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OSCILLATORY PROPERTIES OF SOLUTIONS OF NONLINEAR DIFFERENTIAL SYSTEMS WITH DEVIATING ARGUMENTS

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INTRODUCTION

In this paper we consider the nonlinear differential system with deviating arguments:

$$(S) \quad \begin{aligned} y'_i(t) &= p_i(t) y_{i+1}(t), \quad i = 1, 2, \dots, n - 2, \\ y'_{n-1}(t) &= p_{n-1}(t) f_{n-1}(y_n(h_n(t))), \\ y'_n(t) &= -p_n(t) f_n(y_1(h_1(t))). \end{aligned}$$

The following conditions are always assumed to be fulfilled:

- (1) (a) $p_i: [0, \infty) \rightarrow [0, \infty)$, $i = 1, 2, \dots, n$, are continuous functions and not identically zero on any subinterval of $[a, \infty) \subset [0, \infty)$; $\int^\infty p_i(t) dt = \infty$, $i = 1, 2, \dots, n - 1$.
- (b) $h_i: [0, \infty) \rightarrow R$, $i = 1, n$, are continuous and $\lim_{t \rightarrow \infty} h_i(t) = \infty$;
- (c) $f_i: R \rightarrow R$, $i = n - 1, n$, $f_i(u) \cdot u > 0$ for $u \neq 0$, $f_i(u)$ are nondecreasing in u .

Definition 1. System (S) is called (α_{n-1}, α_n) *superlinear* if there are positive numbers α_{n-1}, α_n such that $\alpha_n \cdot \alpha_{n-1} > 1$ and

$$\frac{|f_i(u)|}{|u|^{\alpha_i}} \geq \frac{|f_i(v)|}{|v|^{\alpha_i}} \quad \text{for } |u| > |v|, \quad u \cdot v > 0, \quad i = n - 1, n.$$

Denote by W the set of all solutions $y(t) = (y_1(t), \dots, y_n(t))$ of the system (S) which exist on some ray $[T_y, \infty) \subset [0, \infty)$ and satisfy $\sup \left\{ \sum_{i=1}^n |y_i(t)| : t \geq T \right\} > 0$ for $T \geq T_y$.

Definition 2. A solution $y \in W$ is called *oscillatory* if each of its components has arbitrarily large zeros. A solution $y \in W$ is called *nonoscillatory* (weakly nonoscillatory) if each of its components (at least one component, respectively) is eventually of a constant sign.

By Lemma 1 [4] it follows that every solution of (S) is either oscillatory or non-oscillatory.

Definition 3. We shall say that the system (S) has the property A, if every solution $y \in W$ is oscillatory for n even, while for n odd it is either oscillatory or y_i ($i = 1, 2, \dots, n$) tend monotonically to zero as $t \rightarrow \infty$.

The oscillation properties of two-dimensional nonlinear differential systems with deviating arguments were studied for example by Kitamura and Kusano [2, 3], Ševelo and Varech [5, 6, 7]. The oscillation results for n -dimensional systems were obtained by Foltynska and Werbowski, and by the present author [4].

In this paper we extend some results established in [7] to the system (S).

OSCILLATION THEOREMS

In what follows we shall use the following notations:

$$\begin{aligned} h_i^*(t) &= \min \{h_i(t), t\}, \quad i = 1, n, \\ \gamma_i(t) &= \sup \{s \geq 0; t > h_i^*(s)\} \quad \text{for } t \geq 0, \quad i = 1, n, \\ \gamma(t) &= \max \{\gamma_1(t), \gamma_n(t)\}. \end{aligned}$$

Let $i_k \in \{1, 2, \dots, n\}$, $k \in \{1, 2, \dots, n-1\}$, $t, s \in [a, \infty)$. We define: $I_0 = 1$,

$$(2) \quad I_k(t, s; p_{i_k}, \dots, p_{i_1}) = \int_s^t p_{i_k}(x) I_{k-1}(x, s; p_{i_{k-1}}, \dots, p_{i_1}) dx.$$

It is not difficult to verify that the following identities hold:

$$(3) \quad \begin{aligned} I_k(t, s; p_{i_k}, \dots, p_{i_1}) &= (-1)^k I_k(s, t; p_{i_1}, \dots, p_{i_k}) = \\ &= (-1)^k \int_t^s p_{i_1}(x) I_{k-1}(x, t; p_{i_2}, \dots, p_{i_k}) dx, \quad k \in \{1, 2, \dots, n-1\}. \end{aligned}$$

In the sequel we shall need the following lemmas [4; Lemma 2, Lemma 4].

