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Output Tracking of Boolean Control Networks Driven by Constant Reference Signal

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ABSTRACT In this paper, output tracking problem for the Boolean control networks (BCNs) under constant reference signal is investigated. A theorem is presented for solving the output tracking problem of BCNs. A set, named the maximum invariant set, is obtained to solve the output tracking problem under shortest time, and based on the invariant set, the number of state feedback matrices which make the output tracking successful is obtained. Compare with the existing results, the computational cost can be dramatically reduced by using our method. Finally, the results presented in this brief is verified by a biological network.

INDEX TERMS Semi-tensor product, optimal control design, Boolean networks, output tracking control.

I. INTRODUCTION

Boolean networks (BNs) were firstly introduced by Kauffman nearly 50 years ago [1] to model gene regulatory networks in efficient way at the system level. The study of BNs is a hot topic now since BNs have been proved to be a powerful modelling tool for many research topics. For example, [2] modeling the lac operon's behavior by Boolean functions. Actually, the nodes in Boolean models can only take one of the Boolean variables (1 and 0) at a time [1], and the value (1 or 0) of the node represents states of nodes being on or off (expressed or unexpressed).

When control inputs are added to the BNs, BNs become Boolean control networks (BCNs). BCNs can be used to the study the optimal regulatory intervention [3] or the therapeutic intervention strategies. By designing control sequence, the controlled system can achieve desirable states [4]. Up to now, many interesting works have been obtained with the help of semi-tensor product (STP) of matrix. STP is a generalization of conventional product of matrices [5], and many interesting works have been obtained, see [6]–[49].

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As is well-known, one of the most important goals in studying BCNs is to design therapeutic intervention strategy to influence the dynamic of the network, and further achieving system stabilizing. Note that in the real case, the state variables sometimes cannot be measured directly. In this situation, one can only measure the outputs of the systems to track a given reference signal (in other literature, it is also called regulation of output trajectory of the system to a constant value). The reference signal is corresponding to some desirable states in BCNs. We can design suitable controller to ensure the output of BCN to a given reference signal by steering the BCN to the corresponding desirable states set. The output tracking strategy is a meaningful research topic in the field of BCNs, since the output control strategy can provide a more suitable control strategy in the therapeutic intervention cases. So, studying the output tracking problem of BCNs is very necessary.

To the best of our knowledge, output trajectory regulation problem with a constant value is first investigated in [9] and a constructive algorithm is presented to realize optimal control. The pioneering work for output tracking problem of BCNs tracking a constant reference is studied in [15]. Actually, our work here is a supplement for the pioneering works [9], [15]. As we can see, in order to solve the output trajectory regulation problem, or the output tracking problem in BCNs, one of the most important step is to calculate the maximum invariant set S_{Max} [10]. The maximum invariant set, i.e., S_{Max} is also denoted by $\mathcal{Z}(y_e)$ in [9], and S in [15] respectively. Actually, more specifically, $S \subset S_{Max} = \mathcal{Z}(y_e)$. However, the authors in [9] do not give the detailed process to calculate $\mathcal{Z}(y_e)$, i.e., S_{Max} in this paper. Therefore, in this paper, the nonempty set S, i.e., S_{Max} can be found by referring [15]. In order to find the proper nonempty set S, according to Remark 2 in [15], one needs to calculate a series of matrices, and this is not that computationally tractable. In this paper, we will provide a more computationally tractable way to calculate the nonempty set S (S_{Max}). Also, some other techniques are applied to reduce the computational cost during designing the state feedback matrices. Actually, the output trajectory regulation problem with a constant value and the output tracking problem tracking constant reference in BCNs are the same problem with different description. Therefore we can only focus on one of their, i.e., the output tracking problem of BCNs under a constant value.

The main contributions of our paper are listed as below:

- New technique is presented to obtain the maximum invariant set S_{Max}. Compared with the existing result, the technique proposed here to calculate S_{Max} can dramatically reduce the computational cost.
- Based on the maximum invariant set S_{Max}, the number of shortest time state feedback matrices K is obtained.

The remainder of this brief is arranged as below. In Section II, necessary preliminaries are presented. The main results are presented in Section III. That is, a novel way to design the output tracking controller is proposed. In Section IV, a numerical example is worked out to show our designed method. Finally, a brief conclusion is presented to end this paper.

