Output feedback stabilization of a class of uncertain nonlinear systems

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***Abstract*— The problem of global output feedback stabilization for a class of nonlinear systems, whose zero dynamics are not necessarily stable, is addressed in this paper. It is shown that, using a novel observer design tool together with standard backstepping and small-gain techniques, it is possible to design a stabilizing output feedback controller, which ensures robustness with respect to dynamic uncertainties. The proposed stabilization method generalizes existing tools in several directions. Finally, the method is illustrated by means of a simple example.**

I. INTRODUCTION

The problem of output feedback stabilization of nonlinear systems has been an active area of research in recent years. Several control methodologies have been proposed, which achieve global or semiglobal results by exploiting certain feedback structures. In particular, the class of systems in "triangular" form has received special attention, see *e.g.* [8], [6] and references therein.

In [6] an adaptive backstepping method, known as tuning functions design, has been introduced and has been used extensively on systems with parametric uncertainties. In [3] this method has been combined with the *nonlinear small-gain theorem* [4] and the notion of *input-to-state stability* [9] to tackle systems with unstructured dynamic uncertainties, described by equations of the form

$$\begin{aligned} \dot{\eta} &= f(\eta, x_1) \\ \dot{x}_1 &= x_2 + \Delta_1(\eta, x_1) \\ &\vdots \\ \dot{x}_i &= x_{i+1} + \Delta_i(\eta, x_1) \\ &\vdots \\ \dot{x}_n &= u + \Delta_n(\eta, x_1) \\ y &= x_1, \end{aligned} \tag{1}$$

where $(\eta, x_1, \ldots, x_n) \in \mathbb{R}^m \times \mathbb{R} \times \cdots \times \mathbb{R}$ is the state of the system, $y = x_1$ is the output and u is the control input. It is assumed that only x_1 is available for measurement and $\Delta_i(\cdot)$ are uncertain functions. Note that in [6] the functions $\Delta_i(\cdot)$ are replaced by the *structured* uncertainty $\phi_i(y)^T\eta$, where η is a vector of unknown parameters (*i.e.* $\dot{\eta} = 0$). Special instances of the system (1) have also been studied in [7], where the matrix $\partial f/\partial\eta$ is constant and Hurwitz.

A common hypothesis in the aforementioned methods is that the zero dynamics of the considered systems possess some strong stability property, *i.e.* they are globally asymptotically stable (GAS) or input-to-state stable (ISS). A method by means of which it is possible to relax this assumption has been recently proposed in [2] and has been shown to achieve semiglobal practical stability. Note that in [2] the functions $\Delta_i(\cdot)$ may depend also on the unmeasured states $x_2, \ldots, x_i$.

The purpose of this paper is to partially extend the results of [3]. In particular, we relax the hypothesis that the η-subsystem is ISS with respect to x_1 and replace it with an input-to-state stabilizability condition (see Assumption 2). This is mainly achieved by extending the reduced-order observer design of [3] using the methodology proposed in [5], which, in turn, is based on the stabilization tools developed in [1].

The paper is organized as follows. In Section II we define the considered class of systems and the assumptions under which the proposed method will be applicable. In Section III we propose a reduced-order observer for the unmeasured states and design the output feedback controller by combining a backstepping construction with a small-gain condition. Some special cases are considered in Section IV. In Section V we apply the proposed method to a paradigm comprising an uncertain system with unstable (linear) zero dynamics and in Section VI we provide some conclusions.

II. PROBLEM FORMULATION

We consider a class of uncertain nonlinear systems described by equations of the form

$$\begin{aligned} \dot{\eta} &= F(x_1)\eta + G(x_1) + \Delta_0(\eta, x_1) \\ \dot{x}_1 &= x_2 + \phi_1(x_1)^T\eta + \Delta_1(\eta, x_1) \\ &\vdots \\ \dot{x}_i &= x_{i+1} + \phi_i(x_1)^T\eta + \Delta_i(\eta, x_1) \\ &\vdots \\ \dot{x}_n &= u + \phi_n(x_1)^T\eta + \Delta_n(\eta, x_1) \\ y &= x_1 \end{aligned} \tag{2}$$

with state $(\eta, x_1, \ldots, x_n) \in \mathbb{R}^m \times \mathbb{R} \times \cdots \times \mathbb{R}$, input u and output $y = x_1$, where $\Delta_i(\cdot)$ are unknown perturbation functions. We assume that the origin is an equilibrium for the system (2) with $u = 0$, *i.e.* $\Delta_i(0, 0) = 0$ and $G(0) = 0$,

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and all functions are sufficiently smooth and we pose the following stabilization problem.

