Ann. Rep. Fac. Educ., Iwate Univ., Vol. 36 (1976)

Oscillatory Property of Second Order Differential Equations

Yoshiyuki HINO*

(Received July 10, 1976)

1. The purpose of this paper is to discuss oscillatory property of solutions of the second order linear differential equation with a constant time lag h, $h > 0$,

$$(E_1) \qquad (r(t)x'(t))' + a(t)x(t-h) = 0,$$

and the second order homogeneous differential equation

$$(E_2) \qquad x''(t) + a(t)f(x(t), x'(t)) = 0,$$

where $a(t)$ is not necessarily nonnegative.

A solution $x(t)$ of (E_i), $i = 1, 2$, which exists in the future is said to be oscillatory if for every $T > 0$ there exists a $t_0 > T$ such that $x(t_0) = 0$. A solution will be nonoscillatory if it is not oscillatory. For any continuous function $\phi(t)$, we define that $\phi^+(t) = \max[\phi(t), 0]$ and $\phi^-(t) = \min[\phi(t), 0]$.

2. In this section, we shall consider system (E_1) where $a(t)$ and $r(t) > 0$ are continuous on I, I $= [0, \infty)$.

Uuder the assumption of $a(t) \geqq 0$, sufficient conditions in order that equation (E_1) is oscillatory have been given by Kung[4]. Recently, without the assumption of $a(t) \geqq 0$, the author[1] has discussed oscillatory property of solutions of the second order differential equation

$$(r(t)x'(t))' + a(t)f(x(t), x(t-h)) = 0,$$

where $f(x, y)$ satisfies the following condition: There exists a $K > 0$ such that $f(x, y) \geqq Kx$ for $x < 0$ and $y < 0$ and $f(x, y) \leqq Kx$ for $x > 0$ and $y > 0$. Clearly, the results in [1] can not be applied to the linear equation (E_1). To make the statements simple, we define that a solution $x(t)$ of (E_1) has Property A if $x(t)$ is nonoscillatory and $\lim_{t\to\infty} x(t) = 0$.

Lemma 1. *Assume that*

(1) $r'(t) \geqq 0$

and

(2) *there exists a* $\tau > 0$ *such that* $a(t) \geqq -M$ *for all* $t \in [\tau, \infty)$ *and for some constant* $M > 0$.

---

*Faculty of Education, Iwate University

Let λ^+ *be the positiwe real root of the equation* $\lambda^2 r(0) = Me^{-\lambda h}$ *and* $x(t)$ *be a nonoscillatory solution of* (E_1) *on* $[\sigma, \infty)$, $\sigma > \tau$.

Then there exists a positive number A *such that*

(3) $|x(t)| \leqq Ae^{\lambda^+ t}$ *for all* $t \in [\sigma, \infty)$.

Proof. Suppose that a solution $x(t)$ of (E_1) is positive on $[\sigma, \infty)$. An analogous argument will hold if $x(t) < 0$. There exists an $A > 0$ such that

(4) $x(t) \leqq Ae^{\lambda^+ t}$ and $x'(t) \leqq A\lambda^+ e^{\lambda^+ t}$ for $t \in [\sigma, \sigma+h]$,

because $x(t)$ and $x'(t)$ are uniformly continuous on $[\sigma, \sigma+h]$ and $\lambda^+ > 0$.
It follows from (1), (2) and (4) that

