m)$
K_{21}	$SO(10) \times SO(2)$	E_6

where $K_{mn}^{\mathbb{R}, \mathbb{C}, \mathbb{H}}$ refers to the Jordan triple system of $n \times m$ matrices over \mathbb{R}, \mathbb{C} and \mathbb{H} . In the case of (2×1) octonionic matrices K_{21} the triple product is modified to be $(abc) = \{(a\bar{b}^T)c + (b\bar{a}^T)c - b(\bar{a}^T c)\} + \{a \leftrightarrow c\}$ due to the non-associativity of octonions.

In the above we have denoted all the Lie algebras L^0 and L with the compact form of the corresponding groups. In general they will be Lie algebras of the non-compact form depending on the underlying Jordan triple system. All simple Lie algebras with a Jordan structure can be constructed from a suitable triple system²⁴⁾.

Now if we denote the groups corresponding to the Lie algebras L and L^0 as G and H then the coset space G/H is a Hermitean symmetric space. Hermitean symmetric spaces are all Kählerian and they can in general be represented in the form of a tensor product²⁵⁾

$$M_0 \times M_1 \times M_2 \dots \times M_r$$

where M_0 is the quotient of a complex Euclidean space by a discrete group of pure translations and each M_i ($i > 0$) is one of the following Riemannian symmetric spaces:

$$SU(p+q)/S(U(p) \times U(q))$$

$$SO(2n)/U(n)$$

$$SO(n+2)/SO(n) \times SO(2)$$

$$E_6 / SO(10) \times SO(2)$$

$$Sp(2n)/U(n)$$

$$E_7 / E_6 \otimes U(1)$$

From this classification it follows that all simple Lie algebras have a Jordan structure with respect to some suitable subalgebra except for the Lie algebras of G_2 , F_4 and E_8 . For the detailed study of the connection between symmetric spaces and Jordan triple systems we refer the reader to Refs. 11) and 26).

All the Lie algebras of the non-compact groups [except for $SO(m,n)$ where both m and n are different from 2] considered in the previous section have a Jordan structure with respect to the Lie algebra of their maximal compact subgroups. Of the remaining Lie algebras of the non-compact groups with a Jordan structure with respect to their maximal compact subgroup, the Lie algebras of $E_{6(-14)}$ and $E_{7(-25)}$ can be similarly constructed from boson operators. In the next section we give a general method for constructing certain classes of UIRs of non-compact groups with a Jordan structure with respect to their maximal compact subgroup.

4. OSCILLATOR-LIKE UNITARY IRREDUCIBLE REPRESENTATIONS OF NON-COMPACT GROUPS WITH A JORDAN STRUCTURE

The Lie algebras of the non-compact groups constructed above have a Jordan structure with respect to the Lie algebra of their maximal compact subgroup:

$$L = L^- \oplus L^0 \oplus L^+ \quad ; \quad (L^+)^{\dagger} \cong L^-, \quad (L^0)^{\dagger} \cong L^0$$

where the L^- and L^+ subspaces correspond to the non-compact generators constructed in terms of diboson annihilation and creation operators. In Ref. 9), we have given a construction of a certain class of UIRs of non-compact groups with a Jordan structure in the case when L^- (and L^+) generators were constructed in terms of diboson annihilation (and creation) operators only. Here we give the same construction in a more general form when we have an arbitrary number of pairs of boson operators $a(k)$, $b(k)$, $k = 1, 2, \dots, N$ instead of a single pair as was done in Ref. 9). On the Lie algebra level this extension is trivial in the sense that it gives us a direct sum of N copies of the same Lie algebra. However, this simple extension enables us to construct larger classes of UIRs of the respective groups²⁷⁾.

Consider now the Fock space constructed from the tensor product of Fock spaces of individual boson operators. The vacuum $|0\rangle$ in our Fock space will be a tensor product of the individual vacua $|0\rangle$.

$$|0\rangle \equiv |0\rangle |0\rangle \dots |0\rangle \quad (4.1)$$

It is annihilated by all the annihilation operators.

$$\alpha_i(\kappa) |0\rangle = 0 = b_i(\kappa) |0\rangle \quad \kappa=1, \dots, N \quad (4.2)$$

Choose a set of states $|\psi_A\rangle$ in our Fock space which is annihilated by all the diboson operators in the L^- space and transform as a certain representation of the maximal compact subgroup generated by L^0 :

$$L^- |\psi_A\rangle = 0 \quad (4.3)$$

Then the infinite number of states generated by applying the operators L^+ on $|\psi_A\rangle$ form the basis of a unitary representation of the non-compact group G :

$$|\psi_A\rangle, L^+ |\psi_A\rangle, (L^+)^2 |\psi_A\rangle, \dots \quad (4.4)$$

Now if $|\psi_A\rangle$ are chosen such that they transform like an irreducible representation of the maximal compact subgroup generated by L^0 , then the corresponding representation of the non-compact group is also irreducible. The proof of this theorem, which was given in Ref. 9), is very simple and uses the Jordan structure of the non-compact group in a crucial manner.