II. PRELIMINARIES

Here, preliminaries about the STP are firstly presented.

Definition 1 [50]: The STP of two matrices $C \in \mathbb{R}^{i\times j}$ and $D \in \mathbb{R}^{k\times l}$ is defined as:

$$C \ltimes D = (C \otimes I_{\alpha/j})(D \otimes I_{\alpha/k}), \tag{1}$$

where $\otimes$ is the Kronecker (tensor) product, α is the least common multiple of j and k.

When $j = k$, $A \ltimes B = AB$, the STP of two matrices becomes the conventional matrices product. In the following, the symbol "$\ltimes$" is simply omitted.

Next, necessary notations are presented.

1) $\Delta_n := \{\delta_n^i | i = 1, \ldots, n\}$, where δ_n^i is the ith column of identity matrix I_n.
2) $\mathbb{N}^+$ is the set of positive integers.
3) $col_i(T)$ denotes the ith column of matrix T, and $Col(T)$ denotes the set of columns of matrix T.
4) $\mathcal{D} := \{1, 0\}$.
5) An $i \times j$ matrix L is called a logical matrix if $Col(L)$ satisfies $Col(L) \subseteq \Delta_i$. $\mathcal{L}_{i\times j}$ denotes the set of $i \times j$ logical matrices.
6) $|\cdot|$ denotes the number of cardinality.
7) Consider a logical matrix $M = [\delta_n^{k_1}\ \delta_n^{k_2}\ \ldots\ \delta_n^{k_s}]$, for compactness, we have $M := \delta_n[k_1\ k_2\ \ldots\ k_s]$.

Let $1 \sim \delta_2^1$ and $0 \sim \delta_2^2$, we have $\mathcal{D} \sim \Delta_2$. Throughout this paper, the vector form δ_2^1 (or δ_2^2) is used to denote logical variable 1 (or 0).

Lemma 1 [50]: For any logical function $h(x_1, x_2, \ldots, x_w) : \mathcal{D}^w \mapsto \mathcal{D}$, there exists a matrix $M_h \in \mathcal{L}_{2\times 2^w}$ such that

$$h(x_1, x_2, \ldots, x_w) = M_h x_1 x_2 \ldots x_w, \tag{2}$$

where M_h is unique and it is called the structural matrix of h, $x_i \in \Delta_2$, $i = 1, 2, \ldots, w$.

III. MAIN RESULTS

A Boolean control network (BCN) with P outputs can be described as below:

$$\begin{cases} x_1(t+1) = f_1(X(t), U(t)), \\ x_2(t+1) = f_2(X(t), U(t)), \\ \qquad\vdots \\ x_n(t+1) = f_n(X(t), U(t)); \\ y_1(t) = h_1(X(t)), \\ y_2(t) = h_2(X(t)), \\ \qquad\vdots \\ y_p(t) = h_p(X(t)), \end{cases} \tag{3}$$

where $X(t) = (x_1(t), x_2(t), \ldots, x_n(t)) \in \mathcal{D}^n$, $U(t) = (u_1(t), u_2(t), \ldots, u_m(t)) \in \mathcal{D}^m$ are states and control inputs of BCN (3), respectively; and $f_i : \mathcal{D}^{m+n} \mapsto \mathcal{D}$, $i = 1, \ldots, n$ and $h_j : \mathcal{D}^n \mapsto \mathcal{D}$, $j = 1, \ldots, p$ are logical functions. Let $Y(t) = (y_1(t), y_2(t), \ldots, y_p(t)) \in \mathcal{D}^p$ be the output of BCN (3), and the output of BCN (3) at t is denoted by $Y(t; X(0), U)$, where U is a given control sequence, $X(0) \in \mathcal{D}^n$ is the initial state.

The output tracking problem investigated in this work is to find/design controllers in the following form

$$\begin{cases} u_1(t) = k_1(X(t)), \\ u_2(t) = k_2(X(t)), \\ \qquad\vdots \\ u_m(t) = k_m(X(t)), \end{cases} \tag{4}$$

such that the output of system (3) and controller (4) is tracking signal $Y_r = (y_1^r, y_2^r, \ldots, y_p^r)$. In other words, for the control sequence U under the designed control law (4), there exists an integer $\lambda \in \mathbb{N}^+$ such that $Y(t; X(0), U) = Y_r$, $\forall t \geqslant \lambda$, $\forall X(0) \in \mathcal{D}^n$. Here $k_i : \mathcal{D}^n \mapsto \mathcal{D}$, $i = 1, 2, \ldots, m$ are to be designed logical functions.

