OUTER GENERALIZED INVERSES IN RINGS

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Abstract

In this paper we formulate basic results on outer generalized inverses of elements in rings. We characterize elements which have the same idempotents related to their particular outer generalized inverses and investigate positive generalized inverses in C^*-algebras.

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1 Introduction

Let $\mathcal{R}$ denote an arbitrary ring and let $a \in \mathcal{R}$. We say that $b \in \mathcal{R}$ is an *outer generalized inverse* of a provided that $bab = b \neq 0$ holds. For such an a we say that it is *outer regular*. In this case ab and ba are idempotents, so called idempotents of a corresponding to its outer generalized inverse b. In general, an arbitrary $a \in R$ need not to be outer regular, even in the case when $\mathcal{R}$ is a Banach algebra [3]. On the other hand, it is a consequence of the Hahn-Banach theorem that every non-zero bounded linear operator on a Banach space is outer regular. The detailed treatment of outer generalized inverses of operators on Banach and Hilbert spaces can be found in [1], [13] and [14].

We say that $c \in \mathcal{R}$ is an *inner generalized inverse* of $a \in \mathcal{R}$, if $aca = a$ holds. In this case a is called *inner regular* (or *relatively regular*). If c is both inner and outer generalized inverse of a, then c is a reflexive generalized inverse of a.

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If c is an inner generalized inverse of a, then cac is a reflexive generalized inverse of a. Hence, inner regularity of a implies the outer regularity of a. Also, if b is an outer generalized inverse of a, then a is an inner generalized inverse of b.

The *Drazin inverse* of $a \in \mathcal{R}$ (see [7]) is the unique $a^D \in \mathcal{R}$ (in the case when it exists) such that the following hold:

$$a^D aa^D = a^D,\ aa^D = a^D a,\ [a(aa^D - 1)]^n = 0,$$

for some non-negative integer n. The least such n is called the *Drazin index* of a, denoted by $\operatorname{ind}(a)$. Obviously, $\operatorname{ind}(a) = 0$ if and only if a is invertible and in this case $a^D = a^{-1}$. If $\operatorname{ind}(a) \leq 1$, then a^D is known as the *group inverse*, denoted by $a^{\#}$.

If there exists a^D, then a is called *Drazin invertible*.

We use $\mathcal{A}$ to denote a complex Banach or C^*-algebra with the unit 1.

In the case when $\mathcal{A}$ is a complex Banach algebra with the unit 1, the element a^d is the *generalized Drazin inverse*, or *Koliha-Drazin inverse* of $a \in \mathcal{A}$ (see [10]), provided that the following hold:

$$a^d aa^d = a^d,\ aa^d = a^d a,\ [a(aa^d - 1)] \text{ is quasinilpotent.}$$

Quasinilpotent elements in a ring $\mathcal{R}$ are defined in the following way (see [8]). For $a \in \mathcal{R}$ consider the following sets of commutators

$$comm(a) = \{b \in \mathcal{R} : ab = ba\}, \quad comm^2(a) = \{b \in \mathcal{R} : bc = cb \text{ for all } c \in comm(a)\}.$$

An element $a \in \mathcal{R}$ is quasinilpotent, if $1 + xa$ is invertible for all $x \in comm(a)$.

Hence, an element $b \in \mathcal{R}$ is the generalized Drazin inverse of $a \in \mathcal{R}$, if the following is satisfied:

$$b \in comm^2(a), \quad bab = b \quad a(ab - 1) \text{ is quasinilpotent.}$$

Here b must double commute with a to ensure its uniqueness. In Banach algebras it is enough to assume simple commutativity. Such an b is denoted by a^d.

If a^d exists, we say that a is *generalized Drazin invertible*.

An element $a \in \mathcal{R}$ has the generalized Drazin inverse, if and only if there exists the element $p = p^2 \in \mathcal{R}$, satisfying the conditions:

$$p \in comm^2(a),\ ap \text{ is quasinilpotent and } a + p \text{ is invertible.}$$

The element p is called the *spectral idempotent* of a, denoted by a^π.

