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Output feedback stabilisation for a cascaded wave PDE-ODE system subject to boundary control matched disturbance

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ABSTRACT
In this paper, we consider output feedback stabilisation for a wave PDE-ODE system with Dirichlet boundary interconnection and external disturbance flowing the control end. We first design a variable structure unknown input type state observer which is shown to be exponentially convergent. Then, we estimate the disturbance in terms of the estimated state, an idea from active disturbance rejection control. These enable us to design an observer-based output feedback stabilising control to this uncertain PDE-ODE system.

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1. Introduction

In recent years, stabilisation for systems described by partial differential equations (PDEs) subject to external disturbance has attracted increasing attention. This is because in many situations, the control is used not only to guarantee the system to be normally operated in an ideal operation environment but also to be normally operated in the environment with uncertainties coming from either as internal or external disturbance. The stability issue of system with disturbance is therefore significant both theoretically and practically. To deal with disturbances, many different approaches have been developed. The backstepping method which was originally applied to stabilise the PDEs without disturbance in Krstic and Smyshlyaev (2008), Smyshlyaev and Krstic (2004), together with the adaptive control method, is designed in Guo and Guo (2013), Krstic (2010) to stabilise one-dimensional (1D) wave equations where the uncertainties are supposed to be unknown parameters in disturbance. The internal model principle for output regulation has been applied in Byrnes, Lauko, Gilliam, and Shubov (2000), Rebarber and Weiss (2003), among many others, to infinite dimensional systems to reject disturbance generated from exogenous system. The Lyapunov functional approach is presented (Ge, Zhang, and He, 2011; He, Zhang, and Ge, 2012) for disturbance rejection, where boundary feedbacks are designed for 1D Euler-Bernoulli beam with spatial and boundary disturbance. Another powerful method in dealing with uncertainties is based on active disturbance rejection control (ADRC) which was first proposed by Han in 2009 . In Guo and Zhou (2014, 2015), the boundary controls are designed by ADRC for multi-dimensional wave equation and Kirchhoff plate equation with control-matched disturbance that depends on both time and spatial variables. Recently, the boundary sliding mode control (SMC) is designed for 1D heat, wave, Euler-Bernoulli, and Schrödinger equations with boundary input disturbance in Cheng, Radisavljevic, and Su (2011), Guo and Jin (2013a, 2013b), Guo and Liu (2014), Pisano, Orlov, and Usai (2011). However, there are a few works on coupled systems. In Krstic (2009, Chapter 16), stabilisation for a cascade system of wave equation with linear time invariant finite-dimensional system is considered, where the backstepping approach is applied but no disturbance is concerned. In Wang, Liu, Ren, and Chen (2015), boundary feedback stabilisation for a cascade system of heat PDE-ODE system with Dirichlet/Neumann interconnection and external disturbance flowing the control end is discussed by the SMC and the backstepping method.

Motivated mainly by Krstic (2009, Chapter 16) and Wang et al. (2015) where only state feedback is concerned, we are concerned with, in this paper, stabilisation for the following wave PDE-ODE cascade system through

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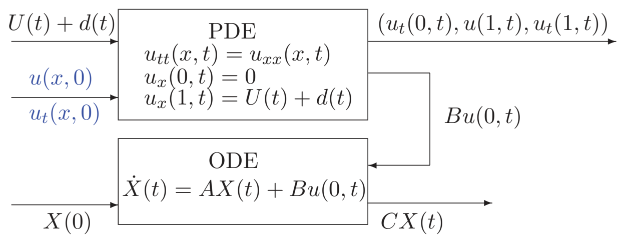

Figure 1. Block diagram of the coupled wave PDE-ODE system (1.1).

Dirichlet interconnection:

$$\begin{cases} \dot{X}(t) = AX(t) + Bu(0,t), \ t > 0, \\ u_{tt}(x,t) = u_{xx}(x,t), \ x \in (0,1), \ t > 0, \\ u_x(0,t) = 0, \ t \geq 0, \\ u_x(1,t) = U(t) + d(t), \ t \geq 0, \\ y_{out}(t) = (CX(t), u_t(0,t), u(1,t), u_t(1,t)), \end{cases} \tag{1.1}$$

where $X \in \mathbb{R}^{n\times 1}$ and $(u, u_t) \in H^1(0,1) \times L^2(0,1)$ are the states of ODE and PDE, respectively, $U \in L^2_{loc}(0,\infty)$ is the control input of the entire system, A is an $n \times n$ matrix, $B \in \mathbb{R}^{n\times 1}$, and $d(t)$ is the external disturbance. It is supposed that $\Sigma(A, B)$ is stabilisable, (C, A) is observable, and $d \in H^1_{loc}(0,\infty)$ is uniformly bounded: $|d(t)| \leq M$ for some $M > 0$ and all $t \geq 0$.

The objective of the present paper is to design an output feedback control that can stabilise system (1.1) depicted in Figure 1 in the state space $\mathcal{H} = \mathbb{R}^n \times H^1(0,1) \times L^2(0,1)$ by rejecting the disturbance $d(t)$.

The inner product of $\mathcal{H}$ is given by

$$\begin{aligned} &\langle (X_1, f_1, g_1), (X_2, f_2, g_2)\rangle \\ &\quad = X_1^\top X_2 + k f_1(1) f_2(1) + \int_0^1 f_1'(x) f_2'(x)dx \\ &\qquad + \int_0^1 g_1(x) g_2(x) dx, \end{aligned} \tag{1.2}$$

for all $(X_1, f_1, g_1), (X_2, f_2, g_2) \in \mathcal{H}$, and $k > 0$ is a positive constant.

We proceed as follows. In Section 2, we design a variable structure unknown input observer for system (1.1) and establish convergence of the observer. The exponential stability of the closed-loop system is presented. In Section 3, we transform system (1.1) into an equivalent target system for which a state feedback is easily designed. By ADRC approach, we design a disturbance estimator to estimate the disturbance. The observer and estimator based output feedback is then designed by compensating the disturbance in feedback. The closed-loop system is shown to be asymptotically stable, followed by concluding remarks in Section 4.

2. State observer

In this section, we first design an unknown type state observer to recover the state of system (1.1) via the output. By the estimated state, we can estimate the disturbance by an idea of extended state observer for ODEs. The disturbance is then compensated in the feedback loop.