Based on the vector form of logical variables, letting $x(t) = x_1(t) \ltimes x_2(t) \ltimes \cdots \ltimes x_n(t) \in \Delta_{2^n}$, $y(t) = y_1(t) \ltimes y_2(t) \ltimes \cdots \ltimes y_p(t) \in \Delta_{2^p}$, and $u(t) = u_1(t) \ltimes u_2(t) \ltimes \cdots \ltimes u_m(t) \in \Delta_{2^m}$ the equivalent algebraic forms of BCN (3) and controller (4) can be obtained respectively by Lemma 1:

$$\begin{cases} x(t+1) = Fx(t)u(t), \\ y(t) = Hx(t), \end{cases} \tag{5}$$

and

$$u(t) = Kx(t) \tag{6}$$

where $F \in \mathcal{L}_{2^n \times 2^{n+m}}$, $K \in \mathcal{L}_{2^m \times 2^n}$ and $H \in \mathcal{L}_{2^p \times 2^n}$. The detailed process to obtain the algebraic form one can refer to [5]. Let the given reference signal Y_r be $y_r = y_1^r \ltimes y_2^r \ltimes \cdots \ltimes y_p^r = \delta_{2^p}^{\alpha}$.

Thus, the output tracking control problem becomes designing matrix K.

Considering the reference signal $y_r = \delta_{2^p}^{\alpha}$, and we define the $\Theta(\delta_{2^p}^{\alpha})$ as

$$\Theta(\delta_{2^p}^{\alpha}) = \{\delta_{2^n}^{i} | col_i(H) = \delta_{2^p}^{\alpha},\ i = 1, 2, \ldots, 2^n\}. \tag{7}$$

Apparently, we have that $0 \leq |\Theta(\delta_{2^p}^{\alpha})| \leq 2^n$. We can see that $\Theta(\delta_{2^p}^{\alpha})$ contains all the states of BCN (5) whose output is y_r. Assuming that $\Theta(\delta_{2^p}^{\alpha}) \neq \emptyset$ in the paper.

From BCN (5), we split the matrix F into 2^n equal blocks

$$F = [F_1\ F_2\ \cdots\ F_{2^n}], \quad F_i \in \mathcal{L}_{2^n \times 2^m}, \tag{8}$$

and for a given state denoted by $\delta_{2^n}^{\Lambda} \in \Delta_{2^n}$,

$$R(\delta_{2^n}^{\Lambda}) = \{\delta_{2^n}^{i} | \delta_{2^n}^{\Lambda} \in Col(F_i),\ i = 1, 2, \ldots, 2^n\}, \tag{9}$$

which implies that $R(\delta_{2^n}^{\Lambda})$ is the set of states which can be controlled to $\delta_{2^n}^{\Lambda}$ in one step, i.e., $R(\delta_{2^n}^{\Lambda})$ is the one step reachable set of $\delta_{2^n}^{\Lambda}$. Further for a set $S \subseteq \Delta_{2^n}$, we have $R(S) = \bigcup_{\delta_{2^n}^{\Lambda} \in S} R(\delta_{2^n}^{\Lambda})$.

The following process provides a method to construct a sequence of vector sets, where those sets are crucial for matrices K. For a given set $S \subseteq \Delta_{2^n}$, we can obtain a sequence of vector sets as follows:

$$\begin{aligned} &1)\ \Omega_0 = S, \\ &2)\ \Omega_1 = R(\Omega_0) \backslash \Omega_0, \\ &3)\ \Omega_2 = R(\Omega_1) \backslash \bigcup_{i=0}^{1} \Omega_i, \\ &\quad \vdots \\ &4)\ \Omega_M = R(\Omega_{M-1}) \backslash \bigcup_{i=0}^{M-1} \Omega_i. \end{aligned} \tag{10}$$

Hence, there exists a positive integer $M < 2^n$, $\Omega_{M+1} = \emptyset$, and $\sum_{i=0}^{M} |\Omega_i| \leqslant 2^n$.

A lemma, which is deduced from [9], is presented here.