If $\mathcal{A}$ is a complex Banach algebra with the unit 1, then $a \in \mathcal{A}$ is generalized Drazin invertible if and only if 0 is not the point of accumulation of the spectrum of a. In this case a^π is the spectral idempotent of a corresponding to the set $\{0\}$.

If $\mathcal{R}$ is a ring with involution, then the *Moore-Penrose inverse* of $a \in \mathcal{R}$ (see [16]) is the unique $a^\dagger \in \mathcal{R}$ (in the case when it exists), such that the following hold:

$$aa^\dagger a = a,\ a^\dagger aa^\dagger = a^\dagger,\ (aa^\dagger)^* = aa^\dagger,\ (a^\dagger a)^* = a^\dagger a.$$

If $a^\dagger$ exists, then a is called *Moore-Penrose invertible.*

In the case when $a \in \mathcal{A}$, where $\mathcal{A}$ is a C^*-algebra, the element $a^\dagger$ exists if and only if there exists some $c \in \mathcal{A}$ such that $aca = a$ ([9] and [12]).

The Drazin and the Moore-Penrose inverse are particular outer generalized inverses. For the detailed treatment of the Drazin and Moore-Penrose inverse in Banach and C^*-algebras see [11]. Notice that the Moore-Penrose and the group inverse are reflexive generalized inverses.

In this paper we investigate generalized inverses with prescribed idempotents. In Section 2 we we find necessary and sufficient conditions for $a, p = p^2, q = q^2 \in \mathcal{R}$, such that there exists an outer generalized inverse $b \in \mathcal{R}$ of a, satisfying $ba = p$ and $1 - ab = q$. In Section 3 selfadjoint and positive generalized inverses in C^*-algebras are investigated. A result from [15] is extended to a more general setting. In Section 4 we consider two different elements $a, c \in \mathcal{R}$ and their generalized inverses $a_1, c_1 \in \mathcal{R}$, such that the corresponding idempotents are equal: $aa_1 = cc_1$ and $a_1 a = c_1 c$. Thus, some recent results from [4], [5], [6], [12], [17], [18] and [19] are extended.

2 Outer generalized inverses with prescribed idempotents

Let X and Y be Banach spaces and let $\mathcal{L}(X,Y)$ denote the set of all bounded operators from X to Y. For $A \in \mathcal{L}(X,Y)$ let $B \in \mathcal{L}(Y,X)$ be its outer generalized inverse, $\mathcal{R}(B) = T$ and $\mathcal{N}(B) = S$. Then we usually write $B = A^{(2)}_{T,S}$. If T and S are given subspaces of X and Y, then it is easy to verify that $A^{(2)}_{T,S}$ exists if and only if the following hold: T and S, respectively, are closed and complemented subspaces of X and Y, the reduction $A_T : T \to A(T)$ is invertible and $A(T) \oplus S = Y$. In this case $A^{(2)}_{T,S}$ is unique. The analogous result for elements of a ring is Theorem 2.1

We use $\mathcal{R}^\bullet$ to denote the set of all idempotents of $\mathcal{R}$.

Definition 2.1 Let $a \in R$ and $p, q \in \mathcal{R}^\bullet$. An element $b \in \mathcal{R}$ satisfying

$$bab = b, \quad ba = p, \quad 1 - ab = q,$$

will be called a (p,q)-*outer generalized inverse* of a, written $a^{(2)}_{p,q} = b$. (The uniqueness of $a^{(2)}_{p,q}$ is provided in the following theorem.)

Theorem 2.1 *Let $a \in \mathcal{R}$ and $p, q \in \mathcal{R}^\bullet$. Then the following statements are equivalent:*

(1) *$a^{(2)}_{p,q}$ exists;*

(2) *$(1-q)a = (1-q)ap$ and there exists some $b \in \mathcal{R}$ such that $pb = b$, $bq = 0$, $bap = p$ and $ab = 1 - q$.*

Moreover, if $a^{(2)}_{p,q}$ exists, then it is unique.

