OUTPUT REGULATION FOR LINEAR DISTRIBUTED

PARAMETER SYSTEMS

by

ISTVAN G. LAUKO, B.S., M.S.

A DISSERTATION

IN

MATHEMATICS

Submitted to the Graduate Faculty
of Texas Tech University in
Partial Fulfillment of
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CONTENTS

ABSTRACT

This dissertation is concerned with the solvability of the output regulation problem for infinite-dimensional linear control systems with bounded control and observation operators. By output regulation we mean a control design problem in which the objective is to achieve tracking, disturbance rejection and internal stability. Two versions of output regulation are considered: the state feedback regulator problem, in which we look for a static state feedback control law and the error feedback regulator problem in which a dynamical controller is sought for which only the tracking error is available to the controller. Under the standard assumption of stabilizability necessary and sufficient conditions are given for the solvability of the state feedback problem and under the additional assumption of detectability necessary and sufficient conditions are given for the solvability of the error feedback problem. The solvability of both problems are characterized in terms of the solvability of a pair of linear regulator equations. This characterization represents an extension of the results obtained by B. Francis for multivariable linear control systems. The approach follows the lines of the geometric theory of output regulation developed by C. I. Byrnes and A. Isidori for finite-dimensional nonlinear systems. The solvability of the regulator equations is shown to be equivalent to the property that the zero dynamics of the composite, formed from the plant and the exosystem, contains isomorphic copies of the exosystem and the plants' zero dynamics. Examples of periodic tracking are presented for parabolic and hyperbolic systems.

LIST OF FIGURES

CHAPTER I

INTRODUCTION

1.1 Output Regulation Problem - Generic Description

We shall outline here the problem that is usually referred to in the control literature as output regulation problem. For the sake of simplicity we will use the finite-dimensional linear control system model to describe the various control objectives that are collected under the name of output regulation. We assume that the *controlled system* (or *controlled plant*) can be modeled by a first order system of linear ordinary differential equations

$$\begin{aligned} &\frac{d}{dt}x(t) = Ax(t) + Bu(t) + d(t), \\ &y(t) = Cx(t), \end{aligned} \tag{1.1}$$

where $x(t) \in I\!R^n$, $u(t) \in I\!R^q$, $y(t) \in I\!R^p$ for $t \geq 0$, represent the state, the input and the output of the system respectively, and where A, B and C are linear operators (in this case matrices) between the corresponding spaces. $d(t)$ in (1.1) represents a disturbance.

In the following we describe some common control design objectives for such a system. By *tracking* we mean that the control input $u(t)$ is designed in such a way that each output variable is tracking a preassigned reference signal:

$$y_i(t) \longrightarrow y_i^r(t),$$

as $t \longrightarrow \infty$ for every $1 \leq i \leq p$, or simply $y(t) \longrightarrow y^r(t)$. We will assume that the reference output y^r is generated as the output of another linear finite-dimensional system,

$$\begin{aligned} &\frac{d}{dt}w^1(t) = S_1 w^1(t), \\ &y^r(t) = R_1 w^1(t), \end{aligned} \tag{1.2}$$

To achieve another important design objective, namely *disturbance rejection*, the designed control has to be such that the disturbance effects on the output are attenuated. A standard assumption which we adopt is that $d(t)$ can be generated by a finite-dimensional linear system

$$\frac{d}{dt}w^2(t) = S_2 w^2(t), \qquad (1.3)$$
$$d(t) = R_2 w^2(t),$$

for some initial condition. It is convenient, and customary, to combine system (1.2) with system (1.3), to obtain a single finite-dimensional system

$$\frac{d}{dt}w(t) = Sw(t), \qquad (1.4)$$
$$y^r(t) = Qw(t),$$
$$d(t) = Pw(t),$$

$S \in \mathcal{L}(\mathbb{R}^k)$, $Q \in \mathcal{L}(\mathbb{R}^k, \mathbb{R}^p)$ and $P \in \mathcal{L}(\mathbb{R}^k, \mathbb{R}^n)$, that produces both the signal to be tracked and the disturbance to be rejected. This system represents the outside world for the controlled plant and is refered to as the *exogenous system*, or *exosystem* for short. With the choice of a suitable initial condition for the exosystem a specific tracking and disturbance rejection task can be initiated and the control problem then is to produce the control input u that carries out this task. In practice, the asymptotic requirements of tracking and disturbance rejection are replaced by that of reaching a certain error level in a given time interval.

An additional, natural requirement for the control design is usually referred to as *internal stability*. It requires the system comprising the plant and the controller to be stable when there is no tracking or disturbance rejection assignment present. We note that the exosystem itself is typically not stable. For example it is often required to produce periodic, quasiperiodic or constant signals.

By *output regulation* we mean the problem of designing a controller so that the output of the resulting closed loop system tracks the reference signal regardless of the plants' initial conditions while maintaining internal stability.

The problem described above, in terms of control objectives is not restricted to the linear finite-dimensional model of (1.1), (1.2) and (1.3). We can formulate the problem similarly when the controlled plant is modeled by nonlinear ordinary differential equations, or for infinite-dimensional models as partial or delay differential equations. We expect however, that the exosystem is chosen to be the simplest system that can produce the perturbations and the required reference signal, and that it is finite-dimensional.

Generally two versions of the output regulation are considered. In the first one, it is assumed that the controller is provided with full information on the state of the plant and exosystem. We will refer to this as the *state feedback regulator problem.* For the multivariable linear state feedback problem the solution is expected to be a linear feedback controller of the form

$$u = Kx + Lw. \tag{1.5}$$

For the second version, only the tracking error

$$e = y - y^r$$

is available for the controller, and in this case the problem is called the *error feedback regulator problem.* The solution in the linear multivariable case consists in finding a finite-dimensional linear error feedback controller

$$\begin{aligned} \dot{X} &= FX + Ge \\ u &= HX. \end{aligned} \tag{1.6}$$

1.2 Outline of the History of the Problem

The regulator theory for finite-dimensional systems is well developed. Among the contributors are Smith and Davison [23], Francis and Wonham [12], Francis [11], Wonham [24], Hautus [14], Hepburn and Wonham [16], Byrnes and Isidori [3]. A characterization of the solvability of the multivariable linear problem was given by Francis in [11] in terms of the solvability of a pair of linear matrix equations, the so called *regulator equations.* This theory is based on the following hypotheses:

h1 $\sigma(S)$ is contained in the closed right half plane,

h2 the pair (A, B) is stabilizable,

h3 the pair

$$\left(\begin{bmatrix} C & -Q \end{bmatrix}, \begin{bmatrix} A & P \\ 0 & S \end{bmatrix} \right)$$

is detectable.

The following results (due to Francis) give necessary and sufficient conditions for the solvability of both the state feedback and error feedback problems.

Theorem 1.1 *Let* **h1** *and* **h2** *hold. The multivariable linear state feedback regulator problem is solvable if and only if there exist matrices* Π *and* Γ *that solve the equations*

$$\begin{aligned} \Pi S &= A\Pi + B\Gamma + P, \\ C\Pi - Q &= 0. \end{aligned} \tag{1.7}$$

Theorem 1.2 *Let* **h1** *and* **h2** *and* **h3** *hold. The multivariable linear error feedback regulator problem is solvable if and only if there exist matrices* Π *and* Γ *that solve the equations*

$$\begin{aligned} \Pi S &= A\Pi + B\Gamma + P, \\ C\Pi - Q &= 0. \end{aligned}$$

This shows that under the assumptions **h1-h3** the state feedback regulator problem is solvable exactly when the error feedback problem is solvable. The solvability of the regulator equations in turn has been characterized in terms of the transmission polynomials of the systems

$$\begin{aligned} \dot{x} &= Ax + Bu + Pw, \\ \dot{w} &= Sw \\ e &= Cx - Qw \end{aligned} \tag{1.8}$$

and

$$\begin{aligned} \dot{x} &= Ax + Bu, \\ \dot{w} &= Sw \\ e &= Cx, \end{aligned} \tag{1.9}$$

with input u and output e. This characterization was proved by Hautus in [14].

Theorem 1.3 *The matrix equations are solvable if and only if the systems (1.8) and (1.9) have the same transmission polynomials.*

These results were generalized to nonlinear multivariable systems by Byrnes and Isidori [3]. In their results the solvability conditions introduced by Francis correspond to the local solvability of a pair of nonlinear equations. Byrnes and Isidori introduced a geometric interpretation of the regulator equations: they point out that the solvability of the regulator equations corresponds to the existence of an error zeroing local manifold (in the linear case a subspace) that is rendered invariant by feedback.

Byrnes and Isidori also generalized Hautus' result given in Theorem 1.3. They express the solvability of the regulator equations in terms of the relationship between the zero dynamics of the plant and the zero dynamics of the composite system (formed from the plant and the exosystem), that has the tracking error as its output. They proved that the regulator equations are solvable exactly when the zero dynamics of the composite system can be decomposed to the exosystem and the zero dynamics of the plant. The Byrnes-Isidori nonlinear theory relies on the center manifold theory as its main tool.

1.3 A Finite Dimensional Example

To illustrate the finite-dimensional regulator theory we will consider a simple example. Let the controlled plant be given by the following system:

$$\begin{aligned} \dot{x}_1 &= x_1 + x_2 + u \\ \dot{x}_2 &= 3x_1 - 3x_2 \\ y &= x_2. \end{aligned} \tag{1.10}$$

With the output we want to track the constant reference signal $y^r \equiv 6$, but we do not consider any disturbance. With

$$A = \begin{bmatrix} 1 & 1 \\ 3 & -3 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 1 \end{bmatrix},$$

and $d(t) = 0$ (1.10) can be written in the form of of (1.1). To produce the reference signal we can use a trivial exogenous system of the form of (1.4), with $w \in I\!R$, $S = 0$, $Q = 1$, $P = 0$ and initial condition $w(0) = 6$. We note that for this system the hypotheses of the linear finite-dimensional regulator theory are satisfied:

h1: $\sigma(S) = 0$,

h2: it is easy to check that (A, B) is stabilizable, indeed with $K = \begin{bmatrix} -2 & -1 \end{bmatrix}$ $\sigma(A + BK) = \{-3, -1\}$,

h3: the detectability of the pair

$$\left(\begin{bmatrix} C & -Q \end{bmatrix}, \begin{bmatrix} A & P \\ 0 & S \end{bmatrix} \right)$$

can be seen if we set, e.g., $G = \begin{bmatrix} 8 & 5 & 1 \end{bmatrix}^T$, and compute that the eigenvalues of

$$\begin{bmatrix} A & P \\ 0 & S \end{bmatrix} - G \begin{bmatrix} C & -Q \end{bmatrix}$$

are $-1, -2$ and -3.

The regulator equations

$$\begin{aligned} \Pi S &= A\Pi + B\Gamma + P, \\ C\Pi - Q &= 0 \end{aligned} \tag{1.11}$$

are also solvable for Π and Γ. A quick computation shows that the pair $\Pi = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\Gamma = -2$ is a solution. Theorem 1.1 and Theorem 1.2 then shows that both the state feedback and the error feedback regulator problems are solvable.

Solutions for both problems can also be found. In the state feedback case the feedback

$$u = -2x_1 - x_2 + 6 \tag{1.12}$$

will result in tracking for any initial condition $\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix}$. In the error feedback case the controller (1.6) with

$$F = \begin{bmatrix} -1 & -8 & 9 \\ 3 & -8 & 5 \\ 0 & -1 & 1 \end{bmatrix}, \quad H = \begin{bmatrix} -2 & -1 & 1 \end{bmatrix}, G = \begin{bmatrix} 8 & 5 & 1 \end{bmatrix}^T,$$

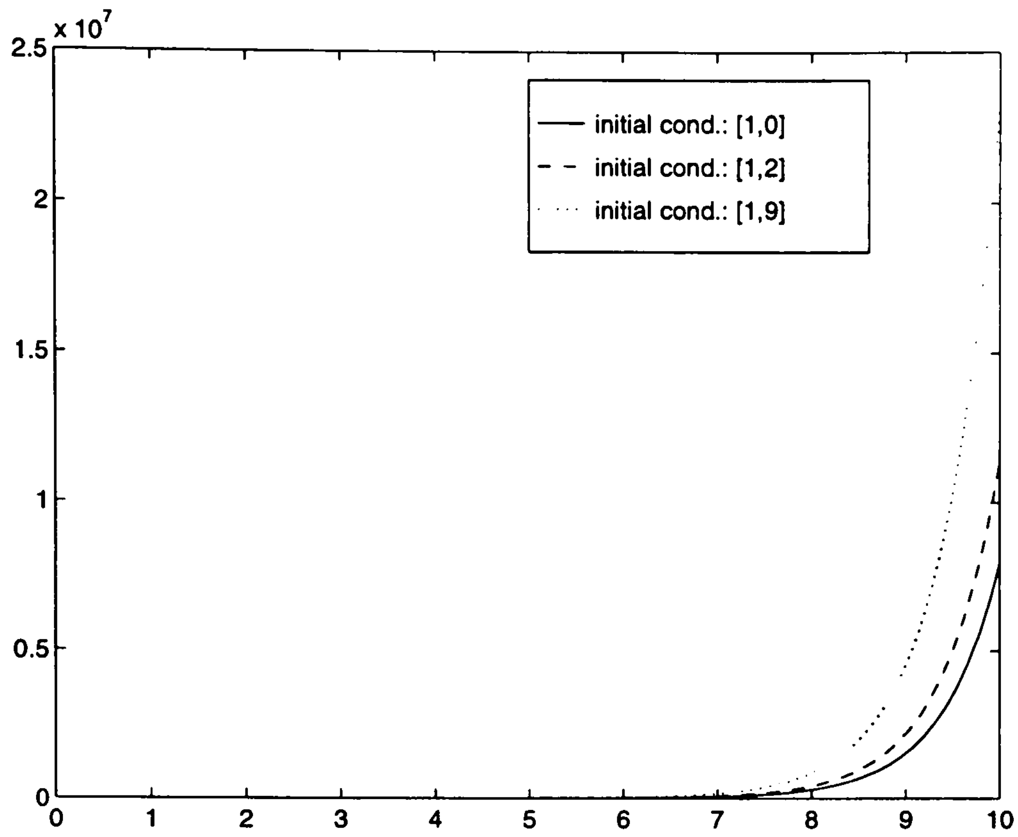

Figure 1.1: Output of the uncontrolled system

i.e., the system

$$\begin{aligned}
\dot{X}_1 &= -X_1 - 8X_2 + 9X_3 + 8e \\
\dot{X}_2 &= 3X_1 - 8X_2 + 5X_3 + 5e \\
\dot{X}_3 &= -X_2 + X_3 + e \\
u &= -2X_1 - X_2 + X_3
\end{aligned}$$

will be a solution.

Figure 1.1 shows a numerical approximation of the uncontrolled output x_2 for various initial conditions of the plant. Numerical results for the output $y = x_2$ for the same initial conditions are shown on Figure 1.2 for the state feedback solution. In Figure 1.3 we can see the tracking with the error feedback solution, starting the plant from the same initial conditions as before, but setting different initial conditions for the controller for each case. This indicates the fact, that for this particular solution for the error feedback problem, the tracking is not sensitive to the initial conditions of the controller, in other words, irrespective of its initial conditions the above controller will solve the tracking problem.

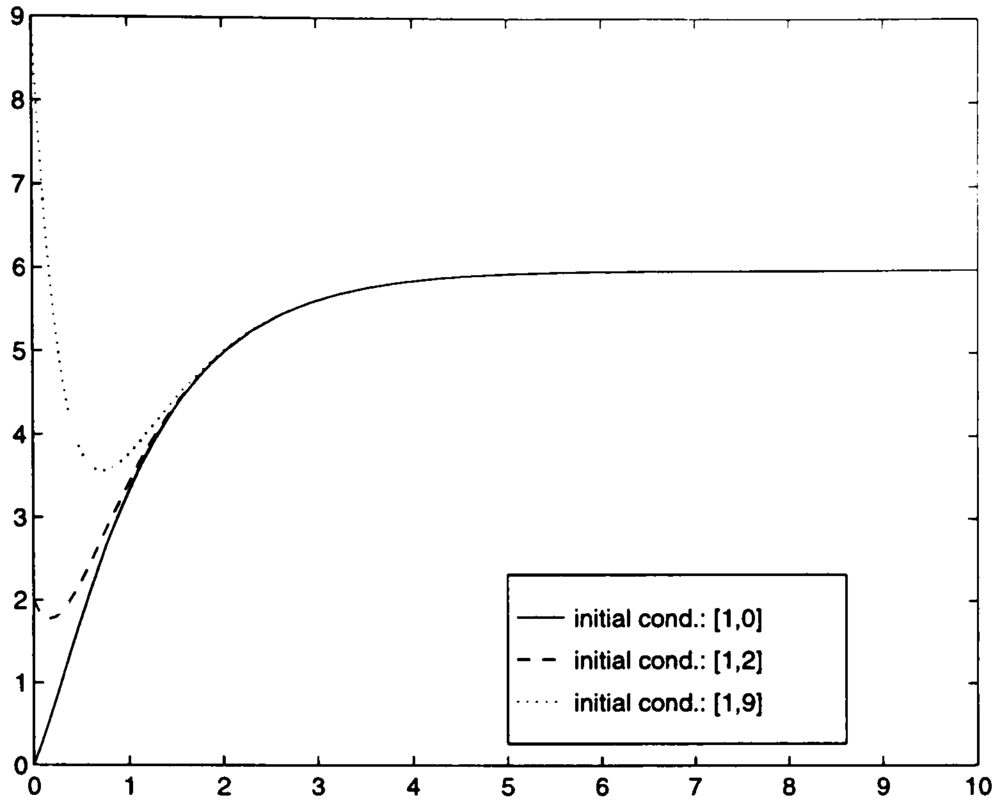

Figure 1.2: Tracking with state feedback controller

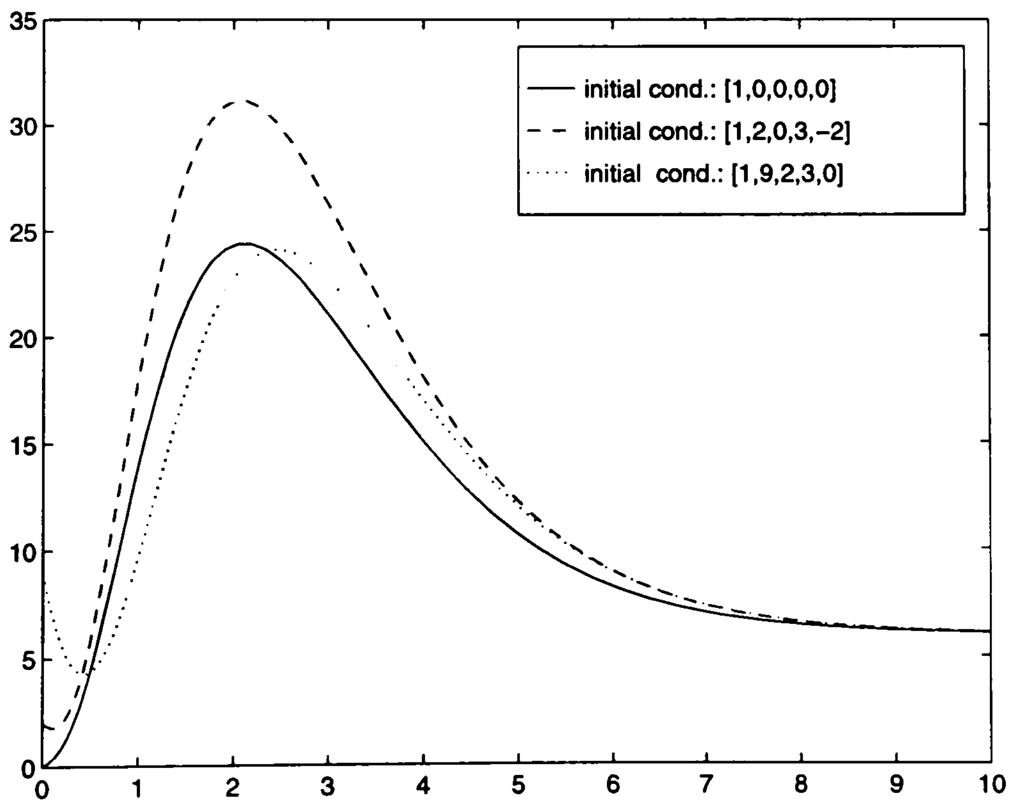

Figure 1.3: Tracking with error feedback controller

1.4 Outline of the Dissertation

In this work we give the generalization of the results outlined above for infinite-dimensional linear control systems assuming that the control and observation operators are bounded. In particular, we obtain results characterizing the solvability of both the state and the error feedback regulator problems in terms of the solvability of certain equations which we again refer to as the regulator equations. We need to address certain difficulties that arise in extending the Byrnes-Isidori approach to the distributed parameter case. Now the phase space is infinite-dimensional, the state operator is unbounded and consequently, only densely defined. There is no direct analog of the Jordan decomposition, so care must be exercised in dealing with the spectra of composite systems. The error zeroing invariant subspace complementary to the plant must be contained in the domain of the state operator, and the resulting regulator equations now become distributed parameter equations. Except for the technical problems, the proofs parallel the development in [3]. Chapter 2 contains a formulation and discussion of basic assumptions, and the presentation of the state and error feedback regulator problems. We give necessary and sufficient conditions for the solvability of both the state and error feedback regulator problems in terms of the solvability of a pair of linear operator equations – the regulator equations. Once a solution of these equations is available, an appropriate state feedback control in the case of the full information problem, or an explicit dynamical controller in the case of error feedback, is obtained that provides the solution to the regulator problem. In Chapter 3, Theorem 2.2 is applied to two specific examples. The first example is the problem of finding a feedback control law for a controlled heat equation so that the output of the closed loop system will track a prescribed periodic trajectory. In the second example we solve the same regulator problem for a damped wave equation. For these examples the regulator equations reduce to a system of linear ordinary differential equations subject to certain constraints. These systems are readily solved off-line. In particular, approximate solutions are easily obtained numerically. In Chapter 4, motivated by Byrnes-Isidori [3], we extend the concept of zero dynamics for linear distributed parameter systems and give a characterization of the solvability of the regulator equations in terms of the structure of the zero dynamics of the system formed from the plant and the exogenous system with the tracking error as output.

