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Output Feedback Control of a Class of Nonlinear Systems: A Nonseparation Principle Paradigm

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***Abstract*—This note considers the problem of global stabilization by output feedback, for a family of nonlinear systems that are dominated by a triangular system satisfying linear growth condition. The problem has remained unsolved due to the violation of the commonly assumed conditions in the literature. Using a feedback domination design method which is *not based on the separation principle*, we explicitly construct a linear output compensator making the closed-loop system globally exponentially stable.**

***Index Terms*—Global robust stabilization, linear growth condition, nonlinear systems, nonseparation principle design, output feedback.**

I. Introduction and Discussion

One of the important problems in the field of nonlinear control is global stabilization by output feedback. Unlike in the case of linear systems, global stabilizability by state feedback plus observability do not imply global stabilizability by output feedback, and therefore, the so-called *separation principle* usually does not hold for nonlinear systems. Perhaps for this reason, the problem is exceptionally challenging and much more difficult than the global stabilization by state feedback. Over the years, several papers have investigated global stabilization of nonlinear systems using output feedback and obtained some interesting results. For example, for a class of detectable bilinear systems [4] or affine and nonaffine systems with *stable-free dynamics* [9], [10], global stabilization via output feedback was proved to be solvable using the input saturation technique [9], [10]. In [7], a necessary and sufficient condition was given for a nonlinear system to be equivalent to an observable linear system perturbed by a vector field that depends only on the output and input of the system. As a consequence, global stabilization by output feedback is achievable for a class of nonlinear systems that are diffeomorphic to a system in the nonlinear observer form [7], [8], and [15].

In [12], counterexamples were given indicating that global stabilization of minimum-phase nonlinear systems via output feedback is usually impossible, without introducing extra growth conditions on the unmeasurable states of the system. Since then, much subsequent research work has been focused on the output feedback stabilization of nonlinear systems under various *structural or growth conditions*. One of the common assumptions is that nonlinear systems should be in an output feedback form [11] or a triangular form with certain growth conditions [5], [3], [14], [1], [13]. The other condition is that the system can nonlinearly depend on the output of the system but is *linear in the unmeasurable states* [1]. The latter was relaxed recently in [13] by only imposing the global Lipschitz-like condition on the unmeasurable states.

In this note, we consider a class of single-input–single-output (SISO) time-varying systems

$$\begin{aligned}\dot{x}_1 &= x_2 + \phi_1(t, x, u)\\ \dot{x}_2 &= x_3 + \phi_2(t, x, u)\\ &\vdots\\ \dot{x}_n &= u + \phi_n(t, x, u)\\ y &= x_1\end{aligned} \tag{1.1}$$

where $x = (x_1, \ldots, x_n)^T \in \mathbb{R}^n$, $u \in \mathbb{R}$ and $y \in \mathbb{R}$ are the system state, input, and output, respectively. The mappings $\phi_i : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$, $i = 1, \ldots, n$, are *continuous* and satisfy the following condition.

Assumption 1.1: For $i = 1, \ldots, n$, there is a constant $c \geq 0$ such that

$$|\phi_i(t, x, u)| \leq c(|x_1| + \cdots + |x_i|). \tag{1.2}$$

Under this hypothesis, it has been shown in [14] that global exponential stabilization of nonlinear systems (1.1) is possible using *linear state* feedback. The objective of this note is to prove that the same growth condition, namely Assumption 1.1, guarantees the existence of a *linear output* dynamic compensator

$$\begin{aligned}\dot{\xi} &= M\xi + Ny, \qquad M \in \mathbb{R}^{n\times n};\ N \in \mathbb{R}^n\\ u &= K\xi, \qquad\qquad K \in \mathbb{R}^{1\times n}\end{aligned} \tag{1.3}$$

such that the closed-loop system (1.1)–(1.3) is globally exponentially stable (GES) at the equilibrium $(x, \xi) = (0, 0)$.

It must be pointed out that systems (1.1) satisfying Assumption 1.1 represent an important class of nonlinear systems that cannot be dealt with by existing output feedback control schemes such as those reported in [14], [11], [1], and [13]. To make this point clearer, in what follows we examine three seemingly simple but nontrivial examples. The first example is a planar system of the form

$$\begin{aligned}\dot{x}_1 &= x_2 + \frac{\ln(1 + u^2 x_1^2 x_2^2)}{1 + u^2 x_2^2}\\ \dot{x}_2 &= u + x_2(1 - \cos(x_2 u))\\ y &= x_1\end{aligned} \tag{1.4}$$

which obviously satisfies Assumption 1.1. However, it is not in an output feedback form (see, e.g., [11]) nor satisfies the structural or growth conditions in [14], [1], and [13]. Therefore, global stabilization of the planar system (1.4) by output feedback appears to be open and unsolvable by existing design methods. Notably, due to a nontriangular structure, uniformly observability of nontriangular systems like (1.4) cannot be guaranteed in general. In particular, the lack of a lower-triangular structure makes the conventional observer-controller based output feedback design not feasible.

Manuscript received August 21, 2001; revised March 12, 2002. Recommended by Associate Editor J. M. A. Scherpen. This work was supported in part by the National Science Foundation under Grants ECS-9875273, ECS-9906218, DMS-9972045, and DMS-0203387.

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Digital Object Identifier 10.1109/TAC.2002.803542.