With the exception of the second construction of $SO(12)^*$ [see Eq. (2.20)] the L^- spaces of all the Lie algebras constructed in Section 2 involve diboson annihilation operators only. The states in our Fock space that are annihilated by L^- involving diboson annihilation operators only, are a linear combination of the states of the form

$$[\alpha^{i(1)}]^{m_1} [\alpha^{j(2)}]^{m_2} \dots [\alpha^{k(N)}]^{m_N} |0\rangle \quad (4.5)$$

and of the form

$$[b^{(1)}]^n [b^{(2)}]^n \dots [b^{(N)}]^n |0\rangle \quad (4.6)$$

These states transform in general like a reducible representation of the maximal compact subgroup H . However, using suitable projection operators one can project out the irreducible components. The possible irreducible representations of H that can be constructed this way depends on the number N pairs of boson operators a and b . For example in the case when $a(K)$ and $b(K)$, ($K = 1, \dots, N$) transform like the fundamental representation of $SU(n)$ then the irreducible representations of $SU(n)$ that one can obtain by this method have Young tableaux with at most N rows. This is simply due to the fact that the largest totally antisymmetric representation of $SU(n)$ that one can construct from N copies of boson operators is of rank N . Of course, if $N \geq n$ any representation of $SU(n)$ can be constructed by repeated application of the creation operators followed by a projection operator. In Young tableaux notation we have

$$[a^{(K)}]^{m_K} |0\rangle \Rightarrow (m_K, 0, 0, \dots) \quad (4.7a)$$

$i, j, \dots = 1, \dots, n$

$$[a^{(K)}]^{m_K} [a^{(L)}]^{m_L} |0\rangle \Rightarrow (m_K + m_L, 0, 0, \dots) + (m_K, m_L, 0, \dots) \quad (4.7b)$$

$$[a^{(1)}]^{m_1} [a^{(2)}]^{m_2} \dots [a^{(N)}]^{m_N} |0\rangle$$

$$\Rightarrow (m_1, 0, \dots) \otimes (m_2, 0, \dots) \otimes \dots \otimes (m_N, 0, 0, \dots) \quad (4.7c)$$

where (m_1, m_2, \dots, m_N) denotes a representation with a Young tableau which has m_i number of boxes in the i^{th} row. The maximal compact subgroups in our case have a $U(1)$ factor whose generator in most cases corresponds to the boson number operator. Each one of the states constructed above has a definite $U(1)$ charge.

The remarkable feature of the unitary representations above is that they are uniquely determined by the initial state $|\psi_A\rangle$ that is annihilated by the L^- space and the irreducibility of the representation follows directly from the irreducibility of $|\psi_A\rangle$ under the maximal compact subgroup H . This is a general property of the representations belonging to the discrete series²⁸⁾. Furthermore the condition $L^-|\psi_A\rangle$ could be interpreted as an holomorphicity condition and thus we would expect the above representations to belong to the holomorphic discrete series.

We should note that the use of boson operators for constructing UIRs of non-compact groups is certainly not new in physics. They have been used from time to time to construct representations of certain non-compact groups of physical interest, using most often one set of boson operators which leads to only two UIRs [see Ref. 12) for a comprehensive list of references]. What our formulation does is to give a unified treatment of the oscillator-like representations of all non-compact groups with a Jordan structure in the most general form. The irreducibility of the resulting representations that was proven in individual cases by the brute force method of calculating all the Casimir operators follows simply from condition (4.3) and the Jordan structure⁹⁾. Furthermore, the use of an arbitrary number of pairs of boson operators enables us to construct infinite classes of UIRs.

5. BOSONIC CONSTRUCTION OF THE LIE ALGEBRA OF NON-COMPACT GROUP $E_{7(7)}$

The Lie algebra of $E_{7(7)}$ with a maximal compact subgroup $SU(8)$ was constructed in Ref. 9) in terms of a pair of boson annihilation and creation operators transforming like the antisymmetric tensor representations of $SU(8)$. Here we give the same construction using an arbitrary number N pairs of boson operators. They satisfy the commutation relations:

$$\begin{aligned} [a_{ij}(K), a^{kl}(L)] &= \delta^{KL} (\delta_i^k \delta_j^l - \delta_j^k \delta_i^l) \\ [b_{ij}(K), b^{kl}(L)] &= \delta^{KL} (\delta_i^k \delta_j^l - \delta_j^k \delta_i^l) \\ [a_{ij}(K), a_{kl}(L)] &= 0 = [b_{ij}(K), b_{kl}(L)] \end{aligned} \tag{5.1}$$