We design a variable structure unknown input type state observer for system (1.1) as follows:

$$\begin{cases} \dot{\widehat{X}}(t) = A\widehat{X}(t) + L(C\widehat{X} - CX) + B\widehat{u}(0,t), \ t > 0, \\ \widehat{u}_{tt}(x,t) = \widehat{u}_{xx}(x,t), \ x \in (0,1), \ t > 0, \\ \widehat{u}_x(0,t) = c_1[\widehat{u}_t(0,t) - u_t(0,t)], \ t \geq 0, \\ \widehat{u}_x(1,t) \in U(t) - c_2[\widehat{u}_t(1,t) - u_t(1,t)] \\ \quad - c_3[\widehat{u}(1,t) - u(1,t)] - M_1 \mathrm{sign}(\widehat{u}_t(1,t) \\ \quad - u_t(1,t)) - M_2 \mathrm{sign}(\widehat{u}(1,t) - u(1,t)), \\ \quad t \geq 0, \\ \widehat{u}(x,0) = \widehat{u}_0(x), \ \widehat{u}_t(x,0) = \widehat{u}_1(x), \end{cases} \tag{2.1}$$

where $M_1 > M$, $M_2 > M + M_1$, and c_1, c_2, c_3 are positive design parameters. The symbolic function is a multi-valued function defined by

$$\mathrm{sign}(x) = \begin{cases} 1, & x > 0, \\ [-1, 1], & x = 0, \\ -1, & x < 0. \end{cases}$$

To ensure the function on the right-hand side of the fourth equation of (2.1) to be measurable, for any $T > 0$ and $f \in L^2(0, T)$, we restrict the set-valued composition of the symbolic function as follows:

$$\mathrm{sign}(f(t)) = \left\{ g(t) \,\middle|\, g(t) = \begin{cases} \mathrm{sign}(f(t)), & f(t) \neq 0, \\ f^{(t)} \in L^2(\Omega), |f^{(t)}| \leq 1, & f(t) = 0, \end{cases} \right\},$$

where $\Omega = \{\tau | f(\tau) = 0\}$. Introduce the variable

$$\begin{aligned} &(\widetilde{X}(t), \widetilde{u}(x,t), \widetilde{u}_t(x,t)) \\ &= (\widehat{X}(t) - X(t), \widehat{u}(x,t) - u(x,t), \widehat{u}_t(x,t) - u_t(x,t)) \end{aligned} \tag{2.2}$$

to be the error. Then $(\widetilde{X}(t), \widetilde{u}(x,t), \widetilde{u}_t(x,t))$ satisfies

$$\begin{cases} \dot{\widetilde{X}}(t) = (A+LC)\widetilde{X}(t) + B\widetilde{u}(0,t), \\ \widetilde{u}_{tt}(x,t) = \widetilde{u}_{xx}(x,t), \\ \widetilde{u}_x(0,t) = c_1\widetilde{u}_t(0,t), \\ \widetilde{u}_x(1,t) \in -c_2\widetilde{u}_t(1,t) - c_3\widetilde{u}(1,t) \\ \quad - M_1\text{sign}(\widetilde{u}_t(1,t)) - M_2\text{sign}(\widetilde{u}(1,t)) - d(t), \\ \widetilde{u}(x,0) = \widetilde{u}_0(x), \widetilde{u}_t(x,0) = \widetilde{u}_1(x). \end{cases} \quad (2.3)$$

For the error system (2.2), we have Theorem 2.1 which shows that the observer (2.1) is convergent.

Theorem 2.1: *For any initial value $(\widetilde{X}(0), \widetilde{u}_0, \widetilde{u}_1) \in \mathbb{R}^n \times H^2(0,1) \times H^1(0,1)$ satisfying compatible conditions:*

$$\begin{cases} \widetilde{u}_0(0) = c_1\widetilde{u}_1(0), \\ \widetilde{u}_0'(1) \in -c_2\widetilde{u}_1(1) - c_3\widetilde{u}_0(1) - M_1\text{sign}(\widetilde{u}_1(1)) \\ \quad - M_2\text{sign}(\widetilde{u}_0(1)) - d(0), \end{cases} \quad (2.4)$$

and $d \in H^1_{loc}(0,\infty)$, (2.1) admits at least one partial differential inclusion solution $(\widetilde{X}(t), \widetilde{u}(\cdot,t), \widetilde{u}_t(\cdot,t)) \in C(0,\infty; \mathbb{R}^n \times H^2(0,1) \times H^1(0,1))$. Moreover, any partial differential inclusion solution $(\widetilde{X}(t), \widetilde{u}(\cdot,t), \widetilde{u}_t(\cdot,t)) \in C(0,\infty; \mathbb{R}^n \times H^2(0,1) \times H^1(0,1))$ is exponentially stable in the sense that

$$E(t) \le C_1 E(0) e^{-\omega_1 t}, \quad (2.5)$$

for some positive constants C_1 and ω_1, where $E(t)$ is given by

$$E(t) := |\widetilde{X}(t)|^2_{\mathbb{R}^n} + \widetilde{u}^2(1,t) + \int_0^1 \widetilde{u}_x^2(x,t)dx + \int_0^1 \widetilde{u}_t^2(x,t)dx.$$

Proof: We first prove the '$\widetilde{u}$ part' in (2.3). Actually, by Guo and Jin (2015, Theorem 1), the '$\widetilde{u}$ part' in (2.3) has at least one solution $(\widetilde{u}(\cdot,t), \widetilde{u}_t(\cdot,t)) \in C(0,\infty; H^2(0,1) \times H^1(0,1))$, which satisfies

$$\widetilde{u}^2(1,t) + \int_0^1 [\widetilde{u}_x^2(x,t) + \widetilde{u}_t^2(x,t)]dx \le C_2 e^{-C_3 t}\left[\widetilde{u}_0^2(1) + \int_0^1 [\widetilde{u}_0'^2(x) + \widetilde{u}_1^2(x)]dx\right]. \quad (2.6)$$

Next we show the '$\widetilde{X}$ part' in (2.3). By the Sobolev embedding $H^1(0,1) \hookrightarrow C(0,1)$,

$$\widetilde{u}^2(0,t) \le C_4\left[\widetilde{u}^2(1,t) + \int_0^1 \widetilde{u}_x^2(x,t)dx\right] \text{ for some constant } C_4 > 0, \quad (2.7)$$

and $\widetilde{u}(0,t) \in C(0,\infty)$. Using the constant variational formula, the solution of '$\widetilde{X}$ part' is given by

$$\widetilde{X}(t) = e^{(A+LC)t}\widetilde{X}(0) + \int_0^t e^{(A+LC)(t-s)}B\widetilde{u}(0,s)ds. \quad (2.8)$$

Since $A + LC$ is Hurwitz, there are positive constants C_5 and $C_6 > 0$ such that

$$\|e^{(A+LC)t}\| \le C_5 e^{-C_6 t}. \quad (2.9)$$

This together with (2.6), (2.7), and (2.8) yields

$$\begin{aligned} |\widetilde{X}(t)|_{\mathbb{R}^n} &\le |e^{(A+LC)t}\widetilde{X}(0)|_{\mathbb{R}^n} + \left|\int_0^t e^{(A+LC)(t-s)}B\widetilde{u}(0,s)ds\right|_{\mathbb{R}^n} \\ &\le C_5 e^{-C_6 t}|X(0)|_{\mathbb{R}^n} + C_2C_4C_5\|B\| \\ &\quad \times \left[\widetilde{u}_0^2(1) + \int_0^1 [\widetilde{u}_0'^2(x) + \widetilde{u}_1^2(x)]dx\right] \\ &\quad \times \int_0^t e^{-C_6(t-s)}e^{-C_3 s}ds. \end{aligned} \quad (2.10)$$