The next example illustrates that a nonlinear system (1.1) with Assumption 1.1, e.g.,

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= u + x_2 \sin x_2 \\ y &= x_1 \end{aligned} \tag{1.5}$$

may fail to satisfy the global Lipschitz-like condition given in [13]. Consequently, global stabilization of (1.5) via output feedback cannot be solved by the approach of [13]. In fact, it is easy to verify that $x_2 \sin x_2$ is *not global Lipschitz* with respect to the unmeasurable state x_2, although Assumption 1.1 holds. For this type of nonlinear systems, most of the existing results on output feedback stabilization are not applicable and a "Luenberger-type" observer, which consists of a copy of (1.5) plus an error correction term, does not seem to work because convergence of the error dynamics is hard to prove.

Finally, in the case when the system under consideration involves parametric uncertainty, the problem of output feedback stabilization becomes even more challenging. Few results are available in the literature dealing with nonlinear systems with uncertainty that is associated with the unmeasurable states. For instance, consider the uncertain system

$$\begin{aligned} \dot{x}_1 &= x_2 + d_1(t)x_1 \\ \dot{x}_2 &= u + d_2(t)\ln(1 + x_2^4)\sin x_2 \\ y &= x_1 \end{aligned} \tag{1.6}$$

which satisfies Assumption 1.1, where $|d_i(t)| \leq 1$, $i = 1, 2$, are *unknown* continuous functions with known bounds (equal to one in the present case). When $d_2(t) \equiv 0$, global stabilization of the uncertain system (1.6) can be easily solved using output feedback. However, when $d_2(t) \neq 0$, all the existing methods cannot be used because the presence of $d_2(t)$ makes the design of a nonlinear observer extremely difficult.

The examples discussed thus far have indicated that nonlinear systems (1.1) with Assumption 1.1 cover a class of nonlinear systems whose global stabilization by output feedback does not seem to be solvable by any existing design method, and therefore is worth of investigation. The main contribution of the note is the development of a feedback domination design approach that enables one to explicitly construct a *linear* dynamic output compensator (1.3), globally exponentially stabilizing the entire family of nonlinear systems (1.1) under the growth condition (1.2). It must be pointed out that our output feedback control scheme is not based on the separation principle. That is, instead of constructing the observer and controller separately, we couple the high-gain linear observer design together with the controller construction. An obvious advantage of our design method is that the precise knowledge of the nonlinearities or uncertainties of the systems needs not to be known. What really needed is the information of the bounding function of the uncertainties, i.e., the constant c in (1.2). This feature makes it possible to stabilize a family of nonlinear systems using a *single* output feedback compensator. In other words, the proposed output feedback controller has a "universal" property. In the case of cascade systems, our design method can deal with an entire family of finite-dimensional minimum-phase nonlinear systems whose dimensions of the zero-dynamics are unknown.

II. Output Feedback Design

In [14], it was proved that a class of nonlinear systems satisfying Assumption 1.1 is globally exponentially stabilizable by *linear state* feedback. When dealing with the problem of global stabilization via output feedback, *stronger conditions* such as lower-triangular structure, differentiability of the vector field $\phi(t, x, u) = (\phi_1(\cdot), \ldots, \phi_1(\cdot))^T$ and the global Lipschitz condition were assumed [14].

In this section, we prove that Assumption 1.1 suffices to guarantee the existence of a globally stabilizing output feedback controller. This is done by using a feedback domination design which explicitly constructs a *linear output* feedback control law. In contrast to the *nonlinear* output feedback controller obtained in [14], the dynamic output compensator we propose is *linear* with a simple structure (1.3).

Theorem 2.1: Under Assumption 1.1, there exists a linear output feedback controller (1.3) making the uncertain nonlinear system (1.1) globally exponentially stable.

Proof: The proof consists of two parts. First of all, we design a *linear* high-gain observer motivated by [2], [5], without using the information of the system nonlinearities, i.e., $\phi_i(t, x, u)$, $i = 1, \ldots, n$. This results in an error dynamics containing some extra terms that prevent convergence of the high-gain observer. We then construct an output controller based on a feedback domination design to take care of the extra terms arising from the observer design. This is accomplished by choosing, step-by-step, the gain parameters of the observer and the virtual controllers in a delicate manner. At the last step, a linear output dynamic compensator can be obtained, making the closed-loop system globally exponentially stable.

Part I—Design of a Linear High-Gain Observer

We begin with by designing the following linear observer

$$\begin{aligned} \dot{\hat{x}}_1 &= \hat{x}_2 + La_1(x_1 - \hat{x}_1) \\ &\vdots \\ \dot{\hat{x}}_{n-1} &= \hat{x}_n + L^{n-1}a_{n-1}(x_1 - \hat{x}_1) \\ \dot{\hat{x}}_n &= u + L^n a_n(x_1 - \hat{x}_1) \end{aligned} \tag{2.1}$$

where $L \geq 1$ is a gain parameter to be determined later, and $a_j > 0$ and $j = 1, \ldots, n$, are coefficients of the Hurwitz polynomial $p(s) = s^n + a_1 s^{n-1} + \cdots + a_{n-1}s + a_n$.