Problem 1 (Output feedback stabilization): For the system (2), find (if possible) a dynamic output feedback control law described by equations of the form

$$\begin{array}{rcl} \dot{\xi} & = & \pi(\xi, y) \\ u & = & \alpha(\xi, y) \end{array} \tag{3}$$

with $\xi \in \mathbb{R}^p$ such that the closed-loop system (2)-(3) is globally asymptotically stable.

Note that the system (2) has relative degree n and its zero dynamics are given by

$$\dot{\eta} = F(0)\eta + \Delta_0(\eta, 0),$$

hence they are not necessarily stable. Moreover, the functions $\Delta_i(\cdot)$ need not be bounded. However, the following conditions must hold.

Assumption 1: There exist known positive-definite and smooth functions $\rho_{i1}(\cdot)$ and $\rho_{i2}(\cdot)$, $i = 0, \dots, n$, such that

$$|\Delta_i(\eta, x_1)|^2 \leq \rho_{i1}(|\eta|) + \rho_{i2}(|x_1|), \tag{4}$$

where $|\cdot|$ denotes the Euclidean norm, the function $\rho_{11}(|\eta|)$ is quadratic and $\rho_{12}(|x_1|) = 0$.

Assumption 2: There exists a smooth function $x_1^\star(\eta)$ such that the system

$$\begin{array}{rcl} \dot{\eta} & = & F(x_1^\star(\eta + d_1) + d_2)\eta + G(x_1^\star(\eta + d_1) + d_2) \\ & & +\Delta_0(\eta, x_1^\star(\eta + d_1) + d_2) \end{array}$$

is ISS with respect to d_1 and d_2, *i.e.* there exists a positive-definite and proper function $V_1(\eta)$ such that

$$\dot{V}_1 \leq -\kappa_{11}(|\eta|) + \gamma_{11}(|d_1|) + \gamma_{12}(|d_2|),$$

where κ_{11}, γ_{11} and γ_{12} are smooth class-$\mathcal{K}_\infty$ functions.

Remark 1: In [3] it is assumed that the η-subsystem is ISS with respect to x_1, *i.e.* Assumption 2 holds for $x_1^\star = 0$. It must be noted that in [3] the functions ρ_i of Assumption 1 are multiplied by *unknown* coefficients, which are estimated on-line using standard Lyapunov techniques.

Remark 2: Assumption 2 is a *robust stabilizability* condition on the zero dynamics. In the linear case, *i.e.* when the matrix F is constant and the vectors G and Δ_0 are linear functions, it is always satisfied, if the pair $(F + \partial\Delta_0/\partial\eta, G + \partial\Delta_0/\partial x_1)$ is stabilizable, or if the pair (F, G) is stabilizable and $\partial\Delta_0/\partial\eta$ and $\partial\Delta_0/\partial x_1$ are sufficiently small.

III. Output feedback stabilization

In this section a solution to the output feedback stabilization problem is proposed based on a reduced-order observer and a combination of backstepping and small-gain ideas. In particular, it is shown that the closed-loop system can be described as an interconnection of ISS subsystems, whose gains can be tuned to satisfy the small-gain theorem.

A. Reduced-order observer design

To begin with, we will construct an observer for the unmeasured states η and $x_2, \dots, x_n$. To this end, we define the estimation errors

$$\begin{array}{rcl} z_1 & = & \hat{\eta} - \eta + \beta_1(x_1) \\ z_2 & = & \hat{x}_2 - x_2 + \beta_2(x_1) \\ & \vdots & \\ z_n & = & \hat{x}_n - x_n + \beta_n(x_1) \end{array}$$

and the update laws

$$\begin{array}{rcl} \dot{\hat{\eta}} & = & F(x_1)\left(\hat{\eta} + \beta_1(x_1)\right) + G(x_1) \\ & & -\dfrac{\partial \beta_1}{\partial x_1}\left[\hat{x}_2 + \beta_2(x_1) + \phi_1(x_1)^T\left(\hat{\eta} + \beta_1(x_1)\right)\right] \\ \dot{\hat{x}}_2 & = & \hat{x}_3 + \beta_3(x_1) + \phi_2(x_1)^T\left(\hat{\eta} + \beta_1(x_1)\right) \\ & & -\dfrac{\partial \beta_2}{\partial x_1}\left[\hat{x}_2 + \beta_2(x_1) + \phi_1(x_1)^T\left(\hat{\eta} + \beta_1(x_1)\right)\right] \\ & \vdots & \\ \dot{\hat{x}}_n & = & u + \phi_n(x_1)^T\left(\hat{\eta} + \beta_1(x_1)\right) \\ & & -\dfrac{\partial \beta_n}{\partial x_1}\left[\hat{x}_2 + \beta_2(x_1) + \phi_1(x_1)^T\left(\hat{\eta} + \beta_1(x_1)\right)\right], \end{array}$$

where $\beta_i(x_1)$ are continuous functions yet to be defined. The "error dynamics" are described by the system

