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Oscillatory properties of solutions of superlinear-sublinear parabolic equations via Picone-type inequalities

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Abstract. Oscillations of solutions of superlinear-sublinear parabolic equations are studied, and the unboundedness of solutions is also investigated as corollaries. The approach used here is to use the Picone-type inequalities for elliptic operators.

In 1962, McNabb [8] established criteria for unboundedness of solutions of linear parabolic equations on the basis of Picone-type identities. His results were extended by Dunninger [4], Kusano and Narita [7] to parabolic differential inequalities, and by Chan [1], Chan and Young [2, 3], Kuks [6] to time-dependent matrix differential inequalities. All of them also contain the results about zeros of solutions or singularities of matrix solutions.

Recently Jaroš, Kusano and Yoshida [5] established Picone-type inequalities which connect a linear elliptic operator with an associated superlinear-sublinear elliptic operator. By extending the Picone-type inequalities to parabolic equations with time-dependent coefficients, we obtain the oscillatory behavior or the unboundedness of solutions of superlinear-sublinear parabolic equations.

We are concerned with the oscillatory behavior of solutions of the non-

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linear parabolic equation

$$\frac{\partial v}{\partial t} - L[v] = 0, \quad (x,t) \in \Omega \equiv G \times (0,\infty), \tag{1}$$

where G is a bounded domain in $\mathbb{R}^n$ with piecewise smooth boundary ∂G and

$$L[v] \equiv \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(A_{ij}(x,t)\frac{\partial v}{\partial x_j}\right) + C(x,t)|v|^{\beta-1}v + D(x,t)|v|^{\gamma-1}v.$$

It is assumed that :

(A_1) $A_{ij}(x,t) \in C(\overline{\Omega};\mathbb{R})$ $(i,j = 1,2,...,n)$ and the matrix $(A_{ij}(x,t))$ is symmetric and positive definite in Ω ;

(A_2) $C(x,t) \in C(\overline{\Omega};[0,\infty))$ and $D(x,t) \in C(\overline{\Omega};[0,\infty))$;

(A_3) β and γ are constants such that $\beta > 1$ and $0 < \gamma < 1$.

Definition 1. The domain $\mathcal{D}_L(\Omega)$ of L is defined to be the set of all functions v of class $C^1(\overline{\Omega};\mathbb{R})$ with the property that $A_{ij}(x,t)\dfrac{\partial v}{\partial x_j} \in C^1(\Omega;\mathbb{R}) \cap C(\overline{\Omega};\mathbb{R})$.

Definition 2. By a *solution* of (1) we mean a function $v \in \mathcal{D}_L(\Omega)$ which satisfies the equation (1).

Definition 3. A solution v of (1) is said to be *oscillatory* on $\overline{\Omega}$ if v has a zero on $\overline{G} \times [t,\infty)$ for any $t > 0$.

We consider the linear differential operator ℓ defined by

$$\ell[u] = \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial u}{\partial x_j}\right) + c(x)u, \tag{2}$$

where the coefficients $a_{ij}(x)$ and $c(x)$ satisfy the following assumptions :

(A_4) $a_{ij}(x) \in C(\overline{G};\mathbb{R})$ $(i,j = 1,2,...,n)$ and the matrix $(a_{ij}(x))$ is symmetric and positive definite in G ;

(A_5) $c(x) \in C(\overline{G}; \mathbb{R})$.

Definition 4. The domain $\mathcal{D}_\ell(G)$ of ℓ is defined to be the set of all functions u of class $C^1(\overline{G}; \mathbb{R})$ with the property that $a_{ij}(x)\dfrac{\partial u}{\partial x_j} \in C^1(G; \mathbb{R}) \cap C(\overline{G}; \mathbb{R})$.

The following theorem is due to Jaroš, Kusano and Yoshida [5, Theorem 8].

Theorem 1. (Picone-type inequality) *Assume that $u \in \mathcal{D}_\ell(G)$, $v \in \mathcal{D}_L(\Omega)$ and $v \neq 0$ in $G \times I$, where I is any interval in $\mathbb{R}$. Then we obtain the following inequality*

$$\begin{aligned}
&\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(u a_{ij}(x)\frac{\partial u}{\partial x_j} - \frac{u^2}{v}A_{ij}(x,t)\frac{\partial v}{\partial x_j}\right) \\
\geq\ & \sum_{i,j=1}^{n} \left(a_{ij}(x) - A_{ij}(x,t)\right)\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j} \\
&\qquad + \left(\frac{\beta-\gamma}{1-\gamma}\left(\frac{\beta-1}{1-\gamma}\right)^{\frac{1-\beta}{\beta-\gamma}} C(x,t)^{\frac{1-\gamma}{\beta-\gamma}} D(x,t)^{\frac{\beta-1}{\beta-\gamma}} - c(x)\right)u^2 \\
&\qquad + \sum_{i,j=1}^{n} A_{ij}(x,t)\left(v\frac{\partial}{\partial x_i}\left(\frac{u}{v}\right)\right)\left(v\frac{\partial}{\partial x_j}\left(\frac{u}{v}\right)\right) + \frac{u}{v}\left(v\ell[u] - uL[v]\right) \qquad (3)
\end{aligned}$$

for $(x,t) \in G \times I$.