Define $\varepsilon_i = (x_i - \hat{x}_i)/L^{i-1}$, $i = 1, \ldots, n$. A simple calculation gives

$$\dot{\varepsilon} = LA\varepsilon + \begin{bmatrix} \phi_1(t, x, u) \\ \dfrac{1}{L}\phi_2(t, x, u) \\ \vdots \\ \dfrac{1}{L^{n-1}}\phi_n(t, x, u) \end{bmatrix} \tag{2.2}$$

where

$$\varepsilon = \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \vdots \\ \varepsilon_n \end{bmatrix} \quad A = \begin{bmatrix} -a_1 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -a_{n-1} & 0 & \cdots & 1 \\ -a_n & 0 & \cdots & 0 \end{bmatrix}.$$

Clearly, A is a Hurwitz matrix. Therefore, there is a positive–definite matrix $P = P^T > 0$ such that

$$A^T P + PA = -I.$$

Consider the Lyapunov function $V_0(\varepsilon) = (n+1)\varepsilon^T P\varepsilon$. By Assumption 1.1, there is a real constant $c_1 > 0$, which is independent of L, such that

$$\dot V_0(\varepsilon) = -(n+1)L\|\varepsilon\|^2 + 2(n+1)\varepsilon^T P \begin{bmatrix} \phi_1(t,x,u) \\ \frac{\phi_2(t,x,u)}{L} \\ \vdots \\ \frac{\phi_n(t,x,u)}{L^{n-1}} \end{bmatrix}$$
$$\le -(n+1)L\|\varepsilon\|^2 + c_1\|\varepsilon\| \left(|x_1| + \frac{1}{L}|x_2| + \cdots + \frac{1}{L^{n-1}}|x_n|\right).$$

Recall that $x_i = \hat x_i + L^{i-1}\varepsilon_i$. Hence

$$\left|\frac{1}{L^{i-1}}x_i\right| \le \left|\frac{1}{L^{i-1}}\hat x_i\right| + |\varepsilon_i|, \qquad i=1,\ldots,n.$$

With this in mind, it is not difficult to deduce that

$$\begin{aligned}\dot V_0(\varepsilon) \le& -\left((n+1)L - c_1\sqrt{n}\right)\|\varepsilon\|^2 \\ &+ c_1\|\varepsilon\|\left(|\hat x_1| + \frac{1}{L}|\hat x_2| + \cdots + \frac{1}{L^{n-1}}|\hat x_n|\right) \\ \le& -\left((n+1)L - c_1\sqrt{n} - \frac{n}{2}c_1\right)\|\varepsilon\|^2 \\ &+ c_1\left(\frac{1}{2}\hat x_1^2 + \frac{1}{2L^2}\hat x_2^2 + \cdots + \frac{1}{2L^{2n-2}}\hat x_n^2\right). \end{aligned} \tag{2.3}$$

Part II—Construction of an Output Feedback Controller

Initial Step: Construct the Lyapunov function $V_1(\varepsilon, \hat x_1) = V_0(\varepsilon) + (\hat x_1^2/2)$. A direct calculation gives

$$\begin{aligned}\dot V_1 \le& -\left((n+1)L - \left(\sqrt{n} + \frac{n}{2}\right)c_1\right)\|\varepsilon\|^2 \\ &+ c_1\left(\frac{1}{2}\hat x_1^2 + \frac{1}{2L^2}\hat x_2^2 + \cdots + \frac{1}{2L^{2n-2}}\hat x_n^2\right) \\ &+ \hat x_1(\hat x_2 + La_1\varepsilon_1) \\ \le& -\left(nL - \left(\sqrt{n} + \frac{n}{2}\right)c_1\right)\|\varepsilon\|^2 \\ &+ c_1\left(\frac{1}{2L^2}\hat x_2^2 + \cdots + \frac{1}{2L^{2n-2}}\hat x_n^2\right) \\ &+ \hat x_1\hat x_2 + \frac{1}{4}La_1^2\hat x_1^2 + \frac{c_1}{2}\hat x_1^2.\end{aligned}$$

Let $L \ge c_1$ and define $\xi_2 = \hat x_2 - \hat x_2^*$ with $\hat x_2^*$ being a virtual control.

Observe that

$$\frac{c_1}{2}\hat x_1^2 \le \frac{L}{2}\hat x_1^2, \qquad c_1\frac{1}{2L^2}\hat x_2^2 \le c_1\frac{1}{L^2}\xi_2^2 + c_1\frac{1}{L^2}\hat x_2^{*2}.$$

With this in mind, we have

$$\begin{aligned}\dot V_1(\varepsilon, \hat x_1) \le& -\left(nL - \left(\sqrt{n} + \frac{n}{2}\right)c_1\right)\|\varepsilon\|^2 \\ &+ c_1\left(\frac{1}{2L^4}\hat x_3^2 + \cdots + \frac{1}{2L^{2n-2}}\hat x_n^2\right) + c_1\frac{1}{L^2}\xi_2^2 \\ &+ c_1\frac{1}{L^2}\hat x_2^{*2} + \hat x_1\xi_2 + \hat x_1\hat x_2^* + L\left(\frac{1}{4}a_1^2 + \frac{1}{2}\right)\hat x_1^2.\end{aligned} \tag{2.4}$$

Choosing the virtual controller

$$\hat x_2^* = -Lb_1\hat x_1, \qquad b_1 := n + \tfrac{1}{4}a_1^2 + \tfrac{1}{2} > 0$$

results in

$$\begin{aligned}\dot V_1 \le& -\left(nL - \left(\sqrt{n} + \frac{n}{2}\right)c_1\right)\|\varepsilon\|^2 - \left(nL - c_1b_1^2\right)\hat x_1^2 \\ &+ c_1\left(\frac{1}{2L^4}\hat x_3^2 + \cdots + \frac{1}{2L^{2n-2}}\hat x_n^2\right) + \frac{c_1}{L^2}\xi_2^2 + \hat x_1\xi_2.\end{aligned} \tag{2.5}$$