Since

$$\int_0^t e^{-C_6(t-s)}e^{-C_3 s}ds = \begin{cases} \dfrac{e^{-C_3 t} - e^{-C_6 t}}{C_6 - C_3}, & C_6 \ne C_3, \\ e^{-C_6 t}t, & C_6 = C_3, \end{cases} \quad (2.11)$$

we obtain (2.5) from (2.6) and (2.10). □

3. Disturbance estimator and output feedback control

In this section, we design an observer-based output stabilising feedback for system (1.1). To this end, we first introduce a transformation $(X, u, u_t) \to (X, v, v_t)$ as follows (Krstic, 2009, Chapter 16):

$$\begin{aligned} X(t) &= X(t), \\ v(x,t) &= u(x,t) - \int_0^x \mu(x-y)u(y,t)dy \\ &\quad - \int_0^x m(x-y)u_t(y,t)dy - \gamma(x)X(t), \\ v_t(x,t) &= u_t(x,t) - KBu(x,t) - \int_0^x \mu(x-y)u_t(y,t)dy \\ &\quad - \int_0^x m''(x-y)u(y,t)dy - \gamma(x)AX(t), \end{aligned} \quad (3.1)$$

where

$$\mu(s) = \int_0^s \gamma(\xi)ABd\xi,\ \ m(s) = \int_0^s \gamma(\xi)Bd\xi,$$
$$\gamma(x) = KM(x),\ \ M(x) = \begin{bmatrix} I & 0 \end{bmatrix} e^{\begin{bmatrix} 0 & A^2 \\ I & 0 \end{bmatrix} x} \begin{bmatrix} I \\ 0 \end{bmatrix}. \quad (3.2)$$

with I being an $n \times n$ identify matrix. This transformation brings system (1.1) into the following system:

$$\begin{cases} \dot{X}(t) = (A + BK)X(t) + Bv(0, t), \\ v_{tt}(x, t) = v_{xx}(x, t), \\ v_x(0, t) = 0,\ \ t \geq 0, \\ v_x(1, t) = U(t) + d(t) - \int_0^1 \mu'(1 - y)u(y, t)dy \\ \qquad - \int_0^1 m'(1 - y)u_t(y, t)dy - \gamma'(1)X(t), \end{cases} \quad (3.3)$$

where K is chosen so that $A + BK$ is Hurwitz. The transformation (3.1) is invertible, that is

$$\begin{aligned} X(t) &= X(t), \\ u(x, t) &= v(x, t) - \int_0^x \sigma(x - y)v(y, t)dy \\ &\quad - \int_0^x n(x - y)v_t(y, t)dy - \rho(x)X(t), \\ u_t(x, t) &= v_t(x, t) + KBv(x, t) - \int_0^x \sigma(x - y)v_t(y, t)dy \\ &\quad - \int_0^x n''(x - y)v(y, t)dy \\ &\quad - \rho(x)AX(t), \end{aligned} \quad (3.4)$$

where

$$\sigma(s) = \int_0^s \rho(\xi)ABd\xi,\ n(s) = \int_0^s \rho(\xi)Bd\xi,$$
$$\rho(x) = -KN(x),\ \ M(x) = \begin{bmatrix} I & 0 \end{bmatrix} e^{\begin{bmatrix} 0 & (A+BK)^2 \\ I & 0 \end{bmatrix} x} \times \begin{bmatrix} I \\ 0 \end{bmatrix}. \quad (3.5)$$

We further introduce $(X, v, v_t) \to (X, w, w_t)$ by

$$\begin{aligned} X(t) &= X(t), \\ w(x, t) &= v(x, t) + c\int_0^x v_t(y, t)dy, \\ w_t(x, t) &= v_t(x, t) + cv_x(x, t), \end{aligned} \quad (3.6)$$

where $c \neq 1$ is positive. We then obtain the target system following:

$$\begin{cases} \dot{X}(t) = (A + BK)X(t) + Bw(0, t),\ \ t > 0, \\ w_{tt}(x, t) = w_{xx}(x, t),\ \ x \in (0, 1),\ \ t > 0, \\ w_x(0, t) = cw_t(0, t),\ \ t \geq 0, \\ w_x(1, t) = U(t) + d(t) - \int_0^1 \mu'(1 - y)u(y, t)dy \\ \qquad - \int_0^1 m'(1 - y)u_t(y, t)dy \\ \qquad - \gamma'(1)X(t) + cu_t(1, t) - cKBu(1, t) \\ \qquad - c\int_0^1 \mu(1 - y)u_t(y, t)dy \\ \qquad - c\int_0^1 m''(1-y)u(y, t)dy - c\gamma(1)AX(t), \\ \qquad t \geq 0. \end{cases} \quad (3.7)$$

The inverse of transformation (3.6) can be found as follows

$$\begin{aligned} X(t) &= X(t), \\ v(x, t) &= w(0, t) + \int_0^x \frac{w_x(y, t) - cw_t(y, t)}{1 - c^2}dy, \\ v_t(x, t) &= \frac{w_t(x, t) - cw_x(x, t)}{1 - c^2}. \end{aligned} \quad (3.8)$$

For the target system (3.7), we can easily design a stabilising state feedback. Since we have state estimation through the observer claimed by Theorem 2.1, we design naturally an output feedback control as follows:

$$\begin{aligned} U(t) &= U_0(t) + \int_0^1 \big(\mu'(1 - y) + cm''(1 - y)\big)\big)\, \widehat{u}(y, t)dy \\ &\quad + \int_0^1 \big(m'(1 - y) + c\mu(1 - y)\big)\, \widehat{u}_t(y, t)dy \\ &\quad + \big(\gamma'(1) + c\gamma(1)A\big)\, \widehat{X}(t) + cKBu(1, t) \\ &\quad - cu_t(1, t) - k\Bigg\{u(1, t) - \int_0^1 \mu(1 - y)\widehat{u}(y, t)dy \\ &\quad - \int_0^1 m(1 - y)\widehat{u}_t(y, t)dy - \gamma(1)\widehat{X}(t) \\ &\quad + c\int_0^1 \Bigg[\widehat{u}_t(x, t) - KB\widehat{u}(x, t) \\ &\quad - \int_0^x \mu(x - y)\widehat{u}_t(y, t)dy \\ &\quad - \int_0^x m''(x - y)\widehat{u}(y, t)dy\Bigg]dx \\ &\quad - c\int_0^1 \gamma(x)dxA\widehat{X}(t)\Bigg\},\ \ t \geq 0, \end{aligned} \quad (3.9)$$