$$\begin{aligned}
x'(t) &= \left\{-\int_{\sigma+h}^{t} a(s)x(s-h)ds + r(\sigma+h)x'(\sigma+h)\right\}/r(t) \\
&= \left\{-\int_{\sigma+h}^{t} a^+(s)x(s-h)ds - \int_{\sigma+h}^{t} a^-(s)x(s-h)ds + r(\sigma+h)x'(\sigma+h)\right\}/r(t) \\
&\leqq \left\{-\int_{\sigma+h}^{t} a^-(s)x(s-h)\,ds\right\}/r(0) + r(\sigma+h)x'(\sigma+h)/r(t) \\
&\leqq \left\{M\int_{\sigma+h}^{t} x(s-h)ds\right\}/r(0) + A\lambda^+ e^{\lambda^+(\sigma+h)} r(\sigma+h)/r(t) \\
&\leqq \left\{MA\int_{\sigma+h}^{t} e^{\lambda^+(s-h)}ds\right\}/r(0) + A\lambda^+ e^{\lambda^+(\sigma+h)} \\
&\leqq MA\,(e^{\lambda^+(t-h)} - e^{\lambda^+\sigma})/\lambda^+ r(0) + A\lambda^+ e^{\lambda^+(\sigma+h)} \\
&\leqq (Ae^{\lambda^+ t} Me^{-\lambda^+ h})/\lambda^+ r(0) - MAe^{\lambda^+\sigma}/\lambda^+ r(0) + A\lambda^+ e^{\lambda^+(\sigma+h)} \\
&\leqq A\lambda^+ e^{\lambda^+ t} - MAe^{\lambda^+\sigma}\lambda^+/Me^{-\lambda^+ h} + A\lambda^+ e^{\lambda^+(\sigma+h)} \\
&\leqq A\lambda^+ e^{\lambda^+ t} - A\lambda^+ e^{\lambda^+(\sigma+h)} + A\lambda^+ e^{\lambda^+(\sigma+h)} \\
&\leqq A\lambda^+ e^{\lambda^+ t}
\end{aligned}$$

for $t \in [\sigma+h, \sigma+2h]$. Thus we have

(5) $x'(t) \leqq A\lambda^+ e^{\lambda^+ t}$ for $t \in [\sigma+h, \sigma+2h]$.

For $t \in [\sigma+h, \sigma+2h]$, we have by (5)

$$\begin{aligned}
x(t) &= \int_{\sigma+h}^{t} x'(s)ds + x(\sigma+h) \\
&\leqq \int_{\sigma+h}^{t} A\lambda^+ e^{\lambda^+ s}ds + x(\sigma+h) \\
&\leqq A\lambda^+(e^{\lambda^+ t} - e^{\lambda^+(\sigma+h)})/\lambda^+ + Ae^{\lambda^+(\sigma+h)} \\
&\leqq Ae^{\lambda^+ t} - Ae^{\lambda^+(\sigma+h)} + Ae^{\lambda^+(\sigma+h)} \\
&\leqq Ae^{\lambda^+ t}
\end{aligned}$$

Thus, by repeating the same arguments, we have (3).

Lemma 2. *Assume that the conditions* (1) *and* (2) *in Lemma* 1 *hold. Suppose that*

$$\int_0^{\infty} 1/r(s)ds=\infty \tag{6}$$

and

$$\int_{\tau}^{\infty} a^-(s)e^{\lambda^+ s}\, ds > -\infty. \tag{7}$$

Let $x(t)$ *be a nonoscillatory solution of* (E_1) *on* $[\sigma, \infty)$, $\sigma > \tau$. *Then*

$$\int_{\sigma+h}^{\infty} a^-(s)\,|x(s-h)|\,ds > -\infty \tag{8}$$

and

$$\int_{\sigma+h}^{\infty} a^+(s)\,|x(s-h)|\,ds < \infty. \tag{9}$$

Moreover, if there exists a sequence $\{s_n\}$, $s_n \longrightarrow \infty$ *as* $n \longrightarrow \infty$ *such that* $x'(s_n) = 0$, *then* $\lim_{t\to\infty} x'(t) = 0$.

Proof. Suppose that a soution $x(t)$ *of* (E_1) *is nonoscillatory on* $[\sigma, \infty)$. By (3) and (7), we have

$$\int_{\sigma+h}^{\infty} a^-(s)\,|\,x(s-h)\,|\,ds \geqq A\int_{\sigma+h}^{\infty} a^-(s)\, e^{\lambda^+(s-h)}ds$$

$$\geqq Ae^{-\lambda^+ h}\int_{\sigma+h}^{\infty} a^-(s)\, e^{\lambda^+ s}\, ds > -\infty$$

for some positive constant A. Thus (8) is proved.