$i, j, k, l \dots = 1, 2, \dots, 8 \quad ; \quad K, L, \dots = 1, \dots, N$

and

$$a_{ij}(K) = -a_{ji}(K) \quad ; \quad b_{ij}(K) = -b_{ji}(K)$$

The SU(8) generators are taken as

$$T_j^i = \vec{\alpha}^{im} \cdot \vec{\alpha}_{jm} + \vec{b}_{jm} \cdot \vec{b}^{im} - \frac{1}{8} \delta_j^i (\vec{\alpha}^{kl} \cdot \vec{\alpha}_{kl} + \vec{b}_{kl} \cdot \vec{b}^{kl}) \quad (5.2)$$

where the dot product again denotes summation over the generation index $K = 1, \dots, N$. They satisfy the commutation relations

$$[T_j^i, T_\ell^k] = \delta_j^k T_\ell^i - \delta_\ell^i T_j^k \quad (5.3)$$

Now the 133 dimensional adjoint representation of E_7 decomposes under the SU(8) subgroup as

$$\underline{133} = \underline{63} \oplus \underline{70}$$

where 63 stands for the adjoint representation of SU(8) and 70 corresponds to the totally antisymmetric rank four tensor representation. This suggests that as the 70 non-compact generators of $E_{7(7)}$ we take

$$V_{ijkl} = \vec{\alpha}_{[ij} \cdot \vec{b}_{kl]} + \frac{1}{4} \epsilon_{ijklmnpq} \vec{\alpha}^{mn} \cdot \vec{b}^{pq} \quad (5.4)$$

where indices inside brackets [] are all antisymmetrized. The operator V_{ijkl} is totally antisymmetric in its indices and satisfies

$$V_{ijkl}^+ = \frac{1}{4!} \epsilon^{ijklmnpq} V_{mnpq} \equiv V^{ijkl} \quad (5.5)$$

which reflects the fact that the representation 70 of SU(8) is self-conjugate. The operators V_{ijkl} do indeed close into the generators of SU(8):

$$[V_{ijkl}, V^{abcd}] = -\frac{1}{6} \epsilon_{ijklmnpqr} T_n^m \epsilon^{abcdnmpqr}$$

$$[T_n^m, V_{ijkl}] = \frac{1}{6} \epsilon_{ijklmnpqr} V^{mnpqr} - \frac{1}{2} \delta_n^m V_{ijkl} \quad (5.6)$$

$$[T_n^m, V^{ijkl}] = -\frac{1}{6} \epsilon^{ijklmnpqr} V_{nmpqr} + \frac{1}{2} \delta_n^m V^{ijkl}$$

$$i, j, k, \dots = 1, \dots, 8$$

That the resulting Lie algebra is that of E_7 follows from the fact that it is the only simple Lie algebra of dimension 133. To determine whether it is the Lie algebra of non-compact $E_{7(7)}$ with the maximal compact subgroup $SU(8)$ one has to look at the Killing form. The Killing metric turns out to be

$$g_{T_j^i, T_\ell^k} = 36 \delta_\ell^i \delta_j^k + 1002 \delta_j^i \delta_\ell^k$$

$$g_{V_{ijkl}, V_{mnpq}} = -36 \epsilon_{ijklmnpq} \quad (5.7)$$

$$g_{T_j^i, V_{mnpq}} = 0$$

showing that in a Hermitean basis it gives us the Lie algebra of $E_{7(7)}$.

The Killing metric determines the quadratic Casimir operator C_2 up to an over-all constant. For $E_{7(7)}$ we choose this constant such that

$$C_2 = \frac{1}{6} \left(\frac{1}{24} V_{ijkl} V^{ijkl} - T_\ell^k T_k^\ell \right) \quad (5.8)$$

This construction of $E_{7(7)}$ from boson operators corresponds to an operator formulation of a realization of E_7 in a 56 dimensional space by H. Freudenthal³⁰⁾. Remarkably enough the Casimir operator C_2 is exactly the quartic symplectic invariant on the 56 dimensional fundamental representation space given by Freudenthal. In our case the operators $a_{ij}(b_{ij})$ and $b^{ij}(a^{ij})$ get transformed into each other under the action of $E_{7(7)}$ and form the 56 dimensional fundamental representation.