Lemma 2: [9] The output of BCN (5) tracks signal $y_r = \delta_{2^n}^{\alpha}$ by (6) if and only if there is a nonempty set $S \subseteq \Theta(\delta_{2^p}^{\alpha})$ such that

(1) $Col(F_{\Lambda}) \bigcap S \neq \emptyset$, for any $\delta_{2^n}^{\Lambda} \in S$,

(2) $\sum_{i=0}^{M} |\Omega_i| = 2^n$, where Ω_i, $i = 0, 1, \ldots, M$ are produced by (10) and $\Omega_0 = S$.

If there exists a set S satisfying conditions (1) and (2) in Lemma 2, the state feedback control matrix K can be further designed.

For the sets Ω_i, $i = 0, 1, \ldots, M$ satisfying condition (2) in Lemma 2, we conclude that for any $\delta_{2^n}^{\Lambda} \notin S$, there exists a unique set Ω_l such that $\delta_{2^n}^{\Lambda} \in \Omega_l$, $l \in \{1, 2, \ldots, M\}$. Let

$$\begin{aligned} P(\Lambda) = \{\delta_{2^m}^{j} | col_j(F_{\Lambda}) \subseteq \Omega_{l-1},\ \delta_{2^n}^{\Lambda} \in \Omega_l, \\ j = 1, 2, \ldots, 2^m\}. \end{aligned} \tag{11}$$

Especially, when $\delta_{2^n}^{\Lambda} \in S$, let

$$P(\Lambda) = \{\delta_{2^m}^{j} | col_j(F_{\Lambda}) \in S,\ j = 1, 2, \ldots, 2^m\}. \tag{12}$$

Therefore, K can be constructed as $K = [\kappa_1\ \kappa_2\ \ldots\ \kappa_{2^n}]$, with $\kappa_{\Lambda} \in P(\Lambda)$, $\Lambda = 1, \ldots, 2^n$.

Theorem 1: The output of BCN (5) can track $y_r = \delta_{2^n}^{\alpha}$ with $u(t) = Kx(t)$, where $K = [\kappa_1\ \kappa_2\ \ldots\ \kappa_{2^n}]$ is constructed as above.

Proof: Denote the state of BCN (5) at t under control $u(t) = Kx(t)$ by $x(t; x(0), u)$, where $x(0) = \delta_{2^n}^{\Lambda} \in \Delta_{2^n}$ is the initial state. We have

$$\begin{aligned} x(1; x(0), u) &= Fx(0)Kx(0) \\ &= F_{\Lambda}\kappa_{\Lambda} \\ &= \delta_{2^n}^{\Lambda_{\kappa_{\Lambda}}} \\ &\in \begin{cases} \Omega_0, & \text{if } \delta_{2^n}^{\Lambda} \in \Omega_0, \\ \Omega_{l-1}, & \text{if } \delta_{2^n}^{\Lambda} \in \Omega_l,\ 1 \leqslant l \leqslant M. \end{cases} \end{aligned}$$

Thus,

$$x(t; x(0), u) \in S, \quad \forall x(0) \in \Delta_{2^n},\ \forall t \geqslant M.$$

It follows from the construction of the set S that

$$y(t; x(0), u) = y_r = Hx(t; x(0), u)$$

for $\forall x(0) \in \Delta_{2^n}$, $\forall t \geqslant M$. This completes the proof.

Up to now, according to Theorem 1 and Lemma 2, we can find set S and construct matrix K which make the output tracking successful. But there is a hidden problem: the proper nonempty set S may not be unique. Hence, another important issue is that we need to find the set S such that the corresponding controller (controllers) can steer the output of BCN (5) tracking the signal in the shortest time (The shortest time control problem discussed here has been investigated in [9] from the aspect of optimal control with the problem of output trajectory regulation. While, the approach presented in our paper to solve the shortest time control problem comes from [13] and [15]). In the following, we will give a new method to find this optimal set S.