Now, suppose that a solution $x(t)$ of (E_1) is positive on $[\sigma, \infty)$ and

$$\int_{\sigma+h}^{\infty} a^+(s)x(s-h)ds = \infty.$$

Since

$$r(t)x'(t) - r(\sigma+h)x'(\sigma+h) = -\int_{\sigma+h}^{t} a(s)x(s-h)ds$$

$$= -\int_{\sigma+h}^{t} a^+(s)x(s-h)ds$$

$$-\int_{\sigma+h}^{t} a^-(s)x(s-h)ds,$$

there exist a $\delta > \sigma$ and $L > 0$ such that $r(t)x'(t) < -L$ for all $t \geqq \delta$ by (8), and hence, by (6),

$$x(t) - x(\delta) < -L\int_{\delta}^{t} 1/r(s)ds \longrightarrow -\infty \quad as\ t \longrightarrow \infty.$$

This contradicts $x(t) > 0$ for $t \geqq \delta$. If $x(t) < 0$ on $[\sigma, \infty)$ and

$$\int_{\sigma+h}^{\infty} a^{+}(s)x(s-h)ds = -\infty,$$

then we have a contradiction again. Thus (9) is proved.

Let $\{s_n\}$, $s_n \longrightarrow \infty$ as $n \longrightarrow \infty$, be a sequence such that $x'(s_n) = 0$. Then, for a nonoscillatory solution $x(t)$, we have $\lim_{t\to\infty} x'(t)=0$ by (8) and (9), because

$$|x'(t)| = \{ | \int_{s_n}^{t} a(s)x(s-h)ds|\} /r(t)$$

$$\leqq \{\int_{s_n}^{t} a^{+}(s)|x(s-h)|ds\} /r(0) - \{\int_{s_n}^{t} a^{-}(s)|x(s-h)|ds\}/r(0).$$

Therem 1. *Assume that the conditons* (1) *and* (2) *in Lemma* 1 *and* (6) *and* (7) *in Lemma* 2 *hold. Suppose that there exists an* $\varepsilon > 0$ *such that if* $\{t_n\}$ *is a sequence with* $0 <\cdots < t_n < t_{n+1} <\cdots \longrightarrow \infty$, *then*

$$(10) \qquad \sum_{n=0}^{\infty} \int_{t_n}^{t_n+\varepsilon} a^{+}(s)ds=\infty.$$

Then, every solution of (E_1) *which exists in the future is oscillatory or has Property A.*

Proof. Let $x(t)$ be a solution of (E_1) which is positive on $[\sigma, \infty)$, where σ is sufficiently large. An analogous argument will hold if $x(t)$ is a negative solution.

First, we shall show that $\liminf_{t\to\infty} x(t)=0$. Suppose not, then there exists an $m > 0$ such that $x(t) \geqq m$ for all $t \geqq \sigma$. It follows from (10) that

$$\int_{\sigma+h}^{\infty} a^{+}(s)x(s-h)ds \geqq m\int_{\sigma+h}^{\infty} a^{+}(s)ds = \infty,$$

which contradicts (9) in Lemma 2.

Next, we shall show that $\limsup_{t\to\infty} x(t) = 0$. Suppose not, then there are sequences $\{s_n\}$, $\{t_n\}$ and $\{r_n\}$ and a constant $K > 0$ such that $s_n \longrightarrow \infty$ as $n \longrightarrow \infty$, $x'(s_n) = 0$, $x(t_n) = K/2$, $x(r_n) = K$, $\cdots < r_n < t_{n+1} < r_{n+1} <\cdots$

$\longrightarrow \infty$ and $K/2 \leqq x(t) \leqq K$ for $t_n \leqq t \leqq r_n$. By Lemma 2 there exists an $N > 0$ such that $x'(t) \leqq K/2\varepsilon$ for $t \geqq t_N$, where ε is the one given in (10), which implies that $r_n \leqq t_n + \varepsilon$. Thus we have

(11) $$x(t) \geqq K/2 \quad \text{on } t_n + \varepsilon \geqq t \geqq t_n$$

for n=N, N+1, N+2,…. It follows from (10) and (11) that

$$\int_{t_N}^{\infty} a^+(s)x(s-h)ds \geqq K/2 \times \sum_{n=N}^{\infty} \int_{t_n}^{t_n+\varepsilon} a^+(s)ds \geqq \infty,$$

which contradicts (9) in Lemma 2. This proves Theorem.