The non-compact group $E_{7(7)}$ is the global invariance group of the bosonic sector of the largest possible supergravity theory ($N=8$) in four dimensions. The natural $SO(8)$ symmetry of $N=8$ supergravity first gets extended to a global $SU(8)$ symmetry via a chiral-dual transformation, i.e., it acts on the spinor fields of the theory as chiral transformations and on the spin one fields as duality rotations which transform the electric and magnetic field strengths into each other¹⁾. Then this compact $SU(8)$ symmetry is enlarged to the non-compact $E_{7(7)}$ which is realized non-linearly over the 70 scalar fields of the theory corresponding to the coset space $E_{7(7)}/SU(8)$ ¹⁾. Thus, it would be interesting to rewrite $E_{7(7)}$ Lie algebra in an $SO(8)$ basis. Doing this turns out to be the same as going to the Hermitean basis:

Consider the following Hermitean linear combinations of $E_{7(7)}$ generators:

$$\begin{aligned} A_{mn} &= i(T_n^m - T_m^n) = -A_{nm} & ; & \quad A_{mn}^\dagger = A_{mn} \\ S_{mn} &= T_n^m + T_m^n = S_{nm} & ; & \quad S_{mn}^\dagger = S_{mn} \\ S_{ijkl} &= V_{ijkl} + V^{ijkl} & ; & \quad S_{ijkl}^\dagger = S_{ijkl} \\ A_{ijkl} &= i(V_{ijkl} - V^{ijkl}) & ; & \quad A_{ijkl}^\dagger = A_{ijkl} \end{aligned} \quad (5.9)$$

Note that since we are working in an $SO(8)$ basis we are not making any distinction between upper and lower indices. These operators satisfy the commutation relations:

$$[A_{ijkl}, A_{\alpha\beta\gamma\delta}] = \frac{1}{6} (\epsilon_{ijklmnpq} \epsilon_{\alpha\beta\gamma\delta}^{\eta\nu\rho\sigma} - \epsilon_{ijklmnpq} \epsilon_{\alpha\beta\gamma\delta\eta\nu\rho\sigma}) A_{mn}$$

$$\begin{aligned}
 [S_{mn}, S_{ijke}] &= \frac{1}{6} \epsilon_{ijklm\nu\sigma} A_{\eta\nu\sigma} + \frac{1}{6} \epsilon_{ijkeln\nu\sigma} A_{m\nu\sigma} \\
 &\quad - \delta_{mn} A_{ijke} \\
 [A_{mn}, S_{ijke}] &= \frac{1}{6} \epsilon_{ijkeln\nu\sigma} S_{m\nu\sigma} - \frac{1}{6} \epsilon_{ijklm\nu\sigma} S_{n\nu\sigma} \\
 [S_{mn}, A_{ijke}] &= \frac{1}{6} \epsilon_{ijkeln\nu\sigma} S_{m\nu\sigma} + \frac{1}{6} \epsilon_{ijklm\nu\sigma} S_{n\nu\sigma} \\
 &\quad - \delta_{mn} S_{ijke} \tag{5.10}
 \end{aligned}$$

$$\begin{aligned}
 [A_{mn}, A_{ijke}] &= -\frac{1}{6} \epsilon_{ijkeln\nu\sigma} A_{m\nu\sigma} + \frac{1}{6} \epsilon_{ijklm\nu\sigma} A_{n\nu\sigma} \\
 [S_{ijke}, S_{\alpha\beta\gamma\delta}] &= -\left(\frac{1}{6} \epsilon_{ijklm\nu\sigma} \epsilon_{\alpha\beta\gamma\delta}^{\eta\nu\sigma} + \frac{1}{6} \epsilon_{ijklm\nu\sigma} \epsilon_{\alpha\beta\gamma\delta\eta\nu\sigma}\right) A_{mn} \\
 [S_{ijke}, A_{\alpha\beta\gamma\delta}] &= \left(-\frac{1}{6} \epsilon_{ijklm\nu\sigma} \epsilon_{\alpha\beta\gamma\delta}^{\eta\nu\sigma} + \frac{1}{6} \epsilon_{ijklm\nu\sigma} \epsilon_{\alpha\beta\gamma\delta\eta\nu\sigma}\right) S_{mn} \\
 i, j, k, \dots, \alpha, \beta, \dots &= 1, \dots, 8; \quad \epsilon_{ijke}^{mnpq} \equiv \frac{1}{24} \epsilon_{ijklm\nu\sigma\lambda} \epsilon^{\mu\nu\sigma\lambda mnpq}
 \end{aligned}$$

The number of independent A_{ijkl} is 35 and so is the number of independent S_{ijkl} . Since $\sum_{n=1}^8 S_{nn} = 0$, there are also 35 independent S_{mn} . Therefore, under the $SO(8)$ subgroup generated by A_{mn} the adjoint representation of E_7 decomposes as

$$\underline{133} = \underline{28} \oplus \underline{35}^V \oplus \underline{35}^L \oplus \underline{35}^R \tag{5.11}$$

which corresponds to the decomposition of the generators as

$$\underline{133} = A_{mn} \oplus S_{mn} \oplus A_{ijke} \oplus S_{ijke} \tag{5.12}$$