where $U_0(t)$ is an auxiliary control. Under feedback control (3.9), the closed-loop of target system (3.7) is

$$\begin{cases} \dot{X}(t) = (A+BK)X(t) + Bw(0,t), \ t>0, \\ w_{tt}(x,t) = w_{xx}(x,t), \ x \in (0,1), \ t>0, \\ w_x(0,t) = cw_t(0,t), \ t \geq 0, \\ w_x(1,t) = -kw(1,t) + U_0(t) + f(t) + d(t), \ t \geq 0, \end{cases} \tag{3.10}$$

where

$$\begin{aligned} f(t) = & \int_0^1 \left(\mu'(1-y) + cm''(1-y))\right) \widetilde{u}(y,t)dy \\ & + \int_0^1 \left(m'(1-y) + c\mu(1-y)\right) \widetilde{u}_t(y,t)dy \\ & + \left(\gamma'(1) + c\gamma(1)A\right) \widetilde{X}(t) \\ & + k\Bigg\{ \int_0^1 \mu(1-y)\widetilde{u}(y,t)dy \\ & + \int_0^1 m(1-y)\widetilde{u}_t(y,t)dy + c\int_0^1 \gamma(x)dxA\widetilde{X}(t) \\ & + \gamma(1)\widetilde{X}(t) - c\int_0^1 \Bigg[\widetilde{u}_t(x,t) - KB\widehat{u}(x,t) \\ & - \int_0^x \mu(x-y)\widetilde{u}_t(y,t)dy \\ & - \int_0^x m''(x-y)\widetilde{u}(y,t)dy \Bigg] dx \Bigg\}, \end{aligned} \tag{3.11}$$

satisfies, by Theorem 2.1, that

$$|f(t)| \leq C_f e^{-\omega_1 t}, \forall\, t \geq 0, \tag{3.12}$$

for some positive constants C_f and ω_1.

The remaining is to design a continuous control $U_0(t)$ to stabilise (3.10) in the presence of disturbances $d(t)$ and $f(t)$. We write (3.10) as

$$\begin{aligned} \frac{d}{dt}\begin{pmatrix} X(t) \\ w(\cdot,t) \\ w_t(\cdot,t) \end{pmatrix} = \mathbb{A}\begin{pmatrix} X(t) \\ w(\cdot,t) \\ w_t(\cdot,t) \end{pmatrix} + \mathbb{B}(U_0(t) \\ + f(t) + d(t)), \end{aligned} \tag{3.13}$$

where $\mathbb{B} = (0, 0, \delta(x-1))^\top$, and $\mathbb{A}$ is a linear operator defined in $\mathbb{R}^n \times H^1(0,1) \times L^2(0,1)$ by

$$\begin{cases} \mathbb{A}\begin{pmatrix} X \\ f \\ g \end{pmatrix} = \begin{pmatrix} (A+BK)X + Bf(0) \\ g \\ f'' \end{pmatrix}, \forall \begin{pmatrix} X \\ f \\ g \end{pmatrix} \in D(\mathbb{A}), \\ D(\mathbb{A}) = \{(X,f,g)^\top \in \mathcal{H} \cap \mathbb{R}^n \times H^2(0,1) \times H^1(0,1) : f'(0) \\ \quad = cg(0),\ f'(1) = -kf(1)\}. \end{cases} \tag{3.14}$$

Lemma 3.1: *The operator $\mathbb{A}$ defined in (3.14) generates an exponential stable C_0-semigroup.*

Proof: It suffices to show that the following system

$$\begin{cases} \dot{X}(t) = (A+BK)X(t) + Bw(0,t), \ t>0, \\ w_{tt}(x,t) = w_{xx}(x,t), \ x \in (0,1), \ t>0, \\ w_x(0,t) = cw_t(0,t), \ t \geq 0, \\ w_x(1,t) = -kw(1,t), \ t \geq 0, \end{cases} \tag{3.15}$$

is exponentially stable. Actually, it is well known that the 'w-part' is exponentially stable in $H^1(0,1) \times L^2(0,1)$, that is, there are constants $C_1, C_2 > 0$ such that

$$\begin{aligned} & w^2(1,t) + \int_0^1 \left[w_x^2(x,t) + w_t^2(x,t)\right]dx \\ & \leq C_1 e^{-C_2 t}\left[w^2(1,0) + \int_0^1 [w_x^2(x,0) + w_t^2(x,0)]dx\right]. \end{aligned} \tag{3.16}$$

By embedding inclusion, $H^1(0,1) \hookrightarrow C(0,1)$,

$$w^2(0,t) \leq C_3\left[w^2(1,t) + \int_0^1 w_x^2(x,t)dx\right] \quad \text{for some constant } C_3 > 0. \tag{3.17}$$

Hence $w(0,\cdot) \in C(0,\infty)$. Apply the constant variational formula to obtain the solution of ODE part as

$$X(t) = e^{(A+BK)t}X(0) + \int_0^t e^{(A+BK)(t-s)}Bw(0,s)ds. \tag{3.18}$$

Since $A+BK$ is Hurwitz, there are positive constants C_4 and $C_5 > 0$ such that

$$\|e^{(A+BK)t}\| \leq C_4 e^{-C_5 t}. \tag{3.19}$$

Since

$$\int_0^t e^{-C_5(t-s)}e^{-C_2 s}ds = \begin{cases} \dfrac{e^{-C_2 t} - e^{-C_5 t}}{C_5 - C_2}, & C_5 \neq C_2, \\ e^{-C_5 t}t, & C_5 = C_2, \end{cases} \tag{3.20}$$

it follows from (3.17), (3.18), and (3.19) that 'X part' in (3.15) is exponentially stable. This completes the proof of the lemma. □

The following result is straightforward (Weiss, Staffans, & Tucsnak, 2001) where for admissibility of control operator, we refer to Weiss (1989).

Proposition 3.1: *The operator $\mathbb{A}$ defined in (3.14) generates a C_0-semigroup of contractions $e^{\mathbb{A}t}$ on $\mathcal{H}$ and $\mathbb{B}$ is admissible to $e^{\mathbb{A}t}$. Therefore, for any initial value $(X(0), w(\cdot,0), \dot{w}(\cdot,0))^\top \in \mathcal{H}$ and control input $U_0 \in$*

$L^2_{loc}(0,\infty)$ *and* $(f,d)\in (L^2_{loc}(0,\infty))^2$, (3.13) *admits a unique solution* $(X(t), w(\cdot,t), \dot w(\cdot,t))^\top \in \mathcal{H}$.