Lemma 1. Let (1a)–(1c) hold. Let $y = (y_1, \dots, y_n) \in W$ be a nonoscillatory solution of (S) on the interval $[a, \infty)$. Then there exist an integer $l \in \{1, 2, \dots, n\}$, $n \equiv l \pmod{2}$, and a $t_0 \geq a$ such that

$$(4) \quad y_i(t) y_1(t) > 0 \quad \text{on } [t_0, \infty) \quad \text{for } i = 1, 2, \dots, l,$$

$$(5) \quad (-1)^{n+i} y_i(t) y_1(t) > 0 \quad \text{on } [t_0, \infty) \quad \text{for } i = l+1, \dots, n.$$

Lemma 2. Let (1a)–(1c) hold. Let $y = (y_1, \dots, y_n) \in W$ be a solution on the interval $[a, \infty)$. Then the following relations hold:

$$(6) \quad \begin{aligned} y_i(t) &= \sum_{j=0}^m (-1)^j y_{i+j}(s) I_j(s, t; p_{i+j-1}, \dots, p_i) + \\ &+ (-1)^{m+1} \int_t^s y_{i+m+1}(x) p_{i+m}(x) I_m(x, t; p_{i+m-1}, \dots, p_i) dx \end{aligned}$$

for $0 \leq m \leq n - i - 2$, $1 \leq i \leq n - 2$, $t, s \in [a, \infty)$;

$$(7) \quad y_i(s) = \sum_{j=0}^{n-i-1} (-1)^j y_{i+j}(t) I_j(t, s; p_{i+j-1}, \dots, p_i) + \\ + (-1)^{n-i} \int_s^t p_{n-1}(x) f_{n-1}(y_n(h_n(x))) I_{n-i-1}(x, s; p_{n-2}, \dots, p_i) dx \\ \text{for } i = 1, 2, \dots, n - 1, \quad t, s \in [a, \infty).$$

The proofs of Lemma 1 and Lemma 2 are found in the paper [4].

Lemma 3. Let (1a)–(1c) hold. Let $y = (y_1, \dots, y_n) \in W$ be a nonoscillatory solution of (S) on the interval $[a, \infty)$ with $y_1(t) > 0$ for $t \geq a$.

Then there exist an integer $l \in \{1, 2, \dots, n\}$, $l \equiv n \pmod{2}$, and a $t_0 \geq a$ such that (4), (5) hold,

$$(8) \quad y_i(t) \geq \int_{t_0}^t H_{i,l-1}(s, t_0) p_{n-1}(s) f_{n-1}(y_n(h_n(s))) ds, \\ \text{for } l \in \{2, 3, \dots, n\}, \quad i = 1, 2, \dots, l - 1, \quad t \geq t_0,$$

where

$$(9) \quad H_{i,l-1}(s, t_0) = \int_{t_0}^s I_{l-i-1}(t, x; p_i, \dots, p_{l-2}) p_{l-1}(x) \times \\ \times I_{n-l-1}(s, x; p_{n-2}, \dots, p_l) dx, \quad l \in \{2, 3, \dots, n - 1\}, \quad s \geq t_0,$$

$$(10) \quad H_{i,n-1}(s, t_0) = I_{n-i-1}(t, s; p_i, \dots, p_{n-2}), \quad l = n, \quad t_0 \leq s \leq t.$$

Proof. We put $m = l - i - 1$, $s = t_0$ in (6) and use (3), (4). Then we have

$$(11) \quad y_i(t) = \sum_{j=0}^{l-i-1} y_{i+j}(t) I_j(t, t_0; p_i, \dots, p_{i+j-1}) + \\ + \int_{t_0}^t y_l(u) p_{l-1}(u) I_{l-i-1}(t, u; p_i, \dots, p_{l-2}) du \geq \\ \geq \int_{t_0}^t y_l(u) p_{l-1}(u) I_{l-i-1}(t, u; p_i, \dots, p_{l-2}) du \quad \text{for } i = 1, 2, \dots, l - 1, \quad t \geq t_0.$$