$$\dot{z} = A(x_1)z - \Delta_r(\eta, x_1) + \Delta_1(\eta, x_1)\frac{\partial \beta}{\partial x_1}, \tag{5}$$

where

$$z = \left[z_1^T, z_2, \dots, z_n\right]^T,$$

$$\Delta_r = \left[\Delta_0^T, \Delta_2, \dots, \Delta_n\right]^T,$$

$$\beta = \left[\beta_1^T, \beta_2, \dots, \beta_n\right]^T$$

and

$$A(x_1) = \left[\begin{array}{c|cccc} F(x_1) & 0 & 0 & \cdots & 0 \\ \hline \phi_2(x_1)^T & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & & \ddots & \\ \phi_{n-1}(x_1)^T & 0 & 0 & \cdots & 1 \\ \phi_n(x_1)^T & 0 & 0 & \cdots & 0 \end{array}\right] - \frac{\partial \beta}{\partial x_1}\left[\begin{array}{c|cccc} \phi_1(x_1)^T & 1 & 0 & \dots & 0 \end{array}\right]. \tag{6}$$

In addition to the estimation error z, we define the *output error*

$$\tilde{x}_1 = x_1 - x_1^\star,$$

where

$$x_1^\star = x_1^\star(\hat{\eta} + \beta_1(x_1)) = x_1^\star(\eta + z_1)$$

verifies Assumption 2.

Consider now the function $V_2(z) = z^T P z$, where P is a constant, positive-definite matrix, and its time-derivative along the trajectories of (5), namely

$$\begin{aligned}\dot{V}_2 &= z^T \left(A(x_1)^T P + P A(x_1) \right) z \\ &\quad -2\Delta_r(\eta, x_1)^T P z + 2\Delta_1(\eta, x_1) \frac{\partial \beta}{\partial x_1}^T P z.\end{aligned}$$

Define the matrix

$$B(x_1) = I + \frac{\partial \beta}{\partial x_1} \frac{\partial \beta}{\partial x_1}^T$$

and note that

$$\begin{aligned}\dot{V}_2 &\leq z^T \left(A(x_1)^T P + P A(x_1) + \frac{P B(x_1) P}{\gamma(x_1)} \right) z \\ &\quad +\gamma(x_1) \left(|\Delta_r(\eta, x_1)|^2 + |\Delta_1(\eta, x_1)|^2 \right),\end{aligned}$$

for any function $\gamma(x_1) > 0$. From Assumption 1 and the definition of $\tilde{x}_1$ it is possible to select functions $\gamma_{21}(\cdot)$, $\gamma_{22}(\cdot)$ and $\gamma_{23}(\cdot)$ such that

$$\begin{aligned}\dot{V}_2 &\leq z^T \left(A(x_1)^T P + P A(x_1) + \frac{P B(x_1) P}{\gamma(x_1)} \right) z \\ &\quad +\gamma_{21}(|\eta|) + \gamma_{22}(|\tilde{x}_1|) + \gamma_{23}(|z_1|).\end{aligned} \tag{7}$$

Consider now the following condition.

Assumption 3: There exist functions $\beta_i(x_1)$ and a class-$\mathcal{K}_\infty$ function $\kappa_{21}(\cdot)$ such that, for any x_1,

$$\begin{aligned}z^T &\left(A(x_1)^T P + P A(x_1) + \frac{P B(x_1) P}{\gamma(x_1)} \right) z \\ &\quad +\gamma_{23}(|z_1|) \leq -\kappa_{21}(|z|).\end{aligned}$$

Remark 3: Assumption 3 is a *robust detectability* condition on the system (2) and can be considered as dual to Assumption 2. In fact, in the linear case, it is a necessary and sufficient condition for detectability when $\Delta_i = 0$ (see Section IV-D).

B. Small-gain condition

Consider again the η-subsystem, which is described by the equation

$$\begin{aligned}\dot{\eta} &= F(x_1^\star(\eta + z_1) + \tilde{x}_1)\eta + G(x_1^\star(\eta + z_1) + \tilde{x}_1) \\ &\quad +\Delta_0(\eta, x_1^\star(\eta + z_1) + \tilde{x}_1),\end{aligned} \tag{8}$$

and note that from Assumption 2 we have

$$\dot{V}_1 \leq -\kappa_{11}(|\eta|) + \gamma_{11}(|z_1|) + \gamma_{12}(|\tilde{x}_1|). \tag{9}$$