These three 35 dimensional representations of $SO(8)$ are all inequivalent and are related by the principle of triality³¹⁾ which generalizes the well-known triality among the three 8 dimensional representations $8^V, 8^L$ and 8^R of $SO(8)$, namely the vectors 8^V , left-handed spinors 8^L and right-handed spinors 8^R , respectively³²⁾. In fact the exceptional group F_4 has the same structure with respect to its $SO(8)$ subgroup, i.e., its adjoint representation 52 decomposes as

$$\underline{52} = \underline{28} \oplus \underline{8}^V \oplus \underline{8}^L \oplus \underline{8}^R \quad (5.13)$$

From the following Kronecker products

$$\begin{aligned} \underline{8}^i \otimes \underline{8}^i &= \underline{1} + \underline{28} + \underline{35}^i, \quad i = V, L, R \\ \underline{8}^i \otimes \underline{8}^j &= \underline{1} + \underline{28} + \underline{35}^k \end{aligned} \quad (5.14)$$

$i, j, k = V, L, R$ taken in cyclic order

it follows that the representations $\underline{35}^V$, $\underline{35}^L$ and $\underline{35}^R$ correspond to symmetric traceless tensors in vector, left-handed spinor and right-handed spinor indices in eight dimensions, respectively. The triality principle also implies that

$$\begin{aligned} \underline{35}^i \otimes \underline{35}^i &= \underline{28} + \underline{35}^i + \dots \\ \underline{35}^i \otimes \underline{35}^j &= \underline{35}^k + \dots \end{aligned} \quad (5.15)$$

$i, j, k = V, L, R$ in cyclic order

A unitary operator representing an element of the group $E_{7(7)}$ can now be written as

$$\begin{aligned} U(g) &= e^{i(\omega_{ij} A_{ij} + v_{ij} S_{ij} + \omega_{ijke} A_{ijke} + v_{ijke} S_{ijke})} \\ U^\dagger(g) U(g) &= 1 \end{aligned} \quad (5.16)$$

where ω_{ij} , v_{ij} , ω_{ijkl} and v_{ijkl} are real parameters with the same tensorial properties as the respective operators, i.e.,

$$\begin{aligned} \omega_{ij} &= -\omega_{ji} \quad ; \quad v_{ij} = v_{ji} \quad + \quad v_{ii} = 0 \\ \omega_{ijke} &= -\tilde{\omega}_{ijke} \equiv -\frac{1}{24} \epsilon_{ijklmnpq} \omega_{mnpq} \\ v_{ijke} &= \tilde{v}_{ijke} \equiv \frac{1}{24} \epsilon_{ijklmnpq} v_{mnpq} \end{aligned} \quad (5.17)$$

Now if we define the boson number operator as $N = \vec{a}^{mn} \cdot \vec{a}_{mn} + \vec{b}^{mn} \cdot \vec{b}_{mn}$ we find that the $SU(8)$ generators have zero boson number

$$[N, T_n^m] = 0 \quad (5.18)$$

and the non-compact generators V_{ijkl} do not have a well-defined boson number since they involve diboson creation as well as annihilation operators. This is a reflection of the exceptional feature of $E_{7(7)}$ whose adjoint representation decomposes as the adjoint plus a real irreducible representation with respect to a maximal compact unitary subgroup. [The only other group with this property is $SO(6,1)$.] The operator N which lies outside of $E_{7(7)}$ does however still generate an automorphism of its Lie algebra.

A maximal rank compact subgroup with respect to which the additional generators in $E_{7(7)}$ split into complex representations is $U(7)$. If we take as the $SU(7)$ generators T_B^A where $A, B = 1, \dots, 7$ and as the $U(1)$ generator $T_8^8 = -T_A^A$ then under this $U(7)$ the Lie algebra of $E_{7(7)}$ decomposes as

$$T_i^j = (T_B^A \oplus T_8^8) \oplus T_8^A \oplus T_A^8 = (\underline{48} + \underline{1}) + \underline{7} + \overline{\underline{7}}$$

$$V_{ijkl} = V_{ABCD} \oplus V_{ABC8} = \underline{35} \oplus \overline{\underline{35}} \quad (5.19)$$

$$i, j, k, l = 1, \dots, 8; \quad A, B, C, D = 1, \dots, 7$$

Denoting the generators V_{ABC8} as A_{ABC} and V^{ABC8} as A^{ABC} we can write the $E_{7(7)}$ Lie algebra in the $U(7)$ basis as a direct sum