CHAPTER II

SOLVABILITY AND THE REGULATOR EQUATIONS

2.1 Definitions, Auxiliary Results and Introductory Comments

To solve the two output regulation problems, the solvability criteria given in terms of the solvability of the regulator equations gives a practical approach. Indeed, once a solution of the regulator equations in Theorems 1.1 and 1.2 is found and the stabilizing feedback and output injection, that appear in the conditions **h2** and **h3**, are available, the controllers that solve the two types of output regulation problems can be explicitly constructed. For the finite-dimensional linear example in the Introduction the state feedback controller was constructed as $u = Kx + (\Gamma - K\Pi)w$, and the error feedback solution was found as

$$H = \begin{bmatrix} K & (\Gamma - K\Pi) \end{bmatrix},$$

$$F = \begin{bmatrix} (A + BK - G_1C) & (P + B(\Gamma - K\Pi) + G_1Q) \\ -G_2C & (S + G_2Q) \end{bmatrix},$$

where $G_1 = \begin{bmatrix} 8 \\ 5 \end{bmatrix}$ and $G_2 = 1$ were parts of the output injection G acting on the state space of the plant and the exosystem, correspondingly. We expect that similar equations give solvability conditions and explicit schemes to construct solutions for distributed parameter systems. In this chapter we will obtain the analog of the finite-dimensional linear theory, but the problem of solving the regulator equations, which was a problem of solving linear matrix equations in finite dimensions, now turns into the solvability problem of linear operator equations frequently involving unbounded operators. As we will see in the next chapter the regulator equations can represent elliptic boundary value problems with extra constraints for plants that are described by partial differential equations.

It is clear from the formulation of the problem that the stabilizability of the pair (A, B) is a necessary condition for the solvability of the state feedback problem. For finite-dimensional linear systems it is also known to be necessary for solving the error feedback problem, as it is shown in Francis [11]. The arguments in [11] can also be carried out for a large class of infinite-dimensional systems that are well represented in applications, namely for linear control systems with finite rank control operator B, finite rank observation operator C and with A that satisfies the so called spectrum decomposition condition at some $\beta < 0$. This is not surprising if we consider that in the error feedback problem the controller is given less information about the plant then in the state feedback case. It is also suggested by the example in the introductory chapter that a necessary step in achieving output regulation should be the stabilization of the plant.

Definition 2.1 *Let the linear operator A be the infinitesimal generator of a C_0 semigroup $T(t)$ on the Hilbert space Z. $T(t)$ is called β-exponentially stable if there exist constants β and $M_\beta > 0$, such that*

$$\|T(t)\| \leq M_\beta e^{\beta t} \qquad for \quad t \geq 0. \tag{2.1}$$

The infimum of all possible values of β for which (2.1) holds is called the stability margin of $T(t)$. We say that $T(t)$ is exponentially stable if its stability margin is negative. Let us additionally assume that $B : U \longrightarrow Z$ is a bounded linear operator from another Hilbert space U to Z. We say that the pair of (A, B) is β-exponentially stabilizable if there exists a bounded operator $K : Z \longrightarrow U$ such that the semigroup $T_{A+BK}(t)$ generated by $A + BK$ is β-exponetially stable and (A, B) is exponentially stabilizable if it is β-exponentially stabilizable for some $\beta < 0$.

In the previous definition we used, as we will do several times in the following, the basic perturbation result for linear operators that bounded perturbations of infinitesimal generators of C_0 semigroups also generate C_0 semigroups. For composite systems in special cases we know the structure of the perturbed semigroup as well. The proof of the following lemma can be found in [10] Section 3.2.

Lemma 2.1 *Let $T_1(t)$ and $T_2(t)$ be C_0 semigroups on the respective Hilbert spaces Z_1 and Z_2, with infinitesimal generators A_1 and A_2. Suppose that $D : Z_2 \longrightarrow Z_1$ is a bounded linear operator, then the operator $A = \begin{bmatrix} A_1 & D \\ 0 & A_2 \end{bmatrix}$ with $\mathcal{D}(A) = \mathcal{D}(A_1) \times \mathcal{D}(A_2)$ is the infinitesimal generator of the C_0 semigroup*

$$T(t)\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} T_1(t)z_1 + \int_0^t T_1(t-s)DT_2(s)z_2\,ds \\ T_2(t)z_2 \end{bmatrix}.$$

The main tool in obtaining the solvability conditions below will be a suitable decomposition of the state space of the closed-loop system – which is simply the product of the plant's and exosystem's state spaces in the state feedback case, but includes the state space of the controller also for the error feedback problem. To obtain such a decomposition we will rely on a result that is originally due to T. Kato [21]. We state here a version of this result for Hilbert spaces from [10] (see Lemma (2.5.7) therein), after defininig the notions of operator and semigroup invariance of subspaces in Hilbert spaces.

Definition 2.2 *Let V be a subspace of a Hilbert space $\mathcal{H}$ and let A be an infinitesimal generator of a C_0 semigroup on $\mathcal{H}$. We say that V is A-invariant if*

$$A(V \cap \mathcal{D}(A)) \subset V.$$

Definition 2.3 *Let V be a subspace of a Hilbert space $\mathcal{H}$ and let $T(t)$ be a C_0 semigroup on $\mathcal{H}$. We say that V is $T(t)$-invariant if*

$$T(t)V \subset V.$$

These two invariance concepts are equivalent for finite-dimensional systems. In general, $T(t)$-invariance implies A-invariance, but not vice versa. However, they are equivalent for closed subspaces in $\mathcal{D}(A)$ as it is shown in [10]:

Lemma 2.2 *Suppose that A is the infinitesimal generator of a C_0 semigroup $T(t)$ on the Hilbert space $\mathcal{H}$. If V is a closed subspace contained in $\mathcal{D}(A)$ and V is A-invariant, then V is $T(t)$-invariant.*

The Kato spectral decomposition result for Hilbert spaces is the following:

Lemma 2.3 *Let A be the infinitesimal generator of a C_0 semigroup $T(t)$ on $\mathcal{H}$. Assume that the spectrum of A is the disjoint union of two parts σ^+ and σ^-, such that a rectifiable, closed , simple curve C can be drawn that encloses an open set containing σ^+ in its interior and σ^- in its exterior. The operator P_C, defined by*

$$P_C h = \frac{1}{2\pi i} \int_C (\lambda I - A)^{-1} h \, d\lambda$$

where C is traversed once in positive direction, is a projection, the so called spectral projection on σ^+. This projection induces a decomposition of the state space

$$\mathcal{H} = \mathcal{H}^+ \oplus \mathcal{H}^-,$$

where $\mathcal{H}^+ = P_C\mathcal{H}$ and $\mathcal{H} = (I - P_C)\mathcal{H}$. The following properties also hold:

i. *$\mathcal{H}^+$ and $\mathcal{H}^-$ are $T(t)$-invariant;*

ii. *$\mathcal{H}^+ = P_C\mathcal{H} \subset \mathcal{D}(A)$, also $\mathcal{H}^+$ and $\mathcal{H}^-$ are A-invariant, i.e., $A\mathcal{H}^+ \subset \mathcal{H}^+$ and $A(\mathcal{H}^- \cap \mathcal{D}(A)) \subset \mathcal{H}^-$;*

iii. *The restriction A^+ of A to $\mathcal{H}^+$ is a bounded operator on $\mathcal{H}^+$ and $\sigma(A^+) = \sigma^+$. The restriction A^- of A to $\mathcal{H}^-$ has spectrum $\sigma(A^-) = \sigma^-$. Furthermore, for $\lambda \in \rho(A)$ we have that $(\lambda I - A^+)^{-1} = (\lambda I - A)^{-1}|_{\mathcal{H}^+}$ and $(\lambda I - A^-)^{-1} = (\lambda I - A)^{-1}|_{\mathcal{H}^-}$;*

iv. *The operators $T^+(t) = T(t)|_{\mathcal{H}^+}$ and $T^-(t) = T(t)|_{\mathcal{H}^-}$ are C_0 semigroups on $\mathcal{H}^+$ and $\mathcal{H}^-$, respectively, and their infinitesimal generators are given by A^+ and A^-, respectively.*

Next we will see that a special case of the assumption on A in the result above, namely, that the spectrum of the operator can be separated into two parts by a simple closed curve, is necessary for the exponential stabilizability of the pair (A, B) if B is finite rank operator. This shows how strong the concept of exponential stabilizability is.

Definition 2.4 *Let $\mathbb{C}_\beta^+ = \{\lambda \in \mathbb{C} \,|Re(\lambda) > \beta\}$, $\mathbb{C}_\beta^- = \{\lambda \in \mathbb{C} \,|Re(\lambda) < \beta\}$,*

$$\sigma_\beta^+(A) := \sigma(A) \cap \overline{\mathbb{C}_\beta^+},$$

and

$$\sigma_\beta^-(A) := \sigma(A) \cap \mathbb{C}_\beta^-.$$

$A : Z \to Z$ satisfies the spectrum decomposition assumption at β if $\sigma_\beta^+(A)$ and $\sigma_\beta^-(A)$ can be separated by a rectifiable, closed , simple curve C_β that encloses an open set containing $\sigma_\beta^+(A)$ in its interior and $\sigma_\beta^-(A)$ in its exterior.

By Lemma 2.3 such a decomposition of the spectrum corresponds to a decomposition of the state space Z. The spectral projection P_β onto $\sigma_\beta^+(A)$ induces the decomposition of Z:

$$Z = P_\beta Z + (I - P_\beta)Z = Z_\beta^+ + Z_\beta^-,$$

and

$$A = \begin{bmatrix} A_\beta^+ & 0 \\ 0 & A_\beta^- \end{bmatrix}, \quad T(t) = \begin{bmatrix} T_\beta^+(t) & 0 \\ 0 & T_\beta^-(t) \end{bmatrix}, \quad \begin{bmatrix} B_\beta^+ \\ B_\beta^- \end{bmatrix}, \tag{2.2}$$

where

$$A_\beta^+ = A|_{Z_\beta^+}, \qquad A_\beta^- = A|_{Z_\beta^-}, \tag{2.3}$$

$$T_\beta^+(t) = T(t)|_{Z_\beta^+}, \qquad T_\beta^-(t) = T(t)|_{Z_\beta^-}, \tag{2.4}$$

$$B_\beta^+ = P_\beta B \qquad B_\beta^- = (I - P_\beta)B. \tag{2.5}$$

Lemma 2.4 *(See [10] Theorem 5.2.6.) Let the linear operator A be the infinitesimal generator of a C_0 semigroup $T(t)$ on the Hilbert space Z. If $B : U \to Z$ is a finite rank linear operator then the following are equivalent:*

i. The pair (A, B) of linear operators is β-exponentially stabilizable;

ii. A satisfies the spectrum decomposition assumption at β. Z_β^+ is finite-dimensional, $T_\beta^-(t)$ is β-exponentially stable, and the pair (A_β^+, B_β^+) is controllable.

In the state feedback problem once the plant is stabilized, instability in the composite system, formed from the plant and the exosystem, is due only to unstable modes in the exosystem itself. We do not want to get rid of this instability, since this system models the disturbances and produces the reference output. To achieve tracking we are free to use the exosystem variables for feedback, additional to the feedback that stabilizes the plant. We design this feedback to shape the composite system's state space so that the unstable part lies in the kernel of the observation operator, i.e., it is output zeroing, while it leaves the exosystem itself intact. If we are able to do this we achieve that the state of the controlled plant asymptotically evolves on an output zeroing subspace. This would mean that the state feedback problem is solved.

The error feedback problem will be approached with the following considerations. Suppose that are able to achieve output regulation with state feedback, but we only have the tracking error available. The objective is to use a observer to approximate the state of the composite system (plant and exosystem) and use the approximate state in the feedback law that provides solution to the correponding state feedback problem. If for example we use a Luenberger observer, as it turns out, this idea works. The approach requires additional conditions, namely, to reconstruct its state with an observer we need the composite system to be detectable. This is the source of hypothesis **h1** for finite-dimensional systems. We will also need the concept of detectability in our settings.

Definition 2.5 *Let A be the infinitesimal generator of the C_0 semigroup $T(t)$ on a Hilbert space Z, $C : Z \longrightarrow Y$ be a bounded linear operator from Z to another Hilbert space Y. We say that the pair of (C, A) is β-exponentially detectable if there exists a bounded linear operator $L : Y \longrightarrow Z$ such that the semigroup $T_{A+LC}(t)$, generated by $A + LC$, is β-exponentially stable and (C, A) is exponentially detectable if it is β-exponentially detectable for some $\beta < 0$.*

Now we will formulate the infinite-dimensional output regulation problems.

2.2 Statement of Problems

Consider a plant described by an abstract distributed parameter control system in Hilbert space:

$$\begin{aligned} &\frac{d}{dt}z(t) = Az(t) + Bu(t) + d(t), \\ &y(t) = Cz(t), \\ &z(0) = z_0, \end{aligned} \tag{2.6}$$

where $z \in Z$ is the state of the system, Z is a separable Hilbert space (state space), $u \in U$ is an input , $y \in Y$ is the measured output, U and Y are Hilbert spaces, the control and output spaces, respectively. The term $d(t)$ represents a disturbance.

S1 A is assumed to be the infinitesimal generator of a strongly continuous semigroup $T(t)$ on the Hilbert space Z, $B \in \mathcal{L}(U, Z)$ and $C \in \mathcal{L}(Z, Y)$.

(Here we use the notation $\mathcal{L}(W_1, W_2)$ to denote the set of all bounded linear operators from a Hilbert space W_1 to a Hilbert space W_2.)

In addition, we will assume that there exists a finite-dimensional linear system, referred to as the exogenous system (or exosystem), that produces a reference output $y_r(t)$ and which is also used to model the disturbance $d(t)$:

$$\frac{d}{dt}w(t) = Sw(t) \tag{2.7}$$

$$y_r(t) = Qw(t) \tag{2.8}$$

$$d(t) = Pw(t) \tag{2.9}$$

$$w(0) = w_0. \tag{2.10}$$

S2 Here $S \in \mathcal{L}(\mathbb{R}^k)$, $Q \in \mathcal{L}(\mathbb{R}^k, Y)$ and $P \in \mathcal{L}(\mathbb{R}^k, Z)$.

We will refer to the difference between the measured and reference outputs as the error

$$e(t) = y(t) - y_r(t) = Cz(t) - Qw(t). \tag{2.11}$$

The two problems we are interested in are formulated as follows.

I. Linear State Feedback Regulator Problem:

Find a feedback control law in the form $u(t) = Kz(t) + Lw(t)$ such that $K \in \mathcal{L}(Z, U)$, $L \in \mathcal{L}(\mathbb{R}^k, U)$ and

I.a $\dot{z}(t) = (A + BK)z(t)$ is stable, i.e., $(A + BK)$ is the infinitesimal generator of an exponentially stable C_0 semigroup.

I.b For the closed-loop system

$$\dot{z}(t) = (A + BK)z(t) + (BL + P)w(t), \tag{2.12}$$

$$\dot{w}(t) = Sw(t)$$

the error

$$e(t) = Cz(t) - Qw(t) \longrightarrow 0$$

as $t \to \infty$, for any initial condition in $Z \times \mathbb{R}^k$.

Since the state of the plant is usually not fully available, we are led to investigate the error feedback regulator problem.

II. Linear Error Feedback Regulator Problem:

Find an error feedback controller of the form

$$\begin{aligned}\frac{d}{dt}X(t) &= FX(t) + Ge(t),\\ u(t) &= HX(t)\end{aligned} \tag{2.13}$$

where $X(t) \in \mathcal{X}$ for $t \geq 0$, $\mathcal{X}$ is a Hilbert space, $G \in \mathcal{L}(Y, \mathcal{X})$, $H \in \mathcal{L}(\mathcal{X}, U)$ and F is the infinitesimal generator of a C_0 semigroup on $\mathcal{X}$ with the properties that

II.a The system

$$\begin{aligned}\frac{d}{dt}z(t) &= Az(t) + BHX(t),\\ \frac{d}{dt}X(t) &= FX(t) + Ge(t)\end{aligned} \tag{2.14}$$

is exponentially stable when $w \equiv 0$, i.e., $\begin{bmatrix} A & BH \\ GC & F \end{bmatrix}$ is the infinitesimal generator of an exponentially stable C_0 semigroup, and

II.b for the closed-loop system

$$\begin{aligned}\frac{d}{dt}z(t) &= Az(t) + BHX(t) + Pw(t),\\ \frac{d}{dt}X(t) &= GCz(t) + FX(t) - GQw(t),\\ \frac{d}{dt}w(t) &= Sw(t)\end{aligned} \tag{2.15}$$

the error $e(t) = Cz(t) - Qw(t) \longrightarrow 0$ as $t \to \infty$, for any initial condition in $Z \times \mathcal{X} \times I\!R^k$.

We impose the following standard assumptions.

Three basic assumptions:

H1 Assume that the spectrum of the exosystem is contained in the closed right half plane, i.e., $\sigma(S) \subset \overline{C_0^+}$;

H2 Assume that the pair (A, B) is exponentially stabilizable, i.e., there exists $K \in \mathcal{L}(Z, U)$ such that $A + BK$ is the infinitesimal generator of an exponentially stable C_0 semigroup;

H3 Assume that the pair

$$\left(\begin{bmatrix} A & P \\ 0 & S \end{bmatrix}, \begin{bmatrix} C & -Q \end{bmatrix} \right) \tag{2.16}$$

is exponentially detectable, i.e., there exists $G \in \mathcal{L}(Y, Z \times \mathbb{R}^k)$,

$$G = \begin{bmatrix} G_1 \\ G_2 \end{bmatrix}, \; G_1 \in \mathcal{L}(Y, Z), \; G_2 \in \mathcal{L}(Y, \mathbb{R}^k),$$

such that

$$\begin{bmatrix} A & P \\ 0 & S \end{bmatrix} - \begin{bmatrix} G_1 \\ G_2 \end{bmatrix} \begin{bmatrix} C & -Q \end{bmatrix}$$

is the infinitesimal generator of an exponentially stable C_0 semigroup.

