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OUTPUT FEEDBACK VARIABLE STRUCTURE CONTROLLERS AND STATE ESTIMATORS FOR UNCERTAIN DYNAMIC SYSTEMS

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ABSTRACT

In this paper we propose a new class of output feedback variable structure controllers and state estimators (observers) for uncertain dynamic systems with hounded uncertainties. No statistical information about the uncertain elements is assumed. **A** variable structure systems (VSS) approach together with the geometric approach to the analysis and synthesis of system zeros are employed in the synthesis of the proposed output feedback controllers and state estimators. The role of system zeros in the output feedback stabilization and state estimation, using the VSS approach, is discussed. Numerical examples included illustrate the feasibility of the proposed stabilization and state estimation schemes.

1. INTRODUCTION

An important problem in control theory is the control of incompletely modeled dynamical systems. Therefore the process of modeling must be incorporated into the controller synthesis process. Hence a fundamental issue in controller design is robustness of the desired system behavior with respect to the modeling uncertainty. For example one common criterion is stability robustness, that is, closed-loop stability under plant parameter variations and neglected dynamics. In many cases, the statistical characterization of the uncertainties and/or nonlinearities in the plant dynamics is not available or prohibitively expensive to assess. However, bounds on the uncertainties may be known. In such cases a deterministic approach to controller synthesis is viable.

The theory of variable structure systems (VSS) [2,3,5,18,19,20,21,22] can be used for the design of feedback control laws for uncertain dynamical systems. VSS theory rests on the concept of changing the structure of the controller in response to the changing state of the system in order to obtain a desired response. This is accomplished by the use of a high speed switching control action which forces the trajectory of the system onto a chosen manifold in the state space, where it is maintained thereafter. The system is insensitive to certain parameter variations and disturbances while the trajectory is on the manifold. In particular, one can show that variable structure controllers are robust with respect to the so-called matched uncertainties/disturbances. The variable structure systems approach has been especially successful in the design of state feedback controllers and stable and robust tracking control ([2,3,5,18,19,21,22,23]). Note that if only the output y is accessible, then one needs to utilize output feedback (see [1] and [8] for output feedback control schemes for linear systems without uncertainties) or construct a state estimator (observer) which estimates the state vector x. White [23], [24] studied the use of output feedback in variable structure systems with no uncertainties for a class of controllers. Asymptotic state estimators which approximately reconstruct the state vector for linear systems without uncertainties were presented by Luenberger ([11],[12]). A combination of Luenberger's ideas of asymptotic state estimation and techniques prevalent in the deterministic approach. to control of uncertain systems led to a new type of observer for nonlinear and/or uncertain dynamical systems as reported by Walcott and Żak in [20]. Alternative approaches to state estimation of nonlinear and/or uncertain systems are reviewed by Misawa and Hedrick in [16]. In this paper we use the VSS approach and the geometric approach to the analysis and synthesis of system zeros in the output feedback control and state estimation synthesis. A synergism of the above mentioned approaches allows one to synthesize a new class of robust output feedback controllers and state estimators.

2. SYSTEM DESCRIPTION AND NOTATION

We consider a class of uncertain dynamic systems modeled by the following equations

$$\dot{x}(t) = Ax(t) + B[u(t) + \xi(t)] , \tag{2.1}$$

$$y(t) = Cx(t) , \tag{2.2}$$

where $x(t) \in \mathbb{R}^n$ is the n-dimensional state vector, $u(t) \in \mathbb{R}^m$, $y(t) \in \mathbb{R}^p$, and the constant matrices A, B, and C are of appropriate dimensions. The vector $\xi(t)$ represents the lumped nonlinearities and/or uncertainties of our system. For the ensuing discussion we will assume the following to be valid:

A.1. There exists a known nonnegative scalar function $\rho(\cdot,\cdot):\mathbb{R} \times \mathbb{R}^p \rightarrow \mathbb{R}$ such that

$$\|\xi(t)\| \leq \rho(t,y(t)) ,$$

where $\|\cdot\|$ denotes standard Euclidean norm.

A.2. The pair (A,B) is controllable and the pair (A,C) is observable with the matrices B and C being of full rank.

A.3. $p \geq m$, that is, the number of output channels is greater than or equal to the number of inputs, and rank (CB) = m. The case when $\text{rank}(CB) < m$ is discussed in Section **6** of the paper.

3. BACKGROUND RESULTS

This section contains some preliminary results related to state feedback which are critical to our discussion of output feedback stabilization control.

3.1 Discussion of System Dynamics on the Switching Surface

Recall that a variable structure control uses a switching control strategy to drive the plant trajectory onto a prespecified switching surface in the state space and maintains the trajectory on this surface for all subsequent time. **A** trajectory confined to such a switching surface is said to be in a sliding mode and system performance is insensitive to matched disturbances.

The design of a variable structure control consists of two steps:

i) The design of the switching surface. The surface is chosen so that the system satisfies certain performance specifications, such as asymptotic stability, while on the surface.

ii) The design of the control strategy to steer the state trajectory to the switching surface.

In this paper we use switching surfaces of the form $\{x \mid Sx = 0\}$ where $S \in \mathbb{R}^{m\times n}$. We also use $Sx = 0$ to denote the switching surface. Let $\sigma(x) = Sx$. Then

$$\sigma(x) = \begin{bmatrix} \sigma_1(x) \\ \vdots \\ \sigma_m(x) \end{bmatrix} = \begin{bmatrix} s_1 x \\ \vdots \\ s_m x \end{bmatrix} \tag{3.1}$$

where $s_i \in \mathbb{R}^{1\times n}$. We say that the system is in a sliding mode if $\sigma(x(t)) = 0$ for $t \geq t_o$, where $x(t)$ is the state trajectory and t_o is a specific time. It follows that in a sliding mode the velocity $\dot{x}$ is tangent to the switching surface. Equivalently,

$$(\nabla\sigma_i)^T \dot{x} = s_i\dot{x} = 0, \quad i = 1,...,m. \tag{3.2}$$

Hence

$$\frac{d}{dt}\sigma(x(t)) = S\dot{x} = 0. \tag{3.3}$$

We can characterize the system in sliding mode by

$$\sigma(x(t)) = 0 \quad \text{and} \quad \dot{\sigma}(x(t)) = 0. \tag{3.4}$$

Consider the plant (2.1),

$$\dot{x} = Ax + Bu + B\xi(t).$$

Combining (2.1) and (3.3) we have

$$SAx + SBu + SB\xi = 0. \tag{3.5}$$

If SB is nonsingular then (3.5) gives

$$u = -(SB)^{-1} SAx - \xi . \tag{3.6}$$

Substituting (3.6) into (2.1) yields

$$\dot{x} = [I_n - B(SB)^{-1}S] Ax . \tag{3.7}$$

The behavior of the system in sliding is therefore governed by

$$\begin{cases} \dot{x} = [I_n - B(SB)^{-1}S]Ax \\ Sx = 0 \end{cases}$$

Note that while in sliding the plant is governed by a reduced set of differential equations and it is not affected by matched uncertainties. An algorithm for the design of switching surface will be given in Section 4.1.