Theorem 2. *Assume that* (A_1) – (A_5) *hold, and that there exists a nontrivial function $u \in \mathcal{D}_\ell(G)$ such that*

$$\begin{aligned}
&\ell[u] = 0 \quad \text{in } G, \\
&u = 0 \quad \text{on } \partial G, \\
&\lim_{t\to\infty} \int_T^t V[u](s)\, ds = \infty \quad \text{for any } T > 0,
\end{aligned}$$

where

$$\begin{aligned}
V[u](t) \equiv \int_G \Bigg[&\sum_{i,j=1}^{n} \left(a_{ij}(x) - A_{ij}(x,t)\right)\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j} \\
&+ \left(\frac{\beta-\gamma}{1-\gamma}\left(\frac{\beta-1}{1-\gamma}\right)^{\frac{1-\beta}{\beta-\gamma}} C(x,t)^{\frac{1-\gamma}{\beta-\gamma}} D(x,t)^{\frac{\beta-1}{\beta-\gamma}} - c(x)\right) u^2\Bigg] dx.
\end{aligned}$$

Let $v \in \mathcal{D}_L(\Omega)$ be a solution of (1) *which is nonoscillatory on $\overline{\Omega}$. Then v satisfies*

$$\lim_{t\to\infty} \int_G u^2 \log |v| \, dx = \infty. \tag{4}$$

Proof. Since $v \in \mathcal{D}_L(\Omega)$ is a nonoscillatory solution on $\overline{\Omega}$, we see that

$$v \neq 0 \quad \text{on } \overline{G} \times [t_0, \infty)$$

for some $t_0 > 0$. Without loss of generality we may assume that $v > 0$ on $\overline{G} \times [t_0, \infty)$. Integrating the Picone-type inequality (3) over G, we obtain

$$\begin{aligned} 0 \;&\geq\; V[u](t) + \int_G \sum_{i,j=1}^{n} A_{ij}(x,t) \left(v \frac{\partial}{\partial x_i} \left(\frac{u}{v} \right) \right) \left(v \frac{\partial}{\partial x_j} \left(\frac{u}{v} \right) \right) dx \\ &\qquad - \int_G \frac{u^2}{v} L[v] \, dx \\ &\geq\; V[u](t) - \int_G \frac{u^2}{v} \frac{\partial v}{\partial t} \, dx, \quad t \geq t_0. \end{aligned} \tag{5}$$

We see from (5) that

$$\int_G u^2 \frac{\partial}{\partial t} \log v \, dx \geq V[u](t), \quad t \geq t_0$$

and therefore

$$\frac{d}{dt} \int_G u^2 \log v \, dx \geq V[u](t), \quad t \geq t_0. \tag{6}$$

Integration of (6) over $[t_0, T]$ yields

$$z(T) - z(t_0) \geq \int_{t_0}^{T} V[u](s) \, ds, \tag{7}$$

where

$$z(t) \equiv \int_G u^2 \log v \, dx.$$

Letting $T \to \infty$ in (7), we observe that $\lim_{T\to\infty} z(T) = \infty$. This completes the proof. □

Corollary 1. *Assume that* (A_1) – (A_5) *hold, and that there exists a nontrivial function* $u \in \mathcal{D}_\ell(G)$ *such that*

$$\begin{aligned}&\ell[u] = 0 \quad \text{in } G,\\ &u = 0 \quad \text{on } \partial G,\\ &\lim_{t\to\infty} \int_T^t V[u](s)\,ds = \infty \quad \text{for any } T > 0.\end{aligned}$$

Then every bounded solution $v \in \mathcal{D}_L(\Omega)$ *of* (1) *is oscillatory on* $\overline{\Omega}$.