On the other hand, we put $i = l$, $s = u$ in (7) and using (5) for $t \geq u$ we then have

$$(12) \quad y_l(t) = \sum_{j=0}^{n-l-1} (-1)^j y_{l+j}(t) I_j(t, u; p_{l+j-1}, \dots, p_l) + \\ + (-1)^{n-l} \int_u^t p_{n-1}(x) f_{n-1}(y_n(h_n(x))) I_{n-l-1}(x, u; p_{n-2}, \dots, p_l) dx \geq \\ \geq \int_u^t p_{n-1}(x) f_{n-1}(y_n(h_n(x))) I_{n-l-1}(x, u; p_{n-2}, \dots, p_l) dx.$$

Substituting (12) into (11), we get

$$\begin{aligned} y_i(t) &\geq \int_{t_0}^t (p_{l-1}(u) I_{l-i-1}(t, u; p_i, \dots, p_{l-2}) \int_u^t p_{n-1}(x) f_{n-1}(y_n(h_n(x))) \times \\ &\quad \times I_{n-l-1}(x, u; p_{n-2}, \dots, p_l) dx) du = \\ &= \int_{t_0}^t H_{i,l-1}(x, t_0) p_{n-1}(x) f_{n-1}(y_n(h_n(x))) dx. \end{aligned}$$

Let $l = n$. Put $t = t_0$, $s = t$ in (7) and use (3) and (4). We get

$$y_i(t) \geq \int_{t_0}^t p_{n-1}(x) I_{n-i-1}(t, x; p_i, \dots, p_{n-2}) f_{n-1}(y_n(h_n(x))) dx \quad \text{for } t \geq t_0.$$

The proof of the lemma is complete.

Let us denote

$$\begin{aligned} \phi_n(t) &= \int_t^\infty p_n(s) ds, \\ J_{k,n}(t, t_0) &= I_{n-1}(t, t_0; p_k, \dots, p_{n-1}), \\ J_{k,l}(t, t_0) &= \int_{t_0}^t H_{k,l-1}(s, t_0) p_{n-1}(s) ds \quad \text{for } l = 1, 2, \dots, n-1. \end{aligned}$$

Theorem 1. *Let there exist a continuous nondecreasing function g on $[a, \infty)$ such that*

$$(13) \quad h_n(t) \leq g(t), \quad g(h_1(t)) \leq t.$$

Let

$$(14) \quad \text{i) } f_n(u \cdot v) \geq K f_n(u) f_n(v) \quad (0 < K = \text{const.});$$

$$\text{ii) } \int_{0^+}^\alpha \frac{dx}{f_n(f_{n-1}(x))} < \infty, \quad \int_{0^-}^{-\alpha} \frac{dx}{f_n(f_{n-1}(x))} < \infty$$

for every constant $\alpha > 0$;

$$(15) \quad \int_{\gamma(T)}^\infty p_n(t) f_n(J_{1,l}(h_1(t), T)) dt = \infty \quad \text{for } l = 2, 3, \dots, n.$$

If n is odd, suppose in addition that for every constant $L > 0$,

$$(16) \quad \int_T^\infty p_{n-1}(t) I_{n-2}(t) f_{n-1}(L \phi_n(h_n(t))) dt = \infty.$$

Then the system (S) has the property A.

Proof. Let $y = (y_1, \dots, y_n) \in W$ be a nonoscillatory solution of (S). Without loss of generality we may suppose that $y_1(t) > 0$, $y_1(h_1(t)) > 0$ for $t \geq t_1 \geq a$. Then the n -th equation of (S) implies that $y_n'(t) \leq 0$ for $t \geq t_1$ and it is not identically zero on any subinterval of $[t_1, \infty)$. Because $y_1(t) > 0$, $y_n'(t) \leq 0$ for $t \geq t_1$, then by Lemma 3, for $t \geq t_2 \geq t_1$ (4), (5) and (8) hold.

I. Let $l \in \{2, 3, \dots, n\}$. For $i = 1$, $t_0 = t_2$, using the monotonicity of y_n, f_{n-1} , (13) and (3), we obtain from (8) that

$$(17) \quad y_1(t) \geq \int_{t_2}^t H_{1,t-1}(s, t_2) p_{n-1}(s) f_{n-1}(y_n(h_n(s))) ds \geq \\ \geq f_{n-1}(y_n(g(t))) J_{1,t}(t, t_2), \quad t \geq t_2.$$

Putting (17) into the n -th equation of (S) and then using (13), (14i), we get

$$(18) \quad y_n'(t) = -p_n(t) f_n(y_1(h_1(t))) \leq \\ \leq -p_n(t) f_n(f_{n-1}(y_n(g(h_1(t)))) J_{1,t}(h_1(t), t_2)) \leq \\ \leq -p_n(t) f_n(f_{n-1}(y_n(t))) J_{1,t}(h_1(t), t_2) \leq \\ \leq -K p_n(t) f_n(f_{n-1}(y_n(t))) f_n(J_{1,t}(h_1(t), t_2))$$

for $t \geq t_3 = \gamma(t_2)$, $l = 2, 3, \dots, n$.