Proof. (1)⇒(2): If $a^{(2)}_{p,q}$ exists, then we can take $b = a^{(2)}_{p,q}$.

(2)⇒(1): We compute in the following way:

$$ba = pb(1-q)a = pb(1-q)ap = bap = p \quad \text{and} \quad bab = pb = p.$$

To prove the uniqueness, suppose that there exist two outer generalized inverses $b, c \in \mathcal{R}$ of a, such that $ba = ca = p$, $ab = ac = 1-q$. Then we have

$$c = cac = cab = b. \ \square$$

If $a^{(2)}_{p,q}$ satisfies $a = aa^{(2)}_{p,q}a$, then $a^{(2)}_{p,q} = a^{(1,2)}_{p,q}$ is called a (p,q)-reflexive generalized inverse of a. It follows that $a^{(1,2)}_{p,q}$ is also unique in the case when it exists.

If $a \in \mathcal{R}$ is generalized Drazin invertible and $p = a^\pi$ is the spectral idempotent of a, then $a^d = a^{(2)}_{1-p,p}$.

If $a \in \mathcal{R}$ is Moore-Penrose invertible, $p = a^\dagger a$ and $q = 1 - aa^\dagger$, then $a^\dagger = a^{(2)}_{p,q}$.

(p,q)-outer generalized inverses are close related to a direct sum of a ring involving principal ideals and annihilators. For $u \in \mathcal{R}$, the right annihilator of u is defined as follows: $u^\circ = \{x \in \mathcal{R} : ux = 0\}$.

Let $a, p = p^2, q = q^2 \in \mathcal{R}$ and $b = a^{(2)}_{p,q}$. If $x \in \mathcal{R}$ is arbitrary, then $x = qx + (1-q)x$. Obviously, $bqx = b(1-ab)x = 0$ and $qx \in b^\circ$. Also, $q(1-q)x = 0$ and $(1-q)x \in q^\circ$. On the other hand, if $y \in b^\circ \cap q^\circ$, then $by = 0$ and $(1-ab)y = 0$, implying $y = 0$. Hence, $\mathcal{R} = b^\circ \oplus q^\circ$. Analogously, the decomposition $\mathcal{R} = b\mathcal{R} \oplus p^\circ$ can be proved.

In the following theorem we give a generalization of the well-known result for the Moore-Penrose inverse of a bounded linear operator with a closed range on Hilbert spaces (see, for example, [2]).

Theorem 2.2 *Let $a, c \in \mathcal{A}$ and $p, q \in \mathcal{A}^\bullet$ such that there exist $a^{(2)}_{p,q}$ and $c^{(1,2)}_{1-q,1-p}$. Then*

$$ind\,(ac) \le 1 \quad \text{and} \quad a^{(2)}_{p,q} = c(ac)^\# = cc^{(1,2)}_{1-q,1-p}a^{(2)}_{p,q}.$$

Proof. For the convenience we write $a' = a^{(2)}_{p,q}$ and $c' = c^{(1,2)}_{1-q,1-p}$.

First we prove that $c'a' = (ac)^\#$. Since $a'a = p$, $aa' = 1-q$, $c'c = 1-q$ and $cc' = p$, we get the following:

$$c'a'ac = c'pc = c'cc'c - p = aa' = apa' = acc'a',$$

$$acc'a'ac = apc = acc'c = ac \quad \text{and} \quad c'a'acc'a' = c'pa'c'a'aa' = c'a'.$$

Hence, $\text{ind}\,(ac) \le 1$ and $(ac)^\# = c'a'$.