Moreover, from Assumption 3 and condition (7) we conclude that the system

$$\begin{aligned}\dot{z} &= A(x_1^\star(\eta + z_1) + \tilde{x}_1) z - \Delta_r(\eta, x_1^\star(\eta + z_1) + \tilde{x}_1) \\ &\quad +\Delta_1(\eta, x_1^\star(\eta + z_1) + \tilde{x}_1) \frac{\partial \beta}{\partial x_1}\end{aligned} \tag{10}$$

is ISS with respect to η and $\tilde{x}_1$, *i.e.*

$$\dot{V}_2 \leq -\kappa_{21}(|z|) + \gamma_{21}(|\eta|) + \gamma_{22}(|\tilde{x}_1|). \tag{11}$$

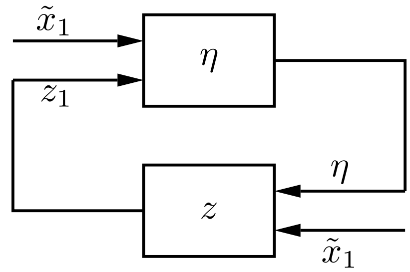

Fig. 1. Block diagram of the interconnected systems (8) and (10).

Thus we have assumed that each of the systems (8) and (10) can be rendered ISS by selecting the functions $\beta_i(x_1)$ and $x_1^\star(\eta + z_1)$ appropriately. In the following we will consider the stability of their interconnection (depicted in Fig. 1) by means of the Lyapunov formulation of the nonlinear small-gain theorem [4].

To this end, define class-$\mathcal{K}_\infty$ functions κ_1, κ_2, γ_1, γ_2 such that

$$\begin{array}{rclcl}\gamma_2^{-1} \circ \gamma_{21}(|\eta|) & \leq & V_1(\eta) & \leq & \kappa_1^{-1} \circ \kappa_{11}(|\eta|) \\ \gamma_1^{-1} \circ \gamma_{11}(|z_1|) & \leq & V_2(z) & \leq & \kappa_2^{-1} \circ \kappa_{21}(|z|)\end{array} \tag{12}$$

and note that conditions (9) and (11) can be written as

$$\begin{aligned}\dot{V}_1 &\leq -\kappa_1(V_1) + \gamma_1(V_2) + \gamma_{12}(|\tilde{x}_1|) \\ \dot{V}_2 &\leq -\kappa_2(V_2) + \gamma_2(V_1) + \gamma_{22}(|\tilde{x}_1|).\end{aligned}$$

We are now ready to state the main result of the paper.

Theorem 1: Consider a system described by equations of the form (2) and such that Assumptions 1, 2 and 3 hold. Let κ_1, κ_2, γ_1 and γ_2 be class-$\mathcal{K}_\infty$ functions satisfying (12) with V_1 as in Assumption 2 and $V_2 = z^T P z$ and suppose that there exist constants $0 < \varepsilon_1 < 1$ and $0 < \varepsilon_2 < 1$ such that

$$\frac{1}{1 - \varepsilon_1} \kappa_1^{-1} \circ \gamma_1 \circ \left(\frac{1}{1 - \varepsilon_2} \kappa_2^{-1} \circ \gamma_2(r) \right) < r, \tag{13}$$

for all $r > 0$. Then the system (8)-(10) with input $\tilde{x}_1$ is ISS. If, in addition, the ISS gain of this system is locally Lipschitz, then there exists a dynamic output feedback control law, described by equations of the form (3), such that the closed-loop system (2)-(3) is GAS.

Remark 4: Theorem 1 states that it is possible to globally asymptotically stabilize the system (2), where $\Delta_i(\cdot)$ satisfy the growth condition (4), provided three subproblems are solvable. The first problem is the robust stabilization of the η-subsystem with input x_1 (Assumption 2). The second problem is the input-to-state stabilization of the observer dynamics with respect to η (Assumption 3). The third problem is the stabilization of the interconnection of the two subsystems, which can be achieved by satisfying the small-gain condition (13). This "reduction" idea is also the basis of the methodology proposed in [2], although therein an entirely different route is followed.

Proof: Using condition (13) and the small-gain theorem in [4] it is straightforward to show that the system (8)-(10) with input $\tilde{x}_1$ is ISS. Since the gain of this system is locally Lipschitz, it suffices to prove that there exists a continuous control law $u(x_1, \hat{x}_2, \ldots, \hat{x}_n, \hat{\eta})$ such that the gain of the system with state $(\tilde{x}_1, \hat{x}_2, \ldots, \hat{x}_n, \hat{\eta})$, output $\tilde{x}_1$ and input (η, z) can be arbitrarily assigned. This can be achieved using a standard backstepping construction. Namely, starting from the dynamics of $\tilde{x}_1$, which are described by the equation