3.2 Sliding Mode and System Zeros

Consider the following square system:

$$\dot{x} = Ax + Bu, \tag{3.8}$$

$$\sigma = Sx : \tag{3.9}$$

where

$$S \in \mathbb{R}^{m\times n}, \quad \text{and} \quad \det(SB) \neq 0 .$$

We form the so-called system matrix corresponding to the plant represented by (3.8) and (3.9)

$$P(\lambda) = \begin{bmatrix} \lambda I - A & -B \\ S & 0 \end{bmatrix} . \tag{3.10}$$

Note that the system matrix $P(\lambda)$ is a square matrix of order (n+m). Its determinant defines the system zeros of the square dynamical system (3.8), (3.9) (see Kouvaritakis and MacFarlane [9]). Let $z(\lambda) = \det P(\lambda)$.

The system zeros are invariant under the following set of transformations (MacFarlane and Karcanias [13]):

(i) nonsingular coordinate transformations in the state space;

(ii) nonsingular transformations of the inputs;

(iii) nonsingular transformations of the outputs;

(iv) state feedback to the inputs;

(v) output feedback to the rates of change of the states.

We also have

$$z(\lambda) = \det(\lambda I - A)\det\{S(\lambda I - A)^{-1}B\} .$$

Hence

$$\det\{S(\lambda I - A)^{-1}B\} = \frac{z(\lambda)}{\det(\lambda I - A)} . \tag{3.11}$$

On the other hand (Verghese et al [19]):

$$\begin{aligned} &\det(\lambda I - A + B(SB)^{-1}SA) \\ &\quad = \det(SB)^{-1}\,\lambda^m \det(\lambda I - A)\det\{S(\lambda I - A)^{-1}B\} . \end{aligned} \tag{3.12}$$

Combining (3.11) and (3.12) yields

$$\det(\lambda I - A + B(SB)^{-1}SA) = \det(SB)^{-1}\,\lambda^m\, z(\lambda) . \tag{3.13}$$

We thus conclude that the dynamics of the system (3.8), (3.9) in sliding along $a = Sx$ is determined by the system zeros of the system represented by the triple (A,B,S). Related observations have also been made by Young et al [25], El-Ghezawi et al [3], and Verghese et al [19]. This observation leads to the following interpretation of the switching surface design. Switching surface design can be viewed as choosing an output matrix S so that the system (3.8), (3.9) has a desired set of system zero locations which in turn govern the dynamics of the system in sliding along $Sx = 0$.

The above interpretation of the switching surface design enables us to use the approach and the theory of system zeros as developed by Kouvaritakis and MacFarlane [9] to the synthesis of output feedback controllers.

4. REGULATION VIA OUTPUT FEEDBACK

In this Section we consider the problem of regulating the states of system (2.1), (2.2) to the origin of the state-space via the use of output feedback.

Suppose we have a system (2.1), (2.2) with $p \geq m$. Our goal is to design a variable structure output feedback controller which drives the system trajectory onto a prespecified switching surface, $\sigma(y) = Fy$, maintains the trajectory on this surface and forces x to go asymptotically to zero in spite of the presence of uncertainties. In forming the feedback loop from output to input via the regulator, a "squaring down" process ([10]) is involved.

4.1. Switching Surface Design

In this section, we present a procedure for the design of a switching surface $Fy = 0$ using only the output variable. This procedure is related to the design of a switching surface $Sx = 0$ for the state variable x via the output equation $y = Cx$. Examples will be given at the end of the section to illustrate the design procedure.

We first give necessary and sufficient conditions for the existence of a state switching surface on which the nominal system has prescribed eigenvalues. **A** method for the design of a state switching surface is contained implicitly in the proof of the following theorem. **A** more explicit description will be given later.

Theorem 4.1.

Suppose we have the nominal system

$$\dot{x} = Ax + Bu$$
$$y = Cx,$$

which satisfies assumptions A.1, A.2, A.3 in Section 2. Then there exits a matrix $S \in \mathbb{R}^{m \times n}$ so that

(1) the system $\dot{x} = Ax + Bu$ restricted to the surface $Sx = 0$ has n–m prescribed distinct, nonzero, real eigenvalues $\{\lambda_1, \ldots, \lambda_{n-m}\}$.

(2) SB is nonsingular

if and only if there exist full rank matrices $W \in \mathbb{R}^{n \times (n-m)}$, $W^g \in \mathbb{R}^{(n-m) \times n}$ so that

(3) $W^g W = I_{n-m}$, $W^g B = 0$, and $W^g AW = \text{diag}\{\lambda_1, \ldots, \lambda_{n-m}\}$.

Proof:

(Necessity) Assume conditions (1) and (2). As shown in Section 3.1, the plant matrix of the given plant on $Sx = 0$ is

$$(I, - B(SB)^{-1}S)A.$$

Let $J = \text{diag}\{\lambda_1, \ldots, \lambda_{n-m}\}$. Since $\lambda_1, \ldots, \lambda_{n-m}$, the eigenvalues of the plant on $Sx = 0$, are distinct, there exists a full rank matrix W so that

$$(I, - B(SB)^{-1}S)AW = WJ . \tag{4.1}$$

Thus

$$SWJ = S(I_n - B(SB)^{-1}S)AW$$
$$= 0 .$$

Since the λ_j's are nonzero, J is nonsingular and we have $SW = 0$. Hence W is a full rank right annihilator of S. Combined with the fact that SB is nonsingular, we have

$$\text{Range } B \cap \text{Range } W = \{0\} .$$

Since B, W have full rank and Range B $\cap$ Range W $= 0$, the matrix $[B \vdots W]$ is invertible. We write its inverse as

$$\begin{bmatrix} B^g \\ W^g \end{bmatrix},$$

with $B^gB = I_m$, $B^gW = 0$, $W^gB = 0$, and $W^gW = I_{n-m}$. Premultiply (4.1) by W^g to obtain

$$W^gAW = J.$$

Condition (3) is proved.