In order to find the optimal set S, we construct a sequence of sets as follows:

$$\begin{aligned} 1)\ & S_0 = \Theta(\delta_{2^n}^{\alpha}), \\ & S_0^{\circ} = \{\delta_{2^n}^{\Lambda_0} | Col(F_{\Lambda_0}) \cap S_0 = \emptyset,\ \text{for any } \delta_{2^n}^{\Lambda_0} \in S_0\}, \\ 2)\ & S_1 = S_0 \backslash S_0^{\circ}, \\ & S_1^{\circ} = \{\delta_{2^n}^{\Lambda_1} | Col(F_{\Lambda_1}) \cap S_1 = \emptyset,\ \text{for any } \delta_{2^n}^{\Lambda_1} \in S_1\}, \\ & \vdots \end{aligned} \tag{13}$$

3) $S_{Max-1} = S_{Max-2} \backslash S^{\circ}_{Max-2}$,
$S^{\circ}_{Max-1} = \{\delta_{2^n}^{\Lambda_{Max-1}} | Col(F_{\Lambda_{Max-1}}) \cap S_{Max-1} = \emptyset$, for any $\delta_{2^n}^{\Lambda_{Max-1}} \in S_{Max-1}\}$,
4) $S_{Max} = S_{Max-1} \backslash S^{\circ}_{Max-1}$,
$S^{\circ}_{Max} = \{\delta_{2^n}^{\Lambda_{Max}} | Col(F_{\Lambda_{Max}}) \cap S_{Max} = \emptyset$, for any $\delta_{2^n}^{\Lambda_{Max}} \in S_{Max}\} = \emptyset$.

There exists a positive integer $Max \leqslant |\Theta(\delta_{2^n}^{\alpha})|$ such that $S^{\circ}_{Max} = \emptyset$, $\Theta(\delta_{2^n}^{\alpha}) = (\bigcup_{i=0}^{Max-1} S^{\circ}_i) \bigcup S_{Max}$.

Note that the obtained $S_{Max} \subset \Theta(\delta_{2^n}^{\alpha})$ in (13) can also be called the maximum invariant set of $\Theta(\delta_{2^n}^{\alpha})$, see [10].

Remark 1: We call S_{Max} the maximum invariant set of $\Theta(\delta_{2^n}^{\alpha})$, satisfying condition (1) in Lemma 2. Apparently, we have that $|S_{Max}| \leqslant |\Theta(\delta_{2^n}^{\alpha})|$. Based on procedure (13), we have that any set S^* satisfying $Col(F_{\Lambda^*}) \bigcap S^* \neq \emptyset$, for any $\delta_{2^n}^{\Lambda^*} \in S^*$ must satisfy $S^* \subseteq S_{Max}$.

Here, a corollary is deduced as below.

Corollary 1: The output of BCN (5) cannot track the signal $y_r = \delta_{2^n}^{\alpha}$ under any control, if the set S_{Max} doesn't satisfy condition (2) in Lemma 2.

Proof. The result can be immediately proved by contradiction, so we omit it here.

Corollary 2: Based on the sets Ω_i, $i = 0, 1, \ldots, M$, where $\Omega_0 = S_{Max}$ and according to (11) and (12), the state feedback control matrices are $K = [\kappa_1 \ \kappa_2 \ \ldots \ \kappa_{2^n}]$, with $\kappa_\Lambda \in P(\Lambda)$, $\Lambda = 1, 2, \ldots, 2^n$. According to Theorem 1, the output of BCN (5) can track the signal $y_r = \delta_{2^n}^{\alpha}$ in the shortest time under $u(t) = Kx(t)$. The total number of K is $\prod_{\Lambda=1}^{2^n} |P(\Lambda)|$ and the shortest time is M.

The state feedback controller $u(t) = Kx(t)$ is designed based the sets Ω_i, $i = 0, 1, \ldots, M$, where $\Omega_0 = S_{Max}$, and $\cup_{i=0}^{M} \Omega_i = \Delta_{2^n}$. Since the sets Ω_i, $i = 0, 1, \ldots, M$ are constructed in the greedy criteria, which result in that all the states in Ω_i can be steered to Ω_0 under some control in the shortest time. Additionally, as $\Omega_0 = S_{Max}$ is also constructed in a greedy criteria, which result in that all the states belong to Δ_{2^n} can tracking the signal y_r in the shortest time under the designed controller $u(t) = Kx(t)$.

Remark 2: We proposed a constructive method for obtaining the optimal output tracking controller for BCN (5). The detailed steps can be summarized as:

1) Calculate $\Theta(\delta_{2^n}^{\alpha})$ (7).
2) Find the non-empty maximum invariant set S_{Max} (13).
3) Calculate sets Ω_i (10), where $\Omega_0 = S_{Max}$.
4) Construct the matrices K (11, 12).
5) Calculate the total number of matrices K (Corollary 2).