$$\begin{aligned}\dot{\tilde{x}}_1 &= \hat{x}_2 + \beta_2(x_1) - z_2 + \phi_1(x_1)^T\left(\hat{\eta} + \beta_1(x_1) - z_1\right)\\ &+\Delta_1(\eta, x_1) - \frac{\partial x_1^\star}{\partial(\eta + z_1)}\Big[F(x_1)\left(\hat{\eta} + \beta_1(x_1)\right)\\ &+G(x_1) + \frac{\partial \beta_1}{\partial x_1}\left(-z_2 - \phi_1(x_1)^T z_1 + \Delta_1(\eta, x_1)\right)\Big],\end{aligned}$$

we consider $\hat{x}_2$ as a virtual control input and define the error $\tilde{x}_2 = \hat{x}_2 - x_2^\star$, where

$$\begin{aligned}x_2^\star &= \lambda_1(x_1, \hat{\eta}) - \beta_2(x_1) - \phi_1(x_1)^T\left(\hat{\eta} + \beta_1(x_1)\right)\\ &+\frac{\partial x_1^\star}{\partial(\eta + z_1)}\Big[F(x_1)\left(\hat{\eta} + \beta_1(x_1)\right) + G(x_1)\Big],\end{aligned}$$

for some function $\lambda_1(\cdot)$ yet to be defined. Continuing with this design philosophy through the dynamics of $\tilde{x}_2, \tilde{x}_3, \ldots, \tilde{x}_n$, we finally select the control u as

$$\begin{aligned}u &= \lambda_n(x_1, \hat{x}_2, \ldots, \hat{x}_n, \hat{\eta}) - \phi_n(x_1)^T\left(\hat{\eta} + \beta_1(x_1)\right)\\ &+\frac{\partial \beta_n}{\partial x_1}\left[\hat{x}_2 + \beta_2(x_1) + \phi_1(x_1)^T\left(\hat{\eta} + \beta_1(x_1)\right)\right]\\ &+\frac{\partial x_n^\star}{\partial x_1}\left[\hat{x}_2 + \beta_2(x_1) + \phi_1(x_1)^T\left(\hat{\eta} + \beta_1(x_1)\right)\right]\\ &+\sum_{i=2}^{n-1}\frac{\partial x_n^\star}{\partial \hat{x}_i}\dot{\hat{x}}_i + \frac{\partial x_n^\star}{\partial \hat{\eta}}\dot{\hat{\eta}}.\end{aligned}$$

A straightforward calculation shows that the functions $\lambda_i(\cdot)$ can be selected in such a way that the $\tilde{x}$-subsystem admits the ISS Lyapunov function $W(\tilde{x}) = |\tilde{x}|^2$. Moreover, *for any* positive constants α, ϵ_1 and ϵ_2,

$$\dot{W} \le -\alpha|\tilde{x}|^2 + \epsilon_1|\eta|^2 + \epsilon_2|z|^2. \tag{14}$$

Note that the quadratic term in η is a result of the perturbation Δ_1 and the fact that the function ρ_{11} in Assumption 1 is quadratic. ■

IV. SPECIAL CASES

In this section, we discuss the applicability of Theorem 1 for special cases of systems described by equations of the form (2). It is worth noting that Theorem 1 is more general than some of the results in [7], [6], [3], although therein unknown parameters are also present.

A. Systems without zero dynamics

Consider the system (2), where η is an empty vector, *i.e.* there are no zero dynamics, and supppose that Assumption 1 holds. Note that, in this case, Assumption 2 is trivially satisfied with $x_1^\star = 0$ (*i.e.* $\tilde{x}_1 = x_1$). Then the matrix (6) can be reduced to a constant $(n-1) \times (n-1)$ Hurwitz matrix by selecting

$$\beta_i(x_1) = k_i x_1, \qquad i = 2, \ldots, n$$

and choosing the constants k_i appropriately. Moreover, we can select $\gamma_{21} = \gamma_{23} = 0$. As a result, Assumption 3 is trivially satisfied for any linear function $\kappa_{21}(\cdot)$ by taking γ sufficiently large and condition (13) holds.

B. Systems with ISS zero dynamics

Consider the system (2) and suppose that Assumptions 1 and 2 hold for $x_1^\star = 0$, *i.e.* the η-subsystem is ISS with respect to x_1. Then condition (9) reduces to

$$\dot{V}_1 \le -\kappa_{11}(\eta) + \gamma_{12}(|x_1|),$$

i.e. $\gamma_{11} = 0$, hence condition (13) holds. Finally, Assumption 3 is simplified with $\gamma_{23} = 0$. Note that, in this case, we could define new perturbation functions

$$\Delta_i'(\eta, x_1) = \phi_i(x_1)^T\eta + \Delta_i(\eta, x_1), \qquad i = 2, \ldots, n$$

and select the functions $\beta_i(x_1)$ as in Section IV-A to yield a constant Hurwitz matrix A, thus recovering the design proposed in [3].

C. Unperturbed systems

Assumption 3 and condition (13) can be relaxed in the case of an unperturbed system, *i.e.* with $\Delta_i(\cdot) = 0$, as the following corollary shows.