Inductive Step: Suppose at step k, there exist a smooth Lyapunov function $V_k(\varepsilon, \xi_1, \ldots, \xi_k)$ which is positive definite and proper, and a set of virtual controllers $\hat x_1^*, \ldots, \hat x_{k+1}^*$, defined by $x_1^* = 0$, $\xi_1 = \hat x_1 - x_1^*$ and

$$\hat x_i^* = -Lb_{i-1}\xi_{i-1} \quad \xi_i = \hat x_i - \hat x_i^*, \qquad i = 2, \ldots, k+1$$

with $b_i > 0$ being *independent of* the gain constant L, such that

$$\begin{aligned}\dot V_k \le& -\left((n+1-k)L - \left(\sqrt{n} + \frac{n}{2}\right)c_1\right)\|\varepsilon\|^2 \\ &- \sum_{j=1}^{k}\frac{1}{L^{2j-2}}\left((n+1-k)L - c_1b_j^2\right)\xi_j^2 \\ &+ c_1\left(\frac{1}{2L^{2k+2}}\hat x_{k+2}^2 + \cdots + \frac{1}{2L^{2n-2}}\hat x_n^2\right) \\ &+ \frac{c_1}{L^{2k}}\xi_{k+1}^2 + \frac{1}{L^{2(k-1)}}\xi_k\xi_{k+1}.\end{aligned} \tag{2.6}$$

Now, consider the Lyapunov function

$$\begin{aligned}&V_{k+1}(\varepsilon, \xi_1, \ldots, \xi_{k+1}) \\ &\quad = V_k(\varepsilon, \xi_1, \ldots, \xi_k) + \frac{1}{2L^{2k}}\xi_{k+1}^2, \qquad \xi_{k+1} = \hat x_{k+1} - \hat x_{k+1}^*.\end{aligned}$$

Observe that

$$\xi_k = \hat x_k + Lb_{k-1}\hat x_{k-1} + L^2b_{k-1}b_{k-2}\hat x_{k-2} + \cdots + L^{k-1}b_{k-1}b_{k-2}\cdots b_1\hat x_1.$$

Then, it is straightforward to show that

$$\begin{aligned}&\frac{d}{dt}\left(\frac{1}{2L^{2k}}\xi_{k+1}^2\right) \\ &= \frac{1}{L^{2k}}\xi_{k+1}\left(\hat x_{k+2} + L^{k+1}a_{k+1}\varepsilon_1 + Lb_k\sum_{i=1}^{k}\frac{\partial \xi_k}{\partial \hat x_i}\left(\hat x_{i+1} + L^ia_i\varepsilon_1\right)\right) \\ &= \frac{1}{L^{2k}}\xi_{k+1}\left(\hat x_{k+2} + L^{k+1}a_{k+1}\varepsilon_1 + \sum_{i=1}^{k}L^{k-i+1}b_k\cdots b_i \cdot \left(\xi_{i+1} - Lb_i\xi_i + L^ia_i\varepsilon_1\right)\right) \\ &= \frac{1}{L^{2k}}\xi_{k+1}\left(\hat x_{k+2} + L^{k+1}d_0\varepsilon_1 + L^{k+1}d_1\xi_1 + L^kd_2\xi_2 + \cdots + Ld_{k+1}\xi_{k+1}\right)\end{aligned} \tag{2.7}$$

where $d_0, \ldots, d_{k+1}$, are suitable real numbers that are *independent of* the gain constant L, and $d_{k+1} > 0$.

Putting (2.6) and (2.7) together, we have

$$\begin{aligned}\dot V_{k+1} \le& -\left((n+1-k)L - \left(\sqrt{n} + \frac{n}{2}\right)c_1\right)\|\varepsilon\|^2 \\ &- \sum_{j=1}^{k}\frac{1}{L^{2j-2}}\left((n+1-k)L - c_1b_j^2\right)\xi_j^2 \\ &+ c_1\left(\frac{1}{2L^{2k+4}}\hat x_{k+3}^2 + \cdots + \frac{1}{2L^{2n-2}}\hat x_n^2\right) \\ &+ \frac{c_1}{L^{2k+2}}\xi_{k+2}^2 + \frac{c_1}{L^{2k+2}}\hat x_{k+2}^{*2} + \frac{1}{L^{2k}}\xi_{k+1}\xi_{k+2} \\ &+ \frac{1}{L^{2k}}\xi_{k+1}\hat x_{k+2}^* + \xi_{k+1}\left(\frac{d_0}{L^{k-1}}\varepsilon_1 + \frac{d_1}{L^{k-1}}\xi_1\right. \\ &+ \frac{d_2}{L^{k-2}}\xi_2 + \cdots + \frac{d_{k-1}}{L^{2k-3}}\xi_{k-1} + \frac{d_k+1}{L^{2k-2}}\xi_k \\ &\left.+ \frac{d_{k+1} + \frac{c_1}{L}}{L^{2k-1}}\xi_{k+1}\right)\end{aligned}$$

$$\begin{aligned}&\le -\left((n-k)L-\left(\sqrt{n}+\frac{n}{2}\right)c_1\right)\|\varepsilon\|^2\\&\quad-\left((n-k)L-c_1b_1^2\right)\xi_1^2-\cdots-\frac{1}{L^{2k-2}}\\&\quad\cdot\left((n-k)L-c_1b_k^2\right)\xi_k^2\\&\quad+c_1\left(\frac{1}{2L^{2k+4}}\hat{x}_{k+3}^2+\cdots+\frac{1}{2L^{2n-2}}\hat{x}_n^2\right)\\&\quad+\frac{c_1}{L^{2k+2}}\xi_{k+2}^2+\frac{c_1}{L^{2k+2}}\hat{x}_{k+2}^{*2}+\frac{1}{L^{2k}}\xi_{k+1}\xi_{k+2}\\&\quad+\frac{1}{L^{2k}}\xi_{k+1}\hat{x}_{k+2}^*+\xi_{k+1}^2\frac{1}{L^{2k-1}}\\&\quad\cdot\left(\frac{d_0^2}{4}+\frac{d_1^2}{4}+\cdots+\frac{d_{k-1}^2}{4}+\frac{(d_k+1)^2}{4}+d_{k+1}+1\right).\end{aligned}\tag{2.8}$$