Remark 3: The major difference in solving the output tracking problem between the method presented in [15] and the method presented in our paper is that the requirement for calculating a set of matrices M_S^k in [15] has been relaxed, and reduced the computational cost is $O((\tau-1)2^{3n})$[1].

[1] In order to obtain the non-empty set S (S_{Max}), one needs to calculate a series of matrices M_S^k, $k = 1, \ldots, \tau$, where M_S^k is rely on a $2^n \times 2^n$ real matrices. Since the computational cost of two $\mathbb{R}^{2^n \times 2^n}$ matrices is $O(2^{3n})$, the total computational cost for calculating the non-empty S (or S_{Max}) is $O((\tau-1)2^{3n})$.

IV. AN EXAMPLE

Consider the following reduced BCN model for the lac operon in [2].

$$\begin{cases} x_1(t+1) = \neg u_1(t) \wedge (x_2(t) \vee x_3(t)), \\ x_2(t+1) = \neg u_1(t) \wedge u_2(t) \wedge x_1(t), \\ x_3(t+1) = \neg u_1(t) \wedge (u_2(t) \vee (u_3(t) \wedge x_1(t))). \end{cases} \tag{14}$$

The output equation is given as follows:

$$\begin{cases} y_1(t) = x_1(t) \vee x_3(t), \\ y_2(t) = x_1(t). \end{cases} \tag{15}$$

The objective in this brief is to design shortest time controller such that the output of BCN (14) can tracking signal $Y_r = (1, 1)$.

Let $y(t) = y_1(t) \ltimes y_2(t)$, $x(t) = x_1(t) \ltimes x_2(t) \ltimes x_3(t)$, and $u(t) = u_1(t) \ltimes u_2(t) \ltimes u_3(t)$, then the algebraic forms of (14), (15) are

$$\begin{cases} x(t+1) = Fx(t)u(t), \\ y(t) = Hx(t), \end{cases} \tag{16}$$

where $H = \delta_4[1\ 1\ 1\ 1\ 2\ 4\ 2\ 4]$, $F = [F_1\ F_2 \ldots F_8]$,

$$\begin{aligned} F_1 &= \delta_8[8\ 8\ 8\ 8\ 1\ 1\ 3\ 4], \\ F_2 &= \delta_8[8\ 8\ 8\ 8\ 1\ 1\ 3\ 4], \\ F_3 &= \delta_8[8\ 8\ 8\ 8\ 1\ 1\ 3\ 4], \\ F_4 &= \delta_8[8\ 8\ 8\ 8\ 5\ 5\ 7\ 8], \\ F_5 &= \delta_8[8\ 8\ 8\ 8\ 3\ 3\ 4\ 4], \\ F_6 &= \delta_8[8\ 8\ 8\ 8\ 3\ 3\ 4\ 4], \\ F_7 &= \delta_8[8\ 8\ 8\ 8\ 3\ 3\ 4\ 4], \\ F_8 &= \delta_8[8\ 8\ 8\ 8\ 7\ 7\ 8\ 8]. \end{aligned} \tag{17}$$

Then, the objective now is to design $u(t) = Kx(t)$, $K \in \mathcal{L}_{8\times 8}$ such that the output of BCN (16) tracks $y_r = \delta_2^1 \ltimes \delta_2^1 = \delta_4^1$.

After a simple calculation, we have $\Theta(y_r) = \{\delta_8^1, \delta_8^2, \delta_8^3, \delta_8^4\}$.

In order to force the output of BCN (16) tracking y_r in the shortest time, we use procedure (13) to find the set S_{Max}:

$$\begin{aligned} &1)\ S_0 = \Theta(\delta_4^1) = \{\delta_8^1, \delta_8^2, \delta_8^3, \delta_8^4\}, \quad S_0^{\circ} = \{\delta_8^4\}, \\ &2)\ S_1 = S_0 \backslash S_0^{\circ} = \{\delta_8^1, \delta_8^2, \delta_8^3\}, \quad S_1^{\circ} = \{\emptyset\} \end{aligned} \tag{18}$$

When $Max = 1$, $S^{\circ}_{Max} = \{\emptyset\}$, and hence we have $S_{Max} = S_1$.