Theorem 2. *Assume that the conditions* (1) *and* (2) *in Lemma* 1 *and* (6) *and* (7) *in Lemma* 2 *hold. Suppose that there exists a function* $\phi(t) \varepsilon C^2([\sigma, \infty), R^I)$, $\sigma \geqq 0$, *such that*

(12) $$\phi(t) > 0 \text{ on } [\sigma, \infty),$$

(13) $$-a^-(t)e^{\lambda^+ t} \leqq (r(t)\phi'(t))' \text{ on } [\sigma, \infty)$$

and

(14) $$\int_{\sigma}^{\infty} a^+(s)\phi(s)ds = \infty,$$

where λ^+ *is the one given in Lemma* 1.

Then, for any nonoscillatory solution $x(t)$ *of* (E_1), *there exists a* $B > 0$ *such that* $|x(t)| \leqq B\phi(t)$ *for all large t. Moreover, if* $\phi(t) \longrightarrow 0$ *as* $t \longrightarrow \infty$, *then every solution of* (E_1) *is oscillatory or has Property* A.

Proof. Let $x(t)$ be a solution of (E_1) satisfing $x(t) > 0$ for $t \geqq T$, where we may assume that $T \geqq \sigma$. An analogous argument will hold if $x(t)$ is negative. By Lemma 1, there exists an $A > 0$ such that $x(t-h) \leqq Ae^{\lambda^+(t-h)}$ for $t \geqq T+h$. Put $Ae^{-\lambda^+ h} = B$, then we have

(15) $$(r(t)x'(t))' = -a(t)x(t-h) \leqq -a^-(t)x(t-h) \leqq -Ae^{\lambda^+(t-h)}a^-(s)$$
$$\leqq -Ae^{-\lambda^+ h}a^-(s)\, e^{\lambda^+ t} = -B\, a^-(s)e^{\lambda^+ t} \leqq B(r(t)\phi'(t))'$$

by (13). It follows from (15) that either $x(t) \geqq B\phi(t)$ for all large t or $x(t) \leqq B\phi(t)$ for all large t. However, if $x(t) \geqq B\phi(t)$ for all $t \geqq T_1$, $T_1 \geqq T$, then we have

$$\int_{T_1}^{\infty} a^+(s)x(s-h)ds \geqq B\int_{T_1}^{\infty} a^+(s)\phi(s)ds = \infty$$

by (14), which contradicts (9) in Lemma 2. Thus Theorem is proved.

Eample. Let $r(t) = 1$ on $[0, \infty)$ and $a(t)$ satisfies the following conditions,

$$-a^-(t) \leqq e^{-t^2} \quad \text{on } [0, \infty)$$

and

$$\int_0^\infty a^+(s)e^{-s}\, ds = \infty.$$

Then, clearly, $r(t)$ satisfisfies the condition (1) in Lemma 1 and (6) in Lemma 2. The function $a^-(t)$ satisfies $a^-(t) \geqq -1$ on $[0, \infty)$. Le λ^+ be the positive real root of the equation $\lambda^2 = e^{-\lambda h}$ and put $\phi(t) = e^{-t}$. Then

$$\int_0^\infty a^-(s)e^{\lambda^+ s}ds \geqq -\int_0^\infty e^{s(\lambda^+ - s)}ds > -\int_0^{\lambda^+ + 1} e^{s(\lambda^+ - s)}ds$$

$$-\int_{\lambda^+ + 1}^\infty e^{-s}ds > -\infty$$

and

$$(r(t)\phi'(t))' = e^{-t} = e^{-t(1+\lambda^+)}e^{\lambda^+ t} \geqq -a^-(t)e^{\lambda^+ t}$$

for all large t. Hence $a^-(t)$ satisfies the condition (7) in Lemma 2 and $\phi(t)$ satisfies the conditions (12), (13) and (14) in Theorem 2. Thus every solution $x(t)$ of (E_1) which exists in the future is oscilllatory or has Property A.