(Sufficiency). Suppose we have (3). Let $S \in \mathbb{R}^{m \times n}$ be a full rank left annihilator of W, or equivalently, $SW = \mathbf{0}$ and $Sz = \mathbf{0}$ if and only if $z \in \text{Range } W$. We first show that SB is nonsingular. Suppose $x \in \mathbb{R}^m$ and $SBx = 0$. Then by the definition of S, there exists $y \in \mathbb{R}^{n-m}$ so that $Bx = Wy$. Hence

$$y = W^gWy = W^g Bx = \mathbf{0} .$$

Thus $Bx = 0$. Since B has full rank by assumption, we have $x = 0$. We conclude that SB is nonsingular, which is condition **(2).** The plant matrix on the surface $Sx = \mathbf{0}$ is $[I, -B(SB)^{-1}S]A$. It follows from condition **(3)** and $SW = \mathbf{0}$ that

$$\text{Range}\left\{[I, \; - B(SB)^{-1}S]AW - WJ\right\} \subset \ker S \cap \ker W^g .$$

As before, $x \in \ker S$ implies that $x = Wy$ for some y. Thus if $x \in \ker S \cap \ker W^g$ we have $\mathbf{0} = W^g x = W^gWy = y$. It follows that $x = 0$. Thus $\ker S \cap \ker W^g = \{0\}$ and

$$[I, \; - B(SB)^{-1}S]AW = WJ .$$

This is equivalent to the fact that $\lambda_1, \ldots, \lambda_{n-m}$ are the eigenvalues of $[I, -B(SB)^{-1}S]A$ on the surface $Sx = \mathbf{0}$ since the columns of W span the surface. The proof is complete.

□

The above theorem gives precise conditions on the existence of a state switching surface. It is reasonable to assume that if a state switching surface cannot be designed to specifications, then an output switching strategy would be equally impossible. Therefore we assume that a state switching surface can be designed on which the nominal system has the desired eigenvalues.

Clearly if $Sx=0$ and F satisfies $FC=S$, then

$$Fy = FCx = Sx = 0$$

defines a switching surface for the output variable y. We will use the solution of $FC=S$ in the design of output feedback controllers.

To complete our design, we need to characterize C and S for which $FC = S$ is solvable. First we prove the following technical lemma.

Lemma 4.1.

Let $F_1, F_2 \in IR^{kxe}$ be full rank left annihilators of $W \in \mathbb{R}^{\ell \times t}$. Then there is $Q \in IR^{kxk}$ so that $F_2 = QF_1$.

Proof:

Since F_1, F_2 are full rank annihilators, we have $\ker F_1 = \ker F_2$. Let $N = \ker F_1 = \ker F_2$. Let $\{e_1, \ldots, e_j\} \subset \mathbb{R}^{\ell}$ be a basis of $N^{\perp}$. Then both $\{F_1 e_1, \ldots, F_1 e_j\}$ and $\{F_2 e_1, \ldots, F_2 e_j\}$ are linearly independent sets. Define Q by $Q(F_1 e_i) = F_2 e_i$ on $F(N^{\perp})$ and $Q(x) = x$ for $x \in [F(N^{\perp})]^{\perp}$. Clearly Q is well defined and we have $F_2 = QF_1$. The proof is complete.

□

We can now characterize the systems and corresponding state switching surfaces that can be factored to give an output switching strategy.

Theorem 4.2.

Let $C \in \mathbb{R}^{p \times n}$, $W \in \mathbb{R}^{n \times (n-m)}$ have full rank. Let $S \in \mathbb{R}^{m \times n}$ be a full rank left annihilator of W. Then there exists $F \in \mathbb{R}^{m \times p}$ so that $S = FC$ if and only if rank $(CW) = p - m$.

Proof:

Recall that $n \geq p \geq m$. Suppose S has full rank and $S = FC$. Then F must also have full rank m. Since $F(CW) = SW = 0$, we have rank $(CW) \leq \dim \ker F = p - m$. On the other hand, by Sylvester's inequality, (Gantmacher [4], p. 66)

$$\begin{aligned} \text{rank } (CW) &\geq \text{rank } C + \text{rank } W - n \\ &= p + n - m - n \\ &= p - m \,. \end{aligned}$$

Thus we have rank $(CW) = p - m$.

Conversely, suppose rank $(CW) = p - m$. Let $\tilde{F} \in \mathbb{R}^{m \times p}$ be a full rank left annihilator of CW. Then clearly $\tilde{F}C \in \mathbb{R}^{m \times n}$ has rank $\leq$ m. Since $\tilde{F} \in \mathbb{R}^{m \times p}$ is a full rank left annihilator of CW and rank $(CW) = p - m$, we have rank $\tilde{F} = m$. By assumption, rank $C = p$. Thus by Sylvester's inequality:

$$\begin{aligned} \text{rank } (\tilde{F}C) &\geq \text{rank } \tilde{F} + \text{rank } C - p \\ &= m + p - p = m. \end{aligned}$$

Hence rank $(\tilde{F}C) = m$ and thus $\tilde{F}C$ is a full rank annihilator of W. Since S is also a full rank left annihilator of W, we have by Lemma 4.1 that

$$S = Q(\tilde{F}C)$$

for some Q. Let $F = Q\tilde{F}$. The proof is complete.

□

We now give an equivalent formulation of condition **(3)** of Theorem 4.1 which is easier to use in practice.

Theorem 4.3.

Let $B \in \mathbb{R}^{n\times m}$, $W \in \mathbb{R}^{n\times(n-m)}$ be full rank matrices. The following conditions are equivalent:

(1) Range B $\cap$ Range W $= \{0\}$, Range $(AW - WJ) \subset$ Range B

(2) there exists a full rank matrix W^g so that $W^gW = I_{n-m}$, $W^gB = 0$, and $W^gAW = J$.

Proof:

Assume (1). Then $[B \vdots W]$ is invertible with inverse

$$\begin{bmatrix} B^g \\ W^g \end{bmatrix}$$

where $B^gB = I,$, $B^gW = O$, $W^gB = O$, and $W^gW = I_{n-m}$. Since $W^gB = 0$ and Range$(AW - WJ) \subset$ Range B, it follows that $W^gAW = J$. Hence (I) implies (2).

Suppose (2) holds. Let $y \in$ Range$(AW - WJ)$. Then $y = (AW - WJ)x$ for some x. Thus

$$\begin{aligned} W^gy &= W^g(AW - WJ)x \\ &= (W^gAW - J)x \\ &= 0. \end{aligned}$$

Hence Range$(AW - WJ) \subset \ker W^g$. Sine W^g is a full rank annihilator of B, we have Range$(AW - WJ) \subset$ Range B. If $y \in$ Range B $\cap$ Range W, then $y = Bx = Wz$. Thus $z = W^gWz = W^gBx = 0$. It follows that $y = Wz = 0$ and Range B $\cap$ Range W $= \{0\}$.

$\square$

The above results lead us to the design method of an output switching surface. We know that choosing the desired poles $\lambda_1, \ldots, \lambda_{n-m}$ of the system in sliding is equivalent to choosing the desired system zeros of the "squared-down" plant (A, B, FC). In the selection process of λ_i, $i=1,\ldots,n-m$, we have to take into account the fact that the system zeros of a non-square system are always system zeros of any "squared-down" system (MacFarlane and Karcanias [13]). Thus we should obey the following rules (Kouvaritakis and MacFarlane [10]):

(i) The matrix J must contain among its diagonal elements all the existing system zeros of the system triple (A,B,C) whose outputs are being squared down.