From the previous inequality, it follows that the linear controller

$$\hat{x}_{k+2}^*=-Lb_{k+1}\xi_{k+1}$$

with

$$b_{k+1}=n-k+\frac{d_0^2}{4}+\frac{d_1^2}{4}+\cdots+\frac{d_{k-1}^2}{4}+\frac{(d_k+1)^2}{4}+d_{k+1}+1>0$$

being independent of L, renders

$$\begin{aligned}\dot{V}_{k+1}\le&-\left((n-k)L-\left(\sqrt{n}+\frac{n}{2}\right)c_1\right)\|\varepsilon\|^2\\&-\left((n-k)L-c_1b_1^2\right)\xi_1^2-\cdots-\frac{1}{L^{2k}}\\&\cdot\left((n-k)L-c_1b_{k+1}^2\right)\xi_{k+1}^2\\&+c_1\left(\frac{1}{2L^{2k+4}}\hat{x}_{k+3}^2+\cdots+\frac{1}{2L^{2n-2}}\hat{x}_n^2\right)\\&+\frac{c_1}{L^{2k+2}}\xi_{k+2}^2+\frac{1}{L^{2k}}\xi_{k+1}\xi_{k+2}.\end{aligned}\tag{2.9}$$

This completes the inductive argument.

Using the inductive argument step by step,[1] at the nth step one can design the linear controller

$$\begin{aligned}u&=-Lb_n\xi_n\\&=-Lb_n(\hat{x}_n+Lb_{n-1}(\hat{x}_{n-1}+\cdots+Lb_2(\hat{x}_2+Lb_1\hat{x}_1)\cdots)\end{aligned}\tag{2.10}$$

with $b_i>0, i=1,\ldots,n$ being real numbers independent of the gain parameter L, such that

$$\begin{aligned}\dot{V}_n\le&-\left(L-\left(\sqrt{n}+\frac{n}{2}\right)c_1\right)\|\varepsilon\|^2-\left(L-c_1b_1^2\right)\xi_1^2\\&-\cdots-\frac{1}{L^{2n-4}}\left(L-c_1b_{n-1}^2\right)\xi_{n-1}^2-\frac{1}{L^{2n-2}}L\xi_n^2\end{aligned}\tag{2.11}$$

where V_n is a positive–definite and proper function defined by

$$V_n(\varepsilon,\xi_1,\ldots,\xi_n)=V_0(\varepsilon)+\sum_{i=1}^{n}\frac{1}{2L^{2(i-1)}}\xi_i^2.$$

If we choose the gain constant $L>L^*:=\max\{1,(\sqrt{n}+(n/2))c_1,c_1b_1^2,\ldots,c_1b_{n-1}^2\}$, the right-hand side of (2.11) becomes negative definite. Therefore, the closed-loop system is globally exponentially stable. ∎

Remark 2.2: The novelty of Theorem 2.1 is two-fold: on one hand, in contrast to the common observer design that usually uses a copy of (1.1), we design only a *linear* observer (2.1) for the uncertain nonlinear system (1.1). Such a construction, under Assumption 1.1, avoids dealing with difficult issues caused by the uncertainties or nonlinearities of the system. On the other hand, the gain parameter L of the observer (2.1) is designed in such a way that it depends on the parameters of the controller [i.e., b_i, $i=1,\ldots,n$, in (2.10)]. In fact, the observer and controller designs in Theorem 2.1 are heavily coupled with each other. This is substantially different from most of the existing work where the designed observer itself can asymptotically recover the state of the controlled plant, regardless of the design of the controller, i.e., the controller design is independent of the observer design—known as the separation principle.

[1]At the last step, the design of the controller u is slightly different from that of inductive argument, because all the junk terms (e.g., $\hat{x}_j^2$, $1\le j\le n$) in (2.6) have already been canceled at Step $n-1$.

Theorem 2.1 has an interesting consequence on output feedback stabilization of a family of time-varying nonlinear systems in the form

$$\begin{aligned}\dot{x}&=\begin{bmatrix}a_{1,1}(t,y)&1&\cdots&0\\\vdots&\vdots&\ddots&\vdots\\a_{n-1,1}(t,y)&a_{n-1,2}(t,y)&\cdots&1\\a_{n,1}(t,y)&a_{n,2}(t,y)&\cdots&a_{n,n}(t,y)\end{bmatrix}x+\begin{bmatrix}0\\\vdots\\0\\1\end{bmatrix}u\\y&=x_1\end{aligned}\tag{2.12}$$

where $a_{i,j}(t,y)$, $k=1,\ldots,i$, $i=1,\ldots,n$, are *unknown continuous* functions uniformly bounded by a *known* constant. Obviously, Assumption 1.1 holds for (2.12). Thus, we have the following result.

Corollary 2.3: For the uncertain time-varying nonlinear system (2.12), there is a linear dynamic output compensator (1.3), such that the closed-loop system (2.12) and (1.3) is globally exponentially stable.

Note that this corollary has recovered the output feedback stabilization theorem in [1], for the time-invariant triangular system

$$\begin{aligned}\dot{x}_1&=x_2+a_{1,1}(y)y\\\dot{x}_2&=x_3+\sum_{k=1}^{2}a_{2,k}(y)x_k\\&\vdots\\\dot{x}_n&=u+\sum_{k=1}^{n}a_{n,k}(y)x_k\\y&=x_1\end{aligned}\tag{2.13}$$

with globally bounded $a_{i,j}(y)$'s (see [1]).