3. In this section, we shall consider system (E_2) where $a(t)$ is continuous on I and $f(x, y)$ is continuous and defined on $R \times R$, $R = (-\infty, \infty)$ and satisfies

(16) $$\operatorname{sgn} f(x, y) = \operatorname{sgn} x$$

and

(17) $f(\lambda x, \lambda y) = \lambda^{2p+1} f(x, y)$ for every $(x, y) \in R^2$, $\lambda \in R$ and some nonnegative integer p.

We shall define that a solution of (E_2) has Property B if there exists a $T > 0$ such that $x(t)$ is monotonous for $t \geqq T$ and tends to zero when t goes to infininity.

Under the assumption of $a(t) \geqq 0$, sufficient conditions in order that the equation (E_2) is oscillatory have been given by Kartosatos [3]. Furthermore, for sufficiently small $g(t)$, the author [2] has discussed asymptotic behavior of the system

$$(r(t)x'(t))' + a(t)f(x(t), x'(t)) = g(t),$$

where $f(x, y)$ satisfies conditions (16) and (17) and $r(t) \geqq k^* $ for some $k^* > 0$ and

$$\int_0^\infty 1/r(s)ds = \infty,$$

Lemma 3. *Let $b(t)$ be a continuous function on $[T, \infty)$, $T > 0$.* If

(18) $$\int_T^\infty b^+(s)ds = \infty$$

and

(19) $$\int_T^\infty b^-(s)ds > -\infty,$$

then every solution of the eguation

(20) $$v'(t) + b(t)f(1, v(t)) + v^2(t) = 0$$

is not bounded on $[T, \infty)$.

Proof. Assume that there exists a solution $v(t)$ of (20) which is bounded on $[T, \infty)$. Then there exists two positive constants K and k by (16) such that

(21) $$\max_{T \leqq t < \infty} f(1, v(t)) = \mathrm{K} \text{ and } \min_{T \leqq t < \infty} f(1, v(t)) = \mathrm{k}.$$

Since $v'(t) \lneqq -b(t)f(1, v(t))$ by (20), we have

$$v(t) - v(T) \leqq -\int_T^t b(s)f(1, v(s))ds$$
$$\leqq -\int_T^t b^+(s)f(1, v(s))ds - \int_T^t b^-(s)f(1, v(s))ds$$
$$\leqq -\mathrm{k}\int_T^t b^+(s)ds - \mathrm{K}\int_T^t b^-(s)ds$$

by (21). Then, there arises a contradiction by (18) and (19), because $v(t)$ is bounded on $[T, \infty)$.

Theorem 3. *Assume that $f(x, y)$ satisfies*

(22) $$|y|/f(1, y) \geqq 1 \text{ for all large } |y|.$$

If

(23) $$\int_0^\infty a^+(s)ds = \infty,$$

(24) $$\int_0^\infty a^-(s)ds > -\infty$$

and

(25) $$\liminf_{t\to\infty} a(t) > -\infty,$$

then every bounded solution of (E_2) *is oscillatory or has Property* B.

Proof. Let $x(t)$ be a bounded solution of (E_2) and suppose that $x(t)$ is not oscillatory. Then $x(t)$ is either positive or negative for all large t. Now assume that $0 < x(t) \leqq L$ for some positive number L and for all $t \geqq t_0$, where t_0 is sufficiently large. An analogous argument will hold if $x(t) < 0$, because $f(x(t), x'(t)) = -f(-x(t), -x'(t))$ by (17). Set

(26) $$v(t) = x'(t)/x(t).$$

(i) The case where there exists a sequence $\{t_n\}$, $t_n \to \infty$ as $n \to \infty$ such that $|v(t_n)| \to \infty$ as $n \to \infty$.

Assume that there exists a subsequence $\{t_{n_k}\}$ of $\{t_n\}$ satiafying $v'(t_{n_k}) \geqq 0$. Since

(27) $$\begin{aligned} v'(t) &= \{x''(t)x(t) - (x'(t))^2\}/x^2(t) \\ &= -a(t)f(x(t), x'(t))/x(t) - (x'(t))^2/(x(t))^2 \\ &= -a(t)x^{2p}(t)f(1, v(t)) - v^2(t) \end{aligned}$$

for $t \geqq t_0$, it follows from (22) and (27) that

$$-La^-(t_{n_k}) \geqq -a^-(t_{n_k})\, x^{2p}(t_{n_k}) \geqq v^2(t_{n_k})/f(1, v(t_{n_k})) \geqq |v(t_{n_k})|$$

for all large k, which contradicts by (25).