Finally, we have

$$c(ac)^\# = cc'a' = pa' = a'aa' = a'. \ \square$$

3 Positive generalized inverses and applications

Let H be a Hilbert space and let $\mathcal{L}(H)$ denote the set of all bounded operators on H. For $A \in \mathcal{L}(H)$ we use $\mathcal{R}(A)$ and $\mathcal{N}(A)$, respectively, to denote the range and the kernel of A. It is well-known that $A^\dagger$ exists if and only if $\mathcal{R}(A)$ is closed. Moreover, if A is selfadjoint, then $A^\dagger$ is selfadjoint also and $A^\dagger = A^\#$.

For outer generalized inverses the following result can be proved.

Theorem 3.1 *Let $\mathcal{R}$ be a ring with involution, $a = a^* \in \mathcal{R}$ and $p, q \in \mathcal{R}^\bullet$, such that $a_{p,q}^{(2)}$ exists and $p + q = 1$. Then the following statements are equivalent:*

(1) $a_{p,q}^{(2)}$ is selfadjoint;
(2) $p = p^*$ and $q = q^*$.

Proof. Let $a = a^*$ and $a' = a_{p,q}^{(2)}$.

(1)⇒(2): Suppose that $a' = (a')^*$. Then

$$p = a'a = (aa')^* = (1-q)^* = p^* \quad \text{and} \quad q = (1-aa') = (1-a'a)^* = (1-p)^* = q^*.$$

(2)⇒(1): Suppose that $p = p^*$ and $q = q^*$. Obviously, $(a')^*$ is an outer generalized inverse of a. We have

$$(a')^* a = (aa')^* = (1-q)^* = p = a'a \quad \text{and} \quad a(a')^* = (a'a)^* = 1 - q = aa'.$$

Since $a' = a_{p,q}^{(2)}$ is unique, it follows that $a' = (a')^*$. $\square$

For Hilbert space operators Theorem 3.1 can be expressed as follows.

Theorem 3.2 *Let H be a Hilbert space, let $A \in \mathcal{L}(H)$ be selfadjoint and let T and S be subspaces of H such that $A_{T,S}^{(2)}$ exists. Then $A_{T,S}^{(2)}$ is selfadjoint if and only if $T = S^\perp$.*

We prove the following result, concerning the positivity of a and $a_{p,q}^{(2)}$ in a C^*-algebra. Let $(\cdot,\cdot)$ denote the inner product in a Hilbert space.

Theorem 3.3 *Let $\mathcal{A}$ be a complex C^*-algebra with the unit 1. Let $a \in \mathcal{A}$ and $p, q \in \mathcal{A}^\bullet$ such that $p + q = 1$ and $a_{p,q}^{(2)}$ exist. Then the following statements are equivalent:*

(1) $a_{p,q}^{(2)} \geq 0$;
(2) $p = p^*$ and $q = q^*$.

Proof. It is enough to prove the implication (2)⇒(1). Let $J : \mathcal{A} \to \mathcal{B} \subset \mathcal{L}(\mathcal{H})$ denote the Gelfand-Naimark-Segal isometric $*$-isomorphism, where $\mathcal{B}$ is a C^*-subalbegra of $\mathcal{L}(H)$ for some Hilbert space H and $J(1) = I$ is the identity operator on H. Denote $a' = a_{p,q,}^{(2)}$. Since $a \geq 0$, it follows that $J(a) \geq 0$ in $\mathcal{B}$ and also in $\mathcal{L}(H)$. By

Theorem 3.1 it follows that $J(a')$ is selfadjoint. Let $T = \mathcal{R}(J(p))$ and $S = \mathcal{R}(J(q))$. Obviously, $T = S^{\perp}$ and, by the uniqueness of the outer generalized inverse with prescribed range and kernel, $J(a') = (J(a))^{(2)}_{T,S}$ and the last operator is selfadjoint. For all $x \in H$ we get

$$((J(a))^{(2)}_{T,S}x, x) = ((J(a))^{(2)}_{T,S}J(a)(J(a))^{(2)}_{T,S}x, x) = (J(a)(J(a))^{(2)}_{T,S}x, (J(a))^{(2)}_{T,S}x) \geq 0.$$

Hence, $(J(a))^{(2)}_{T,S} \geq 0$ in $\mathcal{L}(H)$. Since the spectrum of $(J(a))^{(2)}_{T,S}$ in $\mathcal{B}$ coincides with its spectrum in $\mathcal{L}(H)$, it follows that $J(a') = (J(a))^{(2)}_{T,S} \geq 0$ in $\mathcal{B}$. It follows that $a' \geq 0$ in $\mathcal{A}$. □

As an application, we extend the result of Ogawa [15].