By Proposition 3.1, the (weak) solution of (3.10) satisfies

$$\frac{d}{dt}\left\langle \begin{pmatrix} X\\ w\\ w_t\end{pmatrix}, \begin{pmatrix} Y\\ f\\ g\end{pmatrix}\right\rangle_{\mathcal{H}} = \left\langle \begin{pmatrix} X\\ w\\ w_t\end{pmatrix}, \mathbb{A}^*\begin{pmatrix} Y\\ f\\ g\end{pmatrix}\right\rangle_{\mathcal{H}} + [U_0(t)+f(t)+d(t)]\mathbb{B}^*\begin{pmatrix} Y\\ f\\ g\end{pmatrix}, \quad \forall\ (Y,f,g)^\top\in D(\mathbb{A}^*), \tag{3.21}$$

where $(Y,f,g)^\top\in D(\mathbb{A}^*)$ is called a test function. A simple computation shows that $\mathbb{A}^*$, the adjoint operator of $\mathbb{A}$, is given by

$$\begin{cases} \mathbb{A}^*\begin{pmatrix} X\\ f\\ g\end{pmatrix} = \begin{pmatrix} (A+BK)^\top X\\ -g\\ -f''\end{pmatrix}, \forall \begin{pmatrix} X\\ f\\ g\end{pmatrix}\in D(\mathbb{A}^*),\\ D(\mathbb{A}^*) = \big\{(X,f,g)^\top \in \mathcal{H}\cap\mathbb{R}^n\times H^2(0,1)\\ \quad \times H^1(0,1): f'(0)=-cg(0),\\ \quad f'(1)=-kf(1), B^\top X = 0\big\}, \end{cases} \tag{3.22}$$

By (3.21),

$$\begin{aligned} &\frac{d}{dt}\left[X^\top Y + \int_0^1 [w_x(x,t)f'(x)+w_t(x,t)g(x)]dx + kw(1,t)f(1)\right]\\ &= X^\top (A+BK)^\top Y - \int_0^1 [w_x(x,t)g'(x) + w_t(x,t)f'(x)]dx - kw(1,t)g(1)\\ &\quad + g(1)[U_0+f(t)+d(t)]. \end{aligned} \tag{3.23}$$

Choose $(X, f(x), g(x)) = (0,0,x)\in D(\mathbb{A}^*)$ and substitute into (3.23) to obtain

$$\frac{d}{dt}\int_0^1 w_t(x,t)xdx = -(k+1)w(1,t)+w(0,t)+U_0 + f(t)+d(t). \tag{3.24}$$

Same as $(X,u,u_t)\to(X,w,w_t)$ by (3.1) and (3.6), we have $(\widehat X,\widehat u,\widehat u_t)\to(\widehat X,\widehat w,\widehat w_t)$. Replace (w, w_t) in (3.24) by $(\widehat w,\widehat w_t)$ to have

$$\begin{aligned} &\frac{d}{dt}\int_0^1 \widehat w_t(x,t)xdx\\ &= \frac{d}{dt}\int_0^1 w_t(x,t)xdx + \frac{d}{dt}\int_0^1 \widetilde w_t(x,t)xdx\\ &= -(k+1)w(1,t)+w(0,t)+U_0+f(t)+d(t)\\ &\quad + \frac{d}{dt}\int_0^1 \widetilde w_t(x,t)xdx\\ &= U_0 + f(t)+d(t) - (k+1)\widehat w(1,t)+\widehat w(0,t)\\ &\quad + (k+1)\widetilde w(1,t) - \widetilde w(0,t) + \frac{d}{dt}\int_0^1 \widetilde w_t(x,t)xdx. \end{aligned}$$

Let

$$\begin{cases} z(t) = \int_0^1 \widehat w_t(x,t)xdx,\\ z_1(t) = -(k+1)\widehat w(1,t)+\widehat w(0,t),\\ z_2(t) = (k+1)\widetilde w(1,t) - \widetilde w(0,t) + f(t),\\ z_3(t) = \int_0^1 \widetilde w_t(x,t)xdx. \end{cases}$$

Then

$$\dot z(t) = U_0 + d(t) + z_1(t) + z_2(t) + \dot z_3(t), \tag{3.25}$$

and by Theorem 2.1 and (3.12), there exists a constant $C_z>0$ such that

$$|z_2(t)| + |z_3(t)| \le C_z e^{-\omega_1 t}, \forall\ t\ge 0. \tag{3.26}$$

Now we are in a position to estimate the disturbance $d(t)$ by the active disturbance rejection approach developed in Guo and Zhao (2011) where the constant high gain is used. The extended state observer with time varying high gain for system (3.25) is designed as

$$\begin{cases} \dot{\widehat z}(t) = U_0 + \widehat d(t) + z_1(t) - r(t)[\widehat z(t) - z(t)],\\ \dot{\widehat d}(t) = -r^2(t)[\widehat z(t) - z(t)], \end{cases} \tag{3.27}$$

where $r\in C(\bar{\mathbb{R}}^+,\mathbb{R}^+)$ is a time varying gain that is required to satisfy

$$\begin{cases} \dot r(t) > 0,\ \lim\limits_{t\to\infty} r(t) = \infty,\ \dfrac{\dot r(t)}{r(t)} \le M, \forall\ t\ge 0\\ \qquad \text{for some } M>0,\\ \lim\limits_{t\to\infty} \dfrac{\dot d(t)}{r(t)} = 0,\ \lim\limits_{t\to\infty} r(t)e^{-\omega_1 t} = 0\\ \qquad \text{for } \omega_1>0 \text{ in (3.26)}. \end{cases} \tag{3.28}$$

Lemma 3.2: *Let* $(\widehat z(t), \widehat d(t))$ *be the solution of* (3.27). *Then*

$$\lim_{t\to\infty} [|\widehat z(t) - z(t)| + |\widehat d(t) - d(t)|] = 0. \tag{3.29}$$

Proof: Let

$$\widetilde z(t) = r(t)[\widehat z(t) - z(t)],\ \widetilde d(t) = \widehat d(t) - d(t). \tag{3.30}$$

Then $(\widetilde{z}(t), \widetilde{d}(t))$ satisfies

$$\begin{cases} \dot{\widetilde{z}}(t) = -r(t)\widetilde{z}(t) + r(t)\widetilde{d}(t) + \dfrac{\dot{r}(t)}{r(t)}\widetilde{z}(t) \\ \qquad -r(t)[z_2(t) + \dot{z}_3(t)], \\ \dot{\widetilde{d}}(t) = -r(t)\widetilde{z}(t) - \dot{d}(t). \end{cases} \tag{3.31}$$

Set $\eta(t) = \widetilde{z}(t) + r(t)z_3(t)$. Then (3.31) is equivalent to

$$\begin{cases} \dot{\eta}(t) = -r(t)\eta(t) + r(t)\widetilde{d}(t) + \dfrac{\dot{r}(t)}{r(t)}\eta(t) \\ \qquad -r(t)z_2(t) + r^2(t)z_3(t), \\ \dot{\widetilde{d}}(t) = -r(t)\eta(t) + r^2(t)z_3(t) - \dot{d}(t). \end{cases} \tag{3.32}$$