(ii) No more than $n-m-n_z$ new zeros should be specified, where n_z denotes the number of system zeros of the system represented by the triple (A,B,C).

Observe that if we have $p=m$ and $\det(CB)\neq 0$ then for any nonsingular $F\in\mathbb{R}^{m\times m}$, the system zeros of the system (A,B,C) are the same as the system zeros of the system (A,B,FC). This is because, as we mentioned in Section 3.2, the system zeros are invariant under nonsingular transformations of the outputs. Hence, in the case when $p=m$, output regulation also requires that all the system zeros be located in the open left half complex plane.

We can now summarize Theorems 4.1, 4.2, and 4.3 in the following output switching surface design algorithm.

Output Switching Surface Design Algorithm

Given: A, B, C.

Step **1.** Check if the finite system zeros of the plant (A,B,C) are in the desired locations. If not, modify the input map B and/or output map C so that the systems zeros are in the desired locations.

Step 2. Select desired eigenvalues $\lambda_1, \ldots, \lambda_{n-m}$ and form $J=\text{diag}\{\lambda_1, \ldots, \lambda_{n-m}\}$.

Step 3. Choose **a** full rank matrix $W \in \mathbb{R}^{n \times (n-m)}$ which satisfies:

a) The columns of AW—WJ are in Range B

b) Range B $\cap$ Range W $= \{0\}$.

c) rank CW $= p - m$.

Step 4. Find a full rank $F \in \mathbb{R}^{m \times p}$ such that $FCW = 0$.

Form output switching surface: Fy $= 0$.

We now illustrate the design algorithm with two examples.

Example 4.1.

Consider the following plant model

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & \frac{1}{3} & -1 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & \frac{8}{3} & 1 \\ 4 & \frac{2}{3} & -2 \end{bmatrix} x.$$

We use the algorithm to design an output switching surface.

Step 1. We use the technique proposed by Kouvaritakis and MacFarlane ([10], pp. 168, 169) to compute the finite system zeros. Our plant does not have finite zeros.

Step 2. We choose the poles for the system in sliding to be $\lambda_1 = -1$, λ_2 —–2. We take $J = \text{diag}\{-1, -2)$.

Step 3. We choose $W \in \mathbb{R}^{3 \times 2}$ to satisfy conditions (a), (b), and (c) in step **3** of the algorithm. One can take W to be

$$W = \begin{bmatrix} 1 & 0 \\ -1 & 0 \\ -\frac{1}{3} & 1 \end{bmatrix}$$

Thus

$$CW = \begin{bmatrix} -2 & 1 \\ 4 & -2 \end{bmatrix}.$$

Step 4. A full rank left annihilator of CW is $F = [2 \ \ 1]$. The design of the output switching surface is complete. The output switching surface is $[2 \ 1]\ y = 0$.

We next give an example where $p = m$.

Example 4.2.

Consider the following model of a dynamical system

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ \alpha & 2 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

$$y = [1 \quad 1]\ x,$$

where α is an adjustable parameter.

In this case $p = m = 1$. Thus the design of an output switching surface is reduced to a mere scaling of the output measurement. Such a transformation does not influence the location of system zeros, and hence the dynamics of the system in sliding on the surface $F\,Cx = 0$ is fixed. However, if the system zeros are in the open left hand complex plane then we can take $Cx = 0$ as a switching surface. The system zeros are the eigenvalues of the matrix

$$W^g AW,$$

where W is any matrix which satisfies

$$CW = 0, \quad W^g = 0, \quad W^g W = I.$$

We can take the following matrix W

$$W = \begin{bmatrix} -1 \\ 1 \end{bmatrix} .$$

Hence, for example

$$W^g = [-1 \quad 0],$$

does the job. We have

$$W^g A W = -1 .$$

Thus the system zero is "good".

4.2. Output Feedback Stabilizing Controller Synthesis

After the switching surface design is completed the next step is to synthesize an output feedback control strategy such that the state x converges to $\sigma(\mathbf{x}) = \mathbf{FCx}$ in finite time in the presence of a bounded ξ. Once this switching surface is reached the controller should keep x sliding along $\sigma(\mathbf{x}) = \mathbf{0}$ towards the origin of the state-space.

In general, a variable structure controller varies its structure depending on the position of x relative to the switching surface and may have the form

$$u_i = \begin{cases} u_i^+(x) & \text{if} \quad \sigma_i(x) > 0 \\ u_i^-(x) & \text{if} \quad \sigma_i(x) < 0 \end{cases} .$$

It can be shown (Utkin, [22]), that if $\sigma^T(\mathbf{x})\dot{\sigma}(\mathbf{x}) < 0$ then the system trajectory is directed towards the switching surface. Once the trajectory hits the surface the condition $\sigma^T(\mathbf{x})\dot{\sigma}(\mathbf{x}) < 0$ guarantees that it will be maintained on $\sigma(\mathbf{x})$ thereafter. The motion of the system is not affected by matched uncertainties, that is the uncertainties which influence the system dynamics via the input matrix B like in our case, when the trajectory is on the switching surface; see Section 3.1.

We now focus our attention on choosing the feedback gains using only output measurements such that systems trajectories converge to the switching surface and enter a sliding mode. In general, the matrix FCB is not diagonal. We can then construct a control law

$$\tilde{u} = Q^{-1}(FCB)u \ , \tag{4.2}$$

where $Q \in \mathbb{R}^{m \times m}$ is a nonsingular matrix which can be used to satisfy certain design specifications. In our considerations we assume $Q = I$.

To assure the attractiveness of the sliding surface, it is enough that the following holds

$$\begin{aligned} \sigma^T(y)\dot{\sigma}(y) &= \sigma^T(y)FC\dot{x} = \sigma^T[FCAx + FCB(u + \xi)] \\ &= \sigma^T[FCAx + \tilde{u} + (FCB)\xi] < 0 \ . \end{aligned} \tag{4.3}$$

Our goal is to synthesize an output feedback control strategy such that (4.3) is satisfied. However, one can see that there is the term FCAx in (4.3) in which the state vector is present. If however there exists some matrix $M \in \mathbb{R}^{m \times p}$ such that

$$FCA = MC \tag{4.4}$$

then

$$FCAx = MCx = My \ , \tag{4.5}$$

and (4.3) will take the form

$$\sigma^T \dot{\sigma} = \sigma^T[My + \tilde{u} + (FCB)\xi] < 0 \ . \tag{4.6}$$

Let

$$(M)_i \quad \text{and} \quad (FCB)_i$$

denote the i-th rows of the matrices M and FCB, respectively. Then, if the entries $\tilde{u}_i^+$

and $\tilde{u}_i^-$ are chosen to satisfy

$$\tilde{u}_i^+ < -(M)_i y - (FCB)_i \xi \quad \text{if} \quad \sigma_i(y) > 0 , \tag{4.7a}$$

and

$$\tilde{u}_i^- > -(M)_i y - (FCB)_i \xi \quad \text{if} \quad \sigma_i(y) < 0 , \tag{4.7b}$$

then the sufficient condition for the existence and reachability of sliding mode are satisfied. Note that the conditions (4.7) force each term in the summation $\sigma^T \dot{\sigma} = \sum_{i=1}^{m} \sigma_i \dot{\sigma}_i$ to be negative. Of course, other sufficient conditions for the existence of a sliding mode can also be used during the controller synthesis. We mention here that the control actually implemented is

$$u = (FCB)^{-1} Q \, \tilde{u} . \tag{4.8}$$

The critical condition in the above control synthesis is (4.4). We now give a sufficient condition for solvability of equation (4.4).