In the rest of this section, we use examples to illustrate applications of Theorem 2.1.

Example 2.4: Consider a continuous but nonsmooth planar system of the form

$$\begin{aligned}\dot{x}_1&=x_2+x_1\sin x_2^2\\\dot{x}_2&=u+x_1^{2/3}x_2^{1/3}\\y&=x_1.\end{aligned}\tag{2.14}$$

Due to the presence of $\phi_1(t,x_1,x_2)=x_1\sin x_2^2$, system (2.14) is not in a lower-triangular form. Moreover, $\phi_2(t,x_1,x_2)=x_1^{2/3}x_2^{1/3}$ is only a *non-Lipschitz continuous* function. Neither the differentiability nor the global Lipschitz condition [14], [13] is fulfilled. As a result, global output feedback stabilization of (2.14) cannot be solved by the methods proposed in [14] and [13].

On the other hand, it is easy to verify that

$$|\phi_1(x_1,x_2)|\le|x_1|\quad|\phi_2(x_1,x_2)|\le|x_1|+|x_2|.$$

That is, Assumption 1.1 holds. By Theorem 2.1, the output feedback controller

$$\begin{aligned}\dot{\hat{x}}_1&=\hat{x}_2+L(y-\hat{x}_1)\\\dot{\hat{x}}_2&=u+L^2(y-\hat{x}_1)\\u&=-Lb_2(\hat{x}_2+Lb_1\hat{x}_1)\end{aligned}\tag{2.15}$$

with a suitable choice of the parameters L, b_1 and b_2 (e.g., $b_1=11/4$, $b_2=20$ and $L\ge100$), globally exponentially stabilizes system (2.14). The simulation shown in Fig. 1 demonstrates the GES property of the closed-loop system (2.14) and (2.15).

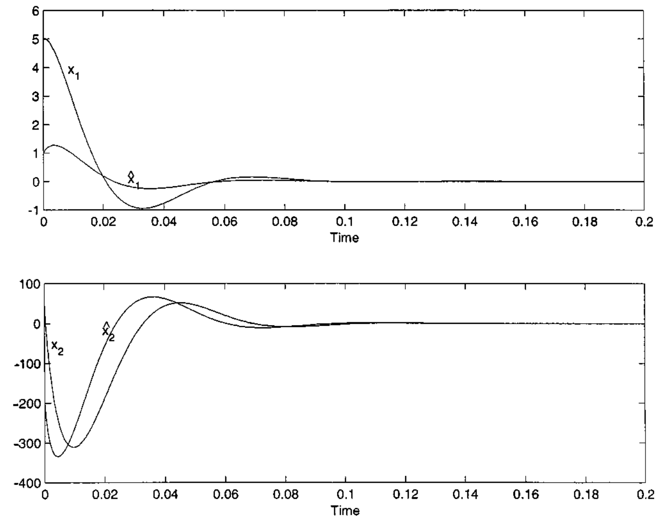

Fig. 1. Transient responses of (2.14) and (2.15) with $(x_1(0),\, x_2(0),\, \hat{x}_1(0),\, \hat{x}_2(0)) = (5,\, 50,\, 1,\, -200)$.

Example 2.5: Consider the three-dimensional system with uncertainty

$$\begin{aligned}\dot{x}_1 &= x_2 + \frac{x_1 x_3^2}{1+u^2+x_3^2}\\ \dot{x}_2 &= x_3 + x_2 \sin x_2\\ \dot{x}_3 &= u + d(t)\ln(1+|x_2 x_3|)\\ y &= x_1\end{aligned} \tag{2.16}$$

where the unknown function $d(t)$ is *continuous,* belonging to a known compact set Ω (e.g., $\Omega = [-1,\, 1]$). Since $\phi_2(t,\, x_1,\, x_2,\, u) \equiv \phi_2(x_2) = x_2 \sin x_2$, $|\phi_2(x_2)| \le |x_2|$. Note that, however, there is no smooth gain function $c(x_1) \ge 0$ satisfying

$$|\phi_2(x_2) - \phi_2(\hat{x}_2)| \le c(x_1)|x_2 - \hat{x}_2|, \qquad \forall\, x_2 \in \mathbb{R},\ \hat{x}_2 \in \mathbb{R}$$

i.e., the global Lipschitz condition required in [13] does not hold and, therefore, the existing output feedback control schemes such as [1], [14], and [13] cannot be applied to (2.16). On the other hand, (2.14) is globally exponentially stabilizable by the linear output feedback controller (2.1)–(2.10) as Assumption 1.1 is obviously satisfied.

Example 2.6: Consider a single-link robot arm system introduced, for instance, in [6]. The state-space model is described by

$$\begin{aligned}\dot{z}_1 &= z_2\\ \dot{z}_2 &= \frac{K}{J_2 N} z_3 - \frac{F_2(t)}{J_2} z_2 - \frac{K}{J_2} z_1 - \frac{mgd}{J_2}\cos z_1\\ \dot{z}_3 &= z_4\\ \dot{z}_4 &= \frac{1}{J_1} u + \frac{K}{J_1 N} z_1 - \frac{K}{J_2 N} z_3 - \frac{F_1(t)}{J_1} z_4\\ y &= z_1\end{aligned} \tag{2.17}$$