Dividing (18) by $f_n(f_{n-1}(y_n(t)))$ and then integrating from t_3 to $u (\geq t_3)$, we get

$$(19) \quad \int_{t_3}^u \frac{y_n'(t)}{f_n(f_{n-1}(y_n(t)))} dt \leq -K \int_{t_3}^u p_n(t) f_n(J_{1,t}(h_1(t), t_2)) dt.$$

From (19) for $u \rightarrow \infty$ we obtain

$$K \int_{t_3}^{\infty} p_n(t) f_n(J_{1,t}(h_1(t), t_2)) dt \leq \int_0^{y_n(t_3)} \frac{dx}{f_n(f_{n-1}(x))} < \infty,$$

which contradicts (15).

Let $l = 1$ (n is odd). Then $y_1(t) \downarrow k$ as $t \uparrow \infty$, where $k \geq 0$. We suppose that $k > 0$. If we put $i = 1$, $s = t_2$ in (7) and use (5), we have

$$(20) \quad y_1(t_2) \geq \int_{t_2}^t p_{n-1}(x) f_{n-1} y_n(h_n(x)) I_{n-2}(x, t_2; p_{n-2}, \dots, p_1) dx \quad \text{for } t \geq t_2.$$

Integrating the n -th equation of (S) from t to ∞ and using $y_1(t) \geq k$ for $t \geq t_2$, we get

$$y_n(t) \geq f_n(k) \int_t^{\infty} p_n(s) ds = L \phi_n(t), \quad \text{where } L = f_n(k) \neq 0.$$

Then in view of the monotonicity of y_n, f_{n-1} and (13), the inequality (20) yields

$$y_1(t_2) \geq \int_{t_2}^t p_{n-1}(x) I_{n-2}(x, t_2; p_{n-2}, \dots, p_1) f_{n-1}(L \phi_n(h_n(x))) dx,$$

which contradicts (16) for $t \rightarrow \infty$.

Therefore $\lim_{t \rightarrow \infty} y_i(t) = 0$ for $i = 1, 2, \dots, n$.

Remark. Theorem 1 extends the results of the author [4; Theorem 3], Kitamura and Kusano [3; Theorem 6], Ševelo and Varech [7; Theorem 1].

Theorem 2. Suppose that (14), (16) hold and

$$(21) \quad h_n(t) \leq t, \quad h_1(t) \geq t \quad \text{on} \quad [a, \infty).$$

If

$$(22) \quad \int_T^\infty p_n(t) f_n(J_{1,l}(t, T)) dt = \infty \quad \text{for} \quad l = 2, 3, \dots, n,$$

then the system (S) has the property A.

Proof. Let $y = (y_1, \dots, y_n) \in W$ be a nonoscillatory solution of (S) such that $y_1(h_1(t)) > 0$ for $t \geq t_1$. Proceeding in the same way as in the proof of Theorem 1 we get (4), (5), (7) and (8) for $t \geq t_2 \geq t_1$.

I. Let $l \in \{2, 3, \dots, n\}$. For $i = 1$, $t_0 = t_2$, using (21) and the monotonicity of y_n, f_{n-1} , we obtain from (8) that

$$y_1(t) \geq f_{n-1}(y_n(t)) J_{1,l}(t, t_2), \quad t \geq t_2.$$

If we put the last inequality into the n -th equation, we get

$$(23) \quad \begin{aligned} y_n'(t) &\leq -p_n(t) f_n(y_1(t)) \leq \\ &\leq -K p_n(t) f_n(f_{n-1}(y_n(t))) f_n(J_{1,l}(t, t_2)) \quad \text{for} \quad l = 2, 3, \dots, n, \quad t \geq t_2. \end{aligned}$$

Dividing (23) by $f_n(f_{n-1}(y_n(t)))$ and then integrating from t_2 to $\tau \rightarrow \infty$ we get a contradiction to (22).