By the local Lipschitz condition of the right-hand side of (3.32), we know that (3.32) has a local classical solution. Now, we show that (3.32) has a global solution by the Lyapunov function presented below. Actually, we can define a Lyapunov function for system (3.32) as follows:

$$V(x_1, x_2) = x_1^2 + 1.5x_2^2 - x_1x_2. \tag{3.33}$$

Then $V(x_1, x_2)$ is positive definite:

$$\frac{1}{2}V(x_1, x_2) \le x_1^2 + x_2^2 \le 2V(x_1, x_2), \ \forall\, x_1, x_2 \in \mathbb{R}.$$

Differentiate $V(\eta(t), \widetilde{d}(t))$ along the solution of (3.32) to obtain

$$\begin{aligned} &\dot{V}(\eta(t), \widetilde{d}(t)) \\ &= 2\eta(t)[-r(t)\eta(t) + r(t)\widetilde{d}(t) + \frac{\dot{r}(t)}{r(t)}\eta(t) - r(t)z_2(t) \\ &\quad + r^2(t)z_3(t)] - [-r(t)\eta(t) + r(t)\widetilde{d}(t) + \frac{\dot{r}(t)}{r(t)}\eta(t) \\ &\quad - r(t)z_2(t) + r^2(t)z_3(t)]\widetilde{d}(t) - \eta(t)[-r(t)\eta(t) \\ &\quad + r^2(t)z_3(t) - \dot{d}(t)] + 3\widetilde{d}(t)[-r(t)\eta(t) \\ &\quad + r^2(t)z_3(t) - \dot{d}(t)] \\ &= \left[-r(t) + 2\frac{\dot{r}(t)}{r(t)}\right]\eta^2(t) - r(t)\widetilde{d}^2(t) + \dot{d}(t)[\eta(t) \\ &\quad - 3\widetilde{d}(t)] - \eta(t)\widetilde{d}(t)\frac{\dot{r}(t)}{r(t)} + r(t)z_2(t)[-2\eta(t)+\widetilde{d}(t)] \\ &\quad + r^2(t)z_3(t)[\eta(t) + 2\widetilde{d}(t)] \\ &\le -\frac{1}{2}\phi(t)V(\eta(t), \widetilde{d}(t)) + \|(\eta(t), \widetilde{d}(t))\|[3r(t)|z_2(t)| \\ &\quad + 3r^2(t)|z_3(t)| + 4|\dot{d}(t)|] \\ &\le -\frac{1}{2}\phi(t)V(\eta(t), \widetilde{d}(t)) + \sqrt{2}[3r(t)|z_2(t)| \\ &\quad + 3r^2(t)|z_3(t)| + 4|\dot{d}(t)|]\sqrt{V(\eta(t), \widetilde{d}(t))}, \end{aligned} \tag{3.34}$$

where

$$\phi(t) = r(t) - 3\sup_{t\ge 0}\left|\frac{\dot{r}(t)}{r(t)}\right|.$$

By assumption (3.28), $\lim_{t\to\infty}\phi(t) = 0$. Re-organising (3.34) gives

$$\begin{aligned} \frac{d\sqrt{V(\eta(t), \widetilde{d}(t))}}{dt} &\le -\frac{1}{4}\phi(t)\sqrt{V(\eta(t), \widetilde{d}(t))} \\ &\quad + [3r(t)|z_2(t)| + 3r^2(t)|z_3(t)| + 4|\dot{d}(t)|], \end{aligned} \tag{3.35}$$

which shows that

$$\begin{aligned} &\sqrt{V(\eta(t), \widetilde{d}(t))} \le \sqrt{V(\eta(0), \widetilde{d}(0))}e^{-\frac{1}{4}\int_0^t \phi(s)ds} \\ &+ \frac{\int_0^t [3r(s)|z_2(s)| + 3r^2(s)|z_3(s)| + 4|\dot{d}(s)|]e^{\frac{1}{4}\int_0^s \phi(\tau)d\tau}ds}{e^{\frac{1}{4}\int_0^t \phi(s)ds}}. \end{aligned} \tag{3.36}$$

Equation (3.36) implies that the local solution never blows up, so the global solution of (3.32) exists. Since $e^{\frac{1}{4}\int_0^t \phi(s)ds} \to \infty$ as $t \to \infty$, we can apply the L'Hospital rule to the second term of the right-hand side of (3.36), to obtain

$$\begin{aligned} &\lim_{t\to\infty} \frac{\int_0^t [3r(s)|z_2(s)| + 3r^2(s)|z_3(s)| + 4|\dot{d}(s)|]e^{\frac{1}{4}\int_0^s \phi(\tau)d\tau}ds}{e^{\frac{1}{4}\int_0^t \phi(s)ds}} \\ &= 4\lim_{t\to\infty} \frac{3r(t)|z_2(t)| + 3r^2(t)|z_3(t)| + 4|\dot{d}(t)|}{\phi(t)} \\ &= 4\lim_{t\to\infty}\left[3|z_2(t)|\frac{r(t)}{\phi(t)} + 3r(t)|z_3(t)|\frac{r(t)}{\phi(t)} + 4\frac{|\dot{d}(t)|}{r(t)}|\frac{r(t)}{\phi(t)}\right] = 0, \end{aligned} \tag{3.37}$$

where in the last equality of (3.37), we used (3.26) and (3.28). It then follows from (3.36) and (3.37) that

$$\lim_{t\to\infty}\sqrt{V(\eta(t), \widetilde{d}(t))} = 0,$$

which implies

$$\lim_{t\to\infty}[|\eta(t)| + |\widetilde{d}(t)|] = 0. \tag{3.38}$$

Furthermore, since $\widetilde{z}(t) = \eta(t) - r(t)z_3(t)$, it follows from (3.28) that $\lim_{t\to\infty}|\widetilde{z}(t)| = 0$. By (3.28) and $\widehat{z}(t) - z(t) = \widetilde{z}(t)/r(t)$, we finally obtain

$$\lim_{t\to\infty} = |\widehat{z}(t) - z(t)| = 0. \tag{3.39}$$

Equation (3.29) then follows from (3.38) and (3.39). □

Since we have an estimation of disturbance and the system (3.10) with $(U_0(t), d(t)) = 0$ is stable ($f(t)$ is a small perturbation by (3.12)), we compensate the disturbance by designing

$$U_0(t) = -\widehat{d}(t). \tag{3.40}$$

It is seen that the term in the right-hand side of (3.40) is used to cancel the effect of the disturbance. This is just the estimation/cancellation nature of ADRC.