These assumptions correspond to the hypotheses **h1-h3**, on which the finite-dimensional linear regulator theory is based. **H1** does not involve loss of generality since the modes of the exosystem that correspond to eigenvalues in the open left half plane decay exponentially to zero, and so asymptotically do not affect the output regulation. It is evident from the formulation of the state feedback problem, that for its solvability **H2** is a necessary condition. For finite-dimensional linear systems it is known that the stabilizability of (A, B) and the detectability of (C, A) are necessary for the solvability of the error feedback problem. The proof of this result, which appeared in [11] can be adjusted to our settings, provided that we make additional assumptions on the system (2.6). Before we do that we will state a result from [10] (Theorem 5.2.11) that, for a class of linear control systems, will give necessary and sufficient conditions for exponential stabilizability and detectability that are generalizations of the finite-dimensional Hautus conditions. We will use notations that were set in Definition 2.4 and in (2.2)-(2.5).

Theorem 2.1 *Consider the linear system with B and C finite rank operators. Suppose that A satisfies the spectrum decomposition assumption at β, $\sigma_\beta^+(A)$ comprises finitely many eigenvalues, with finite multiplicity and $T_\beta^-(t)$ is β-exponentially stable. The pair (A, B) is β-exponentially stabilizable if and only if*

$$Ran(sI - A) + Ran B = Z \qquad for\ all \quad s \in \mathbb{C}_\beta^+. \tag{2.17}$$

The pair (C, A) is β-exponentially detectable if and only if

$$Ker(sI - A) \cap Ker C = \{0\} \qquad for\ all \ \ s \in \mathbb{C}^+_\beta. \tag{2.18}$$

Let us now make the additional assumptions on system (2.6) that the conditions of Theorem 2.1 are satisfied.

Assume that A satisfies the spectrum decomposition assumption at $\beta < 0$, $\sigma^+_\beta(A)$ comprises finitely many eigenvalues, with finite multiplicity and $T^-_\beta(t)$ is β-exponentially stable. Assume also that B and C are finite rank operators. Let there exist a controller of the form (2.13) that solves the error feedback regulator problem. In particular $\tilde{A} = \begin{bmatrix} A & BH \\ GC & F \end{bmatrix}$ is the infinitesimal generator of an exponentially stable C_0 semigroup. It means that the stability margin of $\tilde{A}$,

$$\omega_0 = \lim_{t \to 0} \frac{\log \|T_{\tilde{A}}(t)\|}{t} < 0.$$

But then the spectrum $\sigma(\tilde{A}) \subset \overline{\mathbb{C}^-_{\omega_0}}$, and therefore $\mathbb{C}^+_{\frac{\omega_0}{2}} \subset \rho(\tilde{A})$. This implies that for every $s \in \mathbb{C}^+_{\frac{\omega_0}{2}}$

$$\begin{bmatrix} A - sI & BH \\ GC & F - sI \end{bmatrix}$$

is boundedly invertible in $Z \times \mathcal{X}$ with dense range. This implies that $Ran(sI - A) + Ran(BH)$ is dense in Z and $Ker(sI - A) \cap Ker(GC) = \{0\}$ or, a forteriori, that the subspace

$$Ran(sI - A) + Ran B \tag{2.19}$$

of Z is dense for all $s \in \mathbb{C}^+_\omega$ and

$$Ker(sI - A) \cap Ker C = \{0\} \qquad for \ \ s \in \mathbb{C}^+_\omega, \tag{2.20}$$

with $\omega \doteq \max(\frac{\omega_0}{2}, \beta)$. By Theorem 2.1, (2.20) is equivalent to the ω-exponential detectability of (C, A). Since $\omega < 0$ this gives that the pair (C, A) is exponentially detectable. Considering the statement of Theorem 2.1 it may seem that the density of $Ran(sI - A) + Ran B$ in Z is not sufficient for the ω-exponential stabilizability of (A, B), however scrutinizing the proof of Theorem 2.1 in [10] we find that it can be modified to show that it is indeed sufficient (see Lemma 2.5 below). Since

$\omega < 0$ we can also conclude that (A, B) exponentially stabilizable. Thus, with the following lemma we obtain that for the solvability of the error feedback problem the exponential stabilizability of (A, B) and the exponential detectability of (C, A) are necessary.

Lemma 2.5 *Suppose that the assumptions of Theorem 2.1 hold. If*

$$Ran(sI - A) + RanB$$

is a dense subspace of Z for all $s \in \mathbb{C}_\omega^+(A)$ then (A, B) is ω-exponentially stabilizable.

Proof: We repeat the argument given in [10] page 241 with a slight modification. Since A satisfies the spectrum decomposition assumption at ω, we may consider the spectral decomposition

$$A = \begin{bmatrix} A_\omega^+ & 0 \\ 0 & A_\omega^- \end{bmatrix}, \quad B = \begin{bmatrix} B_\omega^+ \\ B_\omega^- \end{bmatrix}, \quad C = \begin{bmatrix} C_\omega^+ & C_\omega^- \end{bmatrix},$$

where P_ω is the spectral projection onto $\sigma_\omega^+(A)$, $\sigma(A_\omega^+) \subset \mathbb{C}_\omega^+$, $\sigma(A_\omega^-) \subset \mathbb{C}_\omega^-$, using the notations in Definition 2.4, in (2.2)-(2.5) and defining $C_\omega^+ = CP_\omega$, $C_\omega^- = C(I - P_\omega)$. P_ω is projection onto a finite-dimensional subspace Z_ω^+ of Z and the triple $(A_\omega^+, B_\omega^+, C_\omega^+)$ form a finite-dimensional system linear system with state space Z_ω^+ (see Lemma 2.3). By the properties listed in Lemma 2.3 we have for all $s \in \mathbb{C}_\omega^+$ that

$$\begin{aligned} &P_\omega(Ran(sI - A) + RanB) \\ &= P_\omega(Ran(sI - A)) + P_\omega(RanB) \\ &= Ran(P_\omega(sI - A)) + Ran(P_\omega B) \\ &= Ran(sI_{Z_\omega^+} - A_\omega^+) + Ran(B_\omega^+). \end{aligned} \tag{2.21}$$

But also

$$P_\omega(Ran(sI - A) + RanB) = Z_\omega^+. \tag{2.22}$$

We can show (2.22) with the following argument: Clearly $P_\omega(Ran(sI - A) + RanB) \subset Z_\omega^+$, since P_ω is a projection onto the finite-dimensional subspace Z_ω^+.

Suppose that $z \in Z_\omega^+$ is not in the subspace $P_\omega(Ran(sI - A) + RanB)$ of Z_ω^+. Since $Ran(sI - A) + RanB$ is dense in Z we can choose a sequence $\{z_i\}_{i=1}^\infty \subset Ran(sI - A) + RanB$ such that $z_i \to z$ in Z. Since P_ω is continuous this implies that $P_\omega z_i \to P_\omega z = z$. However we have that $\{P_\omega z_i\}_{i=1}^\infty \subset P_\omega(Ran(sI - A) + RanB)$, where $P_\omega(Ran(sI - A) + RanB)$ is finite-dimensional subspace, hence closed and that $z \notin P_\omega(Ran(sI - A) + RanB)$. This contradicts with the convergence $P_\omega z_i \to z$. Now we conclude that the finite-dimensional system $(A_\omega^+, B_\omega^+, C_\omega^+)$ on the state space Z_ω^+ is ω-exponentially stabilizable by the finite-dimensional Hautus criteria (compare equations (2.21) and (2.22)). This implies that the pair (A_ω^+, B_ω^+) is controllable and by Theorem 2.4 the pair (A, B) is exponentially stabilizable. □

2.3 Solvability Conditions for the Regulator Problems

The main results of this section are contained in Theorems 2.2 and 2.3 that give necessary and sufficient conditions for the solvability of the state feedback and error feedback regulator problems, respectively. To obtain these results we prove a series of lemmas.

Lemma 2.6 *Let $S \in \mathcal{L}(\mathbb{R}^k)$ satisfy* **H1**. *Let Y be a normed space and $N \in \mathcal{L}(\mathbb{R}^k, Y)$. If $Ne^{St}w \longrightarrow 0$ as $t \to \infty$ for every $w \in \mathbb{R}^k$ then $N = 0$.*

Proof: Let $Nw = \begin{bmatrix} y_1 & \dots & y_k \end{bmatrix} w$ with some $y_j \in Y$ for $j = 1, 2, \cdots, k$. Without loss of generality we may assume that S is in Jordan normal form since otherwise we can consider

$$Ne^{St}w_0 = (NP)(P^{-1}e^{St}P)(P^{-1}w_0) = (NP)e^{P^{-1}SPt}(P^{-1}w_0)$$

for some nonsingular matrix P and remark that P is bijective and therefore $N = 0$ exactly when $NP = 0$. The semigroup e^{St} has the special form

$$e^{St} = \begin{bmatrix} e^{J_1 t} & 0 & \dots & 0 \\ 0 & e^{J_2 t} & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \dots & e^{J_s t} \end{bmatrix},$$

where

$$e^{J_i t} = e^{\lambda_i t}\begin{bmatrix} 1 & t & \frac{t^2}{2!} & \dots & \frac{t^{r_i-1}}{(r_i-1)!} \\ 0 & 1 & t & \dots & \frac{t^{r_i-2}}{(r_i-2)!} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{bmatrix}, \qquad i = 1, 2, ..., s, \ \sum_{i=1}^{s} r_i = k.$$

We use finite induction on $j = 1, ..., k$. Assume that $y_j \neq 0$, then choosing the standard basis elements in $\mathbb{R}^k$ as initial conditions $w_0 = e_m$ for $m = 1, \cdots, j$, considering the form of the exponential of a matrix in Jordan normal form and taking into account that the eigenvalue λ_j of S have nonnegative real parts we obtain a contradiction to the assumption that $Ne^{St}w_0 \to 0$. This gives that $y_j = 0$. It is true for $j = 1, ..., k$ and hence $N = 0$. □

Lemma 2.7 *Let H be a Hilbert space, $A \in \mathcal{C}(H)$ (closed operators in H), $S \in \mathcal{L}(\mathbb{R}^k)$ and $B \in \mathcal{L}(\mathbb{R}^k, H)$. If $\sigma(A) \cap \sigma(S) = \emptyset$, then $\sigma(F) = \sigma(A) \cup \sigma(S)$ where*

$$F = \begin{bmatrix} A & B \\ 0 & S \end{bmatrix}.$$

Proof: We first show that $\sigma(F) \subset \sigma(A) \cup \sigma(S)$. Suppose that $\lambda \notin \sigma(A)$ and $\lambda \notin \sigma(S)$, then $(\lambda I - A)^{-1}$ and $(\lambda I - S_1)^{-1}$ are bounded operators on H and $\mathbb{R}^k$, respectively. Then the operator

$$R = \begin{bmatrix} (\lambda I - A)^{-1} & (\lambda I - A)^{-1}B(\lambda I - S)^{-1} \\ 0 & (\lambda I - S)^{-1} \end{bmatrix}$$

is a bounded operator on $H \times \mathbb{R}^k$ and by a straightforward calculation using the fact that

$$Ran((\lambda I - A)^{-1}) = \mathcal{D}(\lambda I - A) = \mathcal{D}(A)$$

we have

$$R(\lambda I - F)y = y \qquad \forall \ \ y \in \mathcal{D}(F) = \mathcal{D}(A) \times \mathbb{R}^k,$$
$$(\lambda I - F)Ry = y \qquad \forall \ \ y \in H \times \mathbb{R}^k,$$

which means that $\lambda \notin \sigma(F)$. Next we show that $\sigma(A) \cup \sigma(S) \subset \sigma(F)$. The proof is carried out in two steps:

1. First we show that $\lambda \in \sigma(S)$ implies $\lambda \in \sigma(F)$. Suppose that $Sw = \lambda w$ for some $w \neq 0$. Then

$$F \begin{bmatrix} h \\ w \end{bmatrix} = \lambda \begin{bmatrix} h \\ w \end{bmatrix}$$

is equivalent to

$$\begin{bmatrix} A & B \\ 0 & S \end{bmatrix} \begin{bmatrix} h \\ w \end{bmatrix} = \begin{bmatrix} \lambda h \\ \lambda w \end{bmatrix}$$

which is in turn equivalent to the pair of equations

$$\begin{aligned} Ah + Bw &= \lambda h, \\ Sw &= \lambda w. \end{aligned}$$

The second of these, by assumption, holds for a nonzero w and since $\lambda \in \rho(A)$, the first equation is also solvable for h by $h = (\lambda I - A)^{-1} Bw$.

2. Now we show that $\lambda \in \sigma(A)$ implies $\lambda \in \sigma(F)$: Suppose that $\lambda \in \rho(F)$. Then $(\lambda I - F)^{-1}$ exists, is bounded and is of the form $(\lambda I - F)^{-1} = \begin{bmatrix} P_1 & P_2 \\ P_3 & P_4 \end{bmatrix}$. Let $0 \neq h \in \mathcal{D}(A)$, then

$$\begin{aligned} \begin{bmatrix} h \\ 0 \end{bmatrix} &= (\lambda I - F)^{-1}(\lambda I - F) \begin{bmatrix} h \\ 0 \end{bmatrix} = (\lambda I - F)^{-1} \begin{bmatrix} (\lambda I - A)h \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} P_1 & P_2 \\ P_3 & P_4 \end{bmatrix} \begin{bmatrix} (\lambda I - A)h \\ 0 \end{bmatrix} = \begin{bmatrix} P_1(\lambda I - A)h \\ P_3(\lambda I - A)h \end{bmatrix}. \end{aligned}$$

This implies that $Ran(\lambda I - A) \subset Ker(P_3)$ and $P_1(\lambda I - A)h = h$ for all $h \in \mathcal{D}(A)$. On the other hand,

$$\begin{aligned} \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} &= \begin{bmatrix} (\lambda I - A) & -B \\ 0 & (\lambda I - S) \end{bmatrix} \begin{bmatrix} P_1 & P_2 \\ P_3 & P_4 \end{bmatrix} \\ &= \begin{bmatrix} (\lambda I - A)P_1 - BP_3 & (\lambda I - A)P_2 - BP_4 \\ (\lambda I - S)P_3 & (\lambda I - S)P_4 \end{bmatrix} \end{aligned}$$

The $(2,1)$ position of this matrix identity implies that $Ran(P_3) \subset Ker(\lambda I - S)$. But $Ker(\lambda I - S) = 0$, since $\lambda \in \sigma(A) \subset \rho(S)$. Therefore $P_3 = 0$ and

$(\lambda I - A)P_1 = I_H$, also $P_1 = P(\lambda I - F)^{-1}P$, where P is the projection of $H \times I\!R^k$ onto H. So P_1 is a bounded operator, the inverse of $(\lambda I - A)$, i.e., $\lambda \in \rho(A)$, which is a contradiction. $\square$

Lemma 2.8 *Let $\widetilde{H}$, $\tilde{A}$, $\tilde{S}$, $\tilde{B}$ and $\tilde{\mathcal{F}}$ be as H, A, S, B and F in Lemma 2.7. Assume that $\tilde{A}$ is the infinitesimal generator of an exponentially stable C_0 semigroup and $\tilde{S}$ satisfies* **H1**.

Let $\tilde{T}(t)$ denote the C_0 semigroup in the space $\mathcal{H} = \widetilde{H} \times I\!R^k$ generated by

$$\tilde{\mathcal{F}} = \begin{bmatrix} \tilde{A} & \tilde{B} \\ 0 & \tilde{S} \end{bmatrix}.$$

Then the space $\mathcal{H} = \widetilde{H} \times I\!R^k$ can be decomposed into $\mathcal{H} = \mathcal{H}^+ \oplus \mathcal{H}^-$ where $\mathcal{H}^+$ and $\mathcal{H}^-$ are $\tilde{\mathcal{F}}$-invariant and $\tilde{T}(t)$-invariant complementary subspaces, with the property that if $M : I\!R^k \longrightarrow \widetilde{H}$ is defined by the expression $Graph(M) = \mathcal{H}^+$ then $M \in \mathcal{L}(I\!R^k, \widetilde{H})$ and $Ran(M) \subset \mathcal{D}(\tilde{A})$.

Proof: Applying Lemma 2.7 and using the assumptions on $\tilde{A}$ and $\tilde{S}$, it follows that the part of the spectrum of the operator $\tilde{\mathcal{F}}$ lying in the closed right half plane consists of finitely many eigenvalues of finite multiplicity.

We have that $\tilde{\mathcal{F}}$ satisfies the spectrum decomposition condition so we can apply Lemma 2.3, thus obtaining a decomposition of $\mathcal{H}$ into two $\tilde{\mathcal{F}}$-invariant and $\tilde{T}(t)$-invariant subspaces as $\mathcal{H} = \mathcal{H}^+ \oplus \mathcal{H}^-$. Additional consequences of the spectrum decomposition result include that

$$\mathcal{H}^+ \subset \mathcal{D}(\tilde{\mathcal{F}}).$$

Now define the operator $M : I\!R^k \longrightarrow \widetilde{H}$ by $h = Mw$ for $w \in I\!R^k$, where $\begin{bmatrix} h \\ w \end{bmatrix} \in \mathcal{H}^+$. We will show that M is an operator with the stated properties.

First let us show that M is well defined. It is enough to show that

$$P_\Gamma \begin{bmatrix} h \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

This equality holds since we know from the proof of Lemma 2.7 that when $\lambda \in \Gamma$, $(\lambda I - \widetilde{\mathcal{F}})^{-1}$ has the special form $\begin{bmatrix} P_1 & P_2 \\ 0 & P_4 \end{bmatrix}$ and therefore

$$P_\Gamma \begin{bmatrix} h \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{2\pi i}\int_\Gamma (\lambda I - \widetilde{A})^{-1} h d\lambda \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

The first component of the last equality holds by Cauchy's theorem for vector valued analytic functions since $(\lambda I - \widetilde{A})^{-1} h$ is analytic in the closure of the interior of Γ.

We have just seen that $\begin{bmatrix} h \\ 0 \end{bmatrix}$ is not in $\mathcal{H}^+$ for nonzero h. Next we show that for every $w \in I\!R^k$ there is an element $\begin{bmatrix} h \\ w \end{bmatrix}$ in $\mathcal{H}^+$. For every nonzero $w \in I\!R^k$,

$$P_\Gamma \begin{bmatrix} h \\ w \end{bmatrix} = \begin{bmatrix} * \\ \frac{1}{2\pi i}\int_\Gamma (\lambda I - \widetilde{S})^{-1} w d\lambda \end{bmatrix} = \begin{bmatrix} * \\ w \end{bmatrix},$$

where by "$*$" we denote certain vectors from $\mathcal{H}$ whose precise form is immaterial for us. We see that $\begin{bmatrix} h \\ w \end{bmatrix} \notin \mathcal{H}^-$ and since $\begin{bmatrix} h \\ w \end{bmatrix}$ has a decomposition of the form

$$\begin{bmatrix} h \\ w \end{bmatrix} = P_\Gamma \begin{bmatrix} h \\ w \end{bmatrix} + (I - P_\Gamma) \begin{bmatrix} h \\ w \end{bmatrix} = \begin{bmatrix} * \\ w \end{bmatrix} + \begin{bmatrix} * \\ 0 \end{bmatrix},$$

(where, once again, "$*$" denotes certain vectors from $\mathcal{H}$) and it follows that M is defined on all of $I\!R^k$. The linearity follows from the fact that $\mathcal{H}^+$ in the definition of M is linear subspace and the boundedness of M is obvious since $I\!R^k$ is finite-dimensional. $Ran(M) \subset \mathcal{D}(\widetilde{A})$ follows from Lemma 2.3 since $P_\Gamma \mathcal{H} \subset \mathcal{D}(\widetilde{\mathcal{F}})$, which implies, by the definition of M (defined through values in $\mathcal{H}^+$), that $Mw \in \mathcal{D}(\widetilde{A})$ for every $w \in I\!R^k$. □

Lemma 2.9 *Let* **H1** *hold and assume that for some* $u(t) = Kz(t) + Lw(t)$ **I.a** *holds. Then there exists* $\Pi \in \mathcal{L}(I\!R^k, Z)$ *such that* $Ran(\Pi) \subset \mathcal{D}(A)$ *and*

$$\Pi S = A\Pi + B(K\Pi + L) + P. \tag{2.23}$$

In this case the following equivalence also holds: $e(t) = Cz(t) - Qw(t) \longrightarrow 0$ *as* $t \to \infty$ *for the system (2.12) and for all initial data* $[z_0, w_0]^T \in Z \times \mathbb{R}^k$, *if and only if* $C\Pi = Q$.