II. If $l = 1$ (n is odd) we proceed in the same way as in the case II of the proof of Theorem 1.

Theorem 3. Let the system (S) be (α_{n-1}, α_n) superlinear. Let

$$(24) \quad g_1(t) \leq \min \{h_1(t), t\}, \quad h_n(t) \leq t \quad \text{on} \quad [a, \infty),$$

where g_1 is an increasing function on $[a, \infty)$ and $\lim_{t \rightarrow \infty} g_1(t) = \infty$.

Let

$$(25) \quad \int_a^\infty p_n(t) dt < \infty,$$

$$(26) \quad \int_a^\infty J_{2,l}(g_1(t), a) p_1(g_1(t)) g_1'(t) f_{n-1}(K \phi_n(t)) dt = \infty$$

for any constant $K > 0$, $l = 3, 4, \dots, n$.

In addition we suppose that a) for n even,

$$(27) \quad \int_a^\infty p_1(g_1(t)) g_1'(t) \int_t^\infty p_{n-1}(x) f_{n-1}(K \phi_n(x)) I_{n-3}(x, g_1(t); p_{n-2}, \dots, p_2) dx dt = \infty$$

for any $K > 0$;

b) for n odd, (16) holds.

Then the system (S) has the property A.

Proof. Let $y = (y_1, \dots, y_n) \in W$ be a nonoscillatory solution of (S). Proceeding in the same way as in the proof of Theorem 1, we get (4), (5), (7) and (8). We suppose that $y_1(t) > 0$, $y_1(h_1(t)) > 0$ for $t \geq T_1$. Integrating the n -th equation of (S) from $t (\geq T_1)$ to τ , we get

$$y_n(\tau) - y_n(t) = - \int_t^\tau p_n(s) f_n(y_1(h_1(s))) ds,$$

and then for $\tau \rightarrow \infty$ we have

$$(28) \quad y_n(t) \geq \int_t^\infty p_n(s) f_n(y_1(h_1(s))) ds, \quad t \geq T_1.$$

I. Let $l \geq 2$. Then y_1 is nondecreasing and therefore $y_1(h_1(t)) \geq c$ for some $c > 0$ and $t \geq T_2 \geq T_1$. Using the fact that the system (S) is superlinear, we obtain

$$(29) \quad f_n(y_1(h_1(t))) \geq \frac{f_n(c)}{c^{\alpha_n}} (y_1(h_1(t))) = c^{-\alpha_n} f_n(c) (y_1(h_1(t)))^{\alpha_n} \quad \text{for } t \geq T_3 \geq T_2.$$

Combining (29) with (28) we get

$$(30) \quad y_n(t) \geq c^{-\alpha_n} f_n(c) \int_t^\infty p_n(s) (y_1(h_1(s)))^{\alpha_n} ds, \quad t \geq T_3.$$

Because $y_1(h_1(t)) \geq c$ for $t \geq T_2$, (28) implies

$$(31) \quad y_n(g_1(t)) \geq f_n(c) \int_{g_1(t)}^\infty p_n(s) ds = M \phi_n(g_1(t)), \quad \text{where } M = f_n(c).$$

In view of (30), (24) and the monotonicity of y_n we have

$$(32) \quad y_n(g_1(t)) \geq y_n(t) \geq c^{-\alpha_n} M \int_t^\infty p_n(s) (y_1(h_1(s)))^{\alpha_n} ds.$$

Using the superlinearity of f_{n-1} and (31), we get

$$(33) \quad f_{n-1}(y_n(g_1(t))) \geq \frac{f_{n-1}(M \phi_n(t))}{(M \phi_n(t))^{\alpha_{n-1}}} (y_n(g_1(t)))^{\alpha_{n-1}}.$$

a) Let $l \geq 3$. We put $i = 2$, $T_3 = t_0$ in (8) and using the monotonicity of f_{n-1} , y_n and (24), we obtain

$$(34_i) \quad \begin{aligned} y_2(t) &\geq \int_{T_3}^t H_{2,l-1}(s, T_3) p_{n-1}(s) f_{n-1}(y_n(h_n(s))) \geq \\ &\geq f_{n-1}(y_n(t)) J_{2,l}(t, T_3) \quad (l = 3, 4, \dots, n-1), \end{aligned}$$

and

$$(34_n) \quad \begin{aligned} y_2(t) &\geq \int_{T_3}^t I_{n-3}(t, s; p_2, \dots, p_{n-2}) p_{n-1}(s) f_{n-1}(y_n(h_n(s))) ds \geq \\ &\geq f_{n-1}(y_n(t)) J_{2,n}(t, T_3). \end{aligned}$$