Also, we can use the method presented in [15] to obtain the S_{Max}. According to [15], we have

$$M = [\sum_{i=0}^{2^m} col_i(F_1) \ \ldots \sum_{i=0}^{2^m} col_i(F_{2^n})]$$
$$= \begin{bmatrix} 2 & 2 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 2 & 2 & 2 & 0 \\ 1 & 1 & 1 & 0 & 2 & 2 & 2 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 2 \\ 4 & 4 & 4 & 5 & 4 & 4 & 2 & 6 \end{bmatrix},$$

$$M^2 = \begin{bmatrix} 6 & 6 & 6 & 0 & 4 & 4 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 3 & 3 & 3 & 6 & 2 & 2 & 2 & 4 \\ 3 & 3 & 3 & 6 & 2 & 2 & 2 & 4 \\ 2 & 2 & 2 & 0 & 4 & 4 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 9 & 9 & 9 & 10 & 10 & 10 & 6 & 12 \\ 41 & 41 & 41 & 40 & 42 & 42 & 30 & 40 \end{bmatrix}.$$

According to Theorem 2 in [13], we have that all columns of

$$M^2_{\{\delta_8^1,\delta_8^2,\delta_8^3\}} = \begin{bmatrix} 6 & 6 & 6 & 0 & 4 & 4 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 3 & 3 & 3 & 6 & 2 & 2 & 2 & 4 \end{bmatrix}$$

are nonzero. Thus, the maximal non-empty set S is $S = \{\delta_8^1, \delta_8^2, \delta_8^3\}$, which is the same as S_{Max}. Our method can obtain the S_{Max} without calculating a series of 8×8-dimensional matrices's product.

Based on the set $S_{Max} = \{\delta_8^1, \delta_8^2, \delta_8^3\}$, we have

$$\begin{aligned} &1)\ \Omega_0 = S_{Max} = \{\delta_8^1, \delta_8^2, \delta_8^3\}, \\ &2)\ \Omega_1 = R(\Omega_0)\backslash\Omega_0 = \{\delta_8^5, \delta_8^6, \delta_8^7\}, \\ &3)\ \Omega_2 = R(\Omega_1)\backslash \cup_{i=0}^{1} \Omega_i = \{\delta_8^4, \delta_8^8\}. \end{aligned} \tag{19}$$

When $M = 2$, we have $|\Omega_0|+|\Omega_1|+|\Omega_2| = 8$. According to Lemma 2, we have that the output of BCN (16) tracks $y_r = \delta_4^1$ (The above procedure is briefly displayed in Fig. 1).

According to (11) and (12), we have

$$\begin{aligned} &P(1) = \{\delta_8^5, \delta_8^6, \delta_8^7\}, \quad P(2) = \{\delta_8^5, \delta_8^6, \delta_8^7\}, \\ &P(3) = \{\delta_8^5, \delta_8^6, \delta_8^7\}, \quad P(4) = \{\delta_8^5, \delta_8^6, \delta_8^7\}, \\ &P(5) = \{\delta_8^5, \delta_8^6\}, \quad P(6) = \{\delta_8^5, \delta_8^6\}, \\ &P(7) = \{\delta_8^5, \delta_8^6\}, \quad P(8) = \{\delta_8^5, \delta_8^6\}. \end{aligned} \tag{20}$$

And according to Theorem 1, based on the set S_{Max}, the state feedback control matrices K are $K = [\kappa_1\ \kappa_2\ \ldots\ \kappa_8]$, where $\kappa_\Lambda \in P(\Lambda)$, $\Lambda = 1, 2, \ldots, 8$. Further, based on the set S_{Max}, the total number of matrices K, which make the output tracking successfully in the shortest time, is obtained $\prod_{\Lambda=1}^{8} |P(\Lambda)| = 1296$.