$$L = F_A \oplus A_{ABC} \oplus (T_B^A \oplus T_8^8) \oplus A^{ABC} \oplus F^A$$

$$L = L^{-2} \oplus L^{-1} \oplus L^0 \oplus L^+ \oplus L^{+2} \quad (5.20)$$

where F_A stands for T_A^8 and F^A for T_8^A . We see that with respect to the $U(1)$ generator T_8^8 the Lie algebra has a five dimensional graded structure. This is a more general structure than the Jordan structure and all simple Lie algebras have a five dimensional graded structure with respect to a suitable maximal subalgebra. We shall call this type of a five dimensional graded structure a Kantor structure¹⁴⁾. The construction of Lie (super) algebras from Jordan (super) triple systems has been extended to this more general case^{14), 22)}. The ternary algebra that gives us the Lie algebra of E_7 in a $U(7)$ basis in this more general construction corresponds to antisymmetric tensors of rank three in seven dimensions¹⁴⁾. In fact, Kantor construction yields the Lie algebra of the exceptional group E_d when the underlying ternary algebra is antisymmetric

tensors of rank three in d dimensions with a suitable triple product¹⁵⁾. This construction gives finite dimensional Lie algebras E_d for $d \leq 8$ and leads to infinite dimensional Lie algebras¹⁴⁾ for $d > 8$. Now the $N = 8$ extended supergravity theory in d dimensions has non-compact $E_{d(d)}$ as its global invariance group³³⁾ and all these theories are obtained from the 11 dimensional simple supergravity theory by dimensional reduction. The fundamental boson field that enters this latter theory is an antisymmetric tensor of rank three. Thus the Kantor construction of E series suggests a possible link between the emergence of these groups and the presence of antisymmetric rank three tensor fields in $N = 8$ supergravity theories.

6. UNITARY REPRESENTATIONS OF $E_{7(7)}$

If we apply the methods of Section 4 to $E_{7(7)}$ for the construction of unitary representations we find that the resulting representations are reducible. This can be seen easily as follows. Consider a set of states $|\psi_A\rangle$ transforming as an irreducible representation of the maximal compact subgroup $SU(8)$ that is constructed by acting on the vacuum state with the creation operators \vec{a}^{ij} and b^{ij} . By repeated application of the non-compact generators V_{ijkl} on $|\psi_A\rangle$ we can generate an infinite set of states:

$$|\psi_A\rangle, V_{ijkl}|\psi_A\rangle, V_{ijkl}V_{mnpq}|\psi_A\rangle, \dots \quad (6.1)$$

which form the basis of a unitary representation of $E_{7(7)}$. The V_{ijkl} transform as the self-conjugate representation 70 of $SU(8)$ and the product

$$V_{ijkl}V_{mnpq} = \frac{1}{2}\{V_{ijkl}, V_{mnpq}\} + \frac{1}{2}[V_{ijkl}, V_{mnpq}]$$

transforms as the reducible $(1+720+1764)_{\text{sym.}} + 63_{\text{antisym.}}$ representation of $SU(8)$. The fact that the product contains a singlet of $SU(8)$ means that every irreducible representation of $SU(8)$ that occurs in the infinite set of states (6.1) will reappear again after two applications of the V 's. Thus the multiplicity of an irreducible representation of $SU(8)$ that occurs in the unitary representation defined over the set of states (6.1) is infinite. This means that the resulting unitary representation is infinitely reducible as a consequence of the well-known fact that the multiplicity of an irreducible representation of the maximal compact subgroup inside an UIR of a non-compact group is less than or equal to its dimension³⁴⁾. Though reducible, these representations may still be of relevance for physical applications^{5),16)}.

Application of our method to the second construction of $SO(12)^*$ [see Eqs. (2.20)-(2.22)] gives reducible unitary representations as well. In this case even though we have a Jordan structure there are no states transforming like an irreducible representation of $U(6)$ that is annihilated by the L^- space.

The fact that one gets infinitely reducible unitary representations in the case of $SO(12)^*$ and $E_{7(7)}$ suggests the use of coherent states to construct UIRs of these groups. In fact there is a method due to Gell-Mann for constructing a class of UIRs of some non-compact groups on certain coset spaces of their maximal compact subgroups^{17),18)}. His method does apply to $E_{7(7)}$ and is particularly simple for determining the multiplicities of the irreducible representations of the maximal compact subgroup inside a UIR of the non-compact group. For example, one possible coset space on which to realize UIRs of $E_{7(7)}$ is $SU(8)/Sp(8)$. In this case the multiplicity of an irreducible representation of $SU(8)$ in a UIR of $E_{7(7)}$ constructed by his method is determined by the number of $Sp(8)$ singlets that representation contains¹⁷⁾. The reason why Gell-Mann's method cannot be applied to our construction of $E_{7(7)}$ is due to the fact that the boson operators we use transform linearly under the $SU(8)$ subgroup rather than non-linearly as some coset space of $SU(8)$ satisfying certain criteria¹⁸⁾. On the other hand our construction of the non-compact groups of supergravity parallels very closely their emergence in supergravity and the boson operators we use correspond to the vector fields in these theories. The reducibility of some of the resulting representations seems to be necessary for the compatibility of supersymmetry with the non-compact invariance in $N = 4 - 8$ supergravity theories^{16),35)}.