Theorem 3.4 *Let $\mathcal{A}$ be a C^*-algebra with the unit 1, let $a \in \mathcal{A}$ and $p, q \in \mathcal{A}^\bullet$ be such that $a^{(1,2)}_{p,q}$ exists. If $a \geq 0$ and $a^{(1,2)}_{p,q} \geq 0$, then the following statements are equivalent:*

(1) $(a + bb^*)^{(1,2)}_{p,q} = a^{(1,2)}_{p,q} - a^{(1,2)}_{p,q} b(1 + b^* a^{(1,2)}_{p,q} b)^{-1} b^* a^{(1,2)}_{p,q}$;

(2) $aa^{(1,2)}_{p,q} b = b$.

Proof. Let $0 \leq a^{(1,2)}_{p,q} = a' = t^*t$ for some $t \in \mathcal{A}$. Then $b^*a'b = (tb)^*tb \geq 0$ and $1+b^*a'b$ is invertible. Hence, $y = a' - a'b(1+b^*a'b)^{-1}b^*a'$ exists. Denote $x = a+bb^*$ and we have

$$\begin{aligned} xy &= aa' + [b(1 + b^*a'b) - aa'b - bb^*a'b](1 + b^*a'b)^{-1}b^*a' \\ &= aa' + (b - aa'b)(1 + b^*a'b)^{-1}b^*a'. \end{aligned}$$

and

$$\begin{aligned} yx &= a' + a'b(1 + b^*a'b)^{-1}[(1 + b^*a'b)b^* - b^*a'a - b^*a'bb^*] \\ &= a'a + a'b(1 + b^*a'b)^{-1}(b^* - b^*a'a). \end{aligned}$$

Hence, $yxy = y$ and

$$\begin{aligned} xyx &= a + aa'bb^* + (b - aa'b)(1 + b^*a'b)^{-1}(b^*a'a - b^* + b^* + b^*a'bb^*) \\ &= a + bb^* + (b - aa'b)(1 + b^*a'b)^{-1}(b^*a'a - b^*). \end{aligned}$$

(2)⇒(1): If $aa'b = b$, then obviously $y = x^{(1,2)}_{p,q}$.

(1)⇒(2): Suppose that $y = x^{(1,2)}_{p,q}$. Then

$$\begin{aligned} 0 &= (b - aa'b)(1 + b^*a'b)^{-1}(b^*a'a - b^*) \\ &= (b - aa'b)(1 + b^*a'b)^{-1/2}[(b - aa'b)(1 + b^*a'b)^{-1/2}]^*. \end{aligned}$$

Hence, $(b - aa'b)(1 + b^*a'b)^{-1/2} = 0$ and $aa'b = b$. □

As a corollary, we get the following result.

Corollary 3.1 *Assume that the conditions from Theorem 3.4 are satisfied and let $c \in \mathcal{A}$ be positive and invertible. Then the following statements are equivalent:*

(1) $(a+bcb^*)^{(1,2)}_{p,q} = a^{(1,2)}_{p,q} - a^{(1,2)}_{p,q} b(c^{-1} + b^* a^{(1,2)}_{p,q} b)^{-1} b^* a^{(1,2)}_{p,q}$;

(2) $aa^{(1,2)}_{p,q} b = b$.