Substituting (33) and (32) in (34), we get

$$\begin{aligned} y_2(g_1(t)) &\geq f_{n-1}(y_n(g_1(t))) J_{2,l}(g_1(t), T_3) \geq \\ &\geq \frac{f_{n-1}(M \phi_n(t))}{(M \phi_n(t))^{2n-1}} \left(M c^{-\alpha} \int_t^\infty p_n(s) (y_1(g_1(s)))^{2n} ds \right)^{\alpha n-1} J_{2,l}(g_1(t), T_3) \geq \\ &\geq f_{n-1}(M \phi_n(t)) c^{-\alpha} (y_1(g_1(t)))^\alpha J_{2,l}(g_1(t), T_3), \\ &\text{where } \alpha = \alpha_n \alpha_{n-1} > 1, \quad l = 3, 4, \dots, n. \end{aligned}$$

Multiplying the last inequality by $p_1(g_1(t)) (y_1(g_1(t)))^{-\alpha} g_1'(t)$ and using the first equation of (S), we get

$$(35) \quad \frac{y_1'(g_1(t)) g_1'(t)}{(y_1(g_1(t)))^\alpha} \geq c^{-\alpha} f_{n-1}(M \phi_n(t)) J_{2,l}(g_1(t), T_3) p_1(g_1(t)) g_1'(t).$$

Integrating (35) from $T_4 = \gamma(T_3)$ to τ , we obtain

$$\frac{c^\alpha}{\alpha - 1} [y_1(g_1(T_3))]^{1-\alpha} \geq \int_{T_4}^\tau J_{2,l}(g_1(t), T_3) p_1(g_1(t)) g_1'(t) f_{n-1}(M \phi_n(t)) dt,$$

which contradicts (26) as $\tau \rightarrow \infty$.

Let $l = 2$. We put $i = 2$ in (7) and use (5), obtaining

$$(36) \quad y_2(t) \geq \int_t^\tau p_{n-1}(x) f_{n-1}(y_n(h_n(x))) I_{n-3}(x, t; p_{n-2}, \dots, p_2) dx \quad \text{for } \tau \geq t.$$

Using the superlinearity of f_{n-1} , (24) and (30), we obtain

$$\begin{aligned} y_2(g_1(t)) &\geq \int_{g_1(t)}^\tau p_{n-1}(x) \frac{f_{n-1}(M \phi_n(x))}{(M \phi_n(x))^{2n-1}} (y_n(x))^{2n-1} \\ &\quad \cdot I_{n-3}(x, g_1(t); p_{n-2}, \dots, p_2) dx, \quad t \geq T_3. \end{aligned}$$

Multiplying the last inequality by $p_1(g_1(t)) g_1'(t)$ and using the first equation of (S), (32) and (24), we get

$$\begin{aligned} (37) \quad y_1'(g_1(t)) g_1'(t) &\geq p_1(g_1(t)) g_1'(t) \int_t^\tau p_{n-1}(x) f_{n-1}(M \phi_n(x)) c^{-\alpha} (y_1(g_1(x)))^\alpha \\ &\quad \cdot I_{n-3}(x, g_1(t); p_{n-2}, \dots, p_2) dx \geq \\ &\geq c^{-\alpha} (y_1(g_1(t)))^\alpha p_1(g_1(t)) g_1'(t) \int_t^\tau p_{n-1}(x) f_{n-1}(M \phi_n(x)) \\ &\quad \cdot I_{n-3}(x, g_1(t); p_{n-2}, \dots, p_2) dx, \quad t \geq T_3. \end{aligned}$$

Let $g_1(t) \geq T_3$ for $t \geq T_4$. Multiplying (37) by $c^\alpha (y_1(g_1(t)))^{-\alpha}$ and then integrating from T_4 to u , we get

$$\begin{aligned} \frac{c^\alpha}{\alpha - 1} (y_1(g_1(T_4)))^{1-\alpha} &\geq \int_{T_4}^u (p_1(g_1(t)) g_1'(t)) \int_t^\tau p_{n-1}(x) f_{n-1}(x) f_{n-1}(M \phi_n(x)) \\ &\quad \cdot I_{n-3}(x, g_1(t); p_{n-2}, \dots, p_2) dx dt, \end{aligned}$$

which contradicts (27) as $u \rightarrow \infty, \tau \rightarrow \infty$.

II. Let $l = 1$ (n is odd). Then we proceed in the same way as in the proof of Theorem 1.

This completes the proof of the theorem.

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