Proof. Let $v \in \mathcal{D}_L(\Omega)$ be any bounded solution of (1). Then, we observe that $\log|v|$ is bounded from above, and hence $\int_G u^2 \log|v|\,dx$ is also bounded from above. Therefore (4) does not hold. Theorem 2 implies that the bounded solution v is oscillatory on $\overline{\Omega}$. □

Corollary 2. *Assume that the same hypotheses as those of Theorem* 2 *hold. If* $v \in \mathcal{D}_L(\Omega)$ *is a solution of* (1) *which is nonoscillatory on* $\overline{\Omega}$, *then* v *is unbounded in* Ω.

Proof. Since v is nonoscillatory on $\overline{\Omega}$, v satisfies the condition (4). Hence, $|v|$ cannot be bounded from above in Ω, that is, v is unbounded in Ω. □

The following theorem was established by Jaroš, Kusano and Yoshida [5, Theorem 7].

Theorem 3. *Let* $v \in \mathcal{D}_L(\Omega)$ *and let* $v \neq 0$ *in* $G \times I$, *where* I *is any interval in* $\mathbb{R}$. *Then the following inequality holds for any* $u \in C^1(G;\mathbb{R})$:

$$\begin{aligned}&\sum_{i,j=1}^{n} A_{ij}(x,t)\left(v\frac{\partial}{\partial x_i}\left(\frac{u}{v}\right)\right)\left(v\frac{\partial}{\partial x_j}\left(\frac{u}{v}\right)\right) + \sum_{i,j=1}^{n}\frac{\partial}{\partial x_i}\left(\frac{u^2}{v}A_{ij}(x,t)\frac{\partial v}{\partial x_j}\right)\\ &\quad\le \sum_{i,j=1}^{n} A_{ij}(x,t)\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j} - \frac{\beta-\gamma}{1-\gamma}\left(\frac{\beta-1}{1-\gamma}\right)^{\frac{1-\beta}{\beta-\gamma}} C(x,t)^{\frac{1-\gamma}{\beta-\gamma}} D(x,t)^{\frac{\beta-1}{\beta-\gamma}} u^2\\ &\qquad + \frac{u^2}{v}L[v], \quad (x,t) \in G \times I. \qquad (8)\end{aligned}$$

Theorem 4. *Assume that* (A_1) – (A_3) *hold, and that there exists a nontrivial function* $u \in C^1(\overline{G};\mathbb{R})$ *such that* $u = 0$ *on* ∂G *and*

$$\lim_{t\to\infty} \int_T^t M[u](s)\,ds = -\infty \quad \text{for any } T > 0,$$

where

$$M[u](t) \equiv \int_G \Bigg[\sum_{i,j=1}^{n} A_{ij}(x,t) \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} - \frac{\beta-\gamma}{1-\gamma} \left(\frac{\beta-1}{1-\gamma} \right)^{\frac{1-\beta}{\beta-\gamma}} C(x,t)^{\frac{1-\gamma}{\beta-\gamma}} D(x,t)^{\frac{\beta-1}{\beta-\gamma}} u^2 \Bigg] dx.$$

Let $v \in \mathcal{D}_L(\Omega)$ be a solution of (1) *which is nonoscillatory on $\overline{\Omega}$. Then v satisfies the condition* (4).

Proof. Without loss of generality we may assume that $v > 0$ on $\overline{G} \times [t_0, \infty)$ for some $t_0 > 0$. We integrate the inequality (8) over G to obtain

$$\begin{aligned} 0 &\leq M[u](t) + \int_G \frac{u^2}{v} L[v]\, dx \\ &= M[u](t) + \int_G \frac{u^2}{v} \frac{\partial v}{\partial t}\, dx, \quad t \geq t_0 \end{aligned}$$

or equivalently

$$\int_G u^2 \frac{\partial}{\partial t} \log v\, dx \geq -M[u](t), \quad t \geq t_0.$$

Proceeding as in the proof of Theorem 2, we see that (4) holds. This completes the proof. □

Corollary 3. *Assume that* $(A_1) - (A_3)$ *hold, and that there exists a nontrivial function $u \in C^1(\overline{G}; \mathbb{R})$ such that $u = 0$ on ∂G and*

$$\lim_{t\to\infty} \int_T^t M[u](s)\, ds = -\infty \quad \text{for any } T > 0.$$

Then every bounded solution $v \in \mathcal{D}_L(\Omega)$ of (1) *is oscillatory on $\overline{\Omega}$.*

Proof. The proof follows by using the same arguments as in the proof of Corollary 1. □

Corollary 4. *Assume that the same hypotheses as those of Theorem* 4 *hold. If $v \in \mathcal{D}_L(\Omega)$ is a solution of* (1) *which is nonoscillatory on $\overline{\Omega}$, then v is unbounded in Ω.*

The proof is quite similar to that of Corollary 2, and will be omitted.