Proof: Let us define the operator $\mathcal{F} : Z \times \mathbb{R}^k \to Z \times \mathbb{R}^k$ by

$$\mathcal{F} = \begin{bmatrix} A + BK & P + BL \\ 0 & S \end{bmatrix}.$$

Since $P+BL \in \mathcal{L}(\mathbb{R}^k, Z)$, $\mathcal{F}$ generates a C_0 semigroup $T(t)$ on $Z \times \mathbb{R}^k$. Note that by assumptions **H1** and **I.a**, $\sigma(A + BK) \cap \sigma(S) = \emptyset$. Let $\widetilde{H} = Z$, $\tilde{A} = A + BK$, $\tilde{B} = P+BL$, $\tilde{S} = S$ and $\tilde{\mathcal{F}} = \mathcal{F}$. Lemma 2.8 implies the existence of $M \in \mathcal{L}(\mathbb{R}^k, Z)$ with $\mathcal{H}^+ = Graph(M)$ and $\mathcal{H}^-$ both $\mathcal{F}$-invariant and $T(t)$-invariant. Let $\Pi = M$, then Lemma 2.8 implies that $Ran(\Pi) \subset \mathcal{D}(A)$ and the $\mathcal{F}$-invariance of $Graph(M)$ implies

$$\mathcal{F}\begin{bmatrix} \Pi w \\ w \end{bmatrix} = \begin{bmatrix} A + BK & BL + P \\ 0 & S \end{bmatrix}\begin{bmatrix} \Pi w \\ w \end{bmatrix} = \begin{bmatrix} \Pi S w \\ S w \end{bmatrix}$$

which gives (2.23).

Let us prove the equivalence in the statement. To prove the sufficiency assume that for the closed-loop system

$$\frac{d}{dt}\begin{bmatrix} z \\ w \end{bmatrix} = \mathcal{F}\begin{bmatrix} z \\ w \end{bmatrix},$$

the error $e(t) \longrightarrow 0$ as $t \to \infty$ for every initial condition $\begin{bmatrix} z_0 \\ w_0 \end{bmatrix} \in Z \times \mathbb{R}^k$. Let $\begin{bmatrix} z_0 \\ w_0 \end{bmatrix} \in Graph(M)$ and let $T(t)$ denote the semigroup generated by $\mathcal{F}$. The $T(t)$-invariance of $Graph(M)$ implies that the solution $\begin{bmatrix} z \\ w \end{bmatrix}$ of the initial value problem

$$\frac{d}{dt}\begin{bmatrix} z \\ w \end{bmatrix} = \mathcal{F}\begin{bmatrix} z \\ w \end{bmatrix} \tag{2.24}$$

$$\begin{bmatrix} z(0) \\ w(0) \end{bmatrix} = \begin{bmatrix} z_0 \\ w_0 \end{bmatrix} \tag{2.25}$$

with initial condition in $Graph(M)$ is of the form $\begin{bmatrix} z(t) \\ w(t) \end{bmatrix} = \begin{bmatrix} \Pi w(t) \\ w(t) \end{bmatrix}$ and in this case $e(t) = (C\Pi - Q)e^{St}w_0$. Since **H1** holds and $(C\Pi - Q) \in \mathcal{L}(I\!R^k, Y)$, by Lemma 2.6 $(C\Pi - Q)$ is the zero operator.

Conversely, let us assume that $(C\Pi - Q) = 0$. Consider once again the previous initial value problem with any initial condition in $Z \times I\!R^k$. Decompose the initial condition into components in $\mathcal{H}^+$ and $\mathcal{H}^-$:

$$\begin{bmatrix} z_0 \\ w_0 \end{bmatrix} = \begin{bmatrix} z_1 \\ w_1 \end{bmatrix} + \begin{bmatrix} z_2 \\ 0 \end{bmatrix}.$$

Then the solution can be written as

$$T(t)\begin{bmatrix} z_0 \\ w_0 \end{bmatrix} = T(t)\begin{bmatrix} z_1 \\ w_1 \end{bmatrix} + T(t)\begin{bmatrix} z_2 \\ 0 \end{bmatrix},$$

and

$$e(t) = \begin{bmatrix} C & -Q \end{bmatrix} T(t)\begin{bmatrix} \Pi w_1 \\ w_1 \end{bmatrix} + \begin{bmatrix} C & -Q \end{bmatrix} T(t)\begin{bmatrix} z_2 \\ 0 \end{bmatrix} =$$
$$\begin{bmatrix} C & -Q \end{bmatrix}\begin{bmatrix} T_{A+BK}(t)z_2 \\ 0 \end{bmatrix}.$$

where $T_{A+BK}(t)$ is the semigroup generated by $A + BK$. Since $T_{A+BK}(t)$ is an exponentially stable C_0 semigroup by **I.a**, and C is bounded we see that $e(t)$ converges to 0 as t tends to infinity. □

The following theorem gives necessary and sufficient conditions for the solvability of the state feedback regulator problem:

Theorem 2.2 *Let* **H1** *and* **H2** *hold. The linear state feedback regulator problem is solvable if and only if there exist mappings* $\Pi \in \mathcal{L}(I\!R^k, Z)$ *with* $Ran(\Pi) \subset \mathcal{D}(A)$ *and* $\Gamma \in \mathcal{L}(I\!R^k, U)$ *satisfying the "regulator equations,"*

$$\Pi S = A\Pi + B\Gamma + P, \tag{2.26}$$
$$C\Pi = Q. \tag{2.27}$$

In this case a feedback law solving the state feedback regulator problem is given by $u(t) = Kz(t) + (\Gamma - K\Pi)w(t)$, *where* K *is any exponentially stabilizing feedback for* (A, B).

Proof: Suppose that $u(t) = Kz(t) + Lw(t)$ solves the linear regulator problem, i.e., **I.a** and **I.b** hold. Then by Lemma 2.9 there exists a mapping $\Pi : I\!R^k \to \mathcal{D}(A)$ so that

$$\Pi S = (A + BK)\Pi + (BL + P),$$
$$C\Pi = Q.$$

These are exactly (2.26) and (2.27) with Π and $\Gamma = K\Pi + L$.

Conversely, assume that $\Pi \in \mathcal{L}(I\!R^k, Z)$ and $\Gamma \in \mathcal{L}(I\!R^k, U)$ solve the regulator equations (2.26), (2.27). Let $u(t) = K(z(t) - \Pi w(t)) + \Gamma w(t)$, where K is an exponentially stabilizing feedback. Since **I.a** holds, we can apply Lemma 2.9 again. With K and $L = \Gamma - K\Pi$,

$$\Pi S = A\Pi + B(K\Pi + L) + P = A\Pi + B\Gamma + P,$$

which is (2.26), and $C\Pi - Q = 0$ implies that $e(t) \to 0$ for the closed-loop system (2.12) and for any initial condition. □

We now turn to the error feedback problem.

Lemma 2.10 *Let* **H1** *hold. Let* $G \in \mathcal{L}(Y, \mathcal{X})$, $H \in \mathcal{L}(\mathcal{X}, U)$ *and let* $F : \mathcal{X} \to \mathcal{X}$ *be a possibly unbounded linear operator with dense domain such that* **II.a** *is satisfied. The linear error feedback regulator problem is solvable if and only if there exist mappings* $\Pi \in \mathcal{L}(I\!R^k, Z)$ *and* $\Lambda \in \mathcal{L}(I\!R^k, \mathcal{X})$, *with* $Ran(\Pi) \subset \mathcal{D}(A)$ *and* $Ran(\Lambda) \subset \mathcal{D}(F)$ *such that*

$$\Pi S = A\Pi + BH\Lambda + P, \tag{2.28}$$
$$\Lambda S = F\Lambda, \tag{2.29}$$
$$C\Pi = Q. \tag{2.30}$$

Proof: Consider the closed-loop system with the error feedback controller

$$\frac{d}{dt}\begin{bmatrix} z(t) \\ X(t) \\ w(t) \end{bmatrix} = \mathcal{F}\begin{bmatrix} z(t) \\ X(t) \\ w(t) \end{bmatrix}, \tag{2.31}$$

where

$$\mathcal{F} = \begin{bmatrix} A & BH & P \\ GC & F & -GQ \\ 0 & 0 & S \end{bmatrix}.$$

From **H1** and **II.a** using Lemma 2.7 we get that

$$\sigma(\mathcal{F}) = \sigma(S) \cup \sigma\left(\begin{bmatrix} A & BH \\ GC & F \end{bmatrix}\right).$$

Setting $\widetilde{H} = Z \times \mathcal{X}$, $\tilde{A} = \begin{bmatrix} A & BH \\ GC & F \end{bmatrix}$, $\tilde{B} = \begin{bmatrix} P \\ -GQ \end{bmatrix}$, $\tilde{S} = S$ and $\tilde{\mathcal{F}} = \mathcal{F}$ and applying Lemma 2.8 we obtain $\mathcal{H}^+$, $\mathcal{H}^-$ and $M \in \mathcal{L}(I\!R^k, Z \times \mathcal{X})$ with $\mathcal{H}^+ = Graph(M)$. $\mathcal{H}^+$ and $\mathcal{H}^-$ are $\mathcal{F}$-invariant and $T(t)$-invariant ($T(t)$ denoting the semigroup generated by $\mathcal{F}$). Let the components of M be $\Pi \in \mathcal{L}(I\!R^k, Z)$ and $\Lambda \in \mathcal{L}(I\!R^k, \mathcal{X})$, $M = \begin{bmatrix} \Pi \\ \Lambda \end{bmatrix}$. Then using the $\mathcal{F}$-invariance of $Graph(M)$ we have

$$\mathcal{F}\begin{bmatrix} \Pi w \\ \Lambda w \\ w \end{bmatrix} = \begin{bmatrix} A\Pi w + BH\Lambda w + Pw \\ GC\Pi w + F\Lambda w - GQw \\ Sw \end{bmatrix} = \begin{bmatrix} \Pi S w \\ \Lambda S w \\ Sw \end{bmatrix} \tag{2.32}$$

for all $w \in I\!R^k$. It follows that with this choice of Π and Λ (2.28) holds.

Now suppose that for the closed-loop system (2.31) with any initial condition $\begin{bmatrix} z_0 \\ X_0 \\ w_0 \end{bmatrix} \in Z \times \mathcal{X} \times I\!R^k$ $e(t) \longrightarrow 0$ as $t \to \infty$. Let

$$\begin{bmatrix} \Pi w_0 \\ \Lambda w_0 \\ w_0 \end{bmatrix} \in Graph(M)$$

and let $T(t)$ denote the semigroup generated by $\mathcal{F}$. The $T(t)$-invariance of $Graph(M)$ implies that the solution $\begin{bmatrix} z \\ X \\ w \end{bmatrix}$ of the initial value problem

$$\frac{d}{dt}\begin{bmatrix} z \\ X \\ w \end{bmatrix} = \mathcal{F}\begin{bmatrix} z \\ X \\ w \end{bmatrix}, \quad \begin{bmatrix} z(0) \\ X(0) \\ w(0) \end{bmatrix} = \begin{bmatrix} z_0 \\ X_0 \\ w_0 \end{bmatrix} \tag{2.33}$$

with initial condition in $Graph(M)$ is of the form $\begin{bmatrix} z(t) \\ X(t) \\ w(t) \end{bmatrix} = \begin{bmatrix} \Pi w(t) \\ \Lambda w(t) \\ w(t) \end{bmatrix}$ and in this case $e(t) = (C\Pi - Q)e^{St}w_0$. Again as in the proof of Lemma 2.9 applying Lemma 2.6 we see that $(C\Pi - Q)$ is necessarily the zero operator. This is exactly (2.30), and together with (2.32) it gives (2.29). Thus the necessity is proved.

As for the sufficiency, let us assume that (2.28)-(2.30) hold. We need to argue that **II.b** holds. Consider again the initial value problem (2.33) with initial condition in $Z \times \mathcal{X} \times I\!R^k$. Let us decompose the initial condition into components in $\mathcal{H}^+$ and $\mathcal{H}^-$:

$$\begin{bmatrix} z_0 \\ X_0 \\ w_0 \end{bmatrix} = \begin{bmatrix} z_1 \\ X_1 \\ w_1 \end{bmatrix} + \begin{bmatrix} z_2 \\ X_2 \\ 0 \end{bmatrix}.$$

Then the solution can be written as

$$T(t)\begin{bmatrix} z_0 \\ X_0 \\ w_0 \end{bmatrix} = T(t)\begin{bmatrix} z_1 \\ X_1 \\ w_1 \end{bmatrix} + T(t)\begin{bmatrix} z_2 \\ X_2 \\ 0 \end{bmatrix},$$

and

$$\begin{aligned} e(t) &= \begin{bmatrix} C & 0 & -Q \end{bmatrix} T(t) \begin{bmatrix} \Pi w_1 \\ \Lambda w_1 \\ w_1 \end{bmatrix} + \begin{bmatrix} C & 0 & -Q \end{bmatrix} T(t) \begin{bmatrix} z_2 \\ X_2 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} C & 0 & -Q \end{bmatrix} \begin{bmatrix} S(t)\begin{bmatrix} z_2 \\ X_2 \end{bmatrix} \\ 0 \end{bmatrix}, \end{aligned}$$

where $S(t)$ is the semigroup generated by $\begin{bmatrix} A & BH \\ GC & F \end{bmatrix}$. Since $S(t)$ is an exponentially stable C_0 semigroup by **II.a**, and C is bounded we have that $e(t) \to 0$ as $t \to \infty$. □

Theorem 2.3 *Let* **H1**, **H2** *and* **H3** *hold. The linear error feedback regulator problem is solvable if and only if there exist mappings* $\Pi \in \mathcal{L}(\mathbb{R}^k, Z)$ *and* $\Gamma \in \mathcal{L}(\mathbb{R}^k, U)$ *with* $Ran(\Pi) \subset \mathcal{D}(A)$*, such that*

$$\Pi S = A\Pi + B\Gamma + P \tag{2.34}$$

$$C\Pi = Q. \tag{2.35}$$

With this Π *and* Γ *a controller solving the error feedback regulator problem is given by*

$$\dot{X}(t) = FX(t) + Ge(t),$$
$$u(t) = HX(t),$$

where $X \in \mathcal{X} = Z \times \mathbb{R}^k$,

$$G = \begin{bmatrix} G_1 \\ G_2 \end{bmatrix}, \quad H = \begin{bmatrix} K & (\Gamma - K\Pi) \end{bmatrix},$$

$$F = \begin{bmatrix} (A + BK - G_1 C) & (P + B(\Gamma - K\Pi) + G_1 Q) \\ -G_2 C & (S + G_2 Q) \end{bmatrix},$$

where $K \in \mathcal{L}(Z, U)$ *is an exponentially stabilizing feedback for the pair* (A, B) *and* $\begin{bmatrix} G_1 \\ G_2 \end{bmatrix}$ *is an exponentially stabilizing output injection (such* K *and* G *exist by* **H2** *and* **H3***).*

Proof: Suppose the problem has a solution with controller

$$\frac{d}{dt}X(t) = FX(t) + Ge(t),$$
$$u(t) = HX(t).$$

Since **H1** holds and **II.a** is satisfied, by Lemma 2.10 we see that **II.b** implies the existence of Π and Λ that solve (2.28) and (2.30). Hence Π and $\Gamma = H\Lambda$ solve (2.34)-(2.35).

On the other hand assume that Π and Γ solve (2.34)-(2.35). Let

$$\mathcal{X} = Z \times \mathbb{R}^k, \qquad X \in \mathcal{X},$$

$$G = \begin{bmatrix} G_1 \\ G_2 \end{bmatrix}, \ H = \begin{bmatrix} K & (\Gamma - K\Pi) \end{bmatrix},$$

$$F = \begin{bmatrix} A + BK - G_1C & P + B(\Gamma - K\Pi) + G_1Q \\ -G_2C & S + G_2Q \end{bmatrix},$$

where $K \in \mathcal{L}(Z, U)$ is an exponentially stabilizing feedback for the pair (A, B) and G is an exponentially stabilizing output injection, respectively. To satisfy **II.a** we must show that the closed-loop system

$$\frac{d}{dt}z(t) = Az(t) + BHX(t),$$
$$\frac{d}{dt}X(t) = FX(t) + Ge(t)$$

is exponentially stable when $w \equiv 0$.

To see that **II.a** is satisfied note that the closed-loop system operator can be factored as

$$\begin{bmatrix} A & BK & B(\Gamma - K\Pi) \\ G_1C & A + BK - G_1C & P + B(\Gamma - K\Pi) + G_1Q \\ G_2C & -G_2C & S + G_2Q \end{bmatrix} =$$

$$\begin{bmatrix} I & 0 & 0 \\ I & I & 0 \\ 0 & 0 & I \end{bmatrix} \begin{bmatrix} A + BK & BK & B(\Gamma - K\Pi) \\ 0 & A - G_1C & P + G_1Q \\ 0 & -G_2C & S + G_2Q \end{bmatrix} \begin{bmatrix} I & 0 & 0 \\ -I & I & 0 \\ 0 & 0 & I \end{bmatrix} \equiv J^{-1}EJ.$$

Now the hypotheses **H2**, **H3** and the fact that $\begin{bmatrix} BK & B(\Gamma - K\Pi) \end{bmatrix} \in \mathcal{L}(\mathcal{X}, Z)$ imply that E is the infinitesimal generator of an exponentially stable C_0 semigroup, which we denote by $\mathcal{S}(t)$. It is easy to see that $J^{-1}EJ$ is the generator of $J^{-1}\mathcal{S}(t)J$, and taking into account that J^{-1} and J are bounded, we see that **II.a** is satisfied.

To justify **II.b** it is enough to prove the existence of $\Pi \in \mathcal{L}(\mathbb{R}^k, Z)$ and $\Lambda \in \mathcal{L}(\mathbb{R}^k, U)$ satisfying (2.28) and (2.29) and appeal to Lemma 2.10. With Π given in the hypothesis, let $\Lambda = \begin{bmatrix} \Pi \\ I_{I\!R^k} \end{bmatrix}$, then $BH\Lambda = B\Gamma$ and (2.28) reduces to (2.34). Finally, equation (2.29) is obtained by a straightforward calculation. Namely, the expression on the left in (2.29) is

$$\Lambda S = \begin{bmatrix} \Pi S \\ S \end{bmatrix}.$$

Due to (2.35) and the explicit formulas for F and H, the expression on the right can be written as

$$F\Lambda = \begin{bmatrix} (A + BK - G_1C) & (P + B(\Gamma - K\Pi) + G_1Q) \\ -G_2C & (S + G_2Q) \end{bmatrix} \begin{bmatrix} \Pi \\ I_{I\!R^k} \end{bmatrix}$$

$$= \begin{bmatrix} A\Pi + P + B\Gamma \\ S \end{bmatrix},$$

and using (2.34) we see that (2.29) is satisfied. This completes the proof. □

Beyond establishing necessary and sufficient conditions, both Theorem 2.2 and Theorem 2.3 provide an explicit solution to the respective problem in term of the solutions of the regulator equations and a stabilizing feedback and output injection.

CHAPTER III

EXAMPLES

3.1 Example 1: Periodic Tracking for the Heat Equation

Consider a controlled one-dimensional heat equation on a finite rod (Curtain-Zwart [10]):

$$\begin{aligned} &\frac{d}{dt}z(t) = Az(t) + Bu(t), \\ &y(t) = Cz(t), \\ &z(0) = \phi. \end{aligned} \tag{3.1}$$

Here $A = d^2/dx^2$ in $Z = L^2(0,1)$ is a selfadjoint operator with the domain,

$$\mathcal{D}(A) = \{\phi \in H^2(0,1) \, : \, \phi'(0) = \phi'(1) = 0\}.$$

The spectrum of A consists of $\sigma(A) = \{-j^2\pi^2\}_{j=0}^{\infty}$ with corresponding orthonormal eigenvectors $\psi_0(x) = 1$ and $\psi_j(x) = \sqrt{2}\cos(j\pi x)$ for $j \geq 1$.

In this example, we consider one-dimensional bounded input and output operators B and C so that $Y = U = I\!R$.