Under feedback control (3.40), the closed-loop of system (3.10) becomes

$$\begin{cases} \dot{X}(t) = (A + BK)X(t) + Bw(0,t), \ t > 0, \\ w_{tt}(x,t) = w_{xx}(x,t), \ x \in (0,1), \ t > 0, \\ w_x(0,t) = cw_t(0,t), \ t \geq 0, \\ w_x(1,t) = -kw(1,t) + f(t) - \widetilde{d}(t), \ t \geq 0. \end{cases} \tag{3.41}$$

Lemma 3.3: *For any initial value* $(X(0), w(\cdot,0), w_t(\cdot,0)) \in \mathcal{H}$. *If* $|f(t) - \widetilde{d}(t)| \to 0$ *as* $t \to \infty$, *then* (3.41) *admits a unique solution* $(X, w, \dot{w})^\top \in \mathcal{H}$ *satisfying*

$$\lim_{t\to\infty} \left[|X(t)|^2_{\mathbb{R}^n} + w^2(1,t) + \int_0^1 w_x^2(x,t)dx + \int_0^1 w_t^2(x,t)dx \right] = 0. \tag{3.42}$$

Proof: The existence and uniqueness of solution of system (3.41) has been proved by Proposition 3.1. Since $|f(t) - \widetilde{d}(t)| \to 0$ as $t \to \infty$, for any given $\kappa > 0$, we may suppose that $|f(t) - \widetilde{d}(t)| \leq \kappa$ for all $t > t_0$ for some $t_0 > 0$. Now, we write the solution of (3.41) as

$$\begin{aligned} \begin{pmatrix} X \\ w(\cdot,t) \\ w_t(\cdot,t) \end{pmatrix} &= e^{\mathbb{A}t} \begin{pmatrix} X(0) \\ w(\cdot,0) \\ w_t(\cdot,0) \end{pmatrix} + \int_0^t e^{\mathbb{A}(t-s)} \mathbb{B}(f(s) - \widetilde{d}(s))ds \\ &= e^{\mathbb{A}t} \begin{pmatrix} X(0) \\ w(\cdot,0) \\ w_t(\cdot,0) \end{pmatrix} + e^{\mathbb{A}(t-t_0)} \int_0^{t_0} e^{\mathbb{A}(t_0-s)} \mathbb{B}(f(s) \\ &\quad -\widetilde{d}(s))ds + \int_{t_0}^t e^{\mathbb{A}(t-s)} \mathbb{B}(f(s) - \widetilde{d}(s))ds. \end{aligned} \tag{3.43}$$

The admissibility of $\mathbb{B}$ claimed by Proposition 3.1 implies that

$$\left\| \int_0^t e^{\mathbb{A}(t-s)} \mathbb{B}(f(s) - \widetilde{d}(s))ds \right\|^2_{\mathcal{H}} \leq C_t \|(f - \widetilde{d})\|^2_{L^2(0,t)} \leq C_t t^2 \|(f - \widetilde{d})\|^2_{L^\infty(0,\infty)}, \ \forall\, t > 0,$$

for some constant C_t that is independent of $f(s) - \tilde{d}(s)$. Since by Lemma 3.1 $e^{\mathbb{A}t}$ is exponentially stable, it follows from Remark 2.6 of Weiss (1989) that

$$\begin{aligned} &\left\| \int_{t_0}^t e^{\mathbb{A}(t-s)} \mathbb{B}(f(s) - \widetilde{d}(s))ds \right\|_{\mathcal{H}} \\ &\leq \left\| \int_0^t e^{\mathbb{A}(t-s)} \mathbb{B}(0 \underset{t_0}{\diamondsuit} (f(s) - \widetilde{d}(s))ds \right\|_{\mathcal{H}} \\ &\leq L\|(f - \widetilde{d})\|_{L^\infty(t_0,\infty)} \leq L\kappa, \end{aligned}$$

where L is a constant which is independent of $\tilde{d}(t)$, and

$$(u \underset{\tau}{\diamondsuit} v)(t) = \begin{cases} u(t), 0 \leq t \leq \tau, \\ v(t), t > \tau. \end{cases}$$

Suppose that $\|e^{\mathbb{A}t}\| \leq L_0 e^{-\omega_2 t}$ for some $L_0, \omega_2 > 0$. We have

$$\begin{aligned} \left\| \begin{pmatrix} X(t) \\ w(\cdot,t) \\ w_t(\cdot,t) \end{pmatrix} \right\|_{\mathcal{H}} &\leq L_0 e^{-\omega_2 t} \left\| \begin{pmatrix} X(0) \\ w(\cdot,0) \\ w_t(\cdot,0) \end{pmatrix} \right\|_{\mathcal{H}} \\ &\quad + L_0 C_{t_0} t_0^2 e^{-\omega_2(t-t_0)} \|\tilde{d}\|_{L^\infty(0,t_0,L^2(\Gamma))} \\ &\quad + L\kappa. \end{aligned} \tag{3.44}$$

Passing to the limit as $t \to \infty$ for (3.44), we finally obtain

$$\lim_{t\to\infty} \left\| \begin{pmatrix} X(t) \\ w(\cdot,t) \\ w_t(\cdot,t) \end{pmatrix} \right\|_{\mathcal{H}} \leq L\kappa. \tag{3.45}$$

This proves (3.42). □

Combining (3.9) and (3.40), an output feedback control for the system (1.1) is designed as follows:

$$\begin{aligned} U(t) &= -\widehat{d}(t) + \int_0^1 \left(\mu'(1-y) + cm''(1-y)\right) \widehat{u}(y,t)dy \\ &\quad + \int_0^1 \left(m'(1-y) + c\mu(1-y)\right) \widehat{u}_t(y,t)dy \\ &\quad + \left(\gamma'(1) + c\gamma(1)A\right) \widehat{X}(t) + cKBu(1,t) \\ &\quad - cu_t(1,t) - k \Bigg\{ u(1,t) - \int_0^1 \mu(1-y)\widehat{u}(y,t)dy \\ &\quad - \int_0^1 m(1-y)\widehat{u}_t(y,t)dy - \gamma(1)\widehat{X}(t) \\ &\quad + c \int_0^1 \Bigg[\widehat{u}_t(x,t) - KB\widehat{u}(x,t) \\ &\quad - \int_0^x \mu(x-y)\widehat{u}_t(y,t)dy \\ &\quad \times \int_0^x m''(x-y)\widehat{u}(y,t)dy \Bigg] dx \\ &\quad - c \int_0^1 \gamma(x)dx A\widehat{X}(t) \Bigg\} \ t \geq 0. \end{aligned} \tag{3.46}$$

The closed-loop system of (1.1) then becomes

$$\begin{cases} \dot{X}(t) = AX(t) + Bu(0,t), \ t > 0, \\ u_{tt}(x,t) = u_{xx}(x,t), \ x \in (0,1), \ t > 0, \\ u_x(0,t) = 0, \ t \geq 0, \\ u_x(1,t) = U(t) + d(t), \ t \geq 0, \\ \dot{\widehat{X}}(t) = A\widehat{X}(t) + L(C\widehat{X} - CX) + B\widehat{u}(0,t), \ t > 0, \\ \widehat{u}_{tt}(x,t) = \widehat{u}_{xx}(x,t), \ x \in (0,1), \ t > 0, \\ \widehat{u}_x(0,t) = c_1[\widehat{u}_t(0,t) - u_t(0,t)], \ t \geq 0, \\ \widehat{u}_x(1,t) \in U(t) - c_2[\widehat{u}_t(1,t) - u_t(1,t)] \\ \qquad - c_3[\widehat{u}(1,t) - u(1,t)] - M_1 \text{sign}(\widehat{u}_t(1,t) \\ \qquad - u_t(1,t)) - M_2 \text{sign}(\widehat{u}(1,t) - u(1,t)), \\ \qquad t \geq 0, \\ \dot{\widehat{z}}(t) = z_1(t) - r(t)[\widehat{z}(t) - z(t)], \\ \dot{\widehat{d}}(t) = -r^2(t)[\widehat{z}(t) - z(t)], \\ \widehat{u}(x,0) = \widehat{u}_0(x), \widehat{u}_t(x,0) = \widehat{u}_1(x), 0 \leq x \leq 1, \end{cases} \tag{3.47}$$

where $U(t)$ is given by (3.46).