Theorem 4.2

Let $S = FC$ and let the row space of S be spanned by a set of m left eigenvectors of A labelled $v_1, \ldots, v_r$. Then there exists a matrix $M \in \mathbb{R}^{m \times p}$ such that

$$SA = MC .$$

Proof

We can proceed as in [26]. By the assumption there exists a nonsingular matrix N such that

$$NSA = \begin{bmatrix} v_1 \\ \vdots \\ v_m \end{bmatrix} A = \begin{bmatrix} \lambda_1 & & & 0 \\ & \cdot & & \\ & & \cdot & \\ & & & \cdot \\ 0 & & & \lambda_m \end{bmatrix} \begin{bmatrix} v_1 \\ \vdots \\ v_m \end{bmatrix} = \Lambda NS \ , \tag{4.9}$$

where

$$\Lambda = \text{diag}\ \{\lambda_1,\ldots,\lambda_m\} \quad .$$

Hence

$$SA = N^{-1}\ \Lambda\ N\ S = N^{-1}\ \Lambda\ NFC = MC \ ,$$

where

$$M = N^{-1}\ \Lambda\ NF$$

and the proof is complete.

□

Example 4.1 (continued)

We now attempt to synthesize an output feedback control strategy for the plant whose model is given in Example **4.1.** Recall that the output switching surface we have designed is

$$\sigma(y) = [2 \ \ , \ \ 1]y \ .$$

In order to be able to synthesize the control law of the form **(4.7)** we have to solve equation **(4.4)** for M. In this example

$$FC = [6 \ \ , \ \ 9 \ \ , \ \ 0]$$

and

$$FCA = [0 \ \ , \ \ 6 \ \ , \ \ 9] \ .$$

It is obvious that there is no M such that

$$FCA = MC$$

since FCA $\notin$ Range C. So we cannot proceed with the synthesis of an output feedback controller of the form (4.7). However, we know that the system zeros, or equivalently the poles of the system in sliding along $\sigma(\mathbf{x}) = FC\mathbf{x} \triangleq Sx$ are invariant under output feedback. Hence, if there is no M such that SA = MC one may suggest to solve the following equation

$$S(A - BKC) = \tilde{M}C \tag{4.10}$$

for $\tilde{M}$ and K. Thus the controller would consist of two portions: a linear part $u_1 = -Ky$ and the nonlinear part u_2 of the form (4.7). Unfortunately, this trick cannot be used because if (4.10) is possible then so is (4.4) with $M = SBK + \tilde{M}$.

Observe that for p = m the existence of M which satisfies SA=MC implies that the pair (A,C) is not observable.

Indeed, let S = FC where $\det F \neq 0$. Then FCA = MC can be written as $CA = \hat{M}C$, where $\hat{M} = F^{-1}M$. Hence

$$CA^i = CAA^{i-1} = \hat{M}CA^{i-1} = \ldots = \hat{M}^iC$$

and thus the pair (A,C) is unobservable since the rank of the observability matrix will be equal to $p < n$.

However, the converse is not true as the following example shows.

Let

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \quad , \quad C = [0\ 1\ 0]\ .$$

The pair (A,C) is not observable. But this does not imply that $CA = \hat{M}C$ for some $\hat{M}$. Indeed $CA = [0\ 1\ 1] \notin$ Range C.

The good news is that when $p > m$ then satisfaction of the condition $FCA = MC$ does not necessarily require nor imply the unobservability of the pair (A,C). Indeed, let $p = 2$, $m = 1$, where

$$A = \begin{bmatrix} 1 & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 1 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

$$F = [1 \ \ 0] .$$

The pair (A,C) is observable, however,

$$FCA = [1 \ \ 1 \ \ 0] \in \text{Range } C .$$

When there is no solution to $SA = MC$, one is forced to modify the type of the output feedback control strategy.

However, even when there is no solution to $SA = MC$ the bounded controllers of the form

$$u = -(FCB)^{-1} \begin{bmatrix} \mu_1 \text{ sgn } \sigma_1(y) \\ \\ \mu_m \text{ sgn } \sigma_m(y) \end{bmatrix} = -(FCB)^{-1} \begin{bmatrix} \mu_1 \text{ sgn } s_1 x \\ \vdots \\ \mu_m \text{ sgn } s_m x \end{bmatrix}, \tag{4.11}$$

where $\mu_i > 0$, $i = 1,...,m$, are design parameters, or of the form

$$u = -\mu(FCB)^{-1} \frac{\sigma(y)}{||\sigma(y)||} = -\mu(FCB)^{-1} \frac{FCx}{||FCx||}, \tag{4.12}$$

where $\mu > 0$ is a design parameter, will locally stabilize the closed-loop system. One can also estimate the stability regions for systems driven by the controllers (4.11) or (4.12). The stability regions estimation can be performed using methods proposed by Madani-Esfahani et al in [14] or by Hui and Żak in [7].

In the next Section we discuss the problem of state estimation for plants modeled by (2.1) and (2.2).

5. STATE ESTIMATION OF UNCERTAIN SYSTEMS

In this Section we propose new types of estimators of the state of the plants modeled by (2.1) and (2.2).

5.1. The Full Order State Estimators

Let $\overline{\mathbf{x}}$ be the estimate, obtained from the state estimator, of the plant state x. We denote the estimation error by $e(t)$, that is,

$$e(t) = \overline{x}(t) - x(t) .$$

Let the dynamics of the state estimator be given by

$$\dot{\overline{x}} = A\overline{x} + Bu + Bv , \tag{5.1}$$

where the vector $\mathbf{v}$ will be defined later.

For the above estimator the error satisfies the following equation

$$\dot{e} = Ae - B\xi + Bv . \tag{5.2}$$

We now investigate the stability of the error equation (5.2).