The only basic bosonic fields that transform non-linearly in supergravity theories are the scalar fields which sit on the coset space G/H of the non-compact group G with respect to its maximal compact subgroup. The action of G on the scalar fields can be represented as a generalized linear fractional transformation¹⁾ indicating that the scalars can be considered as a Gelfand-Z-basis of G ³⁶⁾. Gürsey and his collaborators have given an operator formulation of the construction of UIRs in a Z-basis à la Gelfand in the case of some smaller non-compact groups¹⁹⁾. By these operator techniques one can construct new classes of unitary representations of these groups using operators corresponding to the scalar fields in supergravity theories, which may be of relevance as well for physical applications.

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APPENDIX - CURRENT ALGEBRA OF DIBOSON FIELDS

As an illustration of how our boson operators can be related to a field theory context, we construct a current algebra of free boson fields. We will also see in this example that the so far unspecified vectorial character of our operators \vec{a}_i acquires a specific meaning: the components of the vector represent various Fourier modes $a_i(\vec{k})$. Another point of interest is that the construction involves a bosonic version of the Pauli-Gürsey (PG) transformation³⁷⁾.

We first start with a set of boson fields ϕ_i transforming (for simplicity) as an irreducible representation of a global symmetry group H . The bosons can be scalars or vectors, but in the latter case the study of the algebra is simpler if they are taken to be massive, so that gauge related quantization problems are avoided. The generators of H can be then immediately written down as integrated charges obtained from the standard currents, which are always of the form $\phi_i^\dagger \partial_\mu \phi_j$.

Next, we extend H by PG like transformations which mix ϕ_i and ϕ_k^\dagger . This gives rise to new currents of the form $\phi_i^\dagger \partial_\mu \phi_j$ and $\phi_i^\dagger \partial_\mu \phi_j^\dagger$. The original PG of course applies to fermions ψ and leaves the free massless Lagrangian and even certain gauge couplings invariant (up to a total divergence). The new difermions $\int dv \psi \psi$ and $\int dv \psi^\dagger \psi^\dagger$, together with the usual $\int dv \psi^\dagger \psi$ charges, can sometimes close into a new algebra of a compact group. In contrast, our bosonic version of PG and the resulting diboson generators only leave the equal-time free boson field commutators invariant and can lead to both compact and non-compact groups.

The main features of the construction can be understood with the example of a single complex scalar field. The diboson charges are most conveniently written by defining the two-component field $\psi^T = (\psi_1, \psi_2)$ with³⁸⁾

$$\begin{aligned}\psi_1 &= \sqrt{\frac{m}{2}} \left(\phi + \frac{i}{m} \dot{\phi} \right) \\ \psi_2 &= \sqrt{\frac{m}{2}} \left(\phi - \frac{i}{m} \dot{\phi} \right)\end{aligned}\tag{A.1}$$

so that the similarity to fermions becomes more evident. The equal-time commutation relations between ϕ and $\dot{\phi}^\dagger$ now translate into (other combinations vanish)

$$[\psi_\alpha(\bar{x}, 0), \psi_\beta^\dagger(\bar{y}, 0)] = (\tau_3)_{\alpha\beta} \delta^3(\bar{x} - \bar{y})\tag{A.2}$$

indeed displaying a similarity to harmonic oscillator operators or spinor field anticommutation relations: there is no (i) on the right-hand side and the fields ψ and ψ^\dagger appear on the left-hand side, instead of the less symmetrical ϕ, ϕ^\dagger pair. The τ_3 signature on the right-hand side originates from the fact that $\psi_1(\psi_2)$ acts like an annihilation (creation) operator; for example, for the $E^2 = m^2$ modes $\psi_1 \sim a_k$ and $\psi_2 \sim a_k^{c\dagger}$. In this formalism the U(1) charge operator is

$$\begin{aligned} Q_3 &= \frac{1}{2} \int d^3x \left(\psi^\dagger \tau_3 \psi + \psi^\dagger \tau_3 \psi^* \right) \\ &= \frac{1}{2} \sum_k \left(a_k^\dagger a_k + a_k a_k^\dagger - a_k^{c\dagger} a_k^c - a_k^c a_k^{c\dagger} \right) \end{aligned} \quad (A.3)$$