Proof. We have to take $b_1 = bc^{1/2}$ and then replace b in Theorem 4.2 by b_1. □

4 Elements with equal idempotents related to outer generalized inverses

In this section we characterize elements with equal idempotents related to their outer generalized inverses. Notice that in [4] Castro-González, Koliha and Wei characterized matrices with the same eigenprojections, i.e. the same projections corresponding to the Drazin inverses of these matrices. They extended these results to closed operators on Banach spaces in [5]. Results of this type are used to prove error bounds for perturbations of operators with the same eigenprojections. Finally, in [12] Koliha and Patricio proved analogous results for Drazin invertible elements of a ring.

We need the following auxiliary result.

Lemma 4.1 *Let $c, s \in \mathcal{R}$ satisfy $cs = sc$ and $s \in \mathcal{R}^\bullet$. Then c is invertible in $\mathcal{R}$ is and only if cs is invertible in $s\mathcal{R}s$ and $c(1-s)$ is invertible in $(1-s)\mathcal{R}(1-s)$. In this case*

$$c^{-1} = [cs]^{-1}_{s\mathcal{R}s} + [c(1-s)]^{-1}_{(1-s)\mathcal{R}(1-s)}.$$

Now we prove the main result of this section.

Theorem 4.1 *Let $a \in \mathcal{R}$ and let $p, q \in \mathcal{R}^\bullet$ be such that $a^{(2)}_{p,q}$ exists. Then for $b \in \mathcal{R}$ the following statements are equivalent.*

(a) *There exists the generalized inverse $b^{(2)}_{p,q} \in \mathcal{R}$*

(b) $ba^{(2)}_{p,q}a = aa^{(2)}_{p,q}b$ *and there exists the generalized inverse* $(ba^{(2)}_{p,q}a)^{(2)}_{p,q}$.

(c) $ba^{(2)}_{p,q}a = aa^{(2)}_{p,q}b$ *and* $1 + a^{(2)}_{p,q}(b-a)$ *is invertible.*

(d) $ba^{(2)}_{p,q}a = aa^{(2)}_{p,q}b$ *and* $1 + (b-a)a^{(2)}_{p,q}$ *is invertible.*

Moreover, if previous statements are valid, then

$$b^{(2)}_{p,q} = [1 + a^{(2)}_{p,q}(b-a)]^{-1} a^{(2)}_{p,q} = a^{(2)}_{p,q}[1 + (b-a)a^{(2)}_{p,q}]^{-1}.$$

Proof. (a)⇒(b): We immediately obtain $ba^{(2)}_{p,q}a = bb^{(2)}_{p,q}b = aa^{(2)}_{p,q}b$. We also have

$$b^{(2)}_{p,q}(ba^{(2)}_{p,q}a)b^{(2)}_{p,q} = b^{(2)}_{p,q}(bb^{(2)}_{p,q}b)b^{(2)}_{p,q} = b^{(2)}_{p,q}.$$

From

$$b_{p,q}^{(2)}(ba_{p,q}^{(2)}a) = b_{p,q}^{(2)}bb_{p,q}^{(2)}b = b_{p,q}^{(2)}b = p$$

and

$$(ba_{p,q}^{(2)}a)b_{p,q}^{(2)} = bb_{p,q}^{(2)}bb_{p,q}^{(2)} = bb_{p,q}^{(2)} = 1-q$$

we conclude that the equality $(ba_{p,q}^{(2)}a)_{p,q}^{(2)} = b_{p,q}^{(2)}$ holds.

(b)⇒(a): Denote $c = (ba_{p,q}^{(2)}a)_{p,q}^{(2)}$. We have $cba_{p,q}^{(2)}a = p = a_{p,q}^{(2)}a$ and $ba_{p,q}^{(2)}ac = 1-q = aa_{p,q}^{(2)} = aa_{p,q}^{(2)}bc$. Hence, we obtain

$$\begin{aligned} cbc &= c(ba_{p,q}^{(2)}ac)bc(ba_{p,q}^{(2)}ac) \\ &= caa_{p,q}^{(2)}(ba_{p,q}^{(2)}ac) = caa_{p,q}^{(2)} \\ &= caa_{p,q}^{(2)}bc = c. \end{aligned}$$

Also, we have

$$cb = cba_{p,q}^{(2)}acb = caa_{p,q}^{(2)}b = cba_{p,q}^{(2)}a = p$$

and

$$bc = bcba_{p,q}^{(2)}ac = ba_{p,q}^{(2)}ac = 1-q.$$

It follows that $b_{p,q}^{(2)} = (ba_{p,q}^{(2)}a)_{p,q}^{(2)}$.