Hence there exists a $T_1 > t_0$ such that $v'(t) < 0$ for all $t \geqq T_1$ and $v(t) \to -\infty$ as $t \to \infty$, which implies that $x'(t) < 0$ by (26) and there exists an $M > 0$ and a $T > T_1$ such that

(28) $$x'(t)/x(t) \leqq -M$$

for all $t \geqq T$. Integrating (28) from T to t, we have $x(t) \leqq x(T)e^{-M(t-T)}$, and hence $x(t)$ has Property B.

(ii) The case where $v(t)$ is bounded on $[t_0, \infty)$. Then there exist two positive constants K and k satisfing (21). We setting

(29) $$U(t) = v(t)/x^{2p}(t),$$

it follows from (27) that

$$\begin{aligned} U'(t) &= \{v'(t)\, x^{2p}(t) - 2px^{2p-1}(t)x'(t)v(t)\}/x^{4p}(t) \\ &= \{-a(t)x^{4p}(t)f(1, v(t)) - v^2(t)x^{2p}(t) - 2px^{2p}(t)v^2(t)\}/x^{4p}(t) \\ &\leqq -a(t)f(1, v(t)) \\ &\leqq -a^+(t)f(1, v(t)) - a^-(t)f(1, v(t)) \\ &\leqq -ka^+(t) - Ka^-(t) \end{aligned}$$

for $t \geqq t_0$. Hence we have

(30) $$U(t) - U(t_0) \leqq -k\int_{t_0}^{t} a^+(s)ds - K\int_{t_0}^{t} a^-(s)ds.$$

By (23), (24), (26), (29) and (30), there exists a $T > t_0$ such that $x'(t) < 0$ for all $t \geqq T$. Therefore $x(t)$ is monotone decreasing for $t \geqq T$ and there exists an $m \geqq 0$ such that $x(t) \to m$ as $t \to \infty$. Assume that m is positive. Then, since $m \leqq x(t) \leqq L$ for all $t \geqq T$, $a(t)x^{2p}(t)$ satisfies the conditions (18) and (19) in Lemma 3. Hence the equation (27) has not abounded solution by Lemma 3, which contradicts to the boundedness of $v(t)$. Thus $x(t)$ has Property B.

As an example of a function satisfying the conditions (16), (17) and (22), we have $f(x, y) = (x^{2p+1} + x^{2p+1/3} y^{2/3})$, and hence we can apply Theoren 3 to the equation

$$x''(t) + a(t)(x^{2p+1}(t) + x^{2p+1/3}(t)(x'(t))^{2/3}) = 0,$$

if $a(t)$ satisfies the required conditions.

When $p = 0$ in (17), we do not need to assume the condition (22) in Theorem 3, because we have $y^2/f(1, y) \geqq |y|$ for all large $|y|$ by (16) and (17), and we do not also need to consider only bounded solutions in Theorem 3, because $v(t)$ given in (26) is a solution of the equation $v'(t) + a(t)f(1, v(t)) + v^2(t) = 0$. Hence we have the following theorem by the parallel argument to the proof of Theorem 3.

Theorem 4. *Let $p = 0$. Assume that* (23), (24) *and* (25) *in Theorem* 3 *hold good. Then every solution which exists in the future is oscillatory or has Property B.*

References

[1] Y. Hino, On oscillation of the solution of second order functional differential equations, Funkcial. Ekvac., 17(1974), 95 - 105.

[2] Y. Hino, Asymptotic property of nonoscillatory solution of second order differential equations, Funkcial. Ekvac., 18(1975), 219 - 226.

[3] A. G. Kartosatos, Some theorems on oscillations of certain non-linear second order ordinary differential equations, Arch. Math., 18(1969), 425 - 429.

[4] G. C. T. Kung, Oscillation and nonoscillation of differential equations with a time lag, SIAM. J. Appl. Math., 21(1971), 207 - 213.