As mentioned before, massive spin one is no problem: we just let $\phi \rightarrow A_b$, $\dot{\phi} \rightarrow E_b$, $\psi \rightarrow \psi_b$ ($b = 1, 2, 3$) and sum over the space index b in Q_3 . The resulting Q_3 is, of course, nothing but the usual $\int dv (A_b^* E_b + E_b^* A_b)$. Thus it is not surprising that similar but more complicated non-compact symmetries hold for vector bosons in extended supergravity. Since the vector index can be trivially added, we go on with our simple scalar example.

In the ψ basis one can introduce a bosonic version of the Pauli-Gürsey transformation of the form

$$\psi'_\alpha = a \psi_\alpha + b M_{\alpha\beta} \psi_\beta^\dagger \quad (A.4)$$

where a, b are in general complex and $M_{\alpha\beta}$ is a 2×2 matrix. This does not leave the Lagrangian invariant, but it is canonical in the sense of preserving the field commutators when M satisfies³⁹⁾

$$(\tau_3)_{\alpha\beta} = |a|^2 (\tau_3)_{\alpha\beta} - |b|^2 M_{\alpha\mu} (\tau_3)_{\mu\nu} M_{\nu\beta}^\dagger \quad (A.5)$$

This gives rise to two possibilities and conditions:

$$\begin{aligned} a) \quad M &= I \text{ or } \tau_3 & ; \quad |a|^2 - |b|^2 &= 1 \\ b) \quad M &= \tau_1 \text{ or } \tau_2 & ; \quad |a|^2 + |b|^2 &= 1 \end{aligned} \quad (A.6)$$

Conditions (a) and (b) suggest the groups $SU(1,1)$ and $SU(2)$ respectively. Indeed, taking $M = I$, applying Eq. (5) on Eq. (3), and then picking from the results the diboson $SU(1,1)$ generators

$$Q_+ = \int \frac{dV}{\sqrt{2}} \psi^\dagger \tau_3 \psi^* = \frac{1}{\sqrt{2}} \sum_k (a_k^\dagger a_{-k}^\dagger e^{zi\omega t} - a_k^c a_{-k}^c e^{-zi\omega t}) \quad (A.7)$$

$$Q_- = \int \frac{dV}{\sqrt{2}} \psi^\dagger \tau_3 \psi = \frac{1}{\sqrt{2}} \sum_k (a_k a_{-k} e^{-zi\omega t} - a_k^c a_{-k}^c e^{-zi\omega t})$$

this expectation for (a) is verified. $a_k(a_k^c)$ of course represent particle (antiparticle) destruction operators. The exponential time dependence can be absorbed into the operators through $a_k \rightarrow a_k e^{-i\omega t}$, hence it will not be written down again. Note that $M = \tau_3$ duplicates the algebra and $M = \tau_1$ gives charges such as $\int dV \phi \phi$ or $\int dV \dot{\phi} \dot{\phi}$ with different Lorentz transformation properties; we consider them no further. $M = \tau_2$ here gives vanishing charges on account of Bose statistics, but when fields ψ_i representing a $U(n)$ group are considered, $n(n-1)$ combinations of the type $\psi_j i\tau_2 \psi_i$, $\psi^k \tau_2 \psi^l$ become possible, extending the $U(n)$ to compact $SO(2n)$. These have the form

$$Q_{ij} = \int \frac{dV}{\sqrt{2}} \psi_i^\dagger i\tau_2 \psi_j = \frac{1}{\sqrt{2}} \sum_k a_i(k) a_j^c(k) - a_j(k) a_i^c(k)$$

$$Q^{ij} \equiv (Q_{ij})^\dagger$$

The Lie algebra of compact bosonic E_7 can also be represented through such operators. Returning to the set Q_\pm, Q_3 , we find

$$\begin{aligned} [Q_+, Q_-] &= -2Q_3 \\ [Q_3, Q_+] &= Q_+ \\ [Q_3, Q_-] &= -Q_- \end{aligned} \quad (A.8)$$

where the clear $SU(1,1)$ signature is seen in the first commutator. The extension of this current algebra representation to cases where the initial symmetry is bigger than $U(1)$ can obviously be effected by re-interpreting our former expressions: for example, $\vec{a}_i \cdot \vec{b}_j$ could now be taken to mean $\sum_k a_i(\vec{k}) a_j(-\vec{k})$ in the non-compact cases and $\vec{a}_i \cdot \vec{b}_j$ be identified with $\sum_k a_i^\dagger(\vec{k}) a_j^c(\vec{k})$.

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