where $J_1,\, J_2,\, K,\, N,\, m,\, g,\, d$ are known parameters, and $F_1(t)$ and $F_2(t)$ are viscous friction coefficients that may vary continuously with time. Suppose $F_1(t)$ and $F_2(t)$ are unknown but bounded by *known* constants. The control objective is to globally stabilize the equilibrium $(z_1,\, z_2,\, z_3,\, z_4) = (0,\, 0,\, mgdN/K,\, 0)$ by *measurement* feedback. In the current case, z_1—the link displacement of the system is measurable and, therefore, can be used for feedback design. To solve the problem, we introduce a change of coordinates

$$x_1 = z_1,\quad x_2 = z_2,\quad x_3 = \frac{K}{J_2 N} z_3 - \frac{mgd}{J_2},\quad x_4 = \frac{K}{J_2 N} z_4$$

and a prefeedback

$$v = \frac{K}{J_2 N}\left(\frac{1}{J_1} u - \frac{mgd}{J_2}\right)$$

to transform (2.17) into

$$\begin{aligned}\dot{x}_1 &= x_2\\ \dot{x}_2 &= x_3 - \frac{F_2(t)}{J_2} x_2 - \frac{K}{J_2} x_1 - \frac{mgd}{J_2}(\cos x_1 - 1)\\ \dot{x}_3 &= x_4\\ \dot{x}_4 &= v + \frac{K^2}{J_1 J_2 N^2} x_1 - \frac{K}{J_2 N} x_3 - \frac{F_1(t)}{J_1} x_4\\ y &= x_1.\end{aligned} \tag{2.18}$$

Since $F_1(t)$ and $F_2(t)$ are unknown, most of the existing results are not applicable to the output feedback stabilization problem of (2.18). Observe that Assumption 1.1 holds for (2.18) because

$$|\cos x_1 - 1| \le |x_1| \qquad \left|\frac{F_2(t)}{J_2} x_2\right| \le c_1 |x_2|$$

$$\left|\frac{F_1(t)}{J_1} x_4\right| \le c_2 |x_4|, \qquad \text{for constants } c_1,\, c_2.$$

Using Theorem 2.1, it is easy to construct a linear dynamic output compensator of the form (2.1)–(2.10), solving the global stabilization problem for (2.18).

We end this section by pointing out when Assumption 1.1 fails to be satisfied, global stabilization of (1.1) by output feedback may be impossible. For instance, the nonlinear system

$$\begin{aligned}\dot{x}_1 &= x_2\\ \dot{x}_2 &= x_3\\ \dot{x}_3 &= u + x_3^2\\ y &= x_1\end{aligned} \tag{2.19}$$

is not globally stabilizable by any continuous dynamic output compensator. This fact can be proved using an argument similar to the one in [12].

III. Universal Output Feedback Stabilization

From the design procedure of Theorem 2.1, it is clear that there is a *single* linear output feedback controller (1.3) making the entire family of nonlinear systems (1.1) *simultaneously* exponentially stable, as long as they satisfy Assumption 1.1 with the same bound c. This is a nice feature of our output feedback control scheme, due to the use of the feedback domination design.

For example, it is easy to see that the output feedback controller (2.15) designed for the planar system (2.14) also globally exponentially stabilizes the following system:

$$\begin{aligned}\dot{x}_1 &= x_2 + \tfrac{1}{4}(x_1 + x_1 \sin(u x_2))\\ \dot{x}_2 &= u + x_1 \sin(u x_2)\\ y &= x_1\end{aligned} \tag{3.1}$$

which was proved to be globally stabilizable by linear state feedback [14]. However, the problem of output feedback stabilization was not solved in [14] because (3.1) violates the growth conditions (B1)–(B2) of [14].

The universal stabilization idea above can be extended to a family of C^0 minimum-phase nonlinear systems. Specifically, consider m cascade systems with the same relative degree r

$$\begin{aligned}\dot{z}^k &= F^k(z^k) + G^k(t,\, z^k,\, x^k,\, u)\\ \dot{x}_1^k &= x_2^k + \phi_1^k(t,\, z^k,\, x^k,\, u)\\ &\ \vdots\\ \dot{x}_i^k &= x_{i+1}^k + \phi_i^k(t,\, z^k,\, x^k,\, u)\\ &\ \vdots\\ \dot{x}_r^k &= u + \phi_n^k(t,\, z^k,\, x^k,\, u),\\ y &= x_1^k, \qquad k = 1,\, \ldots,\, m\end{aligned} \tag{3.2}$$

where $z^k \in \mathbb{R}^{s^k}$ and s^k is an *unknown* nonnegative integer.

Theorem 3.1: Suppose for each individual system (3.2), $\dot{z}^k = F^k(z^k)$ is GES at $z^k = 0$ and

$$\begin{aligned}|G^k(t,\, z^k,\, x^k,\, u)| &\le \bar{c}^k |x_1^k| \\ |\phi_i^k(t,\, z^k,\, x^k,\, u)| &\le c^k(\|z^k\| + |x_1^k| + \cdots + |x_i^k|), \\ &\qquad k = 1,\, \ldots,\, m.\end{aligned}$$

Then, there is a *universal* linear output feedback controller

$$\begin{aligned}\dot{\xi} &= M\xi + Ny, &\quad M \in \mathbb{R}^{r\times r};\ N \in \mathbb{R}^r \\ u &= K\xi, &\quad K \in \mathbb{R}^{1\times r}\end{aligned} \tag{3.3}$$

that simultaneously exponentially stabilizes the m cascade systems (3.2).