[(a) and (b)] ⇒(c): We compute

$$\begin{aligned} (1+a_{p,q}^{(2)}b - a_{p,q}^{(2)}a)(b_{p,q}^{(2)}a + 1 - a_{p,q}^{(2)}a) &= b_{p,q}^{(2)}a + 1 - a_{p,q}^{(2)}a + a_{p,q}^{(2)}bb_{p,q}^{(2)}a \\ &\quad + a_{p,q}^{(2)}b - a_{p,q}^{(2)}ba_{p,q}^{(2)}a - a_{p,q}^{(2)}ab_{p,q}^{(2)}a - a_{p,q}^{(2)}a + a_{p,q}^{(2)}aa_{p,q}^{(2)}a \\ &= 1. \end{aligned}$$

Analogously, $(b_{p,q}^{(2)}a + 1 - a_{p,q}^{(2)}a)(1 + a_{p,q}^{(2)}b - a_{p,q}^{(2)}a) = 1$ and consequently $(1 + a_{p,q}^{(2)}(b-a))^{-1} = b_{p,q}^{(2)}a + 1 - a_{p,q}^{(2)}a$.

(c)⇒(a): Notice that we have

$$c = 1 + a_{p,q}^{(2)}(b-a) = 1 - a_{p,q}^{(2)}a + a_{p,q}^{(2)}a[a_{p,q}^{(2)}b]a_{p,q}^{(2)}a.$$

From Lemma 4.1 we know that $a_{p,q}^{(2)}a[a_{p,q}^{(2)}b]a_{p,q}^{(2)}a = a_{p,q}^{(2)}b$ is invertible in $a_{p,q}^{(2)}a\mathcal{R}a_{p,q}^{(2)}a = \mathcal{R}_1$. Denote

$$d = c^{-1}a_{p,q}^{(2)} = \left(1 - a_{p,q}^{(2)}a + [a_{p,q}^{(2)}a(a_{p,q}^{(2)}b)a_{p,q}^{(2)}a]_{\mathcal{R}_1}^{-1}\right)a_{p,q}^{(2)} = [a_{p,q}^{(2)}a(a_{p,q}^{(2)}b)a_{p,q}^{(2)}a]_{\mathcal{R}_1}^{-1}a_{p,q}^{(2)}.$$

We prove $d = b_{p,q}^{(2)}$. Notice that

$$dbd = [a_{p,q}^{(2)}b]_{\mathcal{R}_1}^{-1}a_{p,q}^{(2)}b[a_{p,q}^{(2)}a(a_{p,q}^{(2)}b)a_{p,q}^{(2)}a]_{\mathcal{R}_1}^{-1}a_{p,q}^{(2)} = d.$$

Also,

$$db = a_{p,q}^{(2)}a$$

and

$$bd = b[a^{(2)}_{p,q}b]^{-1}_{\mathcal{R}_1}a^{(2)}_{p,q} = aa^{(2)}_{p,q}b[a^{(2)}_{p,q}b]^{-1}_{\mathcal{R}_1}a^{(2)}_{p,q} = aa^{(2)}_{p,q}.$$

Finally, if (a), (b) and (c) be satisfied, form the part (c)⇒(a) it follows that

$$(1 + a^{(2)}_{p,q}(b-a))^{-1}a^{(2)}_{p,q} = b^{(2)}_{p,q}.$$

The proof of all cases involving the part (d) are similar. □

If $a^{(2)}_{p,q}$ and $b^{(2)}_{p,q}$ exist, then $a = b-u = b-(b-a)$ is called the (p,q)-splitting of a (see [6], [17], [18] and [19] for the notion of $\{T,S\}$-splitting of an operator, induced by its outer generalized inverse with prescribed range and kernel). The generalized condition number of a is defined as: $\kappa_{p,q}(a) = \|a\|\|a^{(2)}_{p,q}\|$.