Let

$$\sigma(e) = FCe , \tag{5.3}$$

where the matrix $F \in \mathbb{R}^{m \times p}$ is chosen using methods of Section 4 in such a way that the triple (A, B, FC) has its system zeros in the open left hand complex plane. Thus, if we can find $v \in \mathbb{R}^m$ so that the error reaches the surface $\sigma(e)$ and then enters a sliding mode along this surface then (5.2) restricted to $\sigma(e)$ will be asymptotically stable. Consider now the following condition

$$\sigma^T(e)\dot{\sigma}(e) = \sigma^T(e)(FCAe - FCB\xi + FCBv) \; . \tag{5.4}$$

Suppose the following is satisfied for some matrix M

$$FCA = MC \; . \tag{5.5}$$

Using (5.5) we can represent (5.4) as follows

$$\sigma^T\dot{\sigma} = \sigma^T(MCe - FCB\xi + FCBv) \; . \tag{5.6}$$

Our goal now is to select v so that

$$\sigma^T(e)\dot{\sigma}(e) < 0 \; , \tag{5.7}$$

thus guaranteeing the attractivity of the surface $\sigma(e) = FCe$. Note that while choosing v we can use the elements of Ce since

$$Ce = C\bar{x} - Cx = C\bar{x} - y \; . \tag{5.8}$$

Let $(M)_i$ and $(FCB)_i$ denote the i-th rows of the matrices M and FCB respectively. Let

$$v = (FCB)^{-1}\tilde{v} \; . \tag{5.9}$$

Then, if the components $\tilde{v}_j$ of $\tilde{v}$ are chosen to satisfy

$$\tilde{v}_i^+ < -(M)_i Ce + (FCB)_i \xi \quad \text{if} \quad \sigma_i(e) > 0 \; , \tag{5.10a}$$

and

$$\tilde{v}_i^- > -(M)_i Ce + (FCB)_i \xi \quad \text{if} \quad \sigma_i(e) < 0 \tag{5.10b}$$

then (5.7) is satisfied, and the error equation (5.2) is asymptotically stable towards the origin.

Note that the proposed estimator (5.2) suffers from the same drawbacks as the output feedback controller (4.7). The critical condition in the above construction is the solvability of the equation SA = MC for M. If one attempts to synthesiz the estimator (5.2) for the plant in Example 4.1 then one will fail since the equation SA = MC does not have a solution for this plant. One then may try to modify the estimator (5.2) in the following way

$$\dot{\bar{x}} = (A - LC)\bar{x} + Bu + Bv + Ly \tag{5.11}$$

hoping to be able to find a matrix L so that for some $\overline{M}$

$$FC(A - LC) = \overline{M}C \ . \tag{5.12}$$

However, if (5.12) is possible then $M = FCL + \overline{M}$ will satisfy (5.5).

Another drawback of the estimators (5.2) and (5.11) is the fact that sufficient conditions for their synthesis are also sufficient conditions for the existence of an output feedback stabilizer. This fact makes the above discussed full order estimators impractical. It turns out, however, that we can synthesize reduced order estimators using the theory advanced in Sections **3** and 4. The proposed reduced order estimators do not require (5.5).

5.2. The Reduced Order State Estimators

Consider a dynamical system model given by (2.1) and (2.2). Suppose we were able to find an appropriate output switching surface $\sigma(y) = Fy$ so that the plant (A, B, FC) has asymptotically stable system zeros. The result of the switching surface synthesis are the matrices

$$J = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_{n-m} \end{bmatrix},$$

and

$$W \in \mathbb{R}^{n\times(n-m)} \quad \text{and} \quad W^g \in \mathbb{R}^{(n-m)\times n} \ ,$$

such that

$$W^g W = I_{n-m} \quad , \quad FCW = 0 \quad , \quad W^g B = 0 \; .$$

Let

$$\begin{bmatrix} z \\ \sigma \end{bmatrix} = \begin{bmatrix} W^g \\ FC \end{bmatrix} x \triangleq Tx \; . \tag{5.13}$$

Observe that σ is determined from the output measurements since $Cx = y$. Note that the matrix T in (5.13) is invertible since $\det[W \vdots B] \neq 0$ and

$$\begin{bmatrix} W^g \\ FC \end{bmatrix} [W \vdots B] = \begin{bmatrix} I_{n-m} & 0 \\ 0 & FCB \end{bmatrix} . \tag{5.14}$$

Let us now consider the dynamics of the following reduced order estimator

$$\dot{\bar{z}} = J\bar{z} + Gy \tag{5.15}$$

of $z = W^g x$, where G is to be specified. Thus, we should like to be able to use the outputs to determine m of the xi's and design an estimator of order n-m to estimate the rest. We choose G to satisfy

$$W^g A - JW^g = GC \; . \tag{5.16}$$

It is instructive to note that we arrive at the above equation by setting

$$W^g \dot{x} = \dot{\bar{z}} \tag{5.17}$$

Then since $W^g B = 0$, we have

$$\frac{d}{dt}(\bar{z} - W^g x) = J\bar{z} + GCx - W^g Ax = J\bar{z} - JW^g x \; .$$

Hence

$$\frac{d}{dt}(\bar{z} - W^g x) = J(\bar{z} - W^g x) \; . \tag{5.18}$$

The eigenvalues of J are all negative, thus

$$\bar{z} - W^g x \rightarrow 0 . \tag{5.19}$$

In order to recover all states we invert the equations

$$\left.\begin{aligned} \bar{z} &= W^g x \\ a &= FCx \end{aligned}\right\} . \tag{5.20}$$

The proposed estimation scheme is illustrated in Fig. **5.1.**

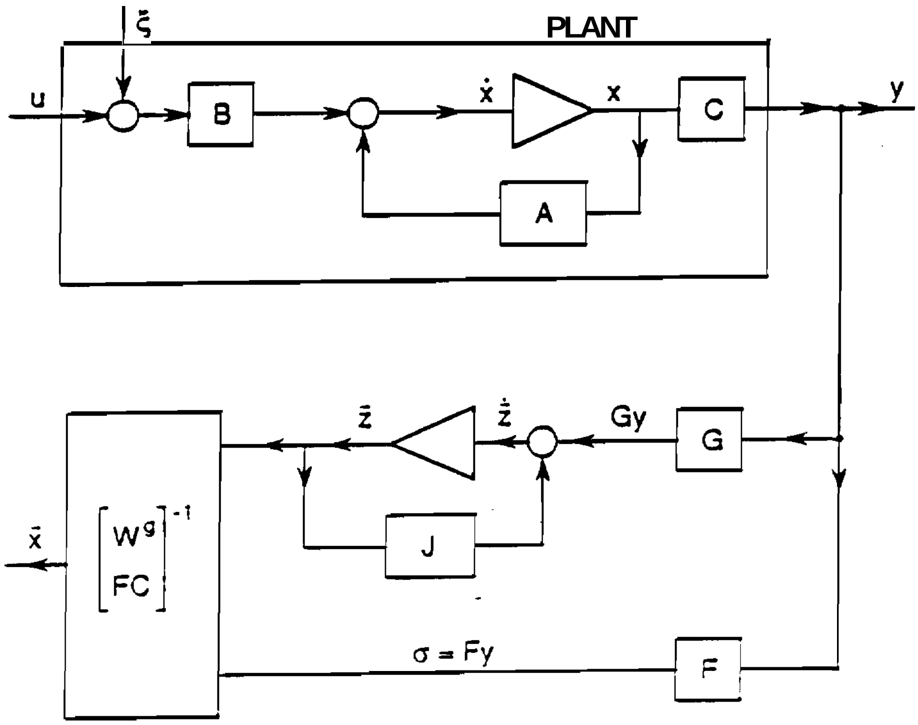

Fig. 5.1. The reduced order state estimator for uncertain dynamic systems.