1. Input operator:

    The input corresponds to a spatially uniform temperature input over a small interval about a fixed point $x_0 \in (0,1)$:

$$Bu = b(x)u, \; b(x) = \frac{1}{2\nu_0}\chi_{[x_0-\nu_0, x_0+\nu_0]}(x). \tag{3.2}$$

2. Output operator:

    The output corresponds to the average temperature over a small interval about a point $x_1 \in (0,1)$.

$$C\phi = \int_0^1 c(x)\phi(x)\,dx, \quad c(x) = \frac{1}{2\nu_1}\chi_{[x_1-\nu_1, x_1+\nu_1]}(x). \tag{3.3}$$

A simple bounded stabilizing feedback law for this problem with $\beta > 0$ is

$$K\phi = -\beta < \phi, 1 > .$$

To see that the closed-loop system with the exosystem turned off is exponentially stable, we note that the spectrum of $(A+BK)$ is $\{-\beta\} \cup \{-j^2\pi^2\}_{j=1}^{\infty}$ and this operator is a discrete Riesz spectral operator in the sense of Curtain and Zwart [10]. In particular, it satisfies the spectrum determined growth condition and therefore generates an exponentially stable semigroup. We note that $(A+BK)$ is not selfadjoint since the eigenfunction associated with the eigenvalue $-\beta$ is not orthogonal to the other eigenfunctions. This eigenfunction is rather complicated. In fact a formula for this normalized eigenfunction, which we again denote by ψ_0 is given in terms of the function

$$\widetilde{\psi}_0(x) = 1 + \beta \sum_{k=1}^{\infty} \frac{\sin(j\pi\nu_0)}{(\lambda_j+\beta)(j\pi\nu_0)} \psi_j(x_0)\psi_j(x)$$

by

$$\psi_0(x) = \|\widetilde{\psi}_0\|^{-1}\widetilde{\psi}_0(x).$$

The eigenfunctions associated with the remainder of the spectrum are the same as for A, namely, for $j \geq 1$, $\psi_j(x) = \sqrt{2}\cos(j\pi x)$.

For this example we are interested in controlling the output $y(t)$ to track a periodic reference trajectory of the form $y_r = M\sin(\alpha t)$. In this case we take the exogenous system to be a harmonic oscillator:

$$\dot{w} = Sw, \quad S = \begin{pmatrix} 0 & \alpha \\ -\alpha & 0 \end{pmatrix}, \quad w(0) = \begin{bmatrix} 0 \\ M \end{bmatrix}.$$

Thus, in terms of our earlier notation, $Q = [1, 0]$ and $S \in \mathcal{L}(I\!R^k)$ with $k = 2$. In order to solve this tracking problem, following Theorem 2.2, we seek mappings $\Pi = [\Pi_1, \Pi_2] \in \mathcal{L}(I\!R^2, Z)$ and $\Gamma = [\gamma_1, \gamma_2] \in I\!R^2$ satisfying the regulator equations

$$\Pi S w = A\Pi w + B\Gamma w \tag{3.4}$$

$$C\Pi w - Qw = 0. \tag{3.5}$$

for all $w \in I\!R^k$. The equation (3.4) can be written as

$$[-\alpha\Pi_2, \alpha\Pi_1]w = [A\Pi_1, A\Pi_2]w + [B\gamma_1, B\gamma_2]w,$$

or as the system of equations

$$A\Pi_1 + \alpha\Pi_2 = -B\gamma_1 \tag{3.6}$$

$$A\Pi_2 - \alpha\Pi_1 = -B\gamma_2. \tag{3.7}$$

This gives rise to a system of second-order ordinary differential equations with boundary conditions

$$\begin{aligned}
\Pi_1'' + \alpha\Pi_2 &= -b\gamma_1 \\
\Pi_2'' - \alpha\Pi_1 &= -b\gamma_2. \\
\Pi_1'(0) = \Pi_1'(1) &= 0 \\
\Pi_2'(0) = \Pi_2'(1) &= 0.
\end{aligned}$$

The equation (3.5) reduces to

$$C\Pi_1 = 1, \quad C\Pi_2 = 0 \tag{3.8}$$

which corresponds to a pair of extra contraints to be imposed on the solutions to the system (3.6), (3.7), in addition to the obvious conditions that Π_1, Π_2 are in the domain of A.

Here we note that B does not map into the domain of A, so to solve the equations (3.6), (3.7), we first regularize (3.6) by applying $A(A^2+\alpha^2)^{-1}$. This produces the equation

$$A^2(A+\alpha^2)^{-1}\Pi_1 + \alpha A(A^2+\alpha^2)^{-1}\Pi_2 = -A(A^2+\alpha^2)^{-1}(B\gamma_1). \tag{3.9}$$

Rewriting equation (3.7) in the form

$$A\Pi_2 = \alpha\Pi_1 - B\gamma_2,$$

and substituting this into (3.9) we obtain

$$A^2(A^2+\alpha^2)^{-1}\Pi_1 + \alpha(A^2+\alpha^2)^{-1}[\alpha\Pi_1 - B\gamma_2] = -A(A^2+\alpha^2)^{-1}(B\gamma_1), \tag{3.10}$$

which simplifies to

$$\Pi_1 = \alpha(A^2+\alpha^2)^{-1}(B\gamma_2) - A(A^2+\alpha^2)^{-1}(B\gamma_1). \tag{3.11}$$

Then from equation (3.6) we have

$$\Pi_2 = -\alpha(A^2+\alpha^2)^{-1}(B\gamma_1) - A(A^2+\alpha^2)^{-1}(B\gamma_2). \tag{3.12}$$

We note that Theorems 2.2 and 2.3 give necessary and sufficient conditions for solvability of the regulator problem in terms of the solvability of the regulator equations. We must now address the question of solvability of these equations. At this point we have found formulas for Π_1 and Π_2 in terms of γ_1 and γ_2 but we must now find γ_1 and γ_2 so that the equations in (3.8) are satisfied. These equations give a system of two equations in two unknowns for γ_1 and γ_2. Setting some notation, let us define

$$R_1 = A(A^2+\alpha^2)^{-1}, \tag{3.13}$$
$$R_2 = \alpha(A^2+\alpha^2)^{-1} \tag{3.14}$$

then the system of equation can be written as

$$-CR_1b\gamma_1 + CR_2b\gamma_2 = 1, \tag{3.15}$$

$$-CR_2b\gamma_1 - CR_1b\gamma_2 = 0. \tag{3.16}$$

Let Δ denote the determinant of the coefficient matrix for the system in (3.15), (3.16)

$$\Delta = \begin{vmatrix} -CR_1b & CR_2b \\ -CR_2b & -CR_1b \end{vmatrix} = (CR_1b)^2 + (CR_2b)^2. \tag{3.17}$$

As we might expect some conditions will have to be met in order to completely solve the regulator equations. For this example, the equations are solvable if and only if $\Delta \neq 0$, i.e., $b \notin Ker(CR_1) \cap Ker(CR_2)$. Assuming that $\Delta \neq 0$ we have

$$\gamma_1 = \frac{-CR_1b}{\Delta}, \quad \gamma_2 = \frac{CR_2b}{\Delta}. \tag{3.18}$$

Combining (3.11)-(3.18) we have

$$\Pi_1 = \frac{CR_1b}{\Delta}R_1b + \frac{CR_2b}{\Delta}R_2b, \tag{3.19}$$
$$\Pi_2 = \frac{CR_1b}{\Delta}R_2b - \frac{CR_2b}{\Delta}R_1b. \tag{3.20}$$

Using this we can introduce the feedback law

$$u = Kz + (\Gamma - K\Pi)w \tag{3.21}$$

to obtain the closed-loop composite system

$$\frac{d}{dt}z(t) = (A+BK)z(t) + B(\Gamma - K\Pi)w(t),$$

$$\frac{d}{dt}w(t) = Sw(t). \tag{3.22}$$

We now turn to the question of approximate numerical solution for this problem. There are of course many methods that could be used to solve the equations (3.6), (3.7), (3.8). We have chosen to introduce the regularization by $A(A^2+\alpha^2)^{-1}$ to solve the equations since, for this example, spectral theory can be employed to obtain explicit formulas for approximations of the feedback law that work quite well numerically. Namely, recall that

$$f(A)(\phi) = \sum_{j=0}^{\infty} f(\lambda_j) < \phi, \psi_j > \psi_j.$$

For each positive integer N define

$$\tilde{R}_1^N = \sum_{\ell=0}^{N} \frac{\lambda_\ell < b, \psi_\ell >}{\lambda_\ell^2 + \alpha^2} \psi_\ell, \quad \tilde{R}_2^N = \sum_{\ell=0}^{N} \frac{\alpha < b, \psi_\ell >}{\lambda_\ell^2 + \alpha^2} \psi_\ell.$$

With these expressions we obtain approximate values for Δ and γ_j defined by

$$\Delta^N = \begin{vmatrix} -C\tilde{R}_1^N b & C\tilde{R}_2^N b \\ -C\tilde{R}_2^N b & -C\tilde{R}_1^N b \end{vmatrix} = (C\tilde{R}_1^N b)^2 + (C\tilde{R}_1^N b)^2, \tag{3.23}$$

$$\gamma_1^N = -\frac{C\tilde{R}_1^N b}{\Delta^N}, \quad \gamma_2^N = \frac{C\tilde{R}_2^N b}{\Delta^N}. \tag{3.24}$$

Combining these approximations we additionally define

$$\Pi_1^N = \frac{C\tilde{R}_1^N b}{\Delta^N}\tilde{R}_1^N b + \frac{C\tilde{R}_2^N b}{\Delta^N}\tilde{R}_2^N b \tag{3.25}$$

$$\Pi_2^N = \frac{C\tilde{R}_1^N b}{\Delta^N}\tilde{R}_2^N b - \frac{C\tilde{R}_2^N b}{\Delta^N}\tilde{R}_1^N b. \tag{3.26}$$

Using these approximate formulas for the feedback control law we consider a numerical example in which we have taken the approximations in (3.23)-(3.26) with $N = 4$ and $x_0 = 2/\pi$, $x_1 = 1/\pi$, $\nu_0 = \nu_1 = 0.05$, $M = 1$, $\alpha = 1$, $\beta = 0.2$, $\phi(x) = 4x^2(3/2 - x)$. In Figure 3.1 we have plotted the exact reference trajectory $y_r = \sin(t)$ and the numerically computed output y using the approximate feedback control law described above. Figure 3.2 contains a plot of the entire numerical solution of (3.22) with this feedback law.

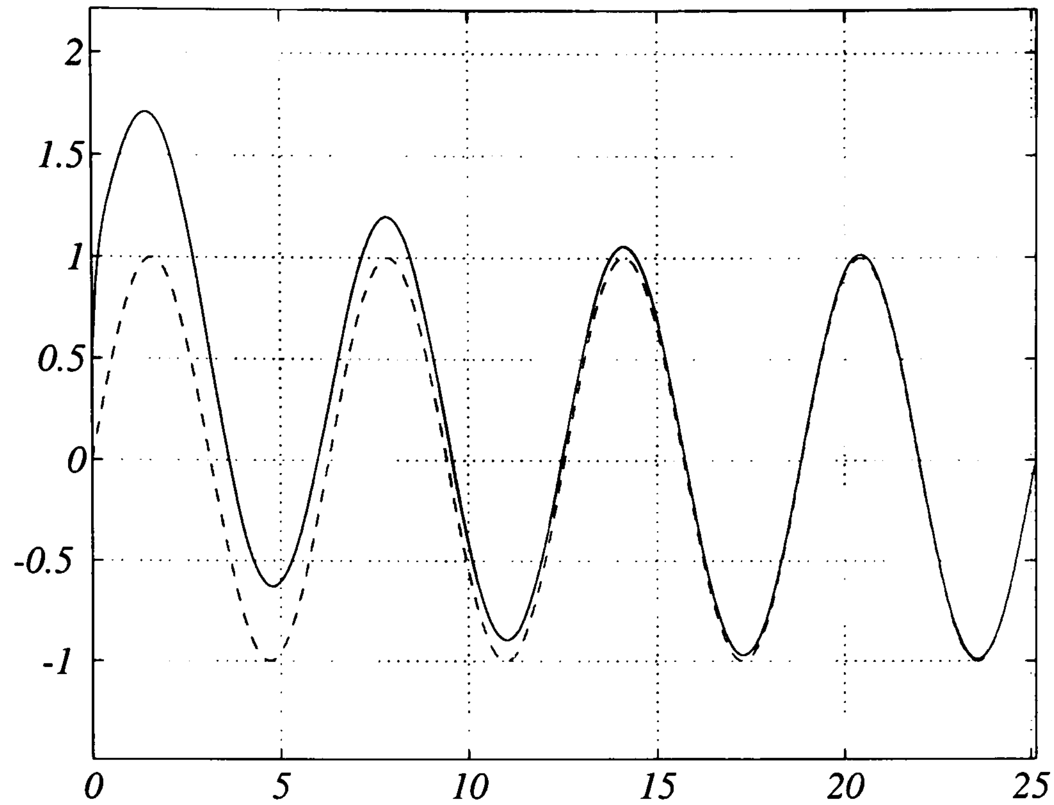

Figure 3.1: Output tracking for the parabolic example

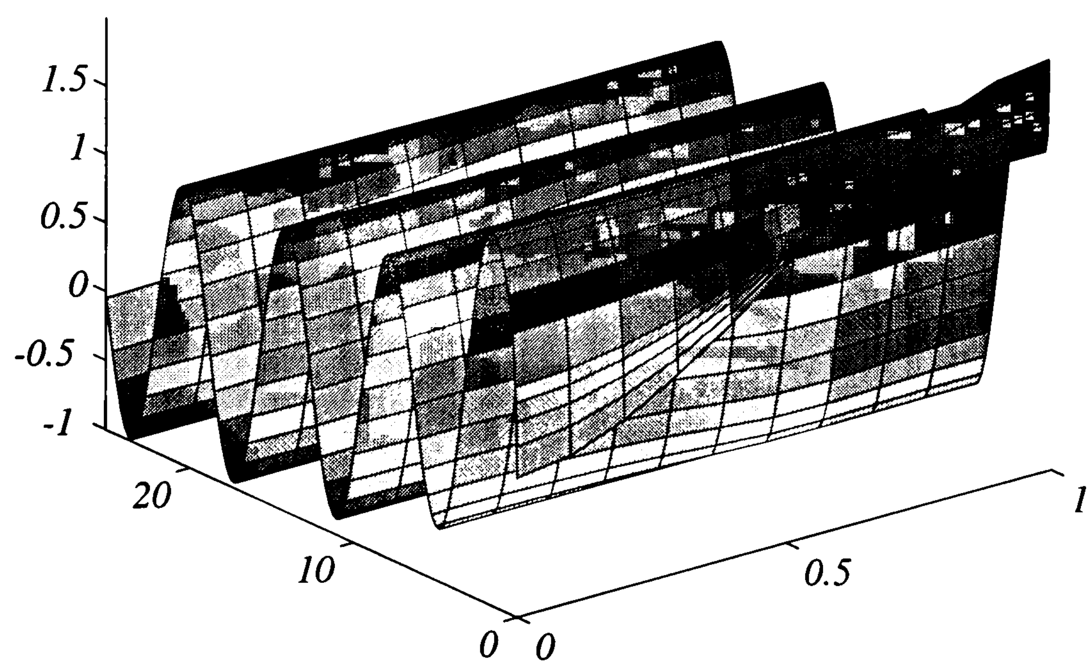

Figure 3.2: Controlled solution surface for the parabolic example

3.2 Example 2: Periodic Tracking for a Damped Wave Equation

Consider a controlled one-dimensional damped wave equation governing the small vibrations of a finite string ($\beta > 0$):

$$\frac{\partial^2}{\partial t^2} z(x,t) = \frac{\partial^2}{\partial x^2} z(x,t) - \beta \frac{\partial}{\partial t} z(x,t) + Bu(t), \tag{3.27}$$

$$\begin{aligned}
&z(0,t) = z(1,t) = 0,\\
&y(t) = Cz(t)\\
&z(0) = \phi \in H^2(0,1)\\
&z_t(0) = \psi \in L^2(0,1).
\end{aligned}$$

Once again our design objective will be to control the average motion of the string over a small fixed interval about a point $x_1 \in (0,1)$ to track a prescribed periodic motion. Just as in the previous example let us choose the output and the control spaces $Y = U = I\!R$, the input operator

$$Bu = b(x)u, \;\; b(x) = \frac{1}{2\nu_0}\chi_{[x_0-\nu_0, x_0+\nu_0]}(x), \tag{3.28}$$

and the output operator as the average value over a small interval,

$$C\phi = \int_0^1 c(x)\phi(x)\,dx, \;\; c(x) = \frac{1}{2\nu_1}\chi_{[x_1-\nu_1, x_1+\nu_1]}(x). \tag{3.29}$$

In order to formulate this problem within the current framework we define $A = d^2/dx^2$ in $Z = L^2(0,1)$ as the selfadjoint operator defined by

$$\mathcal{D}(A) = \{\phi \in H^2(0,1) \,:\, \phi(0) = \phi(1) = 0\} = H^2(0,1) \cap H_0^1(0,1).$$

Now define the space $\mathcal{H} = H_0^1(0,1) \oplus L^2(0,1) = \mathcal{D}((-A)^{1/2}) \oplus L^2(0,1)$ with the graph norm. For $\Phi = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}$, $\Psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \in \mathcal{H}$,

$$\langle \Phi, \Psi \rangle_{\mathcal{H}} = \langle (-A)^{1/2}\phi_1, (-A)^{1/2}\psi_1 \rangle + < \phi_2, \psi_2 > .$$

Next we define the operator $\mathcal{A}$ with $\mathcal{D}(\mathcal{A}) = \mathcal{D}(A) \oplus H_0^1$ by

$$\mathcal{A} = \begin{pmatrix} 0 & I \\ A & -2\beta I \end{pmatrix}.$$

It is easy to compute the adjoint operator

$$\mathcal{A}^* = \begin{pmatrix} 0 & -I \\ -A & -2\beta I \end{pmatrix},$$

with $\mathcal{D}(\mathcal{A}^*) = \mathcal{D}(\mathcal{A})$ and

$$\langle \Phi, \mathcal{A}\Phi \rangle = -2\beta \|\phi_2\|^2 \leq 0, \;\; \Phi \in \mathcal{D}(\mathcal{A}),$$

and

$$\langle \Phi, \mathcal{A}^*\Phi \rangle = -2\beta \|\phi_2\|^2 \leq 0, \;\; \Phi \in \mathcal{D}(\mathcal{A}^*).$$

Thus $\mathcal{A}$ is maximal dissipative and hence generates a contraction semigroup.