Proof: Since $\dot{z}^k = F^k(z^k)$ is globally exponentially stable, by the converse theorem there is a positive definite and proper function $V^k(z^k)$ such that

$$\frac{\partial V^k(z^k)}{\partial z^k} F^k(z^k) \le -\|z^k\|^2, \qquad \left\|\frac{\partial V^k(z^k)}{\partial z^k}\right\| \le \hat{c}^k \|z^k\| \quad \text{with } \hat{c}^k > 0.$$

This, in turn, implies

$$\begin{aligned}&\frac{\partial V^k(z^k)}{\partial z^k}\Big(F^k(z^k) + G^k(t,\, z^k,\, x^k,\, u)\Big) \\ &\quad \le -\tfrac{3}{4}\|z^k\|^2 + \left(\bar{c}^k \hat{c}^k\right)^2 |x_1^k|^2, \qquad k = 1,\, \ldots,\, m.\end{aligned} \tag{3.4}$$

Now, one can construct a *single* r-dimensional observer (2.1) with the gain parameter L to be determined later and a Lyapunov function

$$V_0(\varepsilon^k) = (r+1)(\varepsilon^k)^T P \varepsilon^k \quad \text{with } \varepsilon_i^k = \frac{x_i^k - \hat{x}_i}{L^{i-1}},\ i = 1, \ldots r.$$

Similar to the proof of Theorem 2.1, there is a real constant $\tilde{c}^k > 0$ satisfying

$$\begin{aligned}&\dot{V}^k(z^k) + \dot{V}_0(\varepsilon^k) \\ &\quad \le -\tfrac{3}{4}\|z^k\|^2 + \left(\bar{c}^k \hat{c}^k\right)^2 |x_1^k|^2 - (r+1)L\|\varepsilon^k\|^2 \\ &\qquad + \tilde{c}^k \|\varepsilon^k\| \left(\|z^k\| + |x_1^k| + \frac{|x_2^k|}{L} + \cdots + \frac{|x_r^k|}{L^{r-1}}\right).\end{aligned} \tag{3.5}$$

By the completion of squares, it is easy to see that

$$\tilde{c}^k \|\varepsilon^k\|\,\|z^k\| \le (\tilde{c}^k)^2\|\varepsilon^k\|^2 + \tfrac{1}{4}\|z^k\|^2 \tag{3.6}$$

$$\begin{aligned}\tilde{c}^k \|\varepsilon^k\| \frac{|x_j^k|}{L^{j-1}} &\le \tilde{c}^k \|\varepsilon^k\| \frac{|\hat{x}_j^k|}{L^{j-1}} + \tilde{c}^k \|\varepsilon^k\|\, |\varepsilon_j^k| \\ &\le \frac{\tilde{c}^k}{2}\|\varepsilon^k\|^2 + \tilde{c}^k \frac{(\hat{x}_j)^2}{2L^{2(j-1)}} + \tilde{c}^k \|\varepsilon^k\|\, |\varepsilon_j^k|.\end{aligned} \tag{3.7}$$

Substituting (3.6) and (3.7) into (3.5) yields

$$\begin{aligned}&\dot{V}^k(z^k) + \dot{V}_0(\varepsilon^k) \\ &\quad \le -\tfrac{1}{2}\|z^k\|^2 - (r+1)L\|\varepsilon^k\|^2 \\ &\qquad + \left((\tilde{c}^k)^2 + \tilde{c}^k\sqrt{r} + \frac{r}{2}\tilde{c}^k\right)\|\varepsilon^k\|^2 + (\bar{c}^k\hat{c}^k)^2 \hat{x}_1^2 \\ &\qquad + \tilde{c}^k\left(\frac{1}{2}\hat{x}_1^2 + \frac{1}{2L^2}\hat{x}_2^2 + \cdots + \frac{1}{2L^{2r-2}}\hat{x}_r^2\right) \\ &\quad \le -\frac{1}{2}\|z^k\|^2 - \left((r+1)L - (c_1)^2 - c_1\sqrt{r} - \frac{r}{2}c_1\right)\|\varepsilon^k\|^2 \\ &\qquad + c_1 \sum_{j=1}^{r} \frac{1}{2L^{2j-2}}\hat{x}_j^2\end{aligned} \tag{3.8}$$

where c_1 is a uniform constant defined as

$$c_1 := \max\left\{(\bar{c}^k\hat{c}^k)^2 + \tilde{c}^k,\ \ k = 1,\, \ldots,\, m\right\}.$$

Observe that (3.8) is almost identical to inequality (2.3) in the proof of Theorem 2.1. Using a recursive design procedure similar to the one in Theorem 2.1, we can obtain at the rth step the following globally exponentially stabilizing controller:

$$u = -Lb_r(\hat{x}_r + Lb_{r-1}(\hat{x}_{r-1} + \cdots + Lb_2(\hat{x}_2 + Lb_1\hat{x}_1)\cdots) \tag{3.9}$$

where $b_i > 0$, $i = 1,\, \ldots,\, r$, are real numbers independent of the gain parameter L, and $L > L^* := \max\{1,\, c_1^2 + (\sqrt{r} + (r/2))c_1,\, c_1 b_1^2,\, \ldots,\, c_1 b_{r-1}^2\}$.

Because (3.8) holds uniformly for the m systems with a common constant $c_1 > 0$, it is not difficult to prove that the feedback control law (3.9) together with the *single* r-dimensional observer (2.1) stabilizes the m systems (3.2) simultaneously. ∎

IV. Conclusion

We have presented a new output feedback control scheme for a class of nonlinear systems whose global stabilization problem via output feedback cannot be handled by existing methods. The proposed output dynamic compensator is *linear* and can stabilize *simultaneously* a family of nonlinear systems which are dominated by a chain of integrators perturbed by a triangular vector field with linear growth condition. Moreover, the result can also be applied to a *finite* number of globally exponentially minimum-phase systems (say, for instance, m controlled plants), in which the dimensions of the zero dynamics can be *different* and *unknown*.

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