Corollary 1: Consider a system described by equations of the form (2) with $\Delta_i(\cdot) = 0$, $i = 0, 1, \ldots, n$, and such that Assumption 2 holds. Suppose that there exist functions $\beta_i(x_1)$, $i = 1, \ldots, n$ and a positive-definite matrix P such that

$$z^T\left(A(x_1)^T P + PA(x_1)\right)z \le -\kappa_{21}(|z|),$$

for any x_1. Then there exists a dynamic output feedback control law, described by equations of the form (3), such that the closed-loop system (2)-(3) is GAS.

Proof: We simply verify that Theorem 1 applies. To begin with, note that Assumption 1 is trivially satisfied, and Assumptions 2 and 3 hold by hypothesis. Consider now conditions (12) and note that the function γ_{21} is zero, hence γ_2 can be arbitrarily selected. As a result, condition (13) holds. ■

D. Linear perturbed systems

Consider a linear system described by equations of the form

$$\begin{array}{rcl} \dot{\eta} & = & F_0\eta + Gx_1 + \Delta_0(\eta, x_1) \\ \dot{x}_1 & = & x_2 + F_1^T\eta + \Delta_1(\eta, x_1) \\ & \vdots & \\ \dot{x}_n & = & u + F_n^T\eta + \Delta_n(\eta, x_1) \\ y & = & x_1 \end{array} \tag{15}$$

and suppose that Assumption 1 holds for quadratic functions ρ_{i1} and ρ_{i2} and Assumption 2 holds for a linear function $x_1^\star(\eta + z_1)$. The system (15) can be written in matrix form as

$$\left[\begin{array}{c} \dot{\zeta} \\ \dot{x}_1 \end{array}\right] = \left[\begin{array}{c|c} A_0 & \begin{array}{c} G \\ \vdots \\ 0 \end{array} \\ \hline C_0 & 0 \end{array}\right] \left[\begin{array}{c} \zeta \\ x_1 \end{array}\right] + \left[\begin{array}{c} 0 \\ \vdots \\ 1 \\ \hline 0 \end{array}\right] u + \Delta(\eta, x_1)$$

$$y = \left[\begin{array}{cc} 0 & 1 \end{array}\right] \left[\begin{array}{c} \zeta \\ x_1 \end{array}\right],$$

where $\zeta = \left[\eta^T, x_2, \ldots, x_n\right]^T$, $\Delta = \left[\Delta_0^T, \Delta_2, \ldots, \Delta_n, \Delta_1\right]^T$,

$$A_0 = \left[\begin{array}{ccccc} F_0 & 0 & 0 & \cdots & 0 \\ F_2^T & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & & \ddots & \\ F_n^T & 0 & 0 & \cdots & 0 \end{array}\right],$$

$$C_0 = \left[\begin{array}{ccccc} F_1^T & 1 & 0 & \cdots & 0 \end{array}\right].$$

Define the function

$$\beta(x_1) = Kx_1,$$

where K is a constant vector, and note that the matrix (6) can be written as

$$A = A_0 - KC_0.$$

As a result, Assumption 3 can be replaced by the following.

Assumption 4: There exist a vector K and a constant $\kappa_{21} > 0$ such that

$$z^T \left(A^T P + PA + \frac{P\left(I + KK^T\right)P}{\gamma} \right) z \\ + \gamma_{23}|z_1|^2 \leq -\kappa_{21}|z|^2.$$

Remark 5: If the system (15) is detectable for $\Delta = 0$, then the pair $\{A_0, C_0\}$ is also detectable, hence there exists a positive-definite matrix P such that the matrix $A^TP + PA$ is negative-definite. Note that, since γ_{23} depends on γ, this does not imply (in general) that Assumption 4 holds. However, if the system (15) is also minimum-phase, then $\gamma_{23} = 0$ and Assumption 4 is always satisfied for sufficiently large γ.

Finally, conditions (9) and (11) reduce respectively to

$$\dot{V}_1 \leq -\kappa_{11}|\eta|^2 + \gamma_{11}|z_1|^2 + \gamma_{12}|\tilde{x}_1|^2$$

and

$$\dot{V}_2 \leq -\kappa_{21}|z|^2 + \gamma_{21}|\eta|^2 + \gamma_{22}|\tilde{x}_1|^2.$$

Hence, the small-gain condition (13) reduces to

$$\frac{\gamma_{11}}{(1-\varepsilon_1)\kappa_{11}} \frac{\gamma_{21}}{(1-\varepsilon_2)\kappa_{21}} < 1.$$

V. An illustrative example

In this section we apply the proposed method to a simple example, whose zero dynamics are linear and unstable, hence the result in [3] is not applicable. Consider the system

$$\begin{array}{rcl} \dot{\eta} & = & \eta + x_1 + \delta(t)x_1 \\ \dot{x}_1 & = & x_2 + \left(1 + x_1^2\right)\eta \\ \dot{x}_2 & = & u + \left(2 + x_1^2\right)\eta \\ y & = & x_1, \end{array} \tag{16}$$

where $\delta(t)$ is an unknown disturbance such that $|\delta(t)| \leq p$, for all t, with $p \in [0, 1)$ a known constant. Hence, Assumption 1 is satisfied with $\rho_1(|\eta|) = 0$ and $\rho_2(|x_1|) = p^2x_1^2$.