The spectrum of $\mathcal{A}$ can be expressed in terms of the eigenvalues and eigenfunctions of A as follows: The spectrum of A consists of $\{-j^2\pi^2\}_{j=1}^{\infty}$ with corresponding orthonormal eigenvectors $\phi_j(x) = \sqrt{2}\sin(j\pi x)$ for $j \geq 1$. Now for $j \geq 1$ the spectrum of $\mathcal{A}$ can be expressed as

$$\lambda_{\pm j} = -\beta \pm i\sqrt{\mu_j^2 - \beta^2}, \;\; i = \sqrt{-1}, \;\; \mu_j = j\pi.$$

The corresponding eigenfunctions are given by

$$\Phi_{\pm j}(x) = c_{\pm j} \begin{pmatrix} 1 \\ \lambda_{\pm j} \end{pmatrix} \phi_j(x),$$

where $c_{\pm j} = \lambda_{\pm j}^{-1}$. The spectrum of $\mathcal{A}^*$ is the same as the spectrum of $\mathcal{A}$ but the eigenvectors are given by

$$\Phi_{\pm j}^*(x) = c_{\pm j}^* \begin{pmatrix} 1 \\ -\lambda_{\pm j} \end{pmatrix} \phi_j(x),$$

where $c_{\pm j}^* = \mp \left(2i\sqrt{\mu_j^2 - \beta^2}\right)^{-1}$ with $\mu_j = j\pi$. With the normalizations $c_{\pm j}$ and $c_{\pm j}^*$ the eigenvectors of $\mathcal{A}$ and $\mathcal{A}^*$ form two biorthogonal families, i.e.,

$$\langle \Phi_j, \Phi_k^* \rangle = \begin{cases} 1, & k = -j \\ 0, & k \neq -j \end{cases} \qquad j = \pm 1, \pm 2, \cdots.$$

The operator $\mathcal{A}$ is a Riesz spectral operator for which we have the following spectral functional calculus:

$$\sum_{j=1}^{\infty} \left\{ \langle \Phi, \Phi_{-j}^* \rangle \Phi_j + \langle \Phi, \Phi_j^* \rangle \Phi_{-j} \right\} = \Phi, \;\; \forall \;\; \Phi \in \mathcal{H},$$

$$\mathcal{A}\Phi = \sum_{j=1}^{\infty} \left\{ \lambda_j \langle \Phi, \Phi_{-j}^* \rangle \Phi_j + \lambda_{-j} \langle \Phi, \Phi_j^* \rangle \Phi_{-j} \right\}, \ \forall \ \ \Phi \in \mathcal{D}(\mathcal{A}),$$

$$(\mathcal{A} - \lambda I)^{-1}\Phi = \sum_{j=1}^{\infty} \left\{ \frac{\langle \Phi, \Phi_{-j}^* \rangle}{(\lambda_j - \lambda)} \Phi_j + \frac{\langle \Phi, \Phi_j^* \rangle}{(\lambda_{-j} - \lambda)} \Phi_{-j} \right\}, \ \forall \ \ \Phi \in \mathcal{H}.$$

With this notation the system (3.27) can be reformulated as

$$\dot{Z} = \mathcal{A}Z + \mathcal{B}u, \ Z = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}, \tag{3.30}$$

$$\dot{w} = Sw, \quad S = \begin{pmatrix} 0 & \alpha \\ -\alpha & 0 \end{pmatrix}, \tag{3.31}$$

$$Z(0) = \begin{pmatrix} \phi \\ \psi \end{pmatrix}, \ \ w(0) = \begin{pmatrix} 0 \\ M \end{pmatrix}, \tag{3.32}$$

$$y = \mathcal{C}Z = \begin{bmatrix} C & 0 \end{bmatrix} Z = Cz_1, \ \ \mathcal{B}u = \begin{bmatrix} 0 \\ Bu \end{bmatrix},$$

$$y_r = Qw = \begin{bmatrix} 1 & 0 \end{bmatrix} w = w_1$$

$$e(t) = y(t) - y_r(t).$$

In this example we require the average displacement $y(t)$ over a small interval about x_1 (see (3.29)) to track a sinusoid $y_r(t) = M\sin(\alpha t)$. We note that $y = Cz_1$ is a bounded approximation of $z_1(x_1, t)$.

As we have already noted $\mathcal{A}$ generates a contraction semigroup. In fact more is true, namely, $\mathcal{A}$ generates an exponentially stable semigroup. This can be seen by direct computation or argued as follows: as a Riesz spectral operator $\mathcal{A}$ satisfies the spectrum determined growth condition (see for example Theorem 2.3.5.c in Curtain-Zwart [10]), the spectrum lies along the line $\operatorname{Re} z = -\beta$ therefore the semigroup generated by $\mathcal{A}$ is exponentially stable. Since this semigroup is already stable, in Theorem 3.1 we can choose $K = 0$. Therefore we need only to compute the mappings Π and Γ solving the regulator equations and take the feedback law $u = \Gamma w$.

To this end we seek linear operators $\Pi = \begin{bmatrix} \Pi_1 & \Pi_2 \end{bmatrix} : I\!R^2 \to \mathcal{H}$, and $\Gamma = \begin{bmatrix} \Gamma_1 & \Gamma_2 \end{bmatrix} : I\!R^2 \to U$ satisfying:

$$\Pi S w = \mathcal{A}\Pi w + \mathcal{B}\Gamma w$$

$$\mathcal{C}\Pi w - Q w = 0.$$

Note that since

$$\Pi = \begin{bmatrix} \Pi_1 & \Pi_2 \end{bmatrix}$$

the second regulator equation becomes

$$0 = (\mathcal{C}\Pi - Q)w = \mathcal{C}\Pi_1 w_1 + \mathcal{C}\Pi_2 w_2 - w_1$$

which implies

$$\mathcal{C}\Pi_1 = 1, \quad \mathcal{C}\Pi_2 = 0. \tag{3.33}$$

The first regulator equation gives

$$\alpha\Pi_1 w_2 - \alpha\Pi_2 w_1 = \mathcal{A}\Pi_1 w_1 + \mathcal{A}\Pi_2 w_2 + \mathcal{B}\Gamma_1 w_1 + \mathcal{B}\Gamma_2 w_2,$$

which can be written as

$$\mathcal{A}\Pi_1 + \alpha\Pi_2 = -\mathcal{B}\Gamma_1, \tag{3.34}$$
$$\mathcal{A}\Pi_2 - \alpha\Pi_1 = -\mathcal{B}\Gamma_2. \tag{3.35}$$

From this point we proceed just as in the first example, regularizing the first equation (3.34) by applying $\mathcal{R}_1 = \mathcal{A}(\mathcal{A}^2 + \alpha^2)^{-1}$. Note that $\mathcal{R}_1$ has the following representation

$$\mathcal{R}_1\Phi = \sum_{j=1}^{\infty} \left\{ \frac{\lambda_j \langle \Phi, \Phi_{-j}^* \rangle}{(\lambda_j^2 + \alpha^2)} \Phi_j + \frac{\lambda_{-j} \langle \Phi, \Phi_j^* \rangle}{(\lambda_{-j}^2 + \alpha^2)} \Phi_{-j} \right\}.$$

A simple calculation shows that

$$\lambda_j^2 + \alpha^2 = (2\beta^2 + \alpha^2 - \mu_j^2) - 2\beta i \sqrt{\mu_j^2 - \beta^2} \neq 0$$

since if the imaginary part is zero then the real part equals $(\alpha^2 + \beta^2) \neq 0$. We also define the operator $\mathcal{R}_2$ by

$$\mathcal{R}_2\Phi = \alpha(\mathcal{A}^2 + \alpha^2)^{-1} = \sum_{j=1}^{\infty} \left\{ \frac{\alpha \langle \Phi, \Phi_{-j}^* \rangle}{(\lambda_j^2 + \alpha^2)} \Phi_j + \frac{\alpha \langle \Phi, \Phi_j^* \rangle}{(\lambda_{-j}^2 + \alpha^2)} \Phi_{-j} \right\}.$$

Now combining the regularization of equation (3.34) obtained by multiplying by $\mathcal{R}_1$ just as in Example 1 and using equation (3.35), which implies,

$$\mathcal{A}\Pi_2 = \alpha\Pi_1 - \mathcal{B}\Gamma_2,$$

we arrive at

$$\Pi_1 = -\mathcal{R}_1(\mathcal{B}\Gamma_1) + \mathcal{R}_2(\mathcal{B}\Gamma_2), \tag{3.36}$$

and

$$\Pi_2 = -\mathcal{R}_1(\mathcal{B}\Gamma_2) - \mathcal{R}_2(\mathcal{B}\Gamma_1). \tag{3.37}$$

Now it remains to find Γ_1 and Γ_2 so that the equations in (3.33) are satisfied. The problem reduces to solving a 2×2 system of linear equations for scalars Γ_1, Γ_2. The solvability condition is

$$\Delta = \begin{vmatrix} -\mathcal{C}\mathcal{R}_1(\mathcal{B}1) & \mathcal{C}\mathcal{R}_2(\mathcal{B}1) \\ -\mathcal{C}\mathcal{R}_2(\mathcal{B}1) & -\mathcal{C}\mathcal{R}_1(\mathcal{B}1) \end{vmatrix} = (\mathcal{C}\mathcal{R}_1(\mathcal{B}1))^2 + (\mathcal{C}\mathcal{R}_2(\mathcal{B}1))^2 \neq 0,$$

where $\mathcal{B}1 = \begin{bmatrix} 0 \\ B1 \end{bmatrix} = \begin{bmatrix} 0 \\ b \end{bmatrix}$. If the solvability condition is met, then we obtain

$$\Gamma_1 = -\frac{\mathcal{C}\mathcal{R}_1(\mathcal{B}1)}{\Delta}, \quad \Gamma_2 = \frac{\mathcal{C}\mathcal{R}_2(\mathcal{B}1)}{\Delta}. \tag{3.38}$$

After some lengthy calculations we can simplify the above formulas. Namely, first we note that

$$\frac{\lambda_k}{\lambda_k^2 + \alpha^2} - \frac{\lambda_{-k}}{\lambda_{-k}^2 + \alpha^2} = \frac{2i\sqrt{\mu_k^2 - \beta^2}(-\mu_k^2 + \alpha^2)}{(\mu_k^2 - \alpha^2)^2 + 4\alpha^2\beta^2}$$

$$\frac{1}{\lambda_k^2 + \alpha^2} - \frac{1}{\lambda_{-k}^2 + \alpha^2} = \frac{4i\beta\sqrt{\mu_k^2 - \beta^2}}{|\lambda_k^2 + \alpha^2|^2}.$$

With this we can write,

$$\mathcal{C}\mathcal{R}_1(\mathcal{B}1) = \sum_{k=1}^{\infty} \frac{< b, \phi_k > C\phi_k}{2i\sqrt{\mu_k^2 - \beta^2}} \left\{ \frac{\lambda_k}{\lambda_k^2 + \alpha^2} - \frac{\lambda_{-k}}{\lambda_{-k}^2 + \alpha^2} \right\} \tag{3.39}$$

$$= \sum_{k=1}^{\infty} \frac{(\alpha^2 - \mu_k^2) < b, \phi_k >< c, \phi_k >}{(\alpha^2 - \mu_k^2)^2 + 4\alpha^2\beta^2}$$

and

$$\mathcal{CR}_2(\mathcal{B}1) = \sum_{k=1}^{\infty} \frac{\alpha < b, \phi_k > C\phi_k}{2i\sqrt{\mu_k^2 - \beta^2}} \left\{ \frac{1}{\lambda_k^2 + \alpha^2} - \frac{1}{\lambda_{-k}^2 + \alpha^2} \right\} \tag{3.40}$$

$$= \sum_{k=1}^{\infty} \frac{2\alpha\beta < b, \phi_k >< c, \phi_k >}{(\alpha^2 - \mu_k^2)^2 + 4\alpha^2\beta^2}.$$

Just as in Example 1, we can obtain an approximate formula for the feedback law by truncating the infinite sums in (3.39) and (3.40). As a result we obtain approximate values for Δ and Γ_j.

$$\widetilde{\mathcal{CR}_1(\mathcal{B}1)}^N = \sum_{k=1}^{N} \frac{(\alpha^2 - \mu_k^2) < b, \phi_k >< c, \phi_k >}{(\alpha^2 - \mu_k^2)^2 + 4\alpha^2\beta^2}, \tag{3.41}$$

$$\widetilde{\mathcal{CR}_2(\mathcal{B}1)}^N = \sum_{k=1}^{N} \frac{2\alpha\beta < b, \phi_k >< c, \phi_k >}{(\alpha^2 - \mu_k^2)^2 + 4\alpha^2\beta^2},$$

$$\Delta^N = (\widetilde{\mathcal{CR}_1(\mathcal{B}1)}^N)^2 + (\widetilde{\mathcal{CR}_1(\mathcal{B}1)}^N)^2, \tag{3.42}$$

$$\Gamma_1^N = -\frac{\widetilde{\mathcal{CR}_1(\mathcal{B}1)}^N}{\Delta^N}, \quad \Gamma_2^N = \frac{\widetilde{\mathcal{CR}_2(\mathcal{B}1)}^N}{\Delta^N}. \tag{3.43}$$

Using (3.43), we approximate Γ by

$$\Gamma^N = \begin{bmatrix} \Gamma_1^N & \Gamma_2^N \end{bmatrix}, \tag{3.44}$$

and introduce the feedback law

$$u = \Gamma^N w \tag{3.45}$$

to obtain the closed-loop system

$$\begin{aligned} \frac{d}{dt}Z(t) &= \mathcal{A}Z(t) + \mathcal{B}\Gamma^N w(t) \\ \frac{d}{dt}w(t) &= Sw(t). \end{aligned} \tag{3.46}$$

In our numerical example we have used the approximations in (3.42)-(3.46) with $N = 50$ and, we have set $x_0 = .3$, $x_1 = .5$, $\nu_0 = \nu_1 = 0.1$, $M = 1$, $\alpha = 1$, $\beta = .4$, and chosen initial conditions $\phi(x) = 16x^2(1-x)^2$ and $\psi = .5\sin^2(\pi x)$. Figure 3.3 depicts the reference signal $y_r(t) = \sin(t)$ and the controlled output for the closed-loop system (3.46) given in (3.29) and Figure 3.4 contains the numerical solution for the displacement z for all $x \in [0, 1]$ and $t \in [0, 8\pi]$.

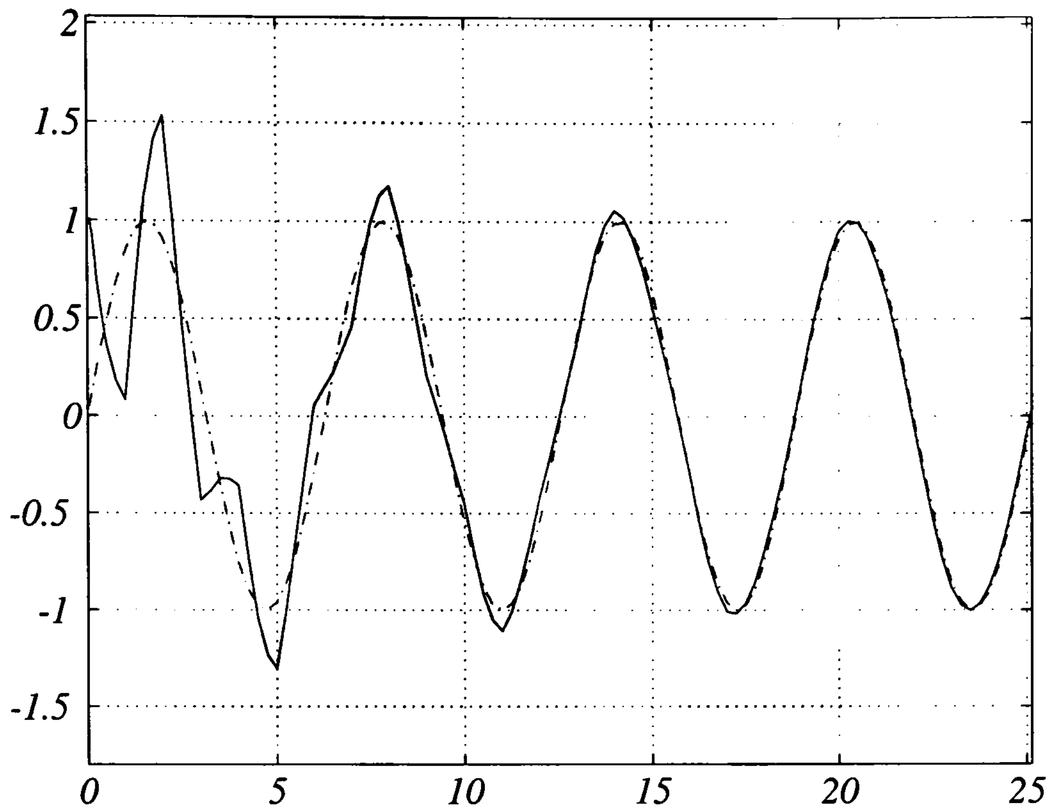

Figure 3.3: Output tracking for the hyperbolic example

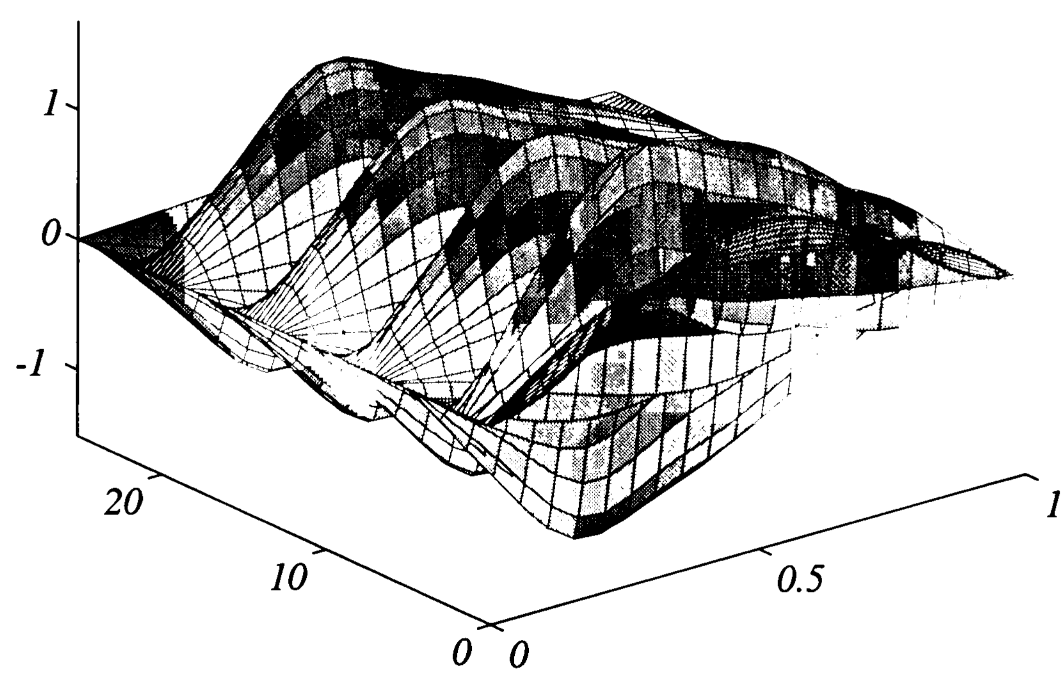

Figure 3.4: Controlled solution surface for the hyperbolic example

CHAPTER IV

ZERO DYNAMICS

4.1 Introduction

The fact that the solvability of the regulator problem is related to the system zeros is well known for finite-dimensional linear systems. We refer here to Theorem 1.3 stated in the introductory chapter. The efforts to find the concept corresponding to the transmission zeroes for nonlinear systems led to the notion of zero dynamics. The nonlinear finite-dimensional theory of zero dynamics was initiated by Isidori-Krener [19], Isidori-Moog [20], and Byrnes-Isidori [4]-[6]. There are attempts in the literature to extend the concept of zero dynamics for infinite-dimensional systems. A zero dynamics concept for a special class of infinite-dimensional systems was proposed by C. I. Byrnes in [2]. The infinite-dimensional version of the so called zero dynamics algorithm, used to compute the zero dynamics submanifold (or zero dynamics subspace for linear case) for the finite-dimensional systems (see Chapter 6. in [18]), has been investigated by Curtain in [8]. The main source of difficulty in exploring this concept for infinite-dimensional systems is that the various controlled invariance notions that coincide in finite dimensions differ for infinite-dimensional systems. The relationship between several invariance concepts for infinite-dimensional linear systems was studied e.g., by Curtain [8] and Zwart [25].

In the first section of this chapter we are going to consider a special case of systems of the form (2.6). Following [2] we will investigate the dynamics for a family of SISO systems with the restriction that the output of the system is identically zero, and define a concept of zero dynamics for these particular systems. The decomposition applied here for relative degree 1 systems (i.e., when $CB \neq 0$) to isolate the subspace on which the dynamics evolves when we restrict the output to zero was also used in [2] for systems with bounded system operator. In this section we will apply this decomposition for systems with an unbounded operator A, which we only assume to be the infinitesimal generator of a C_0 semigroup on state the space Z, and prove that the system restricted to the above mentioned subspace has a well defined dynamics which we will call the zero dynamics of the original system. In the second section we will show that the relationship between the solvability of the regulator problem and the structure of the zero dynamics of the composite system formed from the plant and the exosystem. This result is a direct analog of the result given in Byrnes and Isidori [3] for finite-dimensional systems. Namely, under certain assumptions we show that both the error and state feedback regulator problems are solvable if and only if the zero dynamics of the composite system can be decomposed into isomorphic copies of the plant's zero dynamics and the exosystem.

4.2 Single Input Single Output Systems - A Special Case

A1 We assume that the control input u as well as the measured output y are real scalars so that the plant (2.6) without disturbance takes the form

$$\begin{aligned} &\frac{d}{dt}z(t) = Az(t) + bu(t), \\ &y(t) = Cz(t) \equiv \langle c, z(t)\rangle, \end{aligned} \tag{4.1}$$

where

A is the infinitesimal generator of a C_0 semigroup on Z,

$b \in \mathcal{D}(A)$, $c \in \mathcal{D}(A^*)$ and

$\langle b, c\rangle \neq 0$.