Choosing $\kappa_1 = \delta_8^7$, $\kappa_2 = \delta_8^7$, $\kappa_3 = \delta_8^7$, $\kappa_4 = \delta_8^7$ $\kappa_5 = \delta_8^5$, $\kappa_6 = \delta_8^5$, $\kappa_7 = \delta_8^5$, $\kappa_8 = \delta_8^5$, then $K = \delta_8[7\ 7\ 7\ 7\ 5\ 5\ 5\ 5]$. By following the method presented in Chapter 7.1 in [5], the corresponding logical form for $u(t) = Kx(t)$ is

$$\begin{cases} u_1(t) = 0, \\ u_2(t) = \neg x_1(t), \\ u_3(t) = 1. \end{cases} \tag{21}$$

One can verified that $\Omega_0 = \{\delta_8^3\}$ is also an invariant set of $\mathcal{O}(y_r)$. And Δ_8 can be divided into the following five sets $\Omega_0 = \{\delta_8^3\}$, $\Omega_1 = \{\delta_8^5, \delta_8^6, \delta_8^7\}$, $\Omega_2 = \{\delta_8^4, \delta_8^7, \delta_8^8\}$, $\Omega_3 = \{\delta_8^1\}$, $\Omega_4 = \{\delta_8^2\}$. The designed state feedback matrices K are $K = [\kappa_1\ \kappa_2\ \ldots\ \kappa_8]$, where $\kappa_1 \in \{\delta_8^1, \delta_8^2, \delta_8^3, \delta_8^4, \delta_8^8\}$, $\kappa_2 \in \{\delta_8^5, \delta_8^6\}$, $\kappa_3 = \delta_8^7$, $\kappa_4 \in \{\delta_8^5, \delta_8^6, \delta_8^7\}$, $\kappa_5 \in \{\delta_8^5, \delta_8^6\}$, $\kappa_6 \in \{\delta_8^5, \delta_8^6\}$, $\kappa_7 \in \{\delta_8^5, \delta_8^6\}$, $\kappa_8 \in \{\delta_8^5, \delta_8^6\}$.

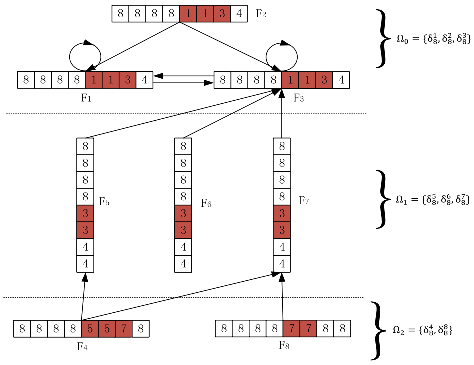

FIGURE 1. The horizontal (vertical) block labelled F_i (i=1,…,8) in this picture is associated with F_i in (17). For example, F_1 in the picture is associated with F_1 in (17). The number 4 in F_1 in the picture corresponds to δ_8^4 in F_1 in (17). In F_1 in (17), there is a column equal to δ_8^3 (which is associated with the shadow part in F_1 in the picture), which means that $x = \delta_8^1$ can be controlled to $x = \delta_8^3$ under some control u. According to (18), we have $\Omega_0 = S_{Max} = \{\delta_8^1, \delta_8^2, \delta_8^3\}$, which means that condition (1) in Lemma 2 is satisfied. Based on (19), Δ_8 was divided into three partitions $\Omega_0, \Omega_1, \Omega_2$, which means that condition (2) in Lemma 2 is satisfied. According to Lemma 2, we conclude that the output of BCN (16) can tracking $y_r = \delta_4^1$.

Choosing $\kappa_1 = \delta_8^1$, $\kappa_2 = \delta_8^5$, $\kappa_3 = \delta_8^7$, $\kappa_4 = \delta_8^7$ $\kappa_5 = \delta_8^5$, $\kappa_6 = \delta_8^5$, $\kappa_7 = \delta_8^5$, $\kappa_8 = \delta_8^5$, then $K = \delta_8[1\ 5\ 7\ 7\ 5\ 5\ 5\ 5]$. And the corresponding logical form for $u(t) = Kx(t)$ is

$$\begin{cases} u_1(t) = x_1 \wedge (x_2 \wedge x_3), \\ u_2(t) = x_1 \wedge x_2 \vee \neg x_1, \\ u_3(t) = 1. \end{cases} \tag{22}$$

One can verify that under controller (22), the system can track the signal y_r in 4 time. While, the shortest time of the system to realize tracking the signal y_r is 2.

Hence, our main results have been well illustrated by the example.

V. CONCLUSION

In this brief, the output tracking control problem of BCNs under constant reference signal has been revised. Our method can be used to calculate the maximum invariant set S_{Max} in a more tractable way. The BCNs can realize the output tracking in the shortest time, and the total number of shortest time state feedback matrices is obtained. Moreover, the computational cost is reduced. Finally, a numerical example is presented to verify our main results.

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