Theorem 3.1: *With the output feedback control (3.46), for any initial value $(X(0), u(\cdot,0), u_t(\cdot,0)) \in \mathbb{R}^n \times H^2(0,1) \times H^1(0,1)$, $(\widehat{X}(0), \widehat{u}(\cdot,0), \widehat{u}_t(\cdot,0)) \in \mathbb{R}^n \times H^2(0,1) \times H^1(0,1)$, $\widehat{z}(0), \widehat{d}(0) \in \mathbb{R}$, any partial differential inclusion solution of closed-loop system (3.47) satisfies*

$$\lim_{t\to\infty} F(t) = 0, \tag{3.48}$$

where

$$F(t) = \|X(t)\|^2_{\mathbb{R}^n} + u^2(1,t) + \int_0^1 u_x^2(x,t)dx + \int_0^1 u_t^2(x,t)dx + |\widehat{X}(t)|^2_{\mathbb{R}^n} + |\widehat{u}(1,t)|^2 + \int_0^1 \widehat{u}_x^2(x,t)dx + \int_0^1 \widehat{u}_t^2(x,t)dx + |\widehat{z}(t)| + |\widehat{d}(t) - d(t)|.$$

Proof: By the error variables $\widetilde{u}(x,t) = \widehat{u}(x,t) - u(x,t)$ in (2.2) and $(\widetilde{z}(t), \widetilde{d}(t)) = (r(t)[\widehat{z}(t) - z(t)], \widehat{d}(t) - d(t))$ in (3.30), the system (3.47) is equivalent to

$$\begin{cases} \dot{X}(t) = AX(t) + Bu(0,t), \ t > 0, \\ u_{tt}(x,t) = u_{xx}(x,t), \ x \in (0,1), \ t > 0, \\ u_x(0,t) = 0, \ t \geq 0, \\ u_x(1,t) = U(t) + d(t), \ t \geq 0, \\ \dot{\widetilde{X}}(t) = (A+LC)\widetilde{X}(t) + B\widetilde{u}(0,t), \ t > 0, \\ \widetilde{u}_{tt}(x,t) = \widetilde{u}_{xx}(x,t), \ x \in (0,1), \ t > 0, \\ \widetilde{u}_x(0,t) = c_1 \widetilde{u}_t(0,t), \ t \geq 0, \\ \widetilde{u}_x(1,t) \in -c_2\widetilde{u}_t(1,t) - c_3\widetilde{u}(1,t) - M_1\text{sign}(\widetilde{u}_t(1,t)) \\ \qquad - M_2 \text{sign}(\widetilde{u}(1,t)) - d(t), \ t \geq 0, \\ \dot{\widetilde{z}}(t) = -r(t)\widetilde{z}(t) + r(t)\widetilde{d}(t) + \dfrac{\dot{r}(t)}{r(t)}\widetilde{z}(t) \\ \qquad - r(t)[z_2(t) + \dot{z}_3(t)], \\ \dot{\widetilde{d}}(t) = -r(t)\widetilde{z}(t) - \dot{d}(t). \end{cases} \tag{3.49}$$

The '$(\widetilde{X}, \widetilde{u})$ part' of (3.49) is just the (2.3) and the '$(\widetilde{z}, \widetilde{d})$ part' of (3.49) is (3.31). By Theorem 2.1 and Lemma 3.2,

$$\int_0^1 [\widetilde{u}_x^2(x,t) + \widetilde{u}_t^2(x,t)]dx + \widetilde{u}^2(1,t) + |\widehat{z}(t) - z(t)| + |\widehat{d}(t) - d(t)| \to 0 \text{ as } t \to \infty. \tag{3.50}$$

Now, we show convergence of the '(X, u) part' of (3.49). To this end, we rewrite '(X, u) part' as follows:

$$\begin{cases} \dot{X}(t) = AX(t) + Bu(0,t), \\ u_{tt}(x,t) = u_{xx}(x,t), \ x \in (0,1), \ t > 0, \\ u_x(0,t) = 0, \\ u_x(1,t) = f(t) - \widetilde{d}(t) + \displaystyle\int_0^1 (\mu'(1-y) \\ \qquad + cm''(1-y)))u(y,t)dy + \displaystyle\int_0^1 (m'(1-y) \\ \qquad + c\mu(1-y))u_t(y,t)dy + (\gamma'(1) \\ \qquad + c\gamma(1)A)X(t) + cKBu(1,t) - cu_t(1,t) \\ \qquad -k\Big\{u(1,t) - \displaystyle\int_0^1 \mu(1-y)u(y,t)dy \\ \qquad - \displaystyle\int_0^1 m(1-y)u_t(y,t)dy - \gamma(1)\widehat{X}(t) \\ \qquad + c\displaystyle\int_0^1 \Big[u_t(x,t) - KBu(x,t) \\ \qquad - \displaystyle\int_0^x \mu(x-y)u_t(y,t)dy \\ \qquad - \displaystyle\int_0^x m''(x-y)u(y,t)dy\Big]dx \\ \qquad - c\displaystyle\int_0^1 \gamma(x)dxAX(t)\Big\}, \end{cases} \tag{3.51}$$

where $f(t)$ is given by (3.11). However, (3.51) is exactly the same as that in the closed-loop system (3.41) under the transformations (3.1) and (3.6). The convergence of (3.51) follows from (3.50), (3.11), and lemma 3.3. This ends the proof of the theorem. □

4. Concluding remarks

In this paper, we consider stabilisation for a wave PDE-ODE cascade system with boundary control and control matched disturbance. We first design an unknown input type observer by variable structure control method. Based on the estimated state, we design, by an idea of extended state observer, a disturbance estimator to estimate the external disturbance. The disturbance is then compensated in the feedback loop. This idea comes from ADRC approach, an emerging control technology. The closed-loop system is shown to be asymptotically stable. The idea is potentially promising for treating other uncertain PDEs.

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