In our applications A is a maximal dissipative (wave type) or maximal accretive (parabolic) operator and hence it generates a C_0 semigroup.

Consider a special decomposition of the state space of the plant, Z. Define $P : Z \to Z$ as the projection

$$P\phi = \frac{\langle \phi, c\rangle}{\langle b, c\rangle} b, \tag{4.2}$$

onto the one-dimensional subspace $span\{b\}$, with $Ker(P) = Ker(C)$, i.e.,

$$Ker(P) = \{\phi \in Z : \langle \phi, c\rangle = 0\}.$$

Note that $Ker(P)$ is a co-dimension 1 closed hyperplane in Z. $(I - P)$ is a projection onto the complementary subspace $Ker(C)$. This gives the decomposition of $Z = \mathcal{S}^1 + \mathcal{S}^2$ into complementary closed subspaces $\mathcal{S}^1 = (I - P)Z = Ker(C)$ and $\mathcal{S}^2 = PZ = span\{b\}$, and for $z \in Z$ we introduce the coordinates

$$\eta = (I - P)z, \qquad \xi b = Pz, \ \xi \in I\!R. \tag{4.3}$$

Notice that $b \in \mathcal{D}(A)$ implies that $Pz \in \mathcal{D}(A)$ and we can write

$$APz = \frac{\langle z, c\rangle}{\langle b, c\rangle} Ab.$$

With the new coordinates $z = \eta + \xi b$ and by the short computation

$$\begin{aligned}&\frac{d}{dt}\eta + b\frac{d}{dt}\xi = \frac{d}{dt}z = (I - P)(Az + bu) + P(Az + bu) = \\ &(I - P)(A\eta + Ab\xi) + P(A\eta + Ab\xi) + bu)\end{aligned}$$

$$y(t) = \langle (I - P)z + Pz, c\rangle = \langle P(\eta + \xi b), c\rangle = \langle P\xi b, c\rangle = \langle b, c\rangle\xi$$

the system (4.1) takes the form

$$\begin{aligned}&\frac{d}{dt}\eta(t) = (I - P)A\eta(t) + (I - P)Ab\xi(t), &(4.4)\\ &\frac{d}{dt}\xi(t)b = PA\eta(t) + PAb\xi(t) + bu(t), &(4.5)\\ &y(t) = \langle Pz(t), c\rangle = \xi\langle b, c\rangle.\end{aligned}$$

In these coordinates A can be written as

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix},$$

here $A_{ij} : \mathcal{S}^i \longrightarrow \mathcal{S}^j$ for $i = 1, 2$ $j = 1, 2$ are linear operators

$$\begin{aligned}&A_{11} = (I - P)A|_{\mathcal{S}^1} \qquad A_{12} = (I - P)A|_{\mathcal{S}^2}\\ &A_{21} = PA|_{\mathcal{S}^1} \qquad A_{22} = PA|_{\mathcal{S}^2}\ .\end{aligned}$$

In particular,

$$A_{21}\eta = \frac{\langle A\eta, c\rangle}{\langle b, c\rangle} b \qquad A_{22}\xi b = \frac{\langle Ab, c\rangle}{\langle b, c\rangle}\xi b.$$

The system becomes

$$\frac{d}{dt}\eta(t) = A_{11}\eta(t) + A_{12}\xi(t)b, \tag{4.6}$$

$$\frac{d}{dt}\xi(t)b = A_{21}\eta(t) + A_{22}\xi b + bu, \tag{4.7}$$

$$y(t) = \xi\langle b, c\rangle.$$

Let us apply the feedback control input

$$u = Kz = J\begin{bmatrix} 0 & 0 \\ -A_{21} & 0 \end{bmatrix}\begin{bmatrix} \eta \\ \xi b \end{bmatrix} = -\frac{\langle A\eta, c\rangle}{\langle b, c\rangle},$$

where J denotes the isomorphism $J\xi b = \xi$ between $\mathcal{S}^2$ and $\mathbb{R}$. We note that by assumption $c \in \mathcal{D}(A^*)$ and therefore A_{21} is a bounded linear operator and so $K : Z \to \mathbb{R}$ is also bounded. With this feedback the system becomes

$$\frac{d}{dt}\eta(t) = A_{11}\eta + A_{12}\xi(t), \tag{4.8}$$

$$\frac{d}{dt}\xi(t)b = A_{22}\xi b, \tag{4.9}$$

$$y(t) = \xi\langle b, c\rangle.$$

We remark that the subspace $\mathcal{S}^1$ of Z is $A + bK$-invariant, i.e.,

$$(A + bK)\left(\mathcal{S}^1 \cap \mathcal{D}(A + bK)\right) \subset \mathcal{S}^1.$$

Now we define the zero dynamics of (4.1) to be the system

$$\frac{d}{dt}\eta = A_{11}\eta, \tag{4.10}$$

obtained by constraining the output to be identically zero: $y(t) \equiv 0$, i.e., $\xi(t) \equiv 0$. Next we will argue that the system we called above zero dynamics does generate dynamics, i.e., A_{11} is the infinitesimal generator of a semigroup on the subspace $\mathcal{S}^1$.

Theorem 4.1 *A_{11} is the infinitesimal generator of a C_0 semigroup on the subspace $\mathcal{S}^1$ of Z.*

Proof: We are going to use the Lumer-Philips theorem (see Chap. 1 Theorem 4.6 in [22]). By our assumptions A is the infinitesimal generator of a C_0 semigroup and the operator

$$\widetilde{A} = A + bK = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}$$

is a bounded perturbation of A we have that $\widetilde{A}$ also generates a C_0 semigroup on Z (see e.g., Theorem 3.2.1 in [10]) which we denote by $T(t)$. Let ω be the growth bound of $T(t)$, then for some positive constant M we have $\|T(t)\| \leq Me^{\omega t}$. The family of bounded operators $S(t) = e^{-\omega t}T(t)$ is also a semigroup on Z, $\|S(t)\| \leq M$ and its infinitesimal generator is $\widetilde{A} - \omega I$. We can see that with an equivalent norm $\widetilde{A} - \omega I$ generates a contraction semigroup on Z and therefore by the Lumer-Philips Theorem $Ran(\lambda I - (\widetilde{A} - \omega I)) = Z$ for all $\lambda > 0$ and $(\widetilde{A} - \omega I)$ is dissipative in Z. We also see that the closed subspace $\mathcal{S}^1$ of Z is $(\widetilde{A} - \omega I)$-invariant, and since dissipativity is a norm condition we can also say that on the subspace $\mathcal{S}^1$

$$(\widetilde{A} - \omega I)|_{\mathcal{S}^1} \tag{4.11}$$

is dissipative. Let us show that there exists a positive λ_0 such that

$$Ran\left(((\lambda_0 - \omega)I - \widetilde{A})|_{\mathcal{S}^1}\right) = \mathcal{S}^1. \tag{4.12}$$

Since we have that $Ran(\lambda I - (\widetilde{A} - \omega I)) = Z$ for all $\lambda > 0$ it is enough to show that for some $\lambda_0 > 0$ there is no element $\overline{z} \in Z \setminus \mathcal{S}^1$ such that $(\lambda_0 I - (\widetilde{A} - \omega I))\overline{z} \in \mathcal{S}^1$. Assume that there is such an element $\overline{z} \in Z \setminus \mathcal{S}^1$. With the notation $\overline{z} = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$, where $z_1 = (I - P)\overline{z}$, $z_2 = P\overline{z}$ we have that

$$[\lambda_0 I - (\widetilde{A} - \omega I)]\overline{z} = [(\lambda_0 + \omega)I - \widetilde{A}] \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} =$$

$$\begin{bmatrix} (\lambda_0 + \omega)z_1 - A_{11}z_1 - A_{12}z_2 \\ (\lambda_0 + \omega)z_2 - A_{22}z_2 \end{bmatrix}$$

is such that the second component $(\lambda_0 + \omega)z_2 - A_{22}z_2 = 0$. This is true only if $(\lambda_0 + \omega) \in \sigma(A_{22}) = \{\frac{\langle Ab,c\rangle}{\langle b,c\rangle}\}$. Therefore for sufficiently large λ_0 we have $Ran\left(((\lambda_0 - \omega)I - \widetilde{A})|_{\mathcal{S}^1}\right) = \mathcal{S}^1$. Since the closed subspace $\mathcal{S}^1$ of Z is Hilbert space referring to Chap.1. Theorem 4.6. in [22] (4.11) and (4.12) implies that $\mathcal{D}\left((\tilde{A}-\omega I)|_{\mathcal{S}^1}\right)$ is dense in $\mathcal{S}^1$, and then using (4.11) and (4.12) again, the Lumer-Philips Theorem gives that $(\tilde{A}-\omega I)|_{\mathcal{S}^1}$ generates a contraction semigroup on $\mathcal{S}^1$. Let us denote it by $\tilde{S}(t)$. Finally $\tilde{T}(t) = e^{\omega t}\tilde{S}(t)$ is a C_0 semigroup on $\mathcal{S}^1$ with infinitesimal generator $A_{11} = \tilde{A}|_{\mathcal{S}^1}$. □

We mention here that for the system (4.1) in the case of periodic tracking there are results that show the connection between the output regulation problem and the transmission zeroes of the controlled plant. Namely, in [1] under additional assumptions on (4.1) the following theorem was proved.

A2 Assume that the transfer function is real, i.e., $g(\overline{s}) = \overline{g(s)}$.

A3 Assume that there are no pole zero cancellations. That is if s_0 is a transmission zero, then $s_0 \in \rho(A)$, the resolvent set of A.

Theorem 4.2 *Let the operator A in (4.1) be a discrete Riesz spectral operator with simple eigenvalues $\sigma(A) = \{\lambda_j\}_{j=1}^{\infty}$, the input be given by $u(t) = \gamma_1 \sin(\alpha t) + \gamma_2 \cos(\alpha t)$ with $\gamma_1^2 + \gamma_2^2 \neq 0$ and let (A, b, c) satisfy* **A2** *and* **A3**. *Then the following results hold.*

1. *There exists periodic solution z to the system (4.1) with period $T = 2\pi/\alpha$ provided*

$$dist(\sigma(A), \{k\alpha i \,|\, k = 0, \pm 1, \pm 2, \cdots\}) > 0.$$

Furthermore, the system supports periodic solutions with all positive periods T (i.e., for any T there exists initial condition z_0 so that the state of the system (4.1) will be a periodic of period T) if

$$dist(\sigma(A), \mathbb{C}^0) > 0$$

where $\mathbb{C}^0 = \{\lambda \in \mathbb{C} : \operatorname{Re}\lambda = 0\}$ denotes the imaginary axis.

2. *In this case, there is a nontrivial periodic output y if and only if $i\alpha$ is not a transmission zero, i.e., $g(i\alpha) \neq 0$.*

3. Let

$$M_u \equiv \sup_{t\in[0,T]} |u(t)| = \sqrt{\gamma_1^2+\gamma_2^2}.$$

denote the amplitude of the periodic input u, then the amplitude of the output y is a linear function of the amplitude of the input u. In particular, the output can be written in the forms

$$\begin{aligned} y(t) &= [\operatorname{Re} g(i\alpha)]u(t) + \frac{1}{\alpha}[\operatorname{Im} g(i\alpha)]\frac{du}{dt}(t) \\ &= M_u|g(i\alpha)|[\tilde{\gamma}_1 \sin(\alpha t) + \tilde{\gamma}_2 \cos(\alpha t)] \\ &= M_u|g(i\alpha)| \sin(\alpha t + \phi) \end{aligned}$$

where $\tilde{\gamma}_1{}^2 + \tilde{\gamma}_2{}^2 = 1$ and we can easily write explicit formulas for $\tilde{\gamma}_1$, $\tilde{\gamma}_2$ and ϕ in terms of γ_1, γ_2 and $g(i\alpha)$. Thus the amplitude M_y of y can be written as

$$M_y \equiv \sup_{t\in[0,T]} |y(t)| = M_u|g(i\alpha)|.$$

4.3 Solvability and Zero Dynamics

In [3] Byrnes and Isidori interpreted the solvability of the regulator problems for nonlinear finite-dimensional systems as a property of the zero dynamics of the composite system formed from the plant and the exogenous system. Namely, under certain assumptions they showed that the regulator problem is solvable if and only if the zero dynamics of the composite system can locally be decomposed into diffeomorphic copies of the exosystem and the plant's zero dynamics. We present here analogous results for linear infinite-dimensional systems with bounded control and observation operators.

For the system

$$\begin{aligned} \frac{d}{dt}z(t) &= Az(t) + Bu(t), \\ z(0) &= z_0, \end{aligned} \tag{4.13}$$

where the assumptions on A, B, z and u are the same as for the system (2.6), we introduce the usual invariance concepts (cf. [8]) :

Definition 4.1 *:*

- *A closed, linear subspace V of the Hilbert space Z is called (A, B)-invariant if*
$$A(V \cap \mathcal{D}(A)) \subset \overline{V + Ran(B)}.$$
- *A closed, linear subspace V of the Hilbert space Z is called feedback (A, B)-invariant if there exists a $K \in \mathcal{L}(Z, U)$ such that*
$$(A + BK)(V \cap \mathcal{D}(A)) \subset V.$$
- *A closed, linear subspace V of the Hilbert space Z is called $T(A, B)$-invariant if there exists a $K \in \mathcal{L}(Z, U)$ such that for all $t \geq 0$*
$$T_{A+BK}(t)V \subset V.$$

While for finite-dimensional systems these three concepts are equivalent, if Z is infinite-dimensional then they form the following hierarchy: $T(A, B)$-invariance implies feedback (A, B)-invariance which in turn implies (A, B)-invariance. For the proof see Curtain [8].

Along with the system (4.13) we consider the observation process $y(t) = Cz(t)$ in (2.6). Motivated by the zero dynamics concept introduced by Byrnes and Isidori for finite-dimensional systems, and by results in the previous section, let us introduce a concept of zero dynamics for infinite-dimensional linear systems.

Z1 For the plant

$$\begin{aligned} &\frac{d}{dt}z(t) = Az(t) + Bu(t), \\ &y(t) = Cz(t) \end{aligned} \tag{4.14}$$

suppose that the largest feedback (A, B)-invariant subspace contained in $Ker(C)$ exists and is denoted by Z^*, and assume that Z^* is also $T(A, B)$-invariant. Let us additionally assume, that any two feedback operators that render Z^* $T(A, B)$-invariant coincide on Z^*. Let A^* denote $(A + BK)|_{Z^*}$, where $K \in \mathcal{L}(Z, U)$ is such that Z^* is $(A + BK)$-invariant and define the system (Z^*, A^*) to be the zero dynamics of the plant.

The assumption that the largest feedback (A, B)-invariant subspace Z^* of $Ker(C)$ is such that there is essentially one feedback that renders it $A + BK$-invariant is part of **Z1**. It is not clear how restrictive is this assumption in general, however in terms of the operator B we can find sufficient conditions for it to hold.

Remark 4.1 *If we assume that* $Ran(B) \cap Z^* = \{0\}$ *and* $Ker(B) = \{0\}$*, then any two bounded feedback operators that render* Z^* $T(A, B)$*-invariant coincide on* Z^*.

This statement can be justified by the following argument. Let $K_1 \in \mathcal{L}(Z, U)$ *and* $K_2 \in \mathcal{L}(Z, U)$ *be two feedback operators that satisfy*

$$T_{(A+BK_i)}Z^* \subset Z^*.$$

Lemma II. 2. in [25] gives that $Ran(B(K_1 - K_2)|_{Z^* \cap \mathcal{D}(A)}) \subset Z^*$*, but since* $Ran(B) \cap Z^* = \{0\}$ *we obtain that* $Ran(B(K_1 - K_2)|_{Z^* \cap \mathcal{D}(A)}) = \{0\}$*. Since* B *is one to one, necessarily* $(K_1 - K_2)|_{Z^* \cap \mathcal{D}(A)} = 0$*, but* $Z^* \cap \mathcal{D}(A)$ *is dense in* Z^* *and therefore* $(K_1 - K_2)|_{Z^*} = 0$.

In the previous remark the assumptions that gave the uniqueness of $K|_{Z^*}$, for feedbacks that render Z^* $T(A, B)$-invariant is related to the assumptions made in [3] (see also Chapter 8,[18]) concerning the definition of zero dynamics for multivariable nonlinear control systems.

Remark 4.2 *The zero dynamics defined in Section 4.1 for a family of single input single output systems is a special case of the concept defined in* **Z1**. *In that case we have seen that* $Z^* = Ker(C)$*,* Z^* *is* $T(A, B)$*-invariant and therefore feedback* (A, B)*-invariant, and also by the previous remark since* $Ran(B) \cap Z^* = \{0\}$ *and* $Ker(B) = \{0\}$*, any two bounded feedback operators that render* Z^* $T(A, B)$*-invariant coincide on* Z^*.

Now we consider the finite-dimensional exogenous system (2.7) with $P = 0$ and the composite system formed from controlled plant and the exosystem. For the composite system

$$\begin{aligned} \frac{d}{dt} z_e(t) &= A_e z_e(t) + B_e u(t), \\ e(t) &= C_e z_e(t), \end{aligned} \tag{4.15}$$

where $Z_e = Z \times I\!R^k$, is a Hilbert space with norm

$$\left\| \begin{bmatrix} z \\ w \end{bmatrix} \right\|^2 = \|z\|_Z^2 + \|w\|_{I\!R^k}^2,$$

$$B_e = \begin{bmatrix} B \\ 0 \end{bmatrix} \in \mathcal{L}(U, Z_e), \qquad C_e = [C \quad -Q] \in \mathcal{L}(Z_e, Y),$$

$$A_e = \begin{bmatrix} A & P \\ 0 & S \end{bmatrix}$$

is the generator of a C_0 semigroup on Z_e, we also consider the set of assumptions and the concept of zero dynamics introduced in **Z1**.

Z2 For the composite system (4.15) suppose that the largest feedback (A_e, B_e)-invariant subspace contained in $Ker(C_e)$ exists and is denoted by Z_e^*, and assume that Z_e^* is also $T(A_e, B_e)$-invariant. Let us additionally assume, that any two feedback operators that render Z_e^* $T(A_e, B_e)$-invariant coincide on Z_e^*. Let A_e^* denote $(A_e + B_e K_e)|_{Z_e^*}$, where $K_e \in \mathcal{L}(Z_e, U)$ is such that Z_e^* is $(A_e + B_e K_e)$-invariant then we define the system (Z_e^*, A_e^*) to be the zero dynamics of the composite system.

Notation 4.1 : *By our assumption in* **Z1** *any two feedback operators* $K_1 \in \mathcal{L}(Z, U)$ *and* $K_2 \in \mathcal{L}(Z, U)$ *that render* Z^* $T(A, B)$*-invariant, coincide in* Z^**. In the following we will only consider the unique restriction* $K_i|_{Z^*}$ *and denote it by* K*. Similarly, by* **Z2** *any two feedback operators* $K_{e_1} \in \mathcal{L}(Z_e, U)$ *and* $K_{e_2} \in \mathcal{L}(Z_e, U)$ *that render* Z_e^* $T(A_e, B_e)$*-invariant, coincide in* Z_e^**. For the rest of this section we will only consider the unique restriction* $(K_{e_i})|_{Z_e^*}$ *and denote it by* K_e*.*

In the next theorem we will identify a copy of the plant's zero dynamics inside the zero dynamics of the composite system, provided that both zero dynamics exist.

Theorem 4.3 *Assume that* **Z1** *and* **Z2** *hold. Let* (Z^*, A^*) *and* (Z_e^*, A_e^*) *denote the zero dynamics of the plant and the composite system, respectively. Then the two subspaces* $Z_e^* \cap (Z \times \{0\})$ *and* $Z^* \times \{0\}$ *of* Z_e^* *coincide and the mapping* Ψ :

$Z_e^* \cap (Z \times \{0\}) \longrightarrow Z^*$ *defined by* $\Psi \begin{bmatrix} z^* \\ 0 \end{bmatrix} = z^*$ *is an isomorphism satisfying* $\Psi A_e^* z_e^* = A^* \Psi z_e^*$ *for all* $z_e^* \in \mathcal{D}(A_e^*)$.