Remark 5.1.

Once a state estimator has been designed, the next step is to combine the controller and estimator. For a discussion of the combined estimator-controller compensator synthesis from the variable structure systems standpoint the reader is referred to Hui and Żak [7].

Example 5.1.

Suppose we are given the following model of a dynamical system

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} (u + \xi)$$

$$y = [1 \;\; 1] \; x \; .$$

We have $p = m = 1$, and $CB = 1 \neq 0$. One can check that the system zero is located at $\lambda_1 = -1$. Thus we can take $\sigma(y) = y$ $(F = 1)$ as the output switching surface. Note, however, that there is no solution to $CA = MC$. Indeed $CA = [0 \, , \, 1]$ while $MC = M[1 \;\; 1]$. Fortunately, the sufficiency condition (5.16) for the existence of the reduced order estimator is satisfied. Indeed, if we take $J = [-1]$, $W^T = [1 \;\; -1]$, $W^g = [1 \;\; 0]$, then $W^g A - JW^g = GC$ becomes $[1 \;\; 1] = G[1 \;\; 1]$. Hence $G = 1$. The reduced order estimator is

$$\dot{\bar{z}} = -\bar{z} + y \; ,$$

and

$$T = \begin{bmatrix} W^g \\ FC \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} .$$

Hence the estimate of the state vector x of the plant is

$$\bar{\mathrm{x}} = \mathrm{T}^{-1}\begin{bmatrix}\bar{z} \\ \sigma\end{bmatrix} = \begin{bmatrix}1 & 0 \\ -1 & 1\end{bmatrix}\begin{bmatrix}\bar{z} \\ \sigma\end{bmatrix}.$$

Having designed a state estimator one can then proceed with a state feedback controller synthesis assuming availability of all the components of the state vector. The estimator has been constructed. The final step consists of combining the controller and the estimator.

Example 4.1 (continued.)

We now design the reduced order state estimator for the plant given in Example **4.1.** Recall that

$$\mathrm{J} = \begin{bmatrix}-1 & 0 \\ 0 & -2\end{bmatrix} \quad \text{and} \quad \mathrm{W}^{g} = \begin{bmatrix}1 & 0 & 0 \\ -\frac{2}{3} & -1 & 1\end{bmatrix}.$$

Hence (5.16) for this example becomes

$$\mathrm{W}^{g}\mathrm{A} - \mathrm{J}\mathrm{W}^{g} = \begin{bmatrix}1 & 1 & 0 \\ -\frac{7}{3} & -\frac{7}{3} & 0\end{bmatrix} = \mathrm{GC} = \begin{bmatrix}\frac{1}{3} & \frac{1}{6} \\ -\frac{7}{9} & -\frac{7}{18}\end{bmatrix}\begin{bmatrix}1 & \frac{8}{3} & 1 \\ 4 & \frac{2}{3} & -2\end{bmatrix}$$

The reduced order estimator then is

$$\dot{\bar{z}} = \begin{bmatrix}-1 & 0 \\ 0 & -2\end{bmatrix}\bar{z} + \begin{bmatrix}\frac{1}{3} & \frac{1}{6} \\ -\frac{7}{9} & -\frac{7}{18}\end{bmatrix}\mathrm{y},$$

and

$$T = \begin{bmatrix} W^g \\ FC \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -\frac{2}{3} & -1 & 1 \\ 6 & 9 & 0 \end{bmatrix} .$$

Hence, the estimate of the state vector x of the plant is

$$\bar{x} = T^{-1} \begin{bmatrix} \bar{z} \\ Fy \end{bmatrix} .$$

A possible variable structure state feedback control law for this example can be synthesized using $\sigma(x) = FCx = Sx$ as the switching surface. The controller gains can be obtained from the condition

$$\sigma^T(x)\dot{\sigma}(x) = \sigma^T(SAx + SBu + SB\xi) < 0 .$$

We have

$$\sigma^T \dot{\sigma} = \sigma(6x_2 + 9x_3 + 9u + 9\xi) < 0$$

if

$$u^+ < -(\frac{2}{3} | x_2 | + | x_3 | + \rho) \quad \text{if} \quad \sigma(x) > 0$$

and

$$u^- > \frac{2}{3} | x_2 | + | x_3 | + \rho \quad \text{if} \quad \sigma(x) < 0 ,$$

where

$$| \xi | \leq \rho .$$

A possible implementation of the controller can have the form

$$u = -(\frac{2}{3} | \bar{x}_2 | + | \bar{x}_3 | + \rho) \operatorname{sgn} (6\bar{x}_1 + 9\bar{x}_2) ,$$

where $\bar{x}_1$, $\bar{x}_2$, and $\bar{x}_3$ are the estimates of state vector components of the plant and are the outputs of the reduced order state estimator. Thus, we have synthesized a nonlinear dynamic, of order **2,** output feedback global robust stabilizer for an unstable uncertain third order plant.

6. GENERALIZATIONS TO THE CASE WHEN $\mathbf{rank(CB)} < \mathbf{m}$

When the matrix CB is not of full rank we take linear combinations of appropriate derivatives of the outputs as suggested in [**15**] and [**19**]. This enables one to use the VSS techniques in a similar fashion as in the case when the matrix CB is of full rank. We proceed as follows:

Let

$$q_0 = \mathbf{rank}(CB) < m \ .$$

Then there exists a nonsingular $p \times p$ matrix U_0 such that

$$U_0 CB = C_0 B = \begin{bmatrix} \bar{C}_0 \\ \tilde{C}_0 \end{bmatrix} B = \begin{bmatrix} D_0 \\ 0 \end{bmatrix} ,$$

where

$C_0 \triangleq U_0 C$, $D_0 = \bar{C}_0 B$ has q_0 rows, rank $D_0 = q_0$, and $\tilde{C}_0 \in \mathbb{R}^{(p-q_0)\times n}$.