Assumption 2 is also satisfied with the function

$$x_1^\star(\eta) = -k_1\eta,$$

for some positive constant k_1. In fact, the time-derivative of the function $V_1(\eta) = \eta^2/2$ along the trajectories of the system

$$\dot{\eta} = \eta + (1 + \delta(t))\left(x_1^\star(\eta + z_1) + \tilde{x}_1\right) \tag{17}$$

is given by

$$\begin{array}{rcl} \dot{V}_1 & = & -\left(k_1\left(1 + \delta(t)\right) - 1\right)\eta^2 - k_1\left(1 + \delta(t)\right)z_1\eta \\ & & + \left(1 + \delta(t)\right)\tilde{x}_1\eta, \end{array}$$

which implies

$$\begin{array}{rcl} \dot{V}_1 & \leq & -\left(k_1\left(1 - p\right) - 1 - \gamma_1\right)\eta^2 + \dfrac{k_1^2(1+p)^2}{2\gamma_1}z_1^2 \\ & & + \dfrac{(1+p)^2}{2\gamma_1}\tilde{x}_1^2, \end{array} \tag{18}$$

for some $\gamma_1 > 0$. Hence, for $k_1 > (1 + \gamma_1)/(1 - p)$, the system (17) is ISS with respect to z_1 and $\tilde{x}_1$. The error dynamics (5) are given by the system

$$\left[\begin{array}{c} \dot{z}_1 \\ \dot{z}_2 \end{array}\right] = \left[\begin{array}{c|c} 1 - \dfrac{\partial\beta_1}{\partial x_1}\left(1 + x_1^2\right) & -\dfrac{\partial\beta_1}{\partial x_1} \\ \hline 2 + x_1^2 - \dfrac{\partial\beta_2}{\partial x_1}\left(1 + x_1^2\right) & -\dfrac{\partial\beta_2}{\partial x_1} \end{array}\right] \left[\begin{array}{c} z_1 \\ z_2 \end{array}\right] \\ - \left[\begin{array}{c} \delta(t)x_1 \\ 0 \end{array}\right].$$

Consider the Lyapunov function $V_2(z) = z^T P z$ with

$$P = \begin{bmatrix} 1 & 0 \\ 0 & a \end{bmatrix},$$

where $a > 1$ is a constant. Assigning the functions $\beta_1(x_1)$ and $\beta_2(x_1)$ so that

$$\frac{\partial \beta_1}{\partial x_1} = \frac{a}{1+x_1^2},$$

$$\frac{\partial \beta_2}{\partial x_1} = \frac{2+x_1^2}{1+x_1^2} - \frac{1}{(1+x_1^2)^2}$$

yields

$$\dot{V}_2 \le -c_1 z_1^2 - c_2 z_2^2 + \gamma_2 p^2 x_1^2 + \frac{1}{4\gamma_2} z_1^2,$$

for some $\gamma_2 > 0$, where $c_2 = 1 + c_1 = a$. Noting that

$$\begin{aligned} x_1^2 \quad \le \quad & (1+d)\,\tilde{x}_1^2 + k_1^2 \left(1+\frac{1}{d}\right)^2 \eta^2 \\ & + k_1^2 \left(1+\frac{1}{d}\right)(1+d)\, z_1^2 \end{aligned}$$

with $d > 0$ yields

$$\begin{aligned} \dot{V}_2 \quad \le \quad & -\left(c_1 - \frac{1}{4\gamma_2} - \gamma_2 p^2 k_1^2 \left(1+\frac{1}{d}\right)(1+d)\right) z_1^2 \\ & - c_2 z_2^2 + \gamma_2 p^2 k_1^2 \left(1+\frac{1}{d}\right)^2 \eta^2 + \gamma_2 p^2 (1+d)\, \tilde{x}_1^2. \end{aligned}$$

Hence, for sufficiently large c_1, Assumption 3 is satisfied. The design is completed by choosing all the constants in the foregoing inequalities to satisfy the small-gain condition. Clearly, such a selection is always possible since the constants c_1 and c_2 can be chosen arbitrarily large. Fig. 2 shows the response of the closed-loop system to the initial conditions $\eta(0) = -1$, $x_1(0) = 0$, $x_2(0) = 0$ for various disturbances $\delta(t)$.