Proof: Let us show first that $Z^* \times \{0\} \subset Z_e^* \cap (Z \times \{0\})$. $Z^* \times \{0\} \subset Z \times \{0\}$, therefore we only need that $Z^* \times \{0\} \subset Z_e^*$. Since Z_e^* is the largest feedback (A_e, B_e)-invariant subspace of Z_e contained in $Ker(C_e)$, it is enough to show that the closed subspace $Z^* \times \{0\}$ of Z_e is also contained in $Ker(C_e)$ and that it is feedback (A_e, B_e)-invariant. $Z^* \times \{0\} \subset Ker(C_e)$ follows from $Z^* \subset Ker(C)$.

Now let $\begin{bmatrix} z^* \\ 0 \end{bmatrix} \in (Z^* \times \{0\}) \cap \mathcal{D}(A_e)$ be arbitrary.

$$(A_e + B_e \tilde{K}_e) \begin{bmatrix} z^* \\ 0 \end{bmatrix} = \begin{bmatrix} (A + BK) z^* \\ 0 \end{bmatrix},$$

where $\tilde{K}_e = [K \; 0]$, and K is the unique feedback for Z^* referred to in Notation 4.1. Since Z^* is $(A+BK)$-invariant, this means that $Z^* \times \{0\}$ is feedback (A_e, B_e)-invariant.

To show that $Z_e^* \cap (Z \times \{0\}) \subset Z^* \times \{0\}$ first note that for any

$$\begin{bmatrix} z^* \\ 0 \end{bmatrix} \in Z_e^* \cap (Z \times \{0\})$$

we must have $\begin{bmatrix} z^* \\ 0 \end{bmatrix} \in Ker(C_e)$, and therefore

$$0 = C_e \begin{bmatrix} z^* \\ 0 \end{bmatrix} = Cz^* - Q0 = Cz^*,$$

i.e., the first component $z^* \in Ker(C)$. Let us define

$$Z' = \left\{ z^* \in Z : \begin{bmatrix} z^* \\ 0 \end{bmatrix} \in Z_e^* \cap (Z \times \{0\}) \right\}.$$

Z' is clearly a subspace of Z and we have seen that $Z' \subset Ker(C)$. Z' is a closed subspace of Z since it is isometrically isomorphic to the closed subspace $Z_e^* \cap (Z \times$

$\{0\})$. We need to show that $Z' \subset Z^*$. Let the unique feedback operator K_e in Z_e^* (see Notation 4.1) be of the form $K_e = [\hat{K}\ \hat{L}]$. Since for every z^* in $Z' \cap \mathcal{D}(A)$, $\begin{bmatrix} z^* \\ 0 \end{bmatrix} \in Z_e^* \cap \mathcal{D}(A_e)$, the A_e^*-invariance of Z_e^* implies that

$$\begin{bmatrix} A + B\hat{K} & P + B\hat{L} \\ 0 & S \end{bmatrix} \begin{bmatrix} z^* \\ 0 \end{bmatrix} = \begin{bmatrix} (A + B\hat{K})z^* \\ 0 \end{bmatrix} \in Z_e^*.$$

This gives that $(A + B\hat{K})z^* \in Z'$ and $(A + B\hat{K})z^* \in Ker(C)$. From this we have that Z' is feedback (A, B)-invariant and contained in $Ker(C)$. But Z^* is the largest subspace of Z with these properties, hence $Z' \subset Z^*$. Thus we have established that $Z_e^* \cap (Z \times \{0\}) = Z' \times \{0\} \subset Z^* \times \{0\}$. Note that we have also obtained $\hat{K} = K$. To finish the proof of Theorem 4.3 we need to show that the mapping Ψ, which is a linear isomorphism satisfies the equation $\Psi A_e^* z_e^* = A^* \Psi z_e^*$. But for every $\begin{bmatrix} z^* \\ 0 \end{bmatrix} \in Z_e^* \cap (Z \times \{0\}) \cap \mathcal{D}(A_e)$ we have:

$$\Psi A_e^* z_e^* = \Psi \begin{bmatrix} A + BK & P + B\hat{L} \\ 0 & S \end{bmatrix} \begin{bmatrix} z^* \\ 0 \end{bmatrix}$$

$$= \Psi \begin{bmatrix} (A + BK)z^* \\ 0 \end{bmatrix} = (A + BK)z^* = A^* \Psi z_e^*.$$

With this the proof is complete. □

The following theorem shows that the zero dynamics of the composite system also contains an isomorphic copy of the exosystem, complementary to the plant's embedded zero dynamics, exactly when the regulator equations are solvable.

Theorem 4.4 *Assume that* **Z1** *and* **Z2** *hold and* (Z^*, A^*) *and* (Z_e^*, A_e^*) *denote the zero dynamics of the plant and the composite system, respectively. Then there exist mappings* $\Pi \in \mathcal{L}(\mathbb{R}^k, Z)$ *and* $\Gamma \in \mathcal{L}(\mathbb{R}^k, U)$, *with* $Ran(\Pi) \subset \mathcal{D}(A)$, *that solve the regulator equations*

$$\Pi S = A\Pi + B\Gamma + P \tag{4.16}$$

$$C\Pi - Q = O \tag{4.17}$$

if and only if there is an A_e^*-invariant k-dimensional subspace Z_s of Z_e^* contained in $\mathcal{D}(A_e^*)$ such that

i) $Z_e^* = Z_s \oplus (Z_e^* \cap (Z \times \{0\}))$,

ii) if for $\Pi_s \in \mathcal{L}(I\!R^k, Z)$, $Z_s = graph(\Pi_s)$, *then the mapping* Φ *defined by* $\Phi w = \begin{bmatrix} \Pi_s w \\ w \end{bmatrix}$ *is a linear isomorphism from* $I\!R^k$ *to* Z_s, *satisfying* $\Phi S w = A_e^* \Phi w$.

In this case we also have that Z_s *is* $T_{A_e^*}(t)$*-invariant and* $\Phi T_S(t) w_0 = T_{A_e^*}(t)\Phi w_0$ *for* $w_0 \in I\!R^k$, *where* $T_S(t) = e^{St}$.

Proof: Suppose that $\Pi \in \mathcal{L}(I\!R^k, Z)$ and $\Gamma \in \mathcal{L}(I\!R^k, U)$ solve the regulator equations (4.16)-(4.17). Then Z_s defined as the $graph(\Pi)$, i.e.

$$Z_s = \left\{ \begin{bmatrix} z \\ w \end{bmatrix} \in Z_e : z = \Pi w \right\}$$

is a k-dimensional subspace of Z_e. We first show that $Z_s \subset Z_e^*$. Note that $Z_s \subset Ker(C_e) \cap \mathcal{D}(A_e)$ by (4.17) and by $Ran(\Pi) \subset \mathcal{D}(A)$. Let $K_s \in \mathcal{L}(Z, U)$ be defined by $K_s = [K \ (\Gamma - K\Pi)]$ where K is the unique feedback referred to in Notation 4.1. Z_s is $(A_e + B_e K_s)$-invariant by the following calculation, where in the last step we use the first regulator equation (4.16):

$$(A_e + B_e K_s) \begin{bmatrix} \Pi w \\ w \end{bmatrix} = \left(\begin{bmatrix} A & P \\ 0 & S \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} [K \ (\Gamma - K\Pi)] \right) \begin{bmatrix} \Pi w \\ w \end{bmatrix}$$

$$= \begin{bmatrix} A\Pi w + Pw + BK\Pi w + B\Gamma w - BK\Pi w \\ Sw \end{bmatrix} = \begin{bmatrix} \Pi S w \\ Sw \end{bmatrix}.$$

Since Z_e^* is the largest feedback (A_e, B_e)-invariant subspace in $Ker(C_e)$, we have that $Z_s \subset Z_e^*$.

Now to prove $Z_s \oplus (Z_e^* \cap (Z \times \{0\})) = Z_e^*$, we need to show that $Z_s \cap (Z_e^* \cap (Z \times \{0\})) = \{0\}$ and $Z_s + (Z_e^* \cap (Z \times \{0\})) = Z_e^*$. The first equality is obvious by the definition of Z_s. Furthermore any $\begin{bmatrix} z \\ w \end{bmatrix} \in Z_e^*$ can be written as

$$\begin{bmatrix} z \\ w \end{bmatrix} = \begin{bmatrix} \Pi w \\ w \end{bmatrix} + \begin{bmatrix} z - \Pi w \\ 0 \end{bmatrix}$$

with
$$\begin{bmatrix} z-\Pi w \\ 0 \end{bmatrix} = \begin{bmatrix} z \\ w \end{bmatrix} - \begin{bmatrix} \Pi w \\ w \end{bmatrix} \in Z_e^* \cap (Z \times \{0\}),$$
and so the second equality also holds.

To prove that ii) holds we need to prove that $Z_e^* \cap (Z \times \{0\})$, the subspace complementary to Z_s in Z_e^*, is also $(A_e + B_e K_s)$-invariant. This, together with the $(A_e + B_e K_s)$-invariance of Z_s implies that Z_e^* is $(A_e + B_e K_s)$-invariant. But this implies, by the uniqueness of K_e, that $K_e = K_s|_{Z_e^*}$ and $A_e^* = A_e + B_e K_s$. Then by the definition of Z_s, the mapping $\Phi : \mathbb{R}^k \longrightarrow Z_s$ defined by $\Phi w = \begin{bmatrix} \Pi w \\ w \end{bmatrix}$ is linear, bijective and by (4.16)
$$A_e^* \Phi w = (A_e + B_e K_s)\Phi w = \left(\begin{bmatrix} A & P \\ 0 & S \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} [K \ (\Gamma - K\Pi)] \right) \begin{bmatrix} \Pi w \\ w \end{bmatrix}$$
$$= \begin{bmatrix} A\Pi w + Pw + BK\Pi w + B\Gamma w - BK\Pi w \\ Sw \end{bmatrix} = \begin{bmatrix} \Pi S w \\ Sw \end{bmatrix} = \Phi S w.$$

It only remains to show that $Z_e^* \cap (Z \times \{0\})$ is $(A_e + B_e K_s)$-invariant. To see this we note that, for any $\begin{bmatrix} z^* \\ 0 \end{bmatrix} \in Z_e^* \cap (Z \times \{0\}) \cap \mathcal{D}(A_e)$, we have $z^* \in Z^*$ by Theorem 4.3. But then
$$(A_e + B_e K_s) \begin{bmatrix} z^* \\ 0 \end{bmatrix} = \left(\begin{bmatrix} A & P \\ 0 & S \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} [K \ (\Gamma - K\Pi)] \right) \begin{bmatrix} z^* \\ 0 \end{bmatrix}$$
$$= \begin{bmatrix} Az^* + BKz^* \\ 0 \end{bmatrix} = \begin{bmatrix} (A+BK)z^* \\ 0 \end{bmatrix} \in Z_e^* \cap (Z \times \{0\})$$

again by Theorem 4.3 and the $(A + BK)$-invariance of Z^*. This completes the proof of the sufficiency.

Now to prove the necessity, let Z_s be an A_e^*-invariant k-dimensional subspace of Z_e^* contained in $\mathcal{D}(A_e^*)$ such that i) and ii) hold. Define $\Pi \equiv \Pi_s : \mathbb{R}^k \longrightarrow Z$ by $Graph(\Pi) = Z_s$. We claim that Π is a well defined linear mapping from $\mathbb{R}^k$ to Z. Note, that by i) any nonzero $z_s \in Z_s$ has the form $\begin{bmatrix} z \\ w \end{bmatrix}$ for some nonzero $w \in \mathbb{R}^k$

and $z \in Z$. If $\begin{bmatrix} z_1 \\ w \end{bmatrix}, \begin{bmatrix} z_2 \\ w \end{bmatrix} \in Z_s$ with $z_1 \neq z_2$ then $\begin{bmatrix} z_1 - z_2 \\ 0 \end{bmatrix} \in Z_e^* \cap (Z \times \{0\})$ which contradicts our remark in the previous sentence, so Π is well defined. Since Z_s is k-dimensional, for every $w \in \mathbb{R}^k$ there exists a $z \in Z$ such that $\begin{bmatrix} z \\ w \end{bmatrix} \in Z_s$, i.e., Π is defined on the whole $\mathbb{R}^k$. Π is also linear which follows from the fact that Z_s is a linear subspace. Since $Z_s \subset \mathcal{D}(A_e^*) = \mathcal{D}(A_e) = \mathcal{D}(A) \times \mathbb{R}^k$ we have $Ran(\Pi) \subset \mathcal{D}(A)$.

To finish the proof of the necessity, we need to show that Π satisfies the regulator equations with some $\Gamma \in \mathcal{L}(\mathbb{R}^k, U)$. We know that $Z_s \subset Z_e^* \cap \mathcal{D}(A_e)$, therefore Z_s is also contained in $Ker(C_e) \cap \mathcal{D}(A_e)$. This means that for every $w \in \mathbb{R}^k$, $C\Pi w - Qw = 0$, which is exactly (4.17). Also we have the equation $\Phi S w = A_e^* \Phi w$ for all $w \in \mathbb{R}^k$, where $A_e^* = \left(\begin{bmatrix} A & P \\ 0 & S \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} [K \ \hat{L}] \right)$ and $[K \ \hat{L}]$ is the unique feedback operator K_e. This equation can be written as

$$\begin{bmatrix} \Pi S w \\ S w \end{bmatrix} = \left(\begin{bmatrix} A & P \\ 0 & S \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} [K \ \hat{L}] \right) \begin{bmatrix} \Pi w \\ w \end{bmatrix}$$

$$= \begin{bmatrix} A\Pi w + Pw + BK\Pi w + B\hat{L}w \\ Sw \end{bmatrix},$$

so by choosing $\Gamma = \hat{L} + K\Pi$, we obtain (4.16).

To finish the proof of the theorem let us note, that Z_s is also $T_{A_e^*}(t)$-invariant (i.e., $T_{A_e + B_e K_s}(t)$-invariant). This can be seen using Lemma 2.5.4 in Curtain-Zwart [10], since Z_s is a closed subspace of Z_e^* and also contained in $\mathcal{D}(A_e^*)$ (since $Ran(\Pi) \subset \mathcal{D}(A)$). Moreover, we have that $\Phi T_S(t) w_0 = T_{A_e^*}(t) \Phi w_0$ for $w_0 \in \mathbb{R}^k$, where $T_S(t) = e^{St}$. Indeed,

$$\Phi T_S(t) w_0 = \begin{bmatrix} \Pi T_S(t) w_0 \\ T_S(t) w_0 \end{bmatrix},$$

and referring to Lemma 3.2.2 in Curtain-Zwart [10] (see also Exercise 3.5 therein) we also have

$$T_{A_e^*}(t)\Phi w_0 = \begin{bmatrix} T_{A+BK}(t)\Pi w_0 + \int_0^t T_{A+BK}(t-s)(P + B(\Gamma - K\Pi))T_S(s) w_0 \, ds \\ T_S(t) w_0 \end{bmatrix}$$

$$= \begin{bmatrix} \Pi T_S(t) w_0 \\ T_S(t) w_0 \end{bmatrix}.$$

In the last equality we have used that $\Phi w \in Z_s$ and Z_s is $T_{A_e^*}(t)$-invariant. □

Remark 4.3 *Since we have shown in Chapter 2 that under suitable hypothesis the solvability of the regulator problems (both the state and the error feedback case) is equivalent to the solvability of the regulator equations, we see that under such hypothesis a necessary and sufficient condition for the solvability of the regulator problems, provided that both zero dynamics exist, is that the zero dynamics of the composite system contains an isomorphic copy of the exosystem.*

For infinite-dimensional systems it is not true in general that the assumptions in **Z1** and **Z2** hold. But as an example we can see for the plant (4.1) that satisfies the condition **A1** and for the exogenous system $\dot{w} = Sw$, $y_r = Qw$ with

$$S = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \qquad Q = [1 \ 0],$$

Z1 and **Z2** can be established.

We have already shown that for the plant (4.1) satisfying **A1** the subspace $Z^* \equiv Ker(C)$ satisfies the assumptions in **Z1** and the zero dynamics (Z^*, A^*) exists (see Remark(4.2)). To establish **Z2** we will show that that the composite system

$$\begin{aligned} &\frac{d}{dt} z_e(t) = A_e z_e(t) + B_e u(t), \\ &e(t) = C_e z_e(t), \end{aligned} \tag{4.18}$$

where

$$Z_e = Z \times I\!R^2, A_e = \begin{bmatrix} A & 0 \\ 0 & S \end{bmatrix}, B_e u = b_e u = \begin{bmatrix} b \\ 0 \end{bmatrix} u, C_e = [C \quad -Q],$$

is also a single input single output system of the form of (4.1) that satisfies **A1**. This together with $Ran(B_e) \cap Ker(C_e) = \{0\}$ and $Ker(B_e) = \{0\}$ implies that **Z1** holds for the composite system (4.18), but this is exactly **Z2**. Indeed, u, e are scalars,

$$b_e = \begin{bmatrix} b \\ 0 \end{bmatrix} \in \mathcal{D}(A) \times I\!R^2 = \mathcal{D}(A_e),$$

$$e = C_e z_e = Cz - Qw = \langle c, z\rangle_Z - \left\langle \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \right\rangle_{I\!R^2}$$

$$= \left\langle \begin{bmatrix} c \\ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \end{bmatrix}, \begin{bmatrix} z \\ \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \end{bmatrix} \right\rangle_{Z_e} = \langle c_e, z_e\rangle_{Z_e},$$

$$c_e = \begin{bmatrix} c \\ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \end{bmatrix} \in \mathcal{D}(A^*) \times I\!R^2 = \mathcal{D}\left(\begin{bmatrix} A^* & 0 \\ 0 & -S \end{bmatrix} \right) = \mathcal{D}(A_e^*),$$

where A^* and A_e^* denotes the adjoint of the operators A and A_e, and

$$\langle c_e, b_e\rangle_{Z_e} = \langle c, b\rangle_Z - \left\langle \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\rangle_{I\!R^2} = \langle c, b\rangle_Z \neq 0$$

show that the system (4.18) satisfies **A1**. Also $Ran(B_e) \cap Ker(C_e) = \{0\}$ because of the last computation and since $b_e \neq 0$ clearly $Ker(B_e) = \{0\}$.

CHAPTER V
CONCLUSIONS

In this dissertation, we presented a treatment of the theory of output regulation for linear control systems in Hilbert spaces, considering only bounded control and bounded observation operators. The two problems we considered are the output regulation with state feedback and the output regulation with error feedback controller. We found that the major results of the finite dimensional output regulation theory can be established in these settings. We proved that under conditions that are analogous to the standard hypotheses of the finite dimensional theory, both problems are solvable if and only if a pair of operator equations, the regulator equations, are solvable. In terms of solutions to the regulator equations and in terms of stabilizing feedback and output injection, the state feedback and error feedback solutions can be explicitly constructed. The solutions that we constructed are not finite dimensional controllers. We gave examples for solving the state feedback problem for single input single output systems that were modeled by parabolic and hyperbolic partial differential equations. For these examples we have seen that the state feedback regulator problem is robust in the sense that a finite dimensional approximation of the controller applied to a finite dimensional approximation of the plant resulted in the required tracking. As we mentioned to construct such solutions, apart from the problem of solving the regulator equations, requires the availability of stabilizing feedback operator and stabilizing output injection for the composite system. To obtain such operators for various types of infinite dimensional systems could be challenging and requires further investigations.

It is also suggested by numerical results, that this theory can be extended to systems with certain types of unbounded control and observation operators. This is a subject of our further investigations.

The zero dynamics concept was introduced as a generalization of transmission zeroes for nonlinear finite dimensional control systems. For such systems it was shown by C. I. Byrnes and A. Isidori that the solvability of the state feedback problem can be interpreted as the property of the zero dynamics of the composite system system (formed from the plant and the exosystem with the tracking error as output) that it can be decomposed to isometric copies of the exosystem and the plant's zero dynamics. We proposed a definition of zero dynamics for infinite dimensional systems, in terms of controlled invariant subspaces, and showed that this result can be generalized to the infinite dimensional output regulation problem. The geometric theory for infinite dimensional systems, in particular the existence and relationship between of various controlled invariant subspaces is very subtle and complicated, and must be considered on a case by case basis. Thus, though this result gives additional insight into the solvability of the output regulation problems, it is not yet clear to what extent it can be used in practice.

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