Consider now the matrix

$$\begin{bmatrix} D_0 \\ \tilde{C}_0 AB \end{bmatrix} .$$

Let

$$q_1 = \text{rank} \begin{bmatrix} D_0 \\ \tilde{C}_0 AB \end{bmatrix} .$$

If $q_1 < m$ then there exists a nonsingular $p \times p$ matrix U_1 such that

$$U_1 \begin{bmatrix} D_0 \\ \tilde{C}_0 AB \end{bmatrix} = \begin{bmatrix} D_1 \\ 0 \end{bmatrix},$$

where

$$D_1 \in \mathbb{R}^{q_1 \times m}, \quad \text{rank } D_1 = q_1 ,$$

$$U_1 = \begin{bmatrix} I_{q_0} & 0 \\ 0 & V_1 \end{bmatrix}, \quad V_1 \tilde{C}_0 AB = C_1 AB = \begin{bmatrix} \overline{C}_1 \\ \tilde{C}_1 \end{bmatrix} AB ,$$

$$\tilde{C}_1 AB = 0, \qquad \overline{C}_1 \in \mathbb{R}^{(q_1 - q_0)\times n}.$$

The remainder of the sequence is defined inductively as follows

$$C_i = V_i \tilde{C}_{i-1} = \begin{bmatrix} \overline{C}_i \\ \tilde{C}_i \end{bmatrix},$$

$$U_i = \begin{bmatrix} I_{q_{i-1}} & 0 \\ 0 & V_i \end{bmatrix}, \quad \text{rank } \overline{C}_i AB = q_i - q_{i-1} ,$$

$$\overline{C}_i \in \mathbb{R}^{(q_i - q_{i-1})\times n}, \quad \tilde{C}_i AB = 0 .$$

If for some $i = \alpha$, $q_\alpha = m$ then we stop.

Thus we obtain

$$UC = U_\alpha \ldots U_0 C \triangleq C_\alpha = \begin{bmatrix} \overline{C}_0 \\ \overline{C}_1 \\ \vdots \\ \overline{C}_\alpha \end{bmatrix},$$

and

$$\text{rank}\begin{bmatrix}\overline{C}_0 B \\ \overline{C}_1 \mathbf{A}\mathbf{B} \\ \cdot \\ \overline{C}_\alpha \mathbf{A}^\alpha \mathbf{B}\end{bmatrix} = m\ .$$

Let

$$Uy = \overline{y} = \begin{bmatrix}\overline{y}_0 \\ \overline{y}_1 \\ \vdots \\ \overline{y}_\alpha\end{bmatrix} .$$

Then the above outlined procedure yields

$$\begin{bmatrix}\dfrac{d\overline{y}_0}{dt} \\ \dfrac{d^2\overline{y}_1}{dt^2} \\ \vdots \\ \dfrac{d^{\alpha+1}\overline{y}_\alpha}{dt^{\alpha+1}}\end{bmatrix} = \begin{bmatrix}\overline{C}_0 A \\ \overline{C}_1 A^2 \\ \vdots \\ \overline{C}_\alpha A^{\alpha+1}\end{bmatrix} x + \begin{bmatrix}\overline{C}_0 B \\ \overline{C}_1 A B \\ \vdots \\ \overline{C}_\alpha A^\alpha B\end{bmatrix} u\ .$$

If we then consider the system represented by the triple **(A, B,** $\tilde{C}$**)**, where

$$\tilde{C} = \begin{bmatrix}\overline{C}_0 \\ \overline{C}_1 A \\ \vdots \\ \overline{C}_\alpha A^\alpha\end{bmatrix} \in \mathbb{R}^{p \times n}$$

and

$$\text{rank}\ (\tilde{C}B) = m$$

then we can proceed as in the case when rank **(CB)** $=$ m. Note that the triple **(A, B,** $\tilde{C}$) can be viewed as one which represents the original system **(A, B, C)** in which the output y is operated upon by a bank of differentiators specified by the operator N where

$$N = \prod_{i=0}^{\alpha} U_{\alpha-i} \, M_{\alpha-i-1} \ ,$$

$$M_{\alpha-i-1} = \begin{bmatrix} I_{q_{\alpha-i-1}} & 0 \\ & \\ 0 & I_{p-q_{\alpha-i-1}} \left(\frac{d}{dt}\right) \end{bmatrix},$$

where

$$M_{-1} = I_p \ .$$

Thus

$$\begin{bmatrix} \bar{y}_0(t) \\ \\ \frac{d\bar{y}_1(t)}{dt} \\ \vdots \\ \frac{d^{\alpha}\bar{y}_{\alpha}(t)}{dt^{\alpha}} \end{bmatrix} = \left(\prod_{i=0}^{\alpha} U_{\alpha-i} \, M_{\alpha-i-1} \right) y(t) \ .$$

Remark 6.1

The above outlined procedure for generating $\tilde{C}$ so that $\mathrm{rank}(\tilde{C}B) = m$ is based on Silverman's Inversion Algorithm [17] for constructing an inverse of a multivariable linear dynamical system. Therefore the procedure will result in an appropriate $\tilde{C}$ if and only if the system (A, B, C) is left invertible in the sense of Silverman [17]. Hence criteria for invertibility given by Silverman can also be utilized in our problem. One can extend the proposed algorithm to a class of nonlinear system using results of Hirschorn [6].

Remark 6.2

Observe that the systems (A, B, C) and (A, B, $\tilde{C}$) may have different system zeros. To illustrate the above observation consider the following system model

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

$$y = [1 \quad 0]\, x\,.$$

The above system does not have any finite system zeros.

Let us now instead of $C = [1 \quad 0]$ consider $\tilde{C} = CA = [0 \quad 1]$. The system model represented by the triple (A, B, $\tilde{C}$) has one system zero at $s = 0$.

Remark 6.3

Having constructed $\tilde{C}$ so that $\mathrm{rank}(\tilde{C}B) = m$ we can then proceed as in the case of a dynamical model (A, B, C) where $\mathrm{rank}(CB) = m$. However working with the new system (A, B, $\tilde{C}$) will result in the switching surface that involves linear combinations of the derivatives of the outputs, that is

$$\sigma = \sigma(\bar{y}_0,\ \frac{d\bar{y}_1}{dt},\ \ldots,\ \frac{d^{\alpha}\bar{y}_{\alpha}}{dt^{\alpha}})\,.$$

7. CONCLUSIONS

A variable structure systems (VSS) approach combined with the geometric approach to the analysis and synthesis of system zeros have been employed in the synthesis of new robust output feedback stabilization schemes and robust state estimators for a class of uncertain dynamic systems. The employment of system zeros in the study of VSS has provided further insight into properties of these systems. Furthermore, it has also revealed an important role of systems zeros in the variable structure control. In particular, we have shown how the system zeros influence the sliding mode behavior and discussed their role in the state estimation. The blending of VSS approach and system zeros was also used in the synthesis of robust variable structure output feedback stabilizers for the class of uncertain systems for which the matrix CB is not of full rank. Numerical examples included have illustrated the feasibility of the proposed stabilizing controllers and state estimators.

The results of Marino [15], see also [6] and [27], are promising in extending the results of this paper to a large class of uncertain nonlinear systems resulting in practical algorithms for designing output feedback stabilizers and state estimators for such systems.

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