Remark 6: Applying the change of co-ordinates $\xi_1 = x_1$, $\xi_2 = x_2 + (1 + x_1^2)\eta$, the system (16) can be transformed into the system

$$\begin{aligned} \dot{\eta} &= \eta + \xi_1 + \delta(t)\xi_1 \\ \dot{\xi}_1 &= \xi_2 \\ \dot{\xi}_2 &= u + \left(3 + 2\xi_1^2 + 2\xi_1\xi_2\right)\eta + \left(1+\xi_1^2\right)\xi_1 \left(1+\delta(t)\right) \\ y &= \xi_1, \end{aligned}$$

for which the result in [2] is applicable. However, its application hinges upon the hypothesis (see Assumption 2 in [2]) that a robust global output feedback stabilizer is available for the auxiliary system

$$\begin{aligned} \dot{\eta} &= \eta + \xi_1 + \delta(t)\xi_1 \\ \dot{\xi}_1 &= u_a \\ y_a &= \left(3 + 2\xi_1^2 + 2\xi_1 u_a\right)\eta + \left(1+\xi_1^2\right)\xi_1 \left(1+\delta(t)\right) \end{aligned}$$

with input u_a and output y_a. Although it may be possible in this case to satisfy this hypothesis, it is certainly not a trivial task.

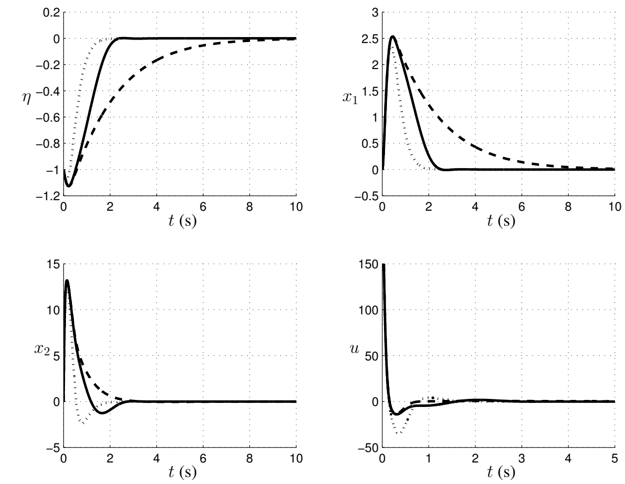

Fig. 2. Initial response of the system for various disturbances. Dotted line: $\delta(t) = 0$. Dashed line: $\delta(t) = -0.4$. Solid line: $\delta(t) = -0.4\cos(t)$.

VI. Conclusions and outlook

The problem of output feedback stabilization of a class of nonlinear systems with dynamic uncertainties has been studied. It has been shown that, by using a novel observer design tool together with a standard backstepping construction and a small-gain condition, it is possible to obtain a globally stabilizing output feedback control law. The proposed method applies to systems with unstable zero dynamics, thus extending the result in [3]. It also allows for cross-terms between the output and the unmeasured states to appear in the system equations, hence it is more general than the observer backstepping method used in [6]. However, in the present context, it does not allow for parametric uncertainties. An extension in such a direction is currently under study.

References

[1] A. Astolfi and R. Ortega. Immersion and invariance: a new tool for stabilization and adaptive control of nonlinear systems. *IEEE Trans. Automatic Control*, 48(4):590–606, 2003.

[2] A. Isidori. A tool for semiglobal stabilization of uncertain non-minimum-phase nonlinear systems via output feedback. *IEEE Trans. Automatic Control*, 45(10):1817–1827, 2000.

[3] Z.-P. Jiang. A combined backstepping and small-gain approach to adaptive output feedback control. *Automatica*, 35(6):1131–1139, 1999.

[4] Z.-P. Jiang, I. Mareels, and Y. Wang. A Lyapunov formulation of the nonlinear small-gain theorem for interconnected ISS systems. *Automatica*, 32(8):1211–1215, 1996.

[5] D. Karagiannis, A. Astolfi, and R. Ortega. Two results for adaptive output feedback stabilization of nonlinear systems. *Automatica*, 39(5):857–866, 2003.

[6] M. Krstić, I. Kanellakopoulos, and P. Kokotović. *Nonlinear and Adaptive Control Design*. John Wiley and Sons, 1995.

[7] R. Marino and P. Tomei. Global adaptive output-feedback control of nonlinear systems, parts I and II. *IEEE Trans. Automatic Control*, 38(1):17–48, 1993.

[8] R. Marino and P. Tomei. *Nonlinear Control Design: Geometric, Adaptive and Robust*. Prentice Hall, 1995.

[9] E.D. Sontag. On the input-to-state stability property. *European Journal of Control*, 1:24–36, 1995.