We are able to prove the following result.

Theorem 4.2 *Let $a,b \in \mathcal{R}$ and let $p,q \in \mathcal{R}^\bullet$ be such that $a^{(2)}_{p,q}$ and $b^{(2)}_{p,q}$ exist. Then the following hold:*

(a) $a^{(2)}_{p,q} - b^{(2)}_{p,q} = b^{(2)}_{p,q}(b-a)a^{(2)}_{p,q} = a^{(2)}_{p,q}(b-a)b^{(2)}_{p,q}$.

(b) *If $\mathcal{R}$ is a Banach algebra and $\|a^{(2)}_{p,q}\|\|(b-a)\| < 1$, then*

$$\frac{\|a^{(2)}_{p,q}(b-a)\|}{\kappa_{p,q}(a)(1+\|a^{(2)}_{p,q}\|\|b-a\|)} \le \frac{\|b^{(2)}_{p,q} - a^{(2)}_{p,q}\|}{\|a^{(2)}_{p,q}\|} \le \frac{\|a^{(2)}_{p,q}(b-a)\|}{1-\|a^{(2)}_{p,q}(b-a)\|} \le \frac{\kappa_{p,q}(a)\|b-a\|/\|a\|}{1-\kappa_{p,q}(a)\|b-a\|/\|a\|}.$$

(c) *If $\mathcal{R}$ is a normed algebra, then*

$$\frac{\|a^{(2)}_{p,q}\|}{1+\|q^{(2)}_{p,q}(b-a)\|} \le \|b^{(2)}_{p,q}\| \le \frac{\|a^{(2)}_{p,q}\|}{1-\|a^{(2)}_{p,q}(b-a)\|}.$$

Proof. The proof of (a) is elementary.

To prove (b), notice that from Theorem 4.1 we have the following:

$$\begin{aligned} b^{(2)}_{p,q} - a^{(2)}_{p,q} &= (1+a^{(2)}_{p,q}(b-a))^{-1}a^{(2)}_{p,q} - a^{(2)}_{p,q} \\ &= \left(\sum_{k=0}^{\infty}(-1)^k(a^{(2)}_{p,q}(b-a))^k - 1\right)a^{(2)}_{p,q} \\ &= \sum_{k=1}^{\infty}(-1)^k(a^{(2)}_{p,q}(b-a))^k a^{(2)}_{p,q}. \end{aligned}$$

Thus, we obtain the second and the third inequality of (b).

To prove the first inequality of (b), we compute:

$$
\begin{aligned}
a_{p,q}^{(2)}(b-a) &= a_{p,q}^{(2)}b - a_{p,q}^{(2)}a \\
&= a_{p,q}^{(2)}b - b_{p,q}^{(2)}b = (a_{p,q}^{(2)} - b_{p,q}^{(2)})b = a_{p,q}^{(2)}(b-a)b_{p,q}^{(2)}b \\
&= a_{p,q}^{(2)}(b-a)a_{p,q}^{(2)}a = a_{p,q}^{(2)}(b-a)a_{p,q}^{(2)}(1+(b-a)a_{p,q}^{(2)})^{-1}(1+(b-a)a_{p,q}^{(2)})a \\
&= a_{p,q}^{(2)}(b-a)b_{p,q}^{(2)}(1+(b-a)a_{p,q}^{(2)})a \\
&= (a_{p,q}^{(2)} - b_{p,q}^{(2)})(1+(b-a)a_{p,q}^{(2)})a
\end{aligned}
$$

The first inequality of (b) follows immediately.

(c) This part is elementary. $\Box$

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