Example 1. We consider the parabolic equation

$$\frac{\partial v}{\partial t} - \left(\frac{\partial}{\partial x} \left(A_0 \frac{\partial v}{\partial x} \right) + C_0 v^3 + C_0 v^{1/3} \right) = 0, \tag{9}$$

$$(x,t) \in (-1,1) \times (0,\infty),$$

where A_0 and C_0 are positive constants satisfying $A_0 < (8/5)3^{-(3/4)}C_0$. Here $n = 1$, $A_{11}(x,t) = A_0 > 0$, $C(x,t) = D(x,t) = C_0 > 0$, $\beta = 3$, $\gamma = 1/3$, $G = (-1,1)$ and $\Omega = (-1,1) \times (0,\infty)$. Letting $u = 1 - x^2$, we see that $u(-1) = u(1) = 0$. An easy computation shows that

$$\begin{aligned} M[u](t) &= \int_{-1}^{1} \left[A_0 (u')^2 - 4 \cdot 3^{-(3/4)} C_0 u^2 \right] dx \\ &= \frac{8}{3} \left(A_0 - \frac{8}{5} 3^{-(3/4)} C_0 \right) < 0. \end{aligned}$$

Hence, it is obvious that

$$\lim_{t\to\infty} \int_T^t M[u](s) ds = -\infty$$

for any $T > 0$. It follows from Theorem 4 that every solution v of (9) which is nonoscillatory on $\overline{\Omega}$ satisfies

$$\lim_{t\to\infty} \int_{-1}^{1} \left(1 - x^2\right)^2 \log |v| \, dx = \infty.$$

Example 2. We consider the parabolic equation

$$\frac{\partial v}{\partial t} - \left(\frac{\partial}{\partial x} \left(A_0 \frac{\partial v}{\partial x} \right) + \frac{1}{2} e^{-4t} v^5 + \frac{1}{2} e^{(2/3)t} v^{1/3} \right) = 0, \tag{10}$$

$$(x,t) \in (0,\pi) \times (0,\infty),$$

where A_0 is a positive constant satisfying $A_0 < (7/2)6^{-(6/7)}$. Here $n = 1$, $A_{11}(x,t) = A_0 > 0$, $C(x,t) = (1/2)e^{-4t}$, $D(x,t) = (1/2)e^{(2/3)t}$, $\beta = 5$, $\gamma = 1/3$, $G = (0,\pi)$ and $\Omega = (0,\pi) \times (0,\infty)$. Letting $u = \sin x$, we find that $u(0) = u(\pi) = 0$ and

$$\begin{aligned} M[u](t) &= \int_0^{\pi} \left[A_0 (u')^2 - \frac{7}{2} 6^{-(6/7)} u^2 \right] dx \\ &= \frac{\pi}{2} \left(A_0 - \frac{7}{2} 6^{-(6/7)} \right) < 0. \end{aligned}$$

Hence, it is clear that

$$\lim_{t\to\infty}\int_T^t M[u](s)ds = -\infty$$

for any $T > 0$. Then, Theorem 4 implies that every solution v of (10) which is nonoscillatory on $[0,\pi]\times[0,\infty)$ satisfies

$$\lim_{t\to\infty}\int_0^\pi \left(\sin^2 x\right)\log|v|\,dx = \infty.$$

One such solution is $v = e^t$.

Example 3. We consider the parabolic equation

$$\frac{\partial v}{\partial t} - \left(\frac{\partial^2 v}{\partial x^2} + 5v^3 + 5v^{1/5}\right) = 0, \quad (x,t)\in(0,\pi/2)\times(0,\infty). \tag{11}$$

Here $n = 1$, $A_{11}(x,t) = 1$, $C(x,t) = D(x,t) = 5$, $\beta = 3$, $\gamma = 1/5$, $G = (0,\pi/2)$ and $\Omega = (0,\pi/2)\times(0,\infty)$. We let $u = x\cos x$ and find that $u(0) = u(\pi/2) = 0$. A direct calculation yields

$$\begin{aligned} M[u](t) &= \int_0^{\pi/2}\left[(u')^2 - 7\left(\frac{5}{2}\right)^{(2/7)} u^2\right]dx \\ &= \frac{\pi}{8}\left(\left(\frac{\pi^2}{6}+1\right) - 7\left(\frac{5}{2}\right)^{(2/7)}\left(\frac{\pi^2}{6}-1\right)\right) < 0. \end{aligned}$$

Since

$$\lim_{t\to\infty}\int_T^t M[u](s)ds = -\infty$$

for any $T > 0$, we conclude from Corollary 3 that every bounded solution v of (11) is oscillatory on $\overline{\Omega}$.

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